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New Higgs Transitions between Dual N=2 String Models

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We describe a new kind of transition between topologically distinct $N = 2$ type II Calabi–Yau vacua through points with enhanced non-abelian gauge symmetries together with fundamental charged matter hyper multiplets. We connect the appearance of matter to the local geometry of the singularity and discuss the relation between the instanton numbers of the Calabi–Yau manifolds taking part in the transition. In a dual heterotic string theory on $K3 \times T^2$ the process corresponds to Higgsing a semi-classical gauge group or equivalently to a variation of the gauge bundle. In special cases the situation reduces to simple conifold transitions in the Coulomb phase of the non-abelian gauge symmetries.

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1. Introduction

During the last few years there has been a great deal of progress in the understanding of non-perturbative phenomena in supersymmetric field theories as well as in various string theories [1]. In particular the idea of duality has proven to be crucial. The basic point here is that one underlying theory might have several descriptions in terms of physical variables. In the current setting these ideas originated in the work by Seiberg and Witten in the context of $N = 2$ supersymmetric Yang-Mills theory [2]. For all gauge groups the global prepotential can be derived from the periods of a suitable auxiliary Riemann-surface [3]. Combining the properties of $N = 2$ gauge theories with the conjectured duality between $N = 4$ string theories in four dimensions [4] it was suggested that also $N = 2$ string theories in four dimensions have a duality structure; the heterotic string theory compactified on $K3 \times T^2$ and the type II theory compactified on a Calabi-Yau manifold, could be dual to each other after including non-perturbative states [5][6].

In [5] concrete pairs of dual $N = 2$ theories were constructed in which non-perturbative properties of the heterotic string can be investigated exactly. The key idea is the absence of neutral couplings between vector multiplets and hyper multiplets in $N = 2$ theories [7]. As the heterotic dilaton sits in a vector multiplet the vector multiplet moduli space can receive space-time perturbative and non-perturbative corrections. Under $N = 2$ string-string-duality it is identified with the vector multiplet moduli space of the Type IIA string, which corresponds to the Kähler moduli space of the compactified Calabi-Yau manifold. The latter has to have Hodge numbers $h_{11} = N_v$ and $h_{21} = N_h - 1$, where N_v, N_h are the number of vector and hyper multiplets and the -1 corresponds to the the dilaton of the type II string. (Note however, that the rank of the gauge group is $N_v + 1$ as the graviphoton contributes a $U(1)$-factor.) As the Type IIA dilaton is in a hyper multiplet the vector multiplet moduli space is exact at tree level and so is not corrected by space-time instantons. It is, however, corrected by worldsheet instantons. By mirror symmetry we can identify it with the hyper multiplet moduli space of the Type IIB theory on the mirror Calabi-Yau manifold, which receives neither world-sheet nor space-time corrections. The upshot is that the exact non-perturbative vector multiplet moduli space of the heterotic string is modelled by the complex structure moduli space of a specific Calabi-Yau manifold [1]. In this sense the complex moduli space of Calabi-Yau manifold replaces the complex moduli space of the auxiliary Riemann surface, which one had in the $N = 2$ Yang-Mills field theory case.

Following the initial work [5][6] a number of consistency checks have been made further establishing the conjectured type II/heterotic string duality; comparison of the perturbative region of the potentials of the heterotic couplings (gauge and gravitational) with that

[1] Similarly, the structure of the non-perturbative hyper moduli space of the type IIA theory, which is a quaternionic manifold [8] can be investigated via the heterotic string.

of the dual type IIA vacuum [5], [9], as well as certain non-perturbative consistency checks in the point-particle limit [10].

Furthermore, and what will be the focus of this paper, aspects of the perturbative enhancements of the gauge symmetry can be studied by considering the Picard lattice of the generic K_3-fiber [11]. Using the Higgs mechanism as a way of lowering the rank of the gauge group and thus finding a way in which the various moduli spaces can be connected[2][12], chains of such transitions have been studied extensively [5][13] [14].

Here we will focus on a particular chain and investigate in detail the transition from the point of view of the local geometry as well as in terms of the the worldsheet instanton sums in the type IIA theory. In particular, we will not only be able to identify the enhanced perturbative gauge symmetry but also the matter representations. The geometrical transition is of a slightly generalized type compared to the conifold transitions on one hand and the strong coupling transitions on the other hand. In addition to having N-1 divisors being contracted to a singular curve C, of genus zero, giving rise to an SU(N) theory with no adjoint hyper multiplets [15], there are singular fibers, which when contracted to C give rise to massless hyper multiplets in the fundamental representation. The general structure of these transitions is discussed in section 2. We then turn to a specific set of models in section 3. This chain of Calabi-Yau manifolds can be viewed as K_3 fibrations over $\mathbf{P}^1$ as well as elliptic fibrations over the Hirzebruch surface F_2. Finally, we end with conclusions and discussions in section 4.

2. Structure of the extremal transitions

2.1. Physical spectrum and flat directions

It will be useful to consider the purely field theoretic problem of which massless particles will appear in the moduli space. At a generic point in the moduli space of vacuum expectation values of the scalar component of the vector multiplet, the gauge symmetry is $U(1)^{N-1}$. By a suitable gauge transformation we can then write the above scalar in the diagonal form $diag(\phi_1, ..., \phi_N)$, where $\sum_{i=1}^{N} \phi_i = 0$ and $\phi_i \neq \phi_j$, for all i, j. Clearly, all the matter fields are massive with masses proportional to ϕ_i. If we now set $\phi_i = 0$, $i = 1, ..., N-1$, the gauge symmetry is enhanced to $SU(N)$ and only matter fields charged with respect to the $SU(N)$ become massless. Let us for simplicity assume that there are M fields which all transform in the fundamental representation. Because of the tracelessness condition there exists a surface for which one can have $SU(N-1) \times U(1)$,

[2] This so called Higgs-branch has to be distinguished from the Higgs breaking mechanism into the Coulomb-branch by vector multiplets, in which the gauge bosons become massive as short vector multiplets under spontaneous generation of central charge, as in the Seiberg-Witten theory.

with $\phi_i = \phi_j \neq 0$, $i, j = 1, ..., N-1$ but without massless charged matter. If we reduce the symmetry enhancement one step further there is however room for massless matter in the fundamental of $SU(N-2)$; e.g. choose $\phi_i = 0$, $i = 1, ..., N-2$ and $\phi_{N-1} = -\phi_N \neq 0$. In general one will have several $SU(k_j)$; however, only the one for which $\phi_i = 0, i = 1, ..., k_j$ will have M massless matter multiplets. In particular, there exists a codimension one surface in which only e.g. $\phi_1 = 0$, such that there is no gauge enhancement. However, because of $\phi_1 = 0$ we will have M massless singlets. It is very gratifying that, as will now be seen, the Calabi-Yau moduli space exactly reproduces this kind of behaviour.

