# KINETIC THEORY

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**Abstract**
The treatment of the kinetic theory presented here concentrates on the physical principles involved and on practical applications. It does not pretend to give rigorous mathematical derivations. From the conservation of the phase-space area - as stated by Liouville's theorem - the Vlasov equation is derived. In the case of stationary conditions this equation states that the phase-space distribution must be a function of the Hamiltonian only. This fact is a very powerful tool for finding stationary distributions in certain conditions. As an application the longitudinal distributions of the particles in a non-linear RF wave form and in the presence of a potential well created by a reactive impedance are derived. For the non-stationary case we often use a perturbation method and write the Vlasov equation for a small deviation from the stationary distribution. In the second part dissipative and random forces are investigated. Using an intuitive approach the effects of such forces on the phase-space area are taken into account and the Fokker-Planck equation is derived. As an application the energy distribution of the particles is evaluated for the dissipative and random effects due to the emission of synchrotron radiation. Often only global parameters such as the rms energy spread are of interest. They can be obtained more directly from Campbell's theorem.

## 1 INTRODUCTION

The kinetic theory is treated here in an intuitive way with emphasis on physical principles and practical applications rather than rigorous derivations. Detailed mathematical treatments of the equations involved can be found in several publications on this subject [1, 2]. We start here with Liouville's theorem which states the invariance of the phase-space density in the immediate neighborhood of a probe particle we follow. Expressing this fact as seen by a stationary observer a simple but not rigorous derivation of the Vlasov equation can be obtained. This equation describes the flow of particles in a way similar to that of a moving liquid. It is therefore only valid when a very large number of particles is involved. The case of stationary distributions is of special interest. It can be shown from the Vlasov equation that in this case the distribution depends only on the Hamiltonian which describes the motion of the individual particles. This can be used to obtain stationary distributions for special cases.

## 2 THE VLASOV EQUATION

We consider now particles moving in phase-space with coordinates $q$ and $p$. Usually we associate $q$ with a space coordinate and $p$ with a momentum or a velocity but the situation can be more general. These coordinates $q$ and $p$ should be canonically conjugate which means that they are derived from a Hamiltonian $H(q,p)$ by the canonical equations

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}\ ,\ \ \dot{p} = -\frac{\partial H(q,p)}{\partial q}. \tag{1}$$

We will not explain these fundamental equations any further here but refer to the standard literature and in particular to the CAS courses [2, 3]. The theorem of Liouville is visualized in Fig. 1 where the phase-space trajectories of three particles are drawn. Their positions and the triangular surface element they determine are shown for two different times $t$. The form of this surface element changes but its area stays constant. The phase-space trajectories can of course not cross each other since (except at a singularity) this would mean that two particles have the same position and velocity at one moment but different values later.

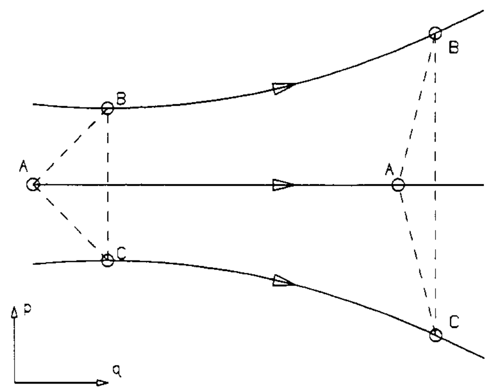


Figure 1: Phase-space trajectories and Liouville's theorem

According to Liouville's theorem the particles in phase space move like an incompressible liquid. This conservation of the phase-space density seems to contradict common experience with accelerators where one often talks about dilution of phase-space due to non-linearities, called filamentation. This paradox is illustrated in Fig. 2. On the left hand side we show a phase space occupied by particles which has the simple form of a rectangle. The instruments observing the distribution of the particles have, in practice, a limited resolution indicated by the grid in the figure. For the left diagram the instruments would localize the beam within seven such resolution elements. After the beam has gone through some non-linear elements (e.g. a mismatched bunch rotating in the non-linear RF wave form) the form of the phase-space occupied by the beam is distorted. Although its actual area has not changed the limited resolution of the instruments indicates now a distribution of particles over many more resolution grid elements. The situation is similar to a liquid which is transformed into foam. On a microscopic level the density is still the same but the global density is much less. For practical purposes, such as for determining the luminosity, the global phase-space density is of course also relevant.

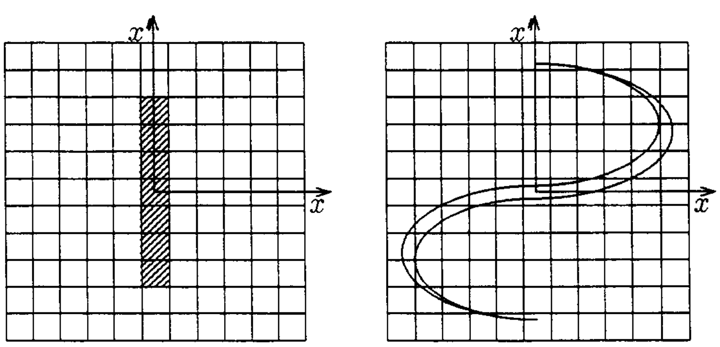

Figure 2: Resolution and phase-space area conservation [6]

The constancy of the phase density $\psi(q,p)$ is expressed by the equation

$$\frac{d\psi(q,p)}{dt} = 0 \tag{2}$$

where the absolute differentiation indicates that one follows the particle while measuring the density of its immediate neighborhood. We would like to know the development of this density as seen by a stationary observer (like a beam monitor) which does not follow the particle. It depends now not only directly on the time $t$ but also indirectly through the coordinates $q$ and $p$ of the moving particles which change with time. Therefore, we have to express the absolute differential with respect to $t$ by the partial differentiations with respect to time as well as with respect to the coordinates $q$ and $p$ multiplied with their time derivative

$$\frac{d\psi(q,p)}{dt} = \frac{\partial\psi(q,p)}{\partial t} + \dot{q}\frac{\partial\psi(q,p)}{\partial q} + \dot{p}\frac{\partial\psi(q,p)}{\partial p} = 0. \tag{3}$$

This expression is the Vlasov equation in its most simple form and is nothing else but an expression for Liouville's conservation of phase-space density seen by a stationary observer. Rigorous derivation of the Vlasov equation can be found in the literature e.g. [1, 2]. In this presentation we apply it to particular accelerator problems. In the first group of applications we search for a stationary distribution which fulfills certain boundary conditions. Such distributions do not depend explicitly on time. For example a stationary bunch looks the same each revolution as it is observed through an intensity monitor. For this stationary case the Vlasov equation becomes

$$\frac{\partial\psi(q,p)}{\partial t} = 0 \;\rightarrow\; \dot{q}\frac{\partial\psi(q,p)}{\partial q} + \dot{p}\frac{\partial\psi(q,p)}{\partial p} = 0. \tag{4}$$

Expressing $\dot{q}$ and $\dot{p}$ with the canonical equations we get

$$\frac{\partial\psi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial\psi}{\partial p}\frac{\partial H}{\partial q} = [\psi, H] = 0. \tag{5}$$

The above expression is called a Poisson bracket. For a system with one variable pair $(p,q)$ a stationary distribution is a function of the Hamiltonian only, $\psi = \psi(H)$. It is easy to show the reverse statement. Assuming $\psi(q,p)$ being a function of the Hamiltonian we get

$$\psi(q,p) = \psi(H) \;\rightarrow\; \frac{\partial\psi}{\partial q} = \frac{d\psi}{dH}\frac{\partial H}{\partial q}\,,\; \frac{\partial\psi}{\partial p} = \frac{d\psi}{dH}\frac{\partial H}{\partial p}, \tag{6}$$

or

$$\frac{\partial\psi/\partial q}{\partial H/\partial q} = \frac{\partial\psi/\partial p}{\partial H/\partial p} \;\rightarrow\; \frac{\partial\psi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial\psi}{\partial p}\frac{\partial H}{\partial q} = 0, \tag{7}$$

which leads again to the stationary Vlasov equation. A stationary distribution is therefore a function of the Hamiltonian only and does not depend explicitly on the coordinates $q$ and $p$ but only implicitly through their relation with the Hamiltonian. This fact is a very powerful tool for finding stationary particle distributions [4, 5].

Before we go to the practical applications we have to make some remarks on the coordinates $q$ and $p$ we use. In Liouville's theorem the phase-space area is only conserved if expressed in canonically conjugate variables $q, p$. The same criterion applies to the validity of the Vlasov equation. Examples of such pairs are position and momentum $x, p$ or time and energy $t, E$. These variables are often not very practical for accelerator applications. We could for example choose the longitudinal position $s$ and the corresponding momentum $p_s$ as coordinates to describe the longitudinal phase-space. However the path length $s$ around the ring is different for particles with different momenta. As a result, individual particles would enter a certain machine element, such as a quadrupole, at different values of the coordinate $s$. From this point of view other coordinate pairs in which the position of the machine element is given for all particles by the same value of one coordinate are more suitable. Such a variable is for example the angle $\theta$ of the motion around the ring. However the conjugate variable is quite complicated [7]

$$W = \int_{E_0}^{E_0+\Delta E} \frac{\Delta E}{\omega} d(\Delta E) \tag{8}$$

where $\Delta E$ is the deviation from the nominal energy $E_0$ and $\omega(\Delta E)$ is the revolution frequency which depends on this energy deviation. In our applications we only use terms up to first order in $\Delta E$ and use as an approximation $\theta$ and $\Delta E$ as variables. However we should always remember that this is a linear approximation in $\Delta E/E_0$ which can become very inaccurate in some cases such as the neighborhood of transition energy. As a further convenience we will us the RF phase angle $\phi$ instead of $\theta$ to which it is related through $\phi = h\theta$ with $h$ being the harmonic number.

## 3 APPLICATIONS OF THE VLASOV EQUATION

### 3.1 Stationary distribution in a non-linear RF wave form

As an introduction we will first go through the derivation of the longitudinal beam dynamics and try to find the longitudinal particle distribution in the presence of the non-linearity represented by the RF wave form. We consider a storage ring with circumference $C$ which depends on the deviation $\Delta p$ from the nominal momentum $p$ or $\Delta E$ from the nominal energy $E$.

$$C = C_0 \left(1 + \alpha \frac{\Delta p}{p}\right) = C_0 \left(1 + \frac{\alpha}{\beta^2} \frac{\Delta E}{E}\right), \tag{9}$$

where $\alpha$ is the momentum compaction factor. The revolution time $T = C/\beta c$ becomes

$$T = T_0 \left(1 + \left(\alpha - \frac{1}{\gamma^2}\right) \frac{1}{\beta^2} \frac{\Delta E}{E}\right) \sim T_0(1 + \alpha \frac{\Delta E}{E}), \tag{10}$$

where an approximation for ultra-relativistic particles ($\gamma \gg 1$ and $\gamma^2 \gg 1/\alpha$) has been made and will be used from now on.

In each revolution a particle goes through an RF cavity which is driven with a frequency $\omega_{RF} = h\omega_0$ and a peak voltage $\hat{V}$ with $\omega_0$ being the nominal revolution frequency $\omega_0 = 2\pi\beta c/C_0$ and the integer $h$ being the harmonic number.

$$V(t) = \hat{V} \sin(h\omega_0 t). \tag{11}$$

We assume now that the particle looses in each turn a certain amount of energy $U_s$ due to the emission of synchrotron radiation

$$U_s = U_{s0} + \frac{dU}{dE} \Delta E. \tag{12}$$

For the nominal energy $E$ we have a stationary condition such that the particle receives at each passage through the cavity the same amount of energy $U_{s0}$ as it looses due to synchrotron radiation. It has to traverse the cavity at the synchronous time $t_s$ such that

$$V(t_s) = U_{s0}/e \rightarrow \sin(h\omega_0 t_s) = \frac{U_{s0}}{e\hat{V}}. \tag{13}$$

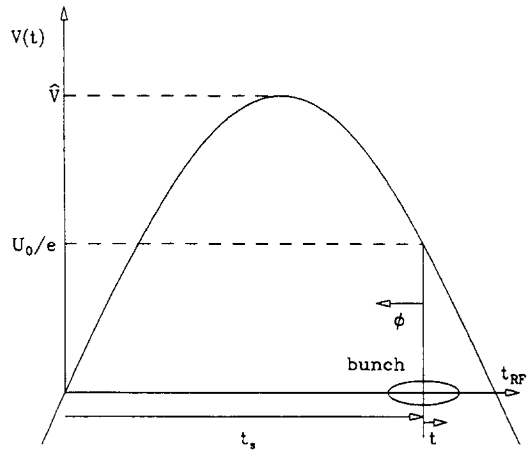


Figure 3: RF acceleration and phase focusing

Instead of the time $t$ we use now the RF phase angle $\phi_{RF}$

$$\phi_{RF} = \pi - h\omega_0 t \ , \quad \phi_s = \pi - h\omega_0 t_s \tag{14}$$

and call the deviation from this synchronous phase angle $\phi_s$

$$\phi = \phi_{RF} - \phi_s \tag{15}$$

as indicated in Fig.3. The energy gain and loss in one turn becomes

$$\delta(\Delta E) = e\hat{V}\sin\phi_{RF} - U_s = e\hat{V}(\cos\phi_s \sin\phi + \sin\phi_s \cos\phi) - \left(U_{s0} + \frac{dU}{dE}\Delta E\right). \tag{16}$$