2.2. Local geometry of the Calabi–Yau singularity

During the recent developments it has become clear that the physical singularities associated to massless solitonic BPS states are essentially encoded in the geometry of the singularity of the compactified manifold. The role played by the geometry can be understood from the interpretation of the massless states as solitonic p-branes wrapped around the vanishing cycles of the singularity; the gauge and Lorentz quantum numbers depend then on some characteristic properties of the homology cycles, in particular their dimension, topology and intersection numbers [15].

The simplest case (in the type IIB picture) is that of a vanishing three-cycle leading to a massless hyper multiplet, the case considered originally by Strominger [16]. On the other hand, if the three-cycle shrinks to a curve, rather than to a point, one obtains enhanced gauge symmetries, as has been argued in [17]. More precisely, if the local geometry is that of an ALE space with A_N singularity over a curve of genus g one obtains an enhanced $SU(N+1)$ gauge symmetry [18] together with g hyper multiplets in the adjoint representation [15]. The case $g = 0$ is exceptional in that the enhanced gauge symmetry is asymptotically free and broken to its abelian factor due to strong coupling effects in the infrared; this case has been considered in [10] [19].

Let us now assume that we have a collection of curves C_i with the transverse space that of an ALE-manifold with A_{N_i} type singularity and consider further a point of intersection between two of these curves [20]. The singularity structure of the transverse space has been analyzed in detail in [21] in the context of elliptic fibrations with the following result: if along the curves C_1 and C_2 the elliptic fiber is of Kodaira type I_{N_1} and I_{N_2} respectively, then above the point where C_1 intersects C_2 the elliptic fiber is of type type $I_{N_1+N_2}$. Indeed the examples we will consider are all ellipticly fibered Calabi–Yau manifolds. This allows in particular for a simple interpretation in terms of 5-branes [17] [15]located at the points where the fibration becomes singular. However this special structure is not necessary; the general configuration is that of a collection of curves C_i with transverse A_{N_i} singularities colliding in a set of M points over which the singularity structure jumps to $A_{N_i+N_j}$.

A simple D-brane arrangement based on a collection of N_i coinciding D-branes intersecting a second collection of N_j coinciding D-branes has been given in [17]. In this picture additional matter in the fundamental representation arises from open string states with one end attached to the first and the other end to the second collection. There is a special configuration in which one of the two collections consists of a single D-brane only, say $N_j = 1$. In this case one expects a single non-abelian factor of $SU(N_i)$ together with matter in the fundamental representation.

This is the physical situation whose realization we will consider in the context of Calabi–Yau compactifications. Specifically the case of $SU(N+1)$ gauge symmetry with M fundamental matter multiplets arises from the following local data: there is a curve C, which in our case will be a P^1, over which one has a bundle structure where the generic fiber is a Hirzebruch-Jung tree of the resolution of the A_N singularity, that is a collection of P^1's, E_i, $i = 1..N$ with intersection matrix proportional to the Cartan matrix of A_N. In addition, above the M exceptional points, the singularity structure of the fiber becomes A_{N+1} because one component of the fiber factorizes. More precisely out of the N generic components $N-1$ are toric divisors D_i of the manifold X_i which are ruled surfaces, while the N-th component, $\hat{E}$ is only birationally ruled, having M degenerate fibers. The last component $\hat{E}$ is actually a *conic bundle*, which means that the fibers are all plane conics [22]. These conics are smooth over a generic points of C while they split into line pairs over the M exceptional points.

Let us describe now how the appearance of this structure will lead to geometrical transitions between two Calabi–Yau manifolds X_i and X_{i-1} (where i denotes the number of $N_V = h_{11}$ of vector multiplets). First we can contract the $N-1$ divisors together with the M degenerate fibers of the conic bundle. According to our previous discussion, the degenerate fibers contain a collection of $N+1$ rational curves E_i with intersection matrix of A_{N+1} which are all contracted. N of them are again associated to the Cartan subalgebra of the gauge group while the additional one is related to the matter hyper multiplets; there is a natural action of the A_N Weyl group on this class which generates the components of the fundamental representation. The gauge quantum numbers of the solitonic p-brane states are determined by the reduction of the Cartan matrix of A_{N+1} to that of A_N; the components of the matter fields arise naturally from wrappings of the cycles $E_{N+1} - E_i$, $i = 0, \ldots, N$; however there is no independent modulus associated to the volume of E_{N+1} and correspondingly no additional vector multiplet[3]. This is expected from the fact that the additional rational curves are isolated rather than being fibers of a ruled surface. After the N surfaces have been contracted to the base the singularities can be simplified by a deformation of the complex structure in such a way that the resulting singularity is the contraction of $N-1$ surfaces arising from X_{i-1}. This completes the transition from X_i to X_{i-1} at the enhanced symmetry points.

[3] Here E_0 is the homology cycle which fulfills the relation $\sum_0^N E_i \sim 0$, reflecting the tracelessness condition of $SU(N+1)$, as discussed in the previous section.

Clearly, in the generic situation one will only contract subsets S_i of the $N-1$ divisors and/or the conic bundle with the M degenerate fibers. The result will depend on whether a given subset S_i contains $\hat{E}$; if it does we will get from that factor an enhanced gauge symmetry $SU(k_i+1)$ together with M fundamental representations; however if it does not, the result is that of a non-abelian gauge symmetry without matter, broken down to the Cartan subalgebra by strong infrared dynamics . In fact if $N>1$ we can contract *only* $\hat{E}$ and the result is M representations of $U(1)$, that is we are back to the familiar case of the conifold singularity.

2.3. String moduli space

We turn now to a discussion of the string moduli spaces involved in the transition. The latter is described by a motion in the vector multiplet moduli M space of the Calabi–Yau manifold X_i to a locus where the vev of the scalar superpartner of a vector field vanishes and then turning on vevs in the new flat directions of the Higgs branch corresponding to a motion in the hyper multiplet moduli space of the Calabi–Yau manifold X_{i-1}; the new Calabi-Yau manifold X_{i-1} will therefore have fewer vector moduli while the number of hyper moduli has increased; the associated change in the Hodge numbers h_{11} and h_{12} indicates that the two manifolds are of different topological type. There are two types of natural coordinates on $\mathcal{M}$, the algebraic coordinates z_n and the special coordinates t_n [23], where the two are related by the mirrormaps $z_m(t_n)$. We are interested in the relations between these two types of coordinates on $\mathcal{M}^i$ and $\mathcal{M}^{i-1}$.

$N=2$ supersymmetry puts strong restrictions on these relations; in particular the special geometry of the vector multiplets in the type IIA compactification constrains the map between the two set of coordinates t_n^i and t_n^{i-1} to be linear. Moreover we will find simple relations between the Gromov–Witten invariants $n_{i_1,\ldots,i_{h_{11}}}$ on $\mathcal{M}^i$ and $\mathcal{M}^{i-1}$, which are defined in terms of the instanton corrected Yukawa couplings y_{abc} as

$$y_{abc}=y^0_{abc}+\sum_{d_1,\ldots,d_{h_{11}}}\frac{n^r_{d_1,\ldots,d_{h_{11}}}d_ad_bd_c}{1-\prod_{n=1}^{h_{11}}q_n^{d_n}}\prod_{n=1}^{h_{11}}q_n^{d_n}$$

where the $n^r_{d_1,\ldots,d_{h_{11}}}$ is the virtual fundamental class of the moduli space of rational curves of multidegree $d_1,\ldots,d_{h_{11}}$. Such a relation between the instanton numbers are of course a special property of the type of singularity we consider and will not be present in other types of transitions proceeding e.g. through non-canonical singularities.