Using the equilibrium for the synchronous particle

$$U_{s0} = e\hat{V}\sin\phi_s \tag{17}$$

we get

$$\delta(\Delta E) = e\hat{V}(\cos\phi_s \sin\phi - \sin\phi_s(1-\cos\phi) - \frac{dU}{dE}\Delta E. \tag{18}$$

In most practical cases the energy gain and loss $\delta(\Delta E)$ per turn is very small compared to the energy $E$ itself. As an approximation we can replace the sudden energy gain while going through the cavity by a smooth acceleration.

$$\Delta\dot{E} = \frac{\omega_0}{2\pi}\left(e\hat{V}(\cos\phi_s \sin\phi - \sin\phi_s(1-\cos\phi)) - \frac{dU}{dE}\Delta E\right). \tag{19}$$

The dependence of the revolution time on the energy deviation $\Delta E$ leads to a change of the revolution time of the particle

$$\frac{\Delta T}{T} = \alpha\frac{\Delta E}{E}. \tag{20}$$

This leads in the smooth approximation to a time derivative of the phase

$$\dot{\phi} = -\omega_0 \alpha h \frac{\Delta E}{E}. \tag{21}$$

Using this relation to replace $\Delta E$ in Eq. (19) gives

$$\ddot{\phi} + \frac{\omega_0}{2\pi}\frac{dU}{dE}\dot{\phi} + \frac{\omega_0^2 \alpha h e\hat{V}}{2\pi E}(\cos\phi_s \sin\phi - \sin\phi_s(1-\cos\phi)) = 0. \tag{22}$$

Introducing the synchrotron frequency $\omega_{s0}$, the synchrotron tune $Q_{s0}$ and the damping rate $\alpha_E$

$$\omega_{s0}^2 = \omega_0^2 \frac{\alpha h e\hat{V}\cos\phi_s}{2\pi E}\ , \ Q_{s0} = \frac{\omega_{s0}}{\omega_0} \text{ and } \alpha_E = \frac{\omega_{s0}}{2\pi}\frac{dU}{dE} \tag{23}$$

in Eq. (22) leads to a more compact expression

$$\ddot{\phi} + 2\alpha_E\dot{\phi} + \omega_{s0}^2\left(\frac{\cos\phi_s\sin\phi - \sin\phi_s(1-\cos\phi)}{\cos\phi_s}\right) = 0. \tag{24}$$

For small amplitudes $\phi \ll 1$ we can linearize the above equation

$$\ddot{\phi} + 2\alpha_E\dot{\phi} + \omega_{s0}^2\phi = 0 \tag{25}$$

which has the solution

$$\phi(t) = \hat{\phi}e^{-\alpha_E t}\cos(\omega_{s0}\sqrt{1-(\alpha_E/\omega_{s0})^2}t + \phi_0), \tag{26}$$

where the amplitude $\hat{\phi}$ and the phase constant $\phi_0$ are given by the initial conditions. Usually $\alpha_E \ll \omega_{s0}$ and the solution is approximately

$$\phi(t) \sim \hat{\phi}e^{-\alpha_E t}\cos(\omega_{s0}t). \tag{27}$$

This is the equation of a damped oscillator with damping rate $\alpha_E$ and resonant frequency $\omega_{s0}\sqrt{1-(\alpha_E/\omega_{s0})^2} \sim \omega_{s0}$. The Vlasov equation applies to systems without dissipative or random forces. We can therefore not use it to describe particles undergoing damped synchrotron oscillations. For our next applications we will neglect the damping and take as approximation $\alpha_E \sim 0$. Strictly speaking, we should, according to Eq. (13), also set the synchronous phase angle $\phi_s$ to zero. We will however carry the term $\cos\phi_s$ along and pretend that the finite synchronous phase angle is here as a compensation for a conservative force.

We neglect now the damping and go back to our two first-order differential equations

$$\Delta\dot{E} = \frac{\omega_0}{2\pi}e\hat{V}(\cos\phi_s\sin\phi - \sin\phi_s(1-\cos\phi))\ ,\ \dot{\phi} = -\omega_0\alpha h\frac{\Delta E}{E} \tag{28}$$

or the second-order differential equation

$$\ddot{\phi} + \frac{\omega_{s0}^2}{\cos\phi_s}(\cos\phi_s\sin\phi - \sin\phi_s(1-\cos\phi) = 0. \tag{29}$$

Multiplying this last equation with $\dot{\phi}$ and integrating once gives

$$\frac{\dot{\phi}^2}{2} + \frac{\omega_{s0}^2}{\cos\phi_s}(\cos\phi_s(1-\cos\phi) - \sin\phi_s(\phi - \sin\phi) = H' = \text{const.} \tag{30}$$

This expression is a constant of motion and must therefore be directly related to the Hamiltonian. We find easily that the Hamiltonian is just

$$H = \frac{EH'}{\omega_0\alpha h} = \omega_0\alpha h E\left[\frac{1}{2}\left(\frac{\Delta E}{E}\right)^2 + \left(\frac{Q_s}{\alpha h}\right)^2\frac{(\cos\phi_s(1-\cos\phi) - \sin\phi_s(\phi-\sin\phi))}{\cos\phi_s}\right] \tag{31}$$

which satisfies the canonical equations

$$\frac{\partial H}{\partial \phi} = \Delta \dot{E} = \frac{\omega_0}{2\pi} e\hat{V}(\cos\phi_s \sin\phi - \sin\phi_s(1-\cos\phi)) = \Delta\dot{E}\ ,\ -\frac{\partial H}{\partial \Delta E} = -\omega_0 \alpha h \frac{\Delta E}{E} = \dot{\phi}. \tag{32}$$

We show later in section 4 that the electron beam has a Gaussian energy distribution with rms value $\sigma_E$ due to the interplay between damping and quantum excitation by the emitted synchrotron radiation

$$\psi(\phi, \Delta E) \propto \exp\left(-\frac{1}{2}\left(\frac{\Delta E/E}{\sigma_E/E}\right)^2\right). \tag{33}$$

On the other hand we also know that the stationary distribution should only be a function of the Hamiltonian. Comparing Eq. (31) with Eq. (33) we find that the phase-space distribution must be of the form

$$\psi(\phi, \Delta E) = C_2 \exp\left(-\frac{\left(\frac{\Delta E}{E}\right)^2 + 2\left(\frac{Q_s}{\alpha h}\right)^2(\cos\phi_s(1-\cos\phi) - \sin\phi_s(\phi - \sin\phi))}{2\left(\frac{\sigma_E}{E}\right)^2}\right). \tag{34}$$

The constant factor can be determined from the normalizing condition

$$\int\int \psi(\phi, \delta E)(\Delta E) d\phi = 1 \tag{35}$$

which has to be solved numerically for the general case. In many practical cases the bunch is much shorter than the RF wavelength, i.e. $\phi \ll 1$, and we can approximate $1 - \cos\phi \sim \phi^2/2$ leading to

$$\psi(\phi, \Delta E) = \frac{1}{2\pi\sigma_\phi \sigma_E/E} \exp\left(-\frac{1}{2}\left(\left(\frac{\Delta E/E}{\sigma_E/E}\right)^2 + \left(\frac{\phi}{\sigma_\phi}\right)^2\right)\right), \tag{36}$$

where $\sigma_\phi$ is the rms bunch length measured in RF phase

$$\sigma_\phi = \frac{\alpha h}{Q_s}\frac{\sigma_E}{E}. \tag{37}$$

In this short-bunch approximation the bunch is Gaussian in energy and in phase.

### 3.2 Stationary distribution in a double RF system

The treatment used above can easily be extended to the case of a double RF system. This system has, besides the normal RF system with frequency $\omega_{RF} = h\omega_0$ and amplitude $\hat{V}$, another system with a frequency $n\omega_{RF}$ being a harmonic of the RF frequency and amplitude $k\hat{V}$, Fig. 4. The total voltage seen by the beam becomes

$$V(\phi) = V_1 + V_n = \hat{V}\left[\sin(\phi_s + \phi) + k\sin(n\phi_n + n\phi)\right]. \tag{38}$$

Such double RF systems are often used to control the bunch length and to produce a large spread in synchrotron frequency to get Landau damping. They are particularly interesting if operated in the bunch-lengthening mode. For this the RF wave form should fulfill two neccessary and one convenient conditions:

a) The energy loss $U_s$ is compensated $V(0) = \hat{V}[\sin\phi_s + k\sin(n\phi_n)] = U_s/e$.

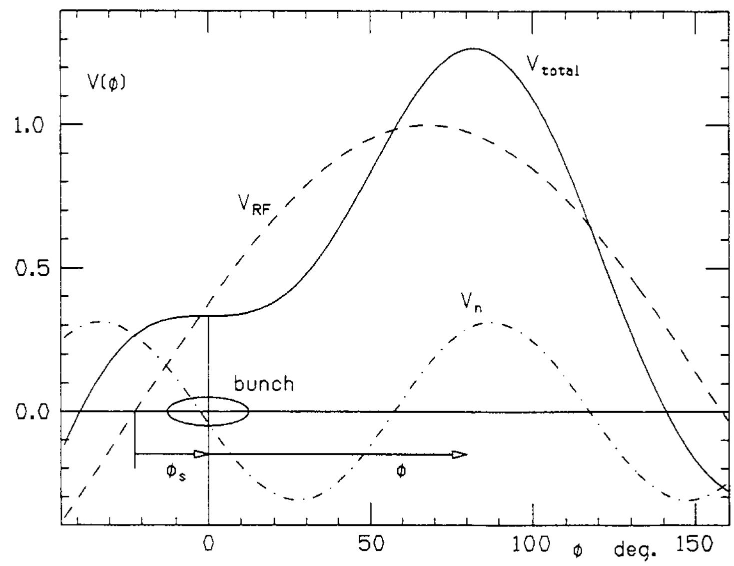


Figure 4: Double RF system

b) The slope vanishes at the bunch center $V'(0) = \hat{V}[\cos\phi_s + kn\cos(n\phi)] = 0$.

c) The curvature vanishes at the center $V''(0) = -\hat{V}[\sin\phi_s + kn^2\sin(n\phi_n)] = 0$.

Using these three conditions and approximating for the case of a bunch short compared to the higher harmonic wave-length we get for the voltage seen by the beam

$$V(\phi) \sim \hat{V}\left[\left(1-\frac{1}{n^2}\right)\sin\phi_s\left(\frac{n^2-1}{6}\right)\phi^3 + \cos\phi_s\right] = \frac{U_s}{e} + \hat{V}\frac{n^2-1}{6}\phi^3. \tag{39}$$

Using the same procedure as in the last subsection we find for the two first-order differential equations

$$\dot{\phi} = -\omega_0\alpha h\frac{\Delta E}{E}\ ,\ \Delta\dot{E} = \frac{\omega_0}{2\pi}e\hat{V}\cos\phi_s(\frac{n^2-1}{6})\phi^3 \tag{40}$$

and for the second-order differential equation

$$\ddot{\phi} + \frac{\omega_0^2\alpha he\hat{V}\cos\phi_s}{2\pi E}\left(\frac{n^2-1}{6}\right)\phi^3 = \ddot{\phi} + \omega_{s0}^2\frac{n^2-1}{6}\phi^3 = 0. \tag{41}$$

where the synchrotron frequency $\omega_{s0}$ of a single RF system is used here as a convenient parameter. This last expression is a non-linear differential equation the solution of which can be expressed with Jacobian elliptic functions [8]. However we are here only interested to find the Hamiltonian. Multiplying it with $\dot{\phi}$ and integrating gives

$$\frac{\dot{\phi}^2}{2} + \omega_{s0}^2\frac{n^2-1}{24}\phi^4 = H' = \text{const.} \tag{42}$$

The Hamiltonian itself is

$$H = \frac{E}{\omega_0\alpha h}H' = \frac{\omega_0\alpha hE}{2}\left[\left(\frac{\Delta E}{E}\right)^2 + \left(\frac{Q_s}{\alpha h}\right)^2\frac{n^2-1}{12}\right]. \tag{43}$$