In a way similar to that the special geometry of the vector moduli space restricts the relations between the coordinates on $\mathcal{M}_V^i$ and $\mathcal{M}_V^{i-1}$, the quaternionic structure of the hyper moduli space of the type IIB theory compactified on the same pair of manifold implies simple relations between the coordinates on the hyper multiplet moduli spaces, ξ_n. They are related to the algebraic moduli z_n by rational functions which in turn depend on the special representation of the Calabi–Yau manifolds. In particular there are in general different reflexive polyhedra describing the same Calabi–Yau space, however in different algebraic coordinates related by rational transformations. It is convenient to choose a preferred representation in which the relation between the algebraic moduli on $\mathcal{M}^i$ and $\mathcal{M}^{i-1}$ becomes particularly simple:

$$z_n^{i-1} = \prod_m (z_m^i)^{\delta_m^n}$$

From the definition of the algebraic coordinates this relation translates to linear relations between the Mori vectors l_n^i, l_n^{i-1}, generating the dual of the Kähler cone. As a consequence the smaller dual polyhedron $\Delta^\star_{i-1}$ is obtained from the larger one $\Delta^\star_i$ by omitting one of the vertices[4].

A new feature of the corresponding transition in the moduli space is that, rather than beeing located at the zero locus of the principal discriminant or a restricted discriminant, it can take place at one of the boundary divisors $z_n = 0$. Such transitions where first discussed in [24]; indeed this is the situation in the examples described in section 3 [5]. Similar examples have been considered independently [26].

In the case when the Calabi–Yau is a K3 fibration [27], the vector moduli space of the pair of Calabi–Yau manifolds can be mapped to that of the dual heterotic theory, the map again being linear by the above mentioned argument. Of course this relation will depend on the special compactification under consideration and will be discussed further in the examples.

3. Global description of the Calabi-Yau spaces and examples

Given a four-dimensional N=2 heterotic vacuum it is far from trivial to find the dual type IIA model. Although there has been some progress recently as far as understanding the perturbative gauge structure on the type IIA side [11][28][25] there is still some guess work to be done in terms of finding a Calabi-Yau manifold that will fit the bill. In fact, recent work [29] indicates that there will in general exist more than one type IIA theory which agrees with the perturbative heterotic vacuum under consideration. From now on, we will assume that there is a set of models which to lowest order corresponds to their counterparts in the heterotic chain, i.e. the number of vector multiplets and singlet hyper multiplets agree, and the models are all K3-fibrations.

[4] This kind of transition was originally studied in [24] and more recently in [25].

[5] A model with similar properties has been observed in [24].

3.1. A chain of connected Calabi–Yau manifolds

We will here consider an example first discussed by Aldazabal et. al. [13]. Let us first briefly review the structure of the chain of heterotic vacua with the least number of vector multiplets, as it is the most suitable for the consideration in this paper; for more details see [13]. Starting from a Z_6 orbifold with a specific embedding in $E_8 \times E_8$ it is possible to Higgs away most of the gauge group, leaving just $SU(5) \times U(1)^4$ [6]. The spectrum consists of 4 **10**, 22 **5** and 118 **1** hyper multiplets. In fact one can further Higgs the $SU(5)$, giving a set of models with the following spectra. (In what follows we will refer to the models by the number of vector multiplets and the number of singlet hyper multiplets.)

(N_v+1, N_h)	gauge group	spectrum	Calabi – Yau
$(8,118)$	$SU(5)\times U(1)^4$	4 **10**+22 **5**+118 **1**	$(\mathbf{P}^5(1,1,2,5,7,9)[14,11])^{7,117}_{-220}$
$(7,139)$	$SU(4)\times U(1)^4$	4 **6**+24 **4**+139 **1**	$(\mathbf{P}^5(1,1,2,6,8,10)[16,12])^{6,138}_{-264}$
$(6,162)$	$SU(3)\times U(1)^4$	30 **3**+162 **1**	$(\mathbf{P}^4(1,1,2,6,8)[18])^{5,161}_{-312}$
$(5,191)$	$SU(2)\times U(1)^4$	28 **2**+191 **1**	$(\mathbf{P}^4(1,1,2,6,10)[20])^{4,190}_{-372}$
$(4,244)$	$U(1)^4$	244 **1**	$(\mathbf{P}^4(1,1,2,8,12)[24])^{3,243}_{-480}$

Table 3.1: Chain of heterotic - type IIA duals

We have analyzed the matter structure and transitions for most of these models. Rather than give this analysis for the hypersurfaces and complete intersections in weighted projective spaces here, we will find it more convenient to report on the transition using Batyrev's construction of Calabi-Yau mirror pairs (X_n, X_n^*) as hypersurfaces[7] in more general toric varieties $(P_{\Delta_n}, P_{\Delta_n^*})$ [31]. An exhaustive list of polyhedra (Δ_n, Δ_n^*) for the models in [13] can be found in [25]. In general there will be several different candidates for (X_n, X_n^*), differing presumably only by birational maps.

3.2. Phases of the Kähler moduli space

Given a singular ambient space P_Δ, we have in general many phases in the associated extended Kähler moduli space of the nonsingular space $\hat{P}_\Delta$. They correspond to the different ways to resolve P_Δ and are defined by the different regular triangulations of

[6] An alternate way of constructing this chain of models starts from the asymmetric instanton embedding $(10,14)$ which breaks $E_8 \times E_8$ to $E_7 \times E_7$, completely Higgsing one of the E_7 and the second E_7 to $SU(5)$[30].

[7] We will indicate the number of vector multiplets of the models by the subscript n.

the polytope Δ^*. If (Δ, Δ^*) are reflexive there is a canonical way[8] to embed a Calabi-hypersurface (X, X^*) in $(\hat{P}_\Delta, \hat{P}_{\Delta^*})$ [31]. Among the phases of $\hat{P}_\Delta$ are the ones that give rise to Calabi-Yau varieties, when the Kähler classes are restricted to X. They correspond to triangulations[9] involving all points of Δ^* on dimension $0, 1, 2$-faces. Even restricting to phases which correspond to manifolds which are K3-fibrations does not in general narrow down the choice to a unique model. Furthermore depending on the particular situation at hand, it may be the case that there exists more than one type IIA vacuum to a given heterotic theory. Finally, there is a technical problem in finding the true Calabi-Yau phases. Frequently the Kähler cones of $\hat{P}_\Delta$ are narrower than the Kähler cone of X, because the former are bounded by curves in $\hat{P}_\Delta$ which vanish on X. We describe in Appendix A how to deal with this situation.