It can easily be checked that it is related to the coordinates $\phi$ and $\Delta E$ through the proper canonical equations. Using again the fact that the energy distribution of an electron beam is Gaussian

$$\psi(\phi, \Delta E) \propto \exp\left(-\frac{1}{2}\left(\frac{\Delta E/E}{\sigma_E/E}\right)^2\right) \tag{44}$$

and that the total distribution is a function of the Hamiltonian only we find the phase-space distribution

$$\psi(\phi, \Delta E) = C_2 \exp\left(-\frac{H}{\omega_0 \alpha h (\sigma_E/E)^2}\right) = C_2 \exp\left(-\frac{\left(\frac{\Delta E}{E}\right)^2 + \left(\frac{Q_s}{\alpha h}\right)^2 \frac{n^2-1}{12}\phi^4}{2\left(\frac{\sigma_E}{E}\right)^2}\right). \tag{45}$$

To get the instantaneous current of this bunch distribution we integrate over $\Delta E$

$$I(\phi) = 2\pi h I_0 \int \psi(\phi, \Delta E) d(\Delta E) = C_3 \exp\left(-\frac{\left(\frac{Q_s}{\alpha h}\right)^2 \frac{n^2-1}{12}\phi^4}{2(\sigma_E/E)^2}\right). \tag{46}$$

This current has no longer a Gaussian distribution. The rms bunch length expressed in RF phase angle is

$$\sigma_\phi = \frac{2\sqrt{\pi}}{\Gamma(1/4)}\sqrt{\sqrt{\frac{3}{n^2-1}}\frac{\alpha h}{Q_s}\frac{\sigma_E}{E}}. \tag{47}$$

Normalizing leads to the final expression for the instantaneous current

$$I(\phi) = \frac{4\pi\sqrt{2\pi} I_0}{\sqrt{\sqrt{2}\Gamma^2(\frac{1}{4})}\sigma_\phi} \exp\left(-\frac{2\pi^2}{\Gamma^2(1/4)}\left(\frac{\phi}{\sigma_\phi}\right)^4\right). \tag{48}$$

From the phase-space and current distribution other properties such as the distribution of synchrotron frequencies, peak current, space-charge effects, etc can be calculated [8, 9].

### 3.3 Potential-well bunch lengthening

We calculate now the longitudinal particle distribution for a beam with a Gaussian energy distribution in the presence of an inductive wall impedance [10]. We consider a bunch in a single RF system which we linearize in the neighborhood of the RF phase angle

$$V_{RF} = \hat{V} \sin\phi_{RF} = \hat{V}\sin(\phi + \phi_s) \sim \hat{V}(\cos\phi_s \phi + \sin\phi_s). \tag{49}$$

We assume that the beam surroundings represent an inductive impedance. This is in fact a reasonable assumption for long bunches. Their spectrum lies mainly below the resonant frequencies of the typical parasitic modes where the impedance is inductive. Taking an inductance $L$ which is approximately constant over the bunch spectrum we get an induced voltage of the form

$$V_i = L\frac{dI}{dt} = -\omega_0 h L \frac{dI}{d\phi}. \tag{50}$$

The impedance of the inductance is

$$Z_L(\omega) = j\omega L. \tag{51}$$

We introduce the absolute value of this impedance divided by the mode number $n = \omega/\omega_0$ which is directly related to the inductance $L$ and the revolution frequency $\omega_0$

$$\left|\frac{Z}{n}\right| = \frac{\omega L}{\omega/\omega_0} = \omega_0 L. \tag{52}$$

With this we find for the induced voltage

$$V_i = -h|Z/n|\frac{dI}{d\phi}. \tag{53}$$

The introduction of the impedance divided by the mode number seems to be just a convenient way to describe the inductance. It is however a good way to characterize the relevant impedance of a ring. It is typically between 10 and 20 Ohms for older machines with many aperture changes and between 1 and 5 Ohms for more modern rings with smooth vacuum chambers.

This induced voltage $V_i$ is seen by the particles in the bunch and has to be subtracted from the RF voltage. We have for the total voltage

$$V(\phi) = V_{RF} - V_i = \hat{V}\left(\cos\phi_s\phi + \sin\phi_s + \frac{h|Z/n|}{\hat{V}\cos\phi_s}\frac{dI}{d\phi}\right). \tag{54}$$

The smoothed energy gain per unit time is

$$\Delta\dot{E} = \frac{\omega_0 e\hat{V}\cos\phi_s}{2\pi}\left(\phi + \frac{h|Z/n|}{\hat{V}\cos\phi_s}\frac{dI}{d\phi}\right) + \frac{\omega_0}{2\pi}(e\hat{V}\sin\phi_s - U_s). \tag{55}$$

For the equilibrium condition $e\hat{V}\sin\phi_s = U_s$ we get the two first-order differential equations

$$\Delta\dot{E} = \frac{\omega_0 e\hat{V}\cos\phi_s}{2\pi}\left(\phi + \frac{h|Z/n|}{\hat{V}\cos\phi_s}\frac{dI}{d\phi}\right) \quad , \quad \dot{\phi} = -\omega_0\alpha h\frac{\Delta E}{E} \tag{56}$$

and the second-order equation

$$\ddot{\phi} + \frac{\omega_0^2\alpha he\hat{V}\cos\phi_s}{2\pi E}\left(\phi + \frac{h|Z/n|}{\hat{V}\cos\phi_s}\frac{dI}{d\phi}\right) = \ddot{\phi} + \omega_{s0}^2\left(\phi + \frac{h|Z/n|}{\hat{V}\cos\phi_s}\frac{dI}{d\phi}\right) \tag{57}$$

where we used with $\omega_{s0}$ the synchrotron frequency without an inductive impedance as a convenient parameter. Multiplying this second-order equation with $\dot{\phi}$ gives

$$\frac{\dot{\phi}^2}{2} + \omega_{s0}^2\frac{\phi^2}{2} + \frac{\omega_{s0}^2 h|Z/n|}{\hat{V}\cos\phi_s}(I(\phi) - I(0)) = H' = \text{const}, \tag{58}$$

from which we get the Hamiltonian

$$H = \frac{E}{\omega_0\alpha h}H' = \frac{\omega_s\alpha hE}{2}\left[\left(\frac{\Delta E}{E}\right)^2 + \left(\frac{Q_s}{\alpha h}\right)^2\left(\phi^2 + \frac{h|Z/n|}{\hat{V}\cos\phi_s}(I(\phi) - I(0))\right)\right]. \tag{59}$$