3.3. The vector moduli space of our examples

We will now use mirror symmetry and toric geometry to investigate the vector moduli space for the models in the chain described in the previous section. Candidates of type II models were constructed as chains of nested polyhedra $\Delta_3^* \subset \ldots \subset \Delta_{10}^*$ $(\Delta_3 \supset \ldots \supset \Delta_{10})$ in [25]. For simplicity, we now turn to studying the extremal transitions connecting the three models with the fewest number of vector multiplets discussed above. By extremal we refer to a general transition obtained by contracting curves corresponding to edges of the Mori cone, and then deforming the resulting singular Calabi-Yau threefold to get a smooth Calabi-Yau manifold. As will be shown this transition is not necessarily of the simple conifold type [34].

We first have to calculate one valid Mori cone for each of the models[10]. Let us therefore start by considering X_5; as we will show, X_4 and X_3 can then be obtained by taking a particular limit in the Kähler moduli space of X_5. Inside the polyhedron Δ_5^* one has the following relevant points (inside dimension $0, 1, 2, 4$-faces) [25]

$$\begin{aligned} &\nu^0 = (0,0,0,0), \nu^1 = (-1,0,2,3), \nu^2 = (0,0,-1,0), \nu^3 = (0,0,0,-1), \nu^4 = (0,0,2,3), \\ &\nu^5 = (0,1,2,3), \nu_6 = (1,2,2,3), \nu^7 = (0,-1,2,3), \nu^8 = (0,-1,1,2), \nu^9 = (0,-1,1,1). \end{aligned} \tag{3.1}$$

The Mori cone can be found by repeated application of the procedure described in Appendix A. This leads to the following set of Mori generators

$$\begin{aligned} &l^{(1)} = (-1; 0,0,0,1,0,0,-2,1,1), l^{(2)} = (0; 1,0,0,0,-2,1,0,0,0), \\ &l^{(3)} = (0; 0,0,0,-2,1,0,1,0,0), l^{(4)} = (-1; 0,1,0,0,0,0,1,-2,1), \\ &l^{(5)} = (-1; 0,0,1,0,0,0,0,1,-1) \end{aligned} \tag{3.2}$$

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[8] Even if we fix the triangulation of Δ^* this does not fix (X_n, X_n^*) uniquely. A simple counterexample with a non-toric phase is the $X_9(3,2,2,1,1)^{2,86}$ case discussed in [24][32] [29].

[9] We have used PUNTOS [33] to find the triangulations

[10] We content ourselves with those phases arising from the triangulations of Δ^*.

The mirror manifold of X_5 is given by the Laurent polynomial [31]

$$P = \sum_{i=0}^{9} \prod_{j=1}^{4} a_i X_i^{\nu_j^i} = 0 \tag{3.3}$$

in $P_{\Delta_5^*}$. A crucial insight [31],[35][36], that the large complex structure variables of the mirror X_5^* are defined by the Mori cone of X, specifically for X_5 above:

$$w_i = (-1)^l \prod_{j=1}^{9} a_j^{l^{(i)}}. \tag{3.4}$$

By the construction of [25] the models X_4, X_3 are given as hypersurfaces in toric varieties whose dual polytopes, $\Delta^*4, 3$ respectively are obtained by deleting the point ν_9 or points (ν_8, ν_9) from Δ_5^*; this corresponds to the restriction of the moduli space of X_5 to $a_9 = 0$ and $a_9 = a_8 = 0$ respectively.

	X_3	X_4	X_5
P	$a_8 = a_9 = 0$	$a_8 = 0$	
Δ^*	$\mathrm{conv}(\nu_1, \ldots, \nu_7)$	$\mathrm{conv}(\nu_1, \ldots, \nu_8)$	$\mathrm{conv}(\nu_1, \ldots, \nu_9)$
$h^{1,1}$	3(0)	4(0)	5(0)
$h^{2,1}$	243(1)	190(1)	161(1)
z_1	$w_1 w_4^2 w_5^3$	$w_1 w_5$	w_1
z_2	w_2	w_2	w_2
z_3	w_3	w_3	w_3
z_4	–	$w_4 w_5$	w_4
z_5	–	–	w_5

Table 3.2: The Calabi-Yau manifolds which correspond to reflexive polyhedra inside Δ_5^*. The polyhedra are specified as convex hulls of the points given in (3.1). Furthermore, we list the number of Kähler $h^{1,1}$ and complex structure deformations $h^{2,1}$ of X_i (the number of non-algebraic deformations is indicated in parentheses) as well as the vanishing coefficients in the Laurent polynomial (3.3) and the canonical large complex structure coordinates z_k of X_i^*, which are related to the Mori cones by (3.4).

Using toric geometry one can calculate the classical intersections corresponding to the Kähler classes, J_i, which are dual to the Mori generators (3.2),

$$\begin{aligned}
&8J_1^3 + 2J_1^2 J_2 + 4J_1^2 J_3 + J_1 J_2 J_3 + 2J_1 J_3^2 + \\
&16J_1^2 J_4 + 4J_1 J_2 J_4 + 8J_1 J_3 J_4 + 2J_2 J_3 J_4 + 4J_3^2 J_4 + 24J_1 J_4^2 + 6J_2 J_4^2 + 12J_3 J_4^2 + 36J_4^3 + \\
&48J_5^2 J_1 + 24J_5 J_1^2 + 36J_5 J_4 J_1 + 6J_5 J_2 J_1 + 12J_5 J_3 J_1 + 6J_5 J_3^2 + 18J_5 J_4 J_3 + \\
&3J_5 J_3 J_2 + 9J_5 J_4 J_2 + 12J_5^2 J_2 + 54J_5 J_4^2 + 24J_5^2 J_3 + 72J_5^2 J_4 + 96J_5^3
\end{aligned} \tag{3.5}$$

as well as the evaluation of the Chern class on the $(1,1)$ forms J_i,

$$c_2(J_1)=92,\quad c_2(J_2)=24,\quad c_2(J_3)=48,\quad c_2(J_4)=132,\quad c_2(J_5)=168 \tag{3.6}$$

Obviously the intersection numbers of X_4 and X_3 are simply given from these by the restriction to the first four respectively three Kähler classes. For further studying the transition we also need the Gromov-Witten invariant for the rational curves. These are obtained from the solutions of the Picard-Fuchs equations using the mirror hypothesis and listed in Appendix B.

3.4. Local geometry and summation of the instanton corrections

Let us begin with X_3, and identify the associated toric variety P_3. [11][12] A partial list on the number of rational curves of low degree, including those of importance for studying the transitions described in this paper, can be found in table A.1.

We will now show that this model is primitive, i.e. it does not admit a geometric transition to a model with fewer Kähler parameters. Let us therefore discuss the edges of the Mori cone one at a time, to determine whether their contraction admits a birational smooth deformation. We start by studying the first edge of the Mori cone. This edge describes curves contained in an elliptic fibration over a surface, so the contraction of these curves is not a birational map (note that $n_{1,0,0}=480$, the negative of the Euler characteristic, as explained in [24]).

For our purposes, it is best to think of F_2 as a complete toric variety with edges $(-1,-2),(1,0),(0,1),(0,-1)$. Its Mori cone is given by $(-2,0,1,0,1),(0,1,-2,1,0)$. As such, it can be thought of as $\mathbb{C}^4-(\{x_1=x_6=0\}\cup\{x_5=x_7=0\})$ (in terms of the x_i of P_3), modulo the $(\mathbb{C}^*)^2$ identification

$$(x_1,x_5,x_6,x_7)\sim(tx_1,st^{-2}x_5,tx_6,sx_7). \tag{3.7}$$

The hyperplane class of F_2 is the toric divisor $x_7=0$, and will be denoted by H. We can also think of F_2 as the minimal desingularization of $\mathbf{P}(1,1,2)$, so it makes sense to talk of the degree of a curve on F_2. Note that a curve in the class dH has degree $2d$. In passing, we note that the exceptional divisor of this blowup, $x_5=0$, is the section of self-intersection -2.

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[11] This model has been studied earlier [37][11].

[12] In this section, we will frequently perform intersection calculations in the Calabi-Yau manifolds X_k. These calculations are often inferred from the Mori cone; at times they may also be performed by Schubert [38].

Returning to the first edge of the Mori cone, we note that the curve is contracted by D_1, D_5, D_6, D_7.[13] The relations $D_1 \cdot D_6 = 0$ and $D_5 \cdot D_7 = 0$, together with the $\mathbb{C}^*$ actions defining the toric variety, show that there is a map $X_3 \to F_2$. The fibers are elliptic curves with typical equation $x_3^2 + x_2^3 + x_2 f_{16} + f_{24} = 0$, where the f_i have degree i in the variables x_1, x_6, x_7 of $\mathbf{P}(1,1,2)$, where x_7 is the variable of degree 2.

The divisor D_5 describes a ruled surface over an elliptic curve. This is the second edge of the Mori cone. In this situation, the Gromov-Witten invariant is $2g - 2 = 0$ [39]. There is no extremal transition in this case, although there is an $SU(2)$ gauge symmetry that is broken after a non-polynomial deformation [15][14].

Now we turn our attention to the divisor D_4. The $K3$ fibration defined by (x_1, x_6) restricts to D_4 to describe D_4 as a ruled surface over a genus 0 curve; the Gromov-Witten invariant is $n_{0,0,1} = 2g - 2 = -2$, and there is no transition.

In summary, the Mori cone of X_3 coincides with that of P_3, and this model is primitive, i.e. does not admit a geometric transition to a model with fewer Kähler parameters.[15] This checks against the heterotic side, where the model with $(n_H, n_V) = (244, 4)$ is at the bottom of the chain.

We next turn to the 4 parameter model X_4. Some of the instanton numbers for this model appear in table A.2. We will see that the fourth edge of the Mori cone is represented by a conic bundle containing 28 line pairs, and that after contraction, there is a transition to X_3. From this transition, the Mori cone of X_3 is the quotient of the Mori cone of X_4 after modding out by the edge $(0,0,0,1)$. It remains to match up the edges from the above geometry. The edges $(0,1,0,0)$ and $(0,1,0)$ correspond to the ruled surface over the elliptic curve, so are to be identified. The elliptic fibration identifies $(1,0,0,2)$ with $(1,0,0)$; or equivalently identifies $(1,0,0,0)$ with $(1,0,0)$ due to the quotient. Finally, the remaining edges $(0,0,1,0)$ and $(0,0,1)$ are identified as ruled surfaces over rational curves. From this, we infer the relation

$$n_{a,b,c} = \sum_k n_{a,b,c,k} \tag{3.8}$$

which checks against the instanton numbers that we have provided, for example $n_{1,0,0} = -2 + 56 + 372 + 56 + -2 = 480$.

[13] Throughout our discussion, we will denote by D_k the restriction to X_i of the toric divisor with equation $x_k = 0$, when the model under discussion is clear from context.

[14] In [40][41] it was shown that turning on the non-polynomial deformation connects the moduli space for X_3 defined as an elliptic fibration over F_2 (our case) with that of an elliptic fibration over F_0.

[15] Recent relevant geometric results about primitive Calabi-Yau threefolds have been given in [42]

As we saw for X_3, the model X_4 also admits a map $\pi : X_4 \to F_2$ defined by (x_1, x_5, x_6, x_7x_8). The fibers have type $(1,0,0,2)$ and are again elliptic. To calculate this type, note that since $x_1 = x_5 = 0$ defines a point of F_2, the same equation defines an elliptic fiber of X_4. We accordingly calculate the intersection numbers $D_1 \cdot D_5 \cdot D_k$ for $1 \le k \le 8$, obtaining the Mori vector $(-6; 0, 2, 3, 1, 0, 0, 0, 0)$, where the -6 arises because the coordinates must sum to 0. In our basis for the Mori cone given in (A.5), this is just $(1,0,0,2)$. Throughout this section, other classes have been computed in this manner; the classes will be given without further comment. A curve C of type $(0,0,0,1)$ is also contracted by π. Since $C \cdot D_8 = -1$, we see that C is necessarily contained in D_8. Since $C \cdot D_7 = 1$, there is a unique fiber of D_7, a curve of type $(1,0,0,0)$ which meets C. This curve is also contracted by π. The fiber of π containing both of these curves contains a third component of type $(0,0,0,1)$; thus there are two curves of type $(0,0,0,1)$ in the same fiber.

Because of $C \subset D_8$, we restrict attention to D_8, which admits a map to $\mathbf{P}^1$ by restricting the $K3$ fibration defined by (x_1, x_6). After restricting to D_8 (hence putting $x_8 = 0$), the equation of X_4 becomes $x_2^2 f_8 + x_3^2 + x_7^2 f_{16} + x_2x_3f_4 + x_2x_7f_{12} + x_3x_7f_8 = 0$ where the f_i have degree i in the variables x_1, x_6 of $\mathbf{P}^1$. We interpret the above as a family of conics in the $\mathbf{P}^2$ with coordinates (x_2, x_3, x_7), with $(x_1, x_6) \in \mathbf{P}^1$ as a parameter. The discriminant of this family has degree 24. Thus the general fiber of D_8 is a smooth conic of type $(0,0,0,2)$, while there are 28 fibers where the conic splits into line pairs, each of type $(0,0,0,1)$. As a check, note that $n_{0,0,0,2} = -2$ and $n_{0,0,0,1} = 56$.