It is interesting to note that the above Hamiltonian depends on the current distribution $I(\phi)$ which is unknown at present. We take the case of electrons which have a Gaussian distribution in energy

$$\psi(\phi, \Delta E) \propto \exp\left(-\frac{1}{2}\left(\frac{\Delta E/E}{\sigma_E/E}\right)^2\right). \tag{60}$$

Furthermore, the distribution is assumed to be stationary and depends therefore only on the Hamiltonian

$$\begin{aligned} \psi(\phi, \Delta E) &= C_4\exp\left(-\frac{H}{\omega_0\alpha h(\sigma_E/E)^2}\right) \\ &= C_4\exp\left(-\frac{\left(\frac{\Delta E}{E}\right)^2 + \left(\frac{Q_s}{\alpha h}\right)^2\left(\phi^2 + \frac{h|Z/n|}{\hat{V}\cos\phi_s}(I(\phi) - I(0))\right)}{2(\sigma_E/E)^2}\right). \end{aligned} \tag{61}$$

Integrating the distribution over $\Delta E$ leads to an implicit expression for the current

$$I(\phi) = C_4 \frac{2\pi h I_0}{\sqrt{2\pi}\sigma_E} \exp\left( -\frac{\left(\frac{Q_s}{\alpha h}\right)^2 \left(\phi^2 + \frac{h|Z/n|}{\hat{V}\cos\phi_s}(I(\phi) - I(0))\right)}{2(\sigma_E/E)^2} \right). \tag{62}$$

This represents a transcendent equation for the current distribution $I(\phi)$. We can write it in a more compact form by introducing the current $I(0)$ at the bunch center in the presence of the impedance $|Z/n|$

$$I(0) = C_4 \frac{2\pi h I_0}{\sqrt{2\pi}\sigma_E}, \tag{63}$$

$$I(\phi) = I(0) \exp\left( \frac{\left(\frac{Q_s}{\alpha h}\right)^2 \left(\phi^2 + \frac{h|Z/n|}{\hat{V}\cos\phi_s}(I(\phi) - I(0))\right)}{2(\sigma_E/E)^2} \right). \tag{64}$$

For a vanishing impedance $|Z/n| \to 0$ this leads to the usual Gaussian expression

$$I(\phi)_{Z\to 0} = I_0(\phi) = I_0(0) \exp\left( -\frac{\left(\frac{Q_s}{\alpha h}\right)^2}{2(\sigma_E/E)^2} \phi^2 \right) = \frac{\sqrt{2\pi} h I_0}{\sigma s0} \exp\left( -\frac{1}{2}\left(\frac{\phi}{\sigma_{s0}}\right)^2 \right) \tag{65}$$

where $I_0(0) = \sqrt{2\pi} h I_0/\sigma_{so}$ is the peak current and

$$\sigma_{so} = \left(\frac{Q_s}{\alpha h}\right)^2 \frac{\sigma_E}{E} \tag{66}$$

is the bunch length measured in RF phase angle in the absence of the impedance. Using these quantities and introducing a further parameter $\xi$ which measures the strength of the induced field relative to the guide field of the RF system

$$\xi = \frac{\sqrt{2\pi} h^2 I_0 |Z/n|}{\hat{V}\cos\phi_s \sigma_{s0}^3} \tag{67}$$

we can write the equation in a more compact form

$$I(\phi) e^{\xi I(\phi)/I_0(0)} = I(0) e^{\xi I(0)/I_0(0)} e^{-\phi^2/2\sigma_{s0}^2}. \tag{68}$$

This implicit and transcendent equation determines the self-consistent current distribution $I(\phi)$ for a Gaussian energy distribution in the presence of an inductive impedance. Since the equation is transcendent it has to be solved numerically. Once a solution is obtained the phase-space distribution $\psi(\phi, \Delta E)$, the induced voltage $V(\phi)$ or the synchrotron frequency distribution can also be found. Figure 5 shows such solutions for different values of the strength parameter $\xi$. Often it is convenient to have an analytic expression for the current distribution even if it is only approximate. We assume now that the impedance strength parameter $\xi$ is small and develop with respect to this parameter

$$I(\phi) \sim \frac{\sqrt{2\pi} h I_0}{\sigma_{s0}} e^{-\frac{\phi^2}{2\sigma_{s0}^2}} \left[ 1 - \xi \frac{2e^{-\frac{\phi^2}{2\sigma_{s0}^2}} - \sqrt{2}}{2} + \xi^2 \frac{2 - \sqrt{3} - 2\sqrt{2} e^{-\frac{\phi^2}{2\sigma_{s0}^2}} + 3e^{-\frac{\phi^2}{\sigma_{s0}^2}}}{2} + \ldots \right] \tag{69}$$

which can be used for further approximate calculations such as synchrotron frequency distributions, etc.

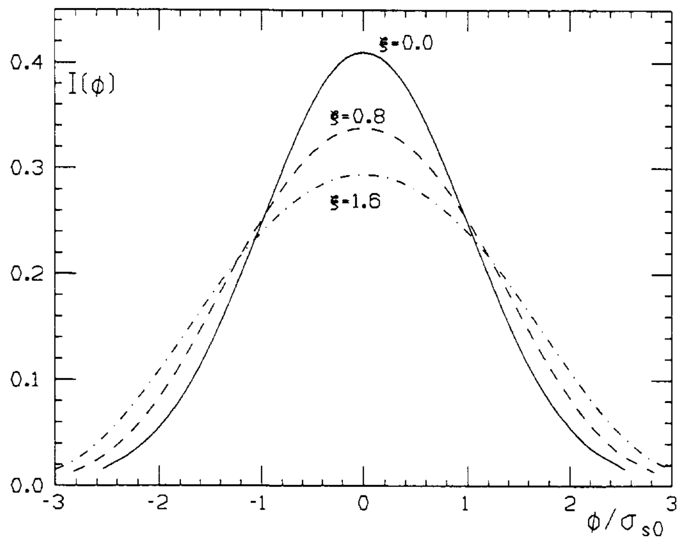


Figure 5: Longitudinal bunch distribution in the presence of an inductive impedance