The transition to X_3 is found by writing down monomials in the variables of P_4 which have intersection number 0 with $(0,0,0,1)$. Choosing them in the order $(x_1, x_2x_8, x_3x_8, x_4, x_5, x_6, x_7x_8)$, we see that the assignment

$$(x_1, \ldots, x_8) \mapsto (x_1, x_2x_8, x_3x_8, x_4, x_5, x_6, x_7x_8)$$

defines a mapping from X_4 to P_3 which contracts the conic bundle, and takes X_4 to a singular form of X_3. The transition is produced simply by deforming the equation of X_3.

As a final comment on this model, we observe that the class of the elliptic fiber is $(1,0,0,2)$. We have seen that the fiber of D_7 is of type $(1,0,0,0)$, while the fiber of D_8 is of type $(0,0,0,2)$. The intersection of D_7 and D_8 meets either fiber in two points. Thus $D_7 \cup D_8$ is a fibration over $\mathbf{P}^1$ whose general fiber is a union of two $\mathbf{P}^1$s intersecting in two points, while there are 28 special fibers which form triangles of curves. By itself, D_8 is contracted to get X_3, and this is the case $N = 2$, $M = 28$ of the geometry described in Section 2.2. We accordingly expect to see 28 **2** hyper multiplets becoming massless at the transition, and that is in perfect agreement with Table 3.1.

Finally, we turn to the 5 parameter model X_5. A partial list on low degree instanton numbers, including those of importance for studying the transitions described in this paper can be found in table A.3.

The key to understanding this transition is the contraction of the curves of type $(0,0,0,0,1)$. From table A.3 we have $n_{0,0,0,0,1} = 30$. This class has Mori vector $(-1,0,0,1,0,0,0,0,1,-1)$, so we see that this curve is contained in $D_9 = 0$ and is contracted by the divisors $D_1, D_2, D_4, D_5, D_6, D_7$. We also note from the first entry of the Mori generators that the equation of X_5 is in the class $J_1 + J_4 + J_5$, where the J_k are the dual generators of the Kähler cone. We accordingly write the equation of X_5 in the form

$$x_3 f + x_8 g, \tag{3.9}$$

where f is a polynomial with cohomology class $J_1 + J_4$ and g is a polynomial with cohomology class $3J_4$. A curve is contracted by the divisors listed above if and only if $f = g = x_9 = 0$. We calculate $D_9 \cdot (J_1 + J_4) \cdot 3J_4 = 30$. Thus $n_{0,0,0,0,1} = 30$. The transition to X_4 is now visible—X_4 is obtained from X_5 by the map $(x_1 \ldots, x_9) \mapsto (x_1, x_2, x_3x_9, x_4, x_5, x_6, x_7, x_8x_9)$.

The Mori cone of X_4 is thus the quotient of the Mori cone of X_5 by the vector $(0,0,0,0,1)$. We now match up the other edges. The edge $(0,1,0,0,0)$ is the fiber of a ruled surface over an elliptic curve, so corresponds to $(0,1,0,0)$. There is again an elliptic fibration over $\mathbf{P}(1,1,2)$; we calculate that the fiber has class $(1,0,0,2,3)$, which is equivalent to $(1,0,0,2,0)$ under the quotient. This must match with the elliptic class $(1,0,0,2)$ of X_4. This implies that $(0,0,0,1,0)$ corresponds to $(0,0,0,1)$, and $(1,0,0,0,0)$ corresponds to $(1,0,0,0)$. The remaining edges $(0,0,1,0,0)$ and $(0,0,1,0)$ are therefore related; they are fibers of ruled surfaces over rational curves.

This gives the formula

$$n_{a,b,c,d} = \sum_k n_{a,c,b,d,k} \tag{3.10}$$

which checks against the instanton numbers that we have provided, for example $n_{1,0,0,0} = -2 + 30 + 30 - 2 = 56$.

As a final comment on this model, we observe that the class of the elliptic fiber is $(1,0,0,2,3)$. We have seen that the fiber of D_7 is of type $(1,0,0,0,0)$, while the fiber of D_8 is of type $(0,0,0,1,0)$. We can also calculate that the fiber of D_9 has class $(0,0,0,1,3)$. The curves of type $(0,0,0,0,1)$ are contained in degenerate fibers of D_9; unlike the X_4 situation, the other component of this fiber is of a different type $(0,0,0,1,2)$. The $(0,0,0,0,1)$ curve meets D_8 but not D_7, while the $(0,0,0,1,2)$ curve meets D_7 but not D_8. Thus $D_7 \cup D_8 \cup D_9$ is a degenerate elliptic fibration over $\mathbf{P}^1$ whose general fiber is a triangle. There are 30 special fibers where there is an extra component, and the fiber is a square of curves. Together, $D_8 \cup D_9$ contract to give X_3 (note that if we contract $(0,0,0,0,1)$ together with D_8 to get X_3, then we are contracting $(0,0,0,1,0)$, hence the entire fibration D_9 with class $(0,0,0,1,0) + 3(0,0,0,0,1)$). This is the case $N = 3,\ M = 30$ of the geometry described in Section 2.2. We accordingly expect to see 30 **3** hyper multiplets becoming massless at the transition, and that is in perfect agreement with Table 3.1.

We can also explicitly see for the X_5 to X_4 transition how the sets of three curves change to sets of two curves as we go to X_4. After contracting $(0,0,0,0,1)$ to get to X_4, the two curves $(0,0,0,1,0)$ and $(0,0,0,1,2)$ pair up to become 30 pairs of $(0,0,0,1)$ curves. In fact, it can be shown that the curve $(0,0,0,1,2)$ is the "partner" of $(0,0,0,0,1)$ under the conic bundle. As the conic bundle gets smoothed out by the deformation process, there are fewer lines pairs left.

We expect similar phenomena to arise for X_6 and X_7. For example, using the $\mathbf{P}^5(1,1,2,6,8,10)[16,12]$ model for X_6, we have checked that the transition to the $\mathbf{P}^4(1,1,2,6,8)[18]$ model for X_5 occurs by projection on the first 5 coordinates, and there are $M = 24$ exceptional curves lying on a birationally ruled surface.

3.5. Physical interpretation of the examples

Let us now try to understand the above described transitions in a physical context. We start with the X_4 model. At the codimension one surface where the conic bundle is contracted we have a singular $\mathbf{P}^1$ of type A_1 with 28 double points of type A_2. As we will now argue, this corresponds to an infrared free theory $SU(2)$ gauge theory with 28 $\mathbf{2}$. First, it has been shown in [18] [15] that a $\mathbf{P}^1$ bundle over a curve with singularity type A_1 gives an enhanced $SU(2)$ gauge symmetry in the type IIA string theory. If the base curve of the family is rational as in our case, it is also shown that there is no new matter [15]. What we find here is that the contraction of the isolated curves, corresponding to solitonic 2-branes wrapped around the curves becoming massless, gives rise to non-abelian charged matter. The non-abelian charges arise from the fact that these isolated curves originate from the same conic bundle as does the continuous family of rational curves leading to the non-abelian gauge bosons. This is the non-abelian generalization of the conifold singularity. As the $SU(2)$ is Higgsed, the rank of the gauge group is reduced by one, while the number of hyper multiplets increase by 53. This is seen very nicely from the expression of the instanton numbers (3.8). Note how the fact that $n_{0,0,0,2} = -2$ fits with losing two of the hyper multiplets as the $W^{\pm}$ of the $SU(2)$ becoming massive. There is of course as usual one hyper multiplet which is "eaten" as the $U(1)$ gauge boson of the Cartan subalgebra of $SU(2)$ become massive. Note how this exactly matches the heterotic description. At the transition from $(191,5)$ to $(244,4)$, the gauge group is $SU(2)\times U(1)^4$, with 28 $\mathbf{2}$ hyper multiplets under the $SU(2)$. The transition occurs by Higgsing the $SU(2)$.

In fact we can give a quite explicit map of the transition of the type II theory to that on the heterotic side by analyzing the physical quantities such as the mirror maps, discriminants and periods which determine the $N = 2$ effective action. The relation between the Mori generators

$$l_1^{(3)} = l_1^{(4)} + 2\, l_4^{(4)},\ l_2^{(3)} = l_2^{(4)},\ l_3^{(3)} = l_3^{(4)} \tag{3.11}$$

implies the following relations between the special and algebraic coordinates, in an obvious notation:

$$\begin{aligned} &z_1^{(3)} = z_1\ z_4^2,\ z_2^{(3)} = z_2,\ z_3^{(3)} = z_3 \\ &t_1^{(3)} = t_1 + 2\ t_4,\ t_2^{(3)} = t_2,\ t_3^{(3)} = t_3\ . \end{aligned} \tag{3.12}$$

From the mirror maps we find

$$z_4 \sim \frac{1}{(1-q_4)^2},\ z_1 \sim (1-q_4)^4$$

that is the transition takes indeed place at the boundary divisor $z_1 = 1/z_4 = 0$. In this limit the data of X_4 such as the periods and discriminants reduce to those of X_3. It is instructive to see the connection to the heterotic moduli. On the heterotic side, which is realized in terms of a simple orbifold construction of $K3$ [13] one has in addition to the dilaton S, the moduli of the torus T, U a Wilson line B. On these moduli acts the perturbative T-duality group which in a similar model has been determined to be [43]:

$$\begin{aligned} &T \to T+1,\ U \to U+1,\ B \to B+1,\ B \to -B,\ T \leftrightarrow U; \\ &B \to B-U,\ T \to T+U-2B,\ U \to U, \end{aligned} \tag{3.13}$$

together with the generalization of the inversion element $T \to -1/T$ which is however realized in a less obvious way on the physical expressions. We expect a similar modular group realized in the present model, possibly up to some coefficients which depend on the details of the compactification lattice. Indeed, matching (3.13) to the symmetries realized on the physical couplings of the Calabi–Yau compactification, we find the identifications

$$t_1 = T,\ t_2 = S-T,\ t_3 = U-T,\ t_4 = B+U$$

where B has in fact period $2\ U$ rather than U. As a check on our physical picture we note that the Weyl symmetry element of the $SU(2)$ subgroup of the E_8 factor is not corrected by non-perturbative string effects contrary to the mirror symmetry of the heterotic torus, as expected.

The situation for X_5 is similar but as the rank is larger there is now room for more interesting phenomena. As for X_4 we can obtain an $SU(2)$ by shrinking down the divisor D_8. As this is not the conic bundle there are no degenerate fibers, i.e. we have a family of curves parameterized by $\mathbf{P}^1$ which is reflected in $n_{0,0,0,1,0} = -2$. Hence, there is no matter, and the unbroken $SU(2)$ is present only in the perturbative theory. This agrees with the general field theoretic picture, as discussed in section 2.1. When the rank of the gauge group is 2, there is an $SU(2)$ for $\phi_1 = \phi_2 \neq 0$, where the ϕ_i are the scalar vevs of the vector multiplets. If we in addition to contracting the instantons of degree $(0,0,0,1,0)$ also shrink those of type $(0,0,0,0,1)$ we get a further enhancement, to $SU(3)$ as well as 30 massless triplets. This can be seen as we are now forced to shrink down any combination of instantons which have degree $(0,0,0,k,l)$. Among the non-zero entries we find three components which all have $n_{(0,0,0,k,l)} = 30$. Thus, 3 times 30 massless particles, forming 30 **3**s under $SU(3)$. As we Higgs the $SU(3)$, the three components split into a set of two which gives the 28 **2**s of the remaining $SU(2)$ and 29 new singlet hyper multiplets. We have then arrived at the $SU(2)$ point of X_4 discussed above. Once more this agrees perfectly with the heterotic picture. At the transition from $(162,6)$ to $(191,5)$, the gauge group is $SU(3) \times U(1)^4$, with 30 **3** hyper multiplets under the $SU(3)$. The transition occurs by breaking the $SU(3)$ to $SU(2)$ just as described above.

Finally, it is possible to avoid the enhanced gauge symmetry, and restrict to a codimension one surface where 30 isolated instantons of degree $(0,0,0,0,1)$ shrink to zero. This is the type IIA analog of the conifold transition in the type IIB string discussed by Greene, Morrison and Strominger [44]. Here, there are 29 flat directions among the 30 hyper multiplets which become massless as the size of the instantons go to zero. Thus, after Higgsing we are left with $U(1)^5$ and 29 new singlets. This IIA transition description was given in [15], and applies as well to a similar transition occuring in [24].

4. Discussion and Conclusions

In this paper we have given strong evidence for extremal transitions between type II Calabi-Yau vacua, where the dual process in the heterotic string corresponds to Higgsing[16]. In particular, we have found evidence for a very nice correspondence between the appearance of enhanced $SU(N)$ gauge symmetries and the corresponding matter structure on one hand, and the existence of a particular type of singularities in the Calabi-Yau manifold. The geometrical structure in question is that of $N-2$ rationally ruled surfaces and a conic bundle with M degenerate fibers. As the fibers, which are $\mathbf{P}^1$s, shrink to zero, particles appearing as BPS-saturated states of 2-branes wrapped around the $\mathbf{P}^1$s become massless. The crucial point is the existence of the degenerate fibers as they are source of the massless matter transforming in the fundamental of the relevant gauge group. The particular models considered in this paper are elliptically fibered Calabi-Yau manifolds. However, it seems as if the existence of the conic bundles is independent of this fact. We thus believe that this scenario is more general, and we are currently investigating such models[17].

[16] In a recent paper, transitions between type II vacua related to the dual $SO(32)$ heterotic string have been discussed [45].

[17] It has become known to us that similar problems are to be discussed in a forthcoming paper by Bershadsky et. al [46].

Finally, recall that the transitions are taking place in the perturbative region of the heterotic string, i.e. in the limit of large base in the K3-fibration in the type II theory. Thus, it still remains the possibility that there exist other K3-fibrations with the same behaviour as the radius of the $\mathbf{P}^1$ becomes large [29].