### 3.4 Perturbation method

In many practical cases we do not look for a stationary solution but for small deviations from it. The most common examples of this are beam instabilities. The stationary distribution is usually known. Small oscillations around this distribution are considered and the forces they introduce through the wall impedance calculated. We then see whether these forces increase the small initial oscillation leading to an instability or whether they reduce the oscillation indicating stability. Starting with the Vlasov equation

$$\frac{\partial\psi(q,p)}{\partial t} + \dot{q}\frac{\partial\psi(q,p)}{\partial q} + \dot{p}\frac{\partial\psi(q,p)}{\partial p} = 0 \tag{70}$$

we split the distribution $\psi(q,p,t)$ into a stationary part $\psi_0(H)$ and an oscillating perturbation $\psi_1(q,p,t)$ which is assumed to be small

$$\psi(q,p,t) = \psi_0(H) + \psi_1(q,p,t). \tag{71}$$

Furthermore the forces $\dot{p}$ consist of a part due to the guide field like the RF system or the focusing magnets and a small part due to fields induced by the oscillating disturbance $\psi_1(q,p,t)$ in the impedance of the beam surroundings

$$\dot{p} = \dot{p}_0 + \dot{p}_1(\psi_1). \tag{72}$$

Introducing this into the Vlasov equation and making use of the fact that the disturbance is small we can linearize it and obtain an expression involving the disturbance $\psi_1$. From this equation we find whether the amplitude of the disturbance is growing (instability) or decaying (stability). Usually this last equation determines the frequency of the oscillating disturbance which can be real or complex. Expressing the oscillation with $\exp(j\omega t)$ the situation is stable if the obtained frequency has a positive imaginary part and unstable for a negative imaginary part. On the other hand the form of the disturbance has to be given as an input and is not obtained from the Vlasov equation. Therefore we have to consider a very general disturbance $\psi_1$ of the stationary distribution, develop it into a complete system of orthogonal modes and check each of them for stability. In practice things are easier since only a few modes are usually relevant. Examples for these procedures can be found in many publications in particular in the CAS lectures [11]

# 4 EFFECTS OF DAMPING AND RANDOM FORCES

## 4.1 The Fokker-Planck equation

In accelerators or storage rings the motion of interest is often periodic and the phase-space trajectories are closed curves. In the absence of dissipative forces the phase-space area of these trajectories as well as the number of particles inside, i.e. the phase-space density, is conserved

$$\frac{d\psi(q,p)}{dt} = \frac{\partial\psi(q,p)}{\partial t} + \dot{q}\frac{\partial\psi(q,p)}{\partial q} + \dot{p}\frac{\partial\psi(q,p)}{\partial p} = 0. \tag{73}$$

However, this is no longer guaranteed if dissipative forces, such as energy loss due to synchrotron radiation, or random forces, like the quantum excitation by the emitted photons, are present. We will use an intuitive approach to consider the effect of such forces on the phase-space density and refer to the standard literature such as CAS lectures [1, 12] for rigorous derivations.

We follow here the method used in Ref. [13] and start with the effect of the energy dissipation due to synchrotron radiation which has already been treated in section 3.1 for the phase oscillation. We found there that both canonical variables $\phi$ and $\Delta E$ follow a damped oscillation

$$\begin{aligned} \phi(t) &= \hat{\phi}e^{-\alpha_E t}\cos\left(\omega_{s0}\sqrt{1-(\alpha_E/\omega_{s0})^2}t + \Phi_0\right) \\ \Delta E(t) &= \Delta E_0 e^{-\alpha_E t}\sin\left(\omega_{s0}\sqrt{1-(\alpha_E/\omega_{s0})^2}t + \Phi_0\right). \end{aligned} \tag{74}$$

The two coordinates $\phi$ and $\Delta E$ spiral inwards to smaller values. Usually the damping is relatively small $\alpha_E \ll \omega_{s0}$ and we can still talk of a phase-space area $A$ covered by this trajectory which decreases now at twice the rate of that of the individual coordinates. As a consequence the phase-space area decreases and the phase-space density increases

$$\frac{dA}{dt} = -2\alpha_E A\ , \ \ \frac{d\psi}{dt} = 2\alpha_E\psi. \tag{75}$$

While the energy dissipation due to synchrotron radiation and its replacement by the RF system leads to damping and therefore to an increase of the phase-space density the random forces due to quantum excitation lead to a diffusion and therefore to a reduction of the phase-space density. Assuming random forces which lead primarily to a change of the momentum coordinate $p$ of the particles, as is the case for the quantum excitation, the resulting diffusion is given by the equation

$$\frac{d\psi}{dt} = D\frac{\partial^2\psi}{\partial E^2}. \tag{76}$$

We have not derived it here and refer to textbooks. However the above equation is rather transparent stating that the diffusion leads to a reduction of the density where its second derivative is negative, i.e. where the distribution is curved such that particles can diffuse into neighboring areas. On the other hand the density increases at locations where the distribution is curved such that diffusion from neighboring areas is more likely. We can now express the effect of the dissipative and the random forces on the phase-space density $\psi$

$$\frac{d\psi(q,p)}{dt} = \frac{\partial\psi(q,p)}{\partial t} + \dot{q}\frac{\partial\psi(q,p)}{\partial q} + \dot{p}\frac{\partial\psi(q,p)}{\partial p} = 2\alpha_E\psi(q,p) + D\frac{\partial^2\psi(q,p)}{\partial p^2}. \tag{77}$$

This is the Fokker-Planck equation in its most simple form which can be regarded as an extension of the Vlasov equation to dissipative and random forces. We write it now for our example of longitudinal motion in a bunch using the coordinates $\phi$ and $\Delta E$

$$\frac{\partial\psi(\phi,\Delta E)}{\partial t} + \dot{\phi}\frac{\partial\psi(\phi,\Delta E)}{\partial\phi} + \Delta\dot{E}\frac{\partial\psi(\phi,\Delta E)}{\partial\Delta E} = 2\alpha_E\psi(\phi,\Delta E) + D\frac{\partial^2\psi(\phi,\Delta E)}{\partial\Delta E^2}. \quad (78)$$

We would like to find a stationary solution of this equation i.e. the distribution for which the damping and the diffusion are in equilibrium and follow the method used in Ref. [13]. Taking the time derivative of the energy deviation from section 3.1 but for a very general RF wave form $V(\phi)$

$$\Delta\dot{E} = \frac{\omega_0}{2\pi}\left(eV(\phi) - \frac{dU}{dE}\Delta E\right) = \Delta\dot{E}_0 - 2\alpha_E\Delta E. \quad (79)$$