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Appendix A. The Kähler Cone: Toric Variety vs. Calabi-Yau Hypersurface

We will now explain the relation between phases, although different when thought of as toric varieties, in fact are identical when restricted to the Calabi-Yau hypersurface. Let us assume that we have two toric varietes, P_I and P_{II}, which are related to each other by a flop, i.e. a surface/curve, C, is blown down on P_I, and when passing through the wall of the Kähler cone where P_I and P_{II} meet C reemerges in P_{II} as a surface/curve, $\tilde{C}$. (In terms of the complex structure moduli space of the mirror theory the flop is merely an analytic continuation beyond the radius of convergence, corresponding to the walls of the Kähler cone.) We are still, however, to restrict this process to that of the hypersurfaces X_I and X_{II}. Indeed, if the restriction of C and $\tilde{C}$ to X_I and X_{II} respectively is empty there is nothing to be flopped and X_I is isomorphic to X_{II}. We then have to consider the new Kähler cone as that of the union of the Kähler cones of the $P_{I,II}$. This process is repeated until we have a distinct set of inequivalent models. (In the example we will consider we always find just one K3-fibration phase after applying the above scheme.)

Let us now apply the above idea to that of toric variety P_4 From the dual polytope Δ_4^* one finds three Calabi-Yau phases which all are $K3$-fibrations. Their respective Mori generators are given by

$$\begin{gathered}(0,0,-1,0,1,0,0,-3,3),(0,1,0,0,0,-2,1,0,0),\\(0,0,0,0,-2,1,0,1,0),(-2,0,1,1,0,0,0,1,-1)\end{gathered}\tag{A.1}$$

$$(0,1,0,0,0,-2,1,0,0),(-2,0,0,1,1,0,0,-2,2),$$
$$(0,0,1,0,-1,0,0,3,-3),(0,0,-1,0,-5,3,0,0,3) \tag{A.2}$$

and

$$(0,0,1,0,5,-3,0,0,-3),(0,1,0,0,0,-2,1,0,0),$$
$$(0,0,0,0,-2,1,0,1,0),(-2,0,0,1,1,0,0,-2,2). \tag{A.3}$$

The second toric variety is obtained from the third one by a birational transformation which contracts the surface $x_2 = x_4 = 0$ on the second toric variety and resolves the resulting singularity to the surface $x_5 = x_8 = 0$ on the third toric variety. But on the second model for X_4 we have $D_2 \cdot D_4 = 0$, and on the second we have $D_5 \cdot D_8 = 0$. This says that the birational tranformation does not affect the hypersurface, which are therefore isomorphic. So there is really just one Calabi-Yau phase coming from the two toric varieties described by (A.2) and (A.3). Therefore the Kähler cone of the hypersurface is the union of the two Kähler cones. Since the Mori cone is dual to the Kähler cone, we conclude that the actual Mori cone is the intersection of the two Mori cones (A.2) and (A.3), which we calculate to be

$$(0,1,0,0,0,-2,1,0,0),(-2,0,0,1,1,0,0,-2,2),$$
$$(0,0,0,0,-2,1,0,1,0),(0,0,1,0,-1,0,0,3,-3). \tag{A.4}$$

However, this new toric variety is related to that of (A.1) by a flop as well; contracting $x_4 = x_8 = 0$ in the above phase and then resolving the surface $x_2 = x_7 = 0$ in phase I. However, just as in the previous case $D_4 \cdot D_8 = 0$ when restricted to the Calabi-Yau hypersurface in (A.4) and $D_2 \cdot D_7 = 0$ on the hypersurface in phase I. Thus we are left with just one Calabi-Yau phase given as a hypersurface in a toric variety where the Mori cone is generated by

$$(-2,0,0,1,1,0,0,-2,2),(0,1,0,0,0,-2,1,0,0)$$
$$(0,0,0,0,-2,1,0,1,0),(-2,0,1,1,0,0,0,1,-1). \tag{A.5}$$

Appendix B. The Gromov-Witten invariants for $X_{3,4,5}$

[0, 0, 1]	−2	[0, 1, 1]	−2	[0, 1, 2]	−4	[0, 1, 3]	−6
[0, 1, 4]	−8	[0, 1, 5]	−10	[0, 2, 3]	−6	[0, 2, 4]	−32
[1, 0, 0]	480	[1, 0, 1]	480	[1, 1, 1]	480	[1, 1, 2]	1440
[1, 1, 3]	2400	[1, 1, 4]	3360	[1, 2, 3]	2400	[2, 0, 0]	480
[2, 0, 2]	480	[2, 2, 2]	480	[3, 0, 0]	480	[3, 0, 3]	480
[4, 0, 0]	480	[5, 0, 0]	480	[6, 0, 0]	480	[0, 1, 0]	0

Table B.1: A list of instanton numbers for rational curves of degree $[a_1, a_2, a_3]$ on X_3.

$[0,0,0,1]$	56	$[0,0,0,2]$	-2	$[0,0,0,3]$	0	$[0,0,1,0]$	-2
$[0,1,0,0]$	0	$[0,1,1,0]$	-2	$[0,1,2,0]$	-4	$[0,1,3,0]$	-6
$[0,1,4,0]$	-8	$[0,1,5,0]$	-10	$[0,2,3,0]$	-6	$[0,2,4,0]$	-32
$[0,2,5,0]$	-110	$[0,2,6,0]$	-288	$[0,3,4,0]$	-8	$[0,3,4,0]$	-8
$[0,3,5,0]$	-110	$[1,0,0,0]$	-2	$[1,0,0,1]$	56	$[1,0,0,2]$	372
$[1,0,0,3]$	56	$[1,0,0,4]$	-2				0

Table B.2: A list of instanton numbers for rational curves of degree $[a_1, a_2, a_3, a_4]$ on X_4.

$[0,0,0,0,1]$	30	$[0,0,0,0,2]$	0	$[0,0,0,1,0]$	-2	$[0,0,0,1,1]$	30
$[0,0,0,1,2]$	30	$[0,0,0,1,3]$	-2	$[0,0,1,0,0]$	-2	$[0,1,0,0,0]$	0
$[0,1,1,0,0]$	-2	$[0,1,2,0,0]$	-4	$[0,1,3,0,0]$	-6	$[0,1,4,0,0]$	-8
$[0,2,3,0,0]$	-6	$[0,2,4,0,0]$	-32	$[0,2,5,0,0]$	-110	$[1,0,0,0,0]$	-2
$[1,0,0,1,0]$	-2	$[1,0,0,1,1]$	30	$[1,0,0,1,2]$	30	$[1,0,0,1,3]$	-2
$[1,0,0,2,2]$	30	$[1,0,0,2,3]$	312	$[1,0,0,2,4]$	30	$[1,0,1,0,0]$	-2
$[1,0,1,1,0]$	-2	$[1,0,1,1,1]$	30	$[1,0,1,1,2]$	30	$[1,0,1,1,3]$	-2

Table B.3: A list of instanton numbers for rational curves of degree $[a_1, a_2, a_3, a_4, a_5]$ on X_5.

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