We introduce this into the Fokker-Planck equation and separate the damping and diffusion terms

$$\frac{\partial\psi}{\partial t} + \dot{\phi}\frac{\partial\psi}{\partial\phi} + \Delta\dot{E}_0\frac{\partial\psi}{\partial\Delta E} = 2\alpha_E\Delta E\frac{\partial\psi}{\partial\Delta E} + 2\alpha_E\psi + D\frac{\partial^2\psi}{\Delta E^2}. \quad (80)$$

The left hand side of the above equation has no damping and diffusion terms and is just the Vlasov equation as we used it in the earlier section but for a general RF wave form and must therefore vanish. For the right hand side we get

$$2\alpha_E\left(\psi + \Delta E\frac{\partial\psi}{\partial\Delta E} + \frac{D}{2\alpha_E}\frac{\partial^2\psi}{\partial E^2}\right) = 2\alpha_E\frac{\partial}{\partial\Delta E}\left(\Delta E\psi + \frac{D}{2\alpha_E}\frac{\partial\psi}{\partial\Delta E}\right) = 0. \quad (81)$$

From this it follows that the last expression in the brackets does not depend on $\Delta E$

$$\frac{\partial\psi}{\partial\Delta E} + 2\frac{\alpha_E}{D}\psi\Delta E = C_5 = f(\phi). \quad (82)$$

It has the solution

$$\psi(\phi,\Delta E) = F(\phi)\exp\left(-\frac{1}{2}\left(\frac{\Delta E}{\sigma_E}\right)^2\right) \text{ with } \sigma_E^2 = \frac{D}{2\alpha_E}. \quad (83)$$

The above derivation was carried out for a very general RF wave form and resulted in a Gaussian distribution in energy determined only by damping and diffusion, a result we used already before. The longitudinal distribution in $\phi$, however, is not determined by damping and diffusion alone but depends on the RF wave form. These statements are only correct if the random force effects the energy directly, as in the case of the emission of a synchrotron radiation photon, and if neither the damping nor the quantum excitation depends on the energy deviation $\Delta E$ itself. For all the examples treated here these conditions are fulfilled in very good approximation. There are some exotic cases where this is not so, like the (usually very weak) quantum excitation in quadrupoles which depend on the displacement of the particle, or for some non-linear wigglers in which the damping becomes a function of the displacement [12, 14]

In the above derivation of the Gaussian distribution we did not calculate the rms energy spread $\sigma_E$ but related it just to the diffusion constant $D$ which should be determined from the properties of the synchrotron radiation. We will do that in the next subsection using a different method.

### 4.2 Campbell's theorem - energy spread due to synchrotron radiation emission

In the previous section it was shown that the quantum fluctuation and the damping provided by the synchrotron radiation leads to a Gaussian energy distribution of an electron beam in a storage ring. We will now use a direct method which does not give the form of the distribution but rather its variance or its rms value. This will be done by using Campbell's theorem [1, 15]. We will not prove the theorem here but refer to the references mentioned above. We follow the method used by M. Sands [16] who first derived the energy spread in electron storage rings.

We assume that we have a linear system and know its response to a $\delta$-pulse excitation of amplitude $a$ at the time $t_0$

$$x(t) = ag(t - t_0) \ , \ \ g(\tau) = 0 \text{ for } \tau < 0. \tag{84}$$

If such excitations occur randomly in time with average frequency $f$ but with the same amplitude $a$, the resulting average of the square of the excursion (variance) is

$$\langle x^2 \rangle = a^2 f \int_{-\infty}^{\infty} g^2(t - t_0) dt. \tag{85}$$

This can be generalized for the case where the amplitude also has a random distribution of the form $\dot{n}(a)$

$$\langle x^2 \rangle = \int_0^{\infty} a^2 \dot{n}(a) da \int_{-\infty}^{\infty} g^2(t - t_0) dt. \tag{86}$$

This is Campbell's theorem which we now wish to apply to the case of quantum excitation by synchrotron radiation.

We have first to recapitulate some properties of the synchrotron radiation. The total power emitted by a particle of charge $e$ and energy $m_0 c^2 \gamma$ going on a trajectory with curvature $1/\rho$ through a magnet is

$$P_\gamma = \frac{2 c r_e m_0 c^2 \gamma^4}{3 \rho^2}, \tag{87}$$

where $r_e$ is the classical electron radius. The spectral distribution in photon energy $\epsilon$ is

$$\frac{d\dot{n}(\epsilon)}{d\epsilon} = \frac{P_\gamma}{\epsilon_c^2} \frac{\epsilon_c}{\epsilon} S(\epsilon/\epsilon_c) \text{ with } S(\epsilon/\epsilon_c) = \frac{9\sqrt{3}}{8\pi} \frac{\epsilon}{\epsilon_c} \int_{\epsilon/\epsilon_c}^{\infty} K_{5/3}(x) dx, \tag{88}$$

where we used the critical energy $\epsilon_c$

$$\epsilon_c = \frac{3}{4\pi} \frac{\lambda_{Comp}}{\rho} m_0 c^2 \gamma^3 \ , \ \ \lambda_{Comp} = \frac{h}{m_0 c}, \tag{89}$$

the Compton wave length $\lambda_{Comp}$, Planck's constant $h$ and the modified Bessel function $K_{5/3}$ of order 5/3. We have to know also the variance of this spectrum

$$\int_0^{\infty} \epsilon^2 \dot{n}(\epsilon) d\epsilon = \frac{55}{8\, 3^{3/2}} P_\gamma \epsilon_c \tag{90}$$

for which we give here only the result.

Next we need the response of the particle to a $\delta$-pulse excitation which we can get from Eq. (74) by setting the initial energy deviation to the loss due to the emission of a photon $\Delta E = -\epsilon$ and assume weak damping

$$g(t - t_0) = \Delta E = -\epsilon e^{-\alpha_E (t - t_0)} \cos(\omega_{s0}(t - t_0)) \tag{91}$$

the square of which can easily be integrated

$$\int_{t_0}^{\infty} g^2(t-t_0)dt = \int_0^{\infty} g^2(t)dt = \frac{1}{4\alpha_E}. \quad (92)$$

To get the variance of the energy distribution we have to insert the variance of the photon energy spectrum and the integral over the square of the response function into Campbell's theorem

$$\sigma_E^2 = \langle \Delta E^2 \rangle = \int_0^{\infty} \epsilon^2 \dot{n}(\epsilon) d\epsilon \int_0^{\infty} g^2(t)dt = \frac{55}{8\,3^{3/2}} \frac{P_\gamma \epsilon_c}{4\alpha_E} \quad (93)$$

which is the well known expression for the energy spread in an electron storage ring.

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