BEAM INSTABILITIES

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Abstract

Collective effects are introduced in the example of the longitudinal Robinson instability which can be generalized to cover all coupled bunch modes. We first introduce the concept of the impedance by approximating a cavity resonance by a RLC circuit. The response of such a resonator to pulse excitation gives the wake potential or Green function while a harmonic excitation reveals the concept and properties of the impedance. The interaction of a stationary circulating bunch with an impedance leads to an energy loss and to a shift of the incoherent synchrotron frequency. A bunch executing a synchrotron oscillation has spectral lines and harmonics of the revolution frequency with side bands spaced by the synchrotron frequency. The voltage induced in an impedance by these spectral lines can act back on the beam. A simple expression involving the impedances at the upper and lower side bands gives the growth rate of the Robinson instability. This can be generalized for a more complicated impedance, for the case of many bunches and also for higher modes of longitudinal oscillations.

1   INTRODUCTION

The motion of a single particle in a storage ring is determined by the external guide fields created by the dipole and quadrupole magnets and the RF system, by the initial conditions, and by the synchrotron radiation. The many particles contained in a high intensity beam represent a sizeable charge and current which act as a source of electro-magnetic fields called *self fields*. These fields are modified by the boundary conditions imposed by the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a *frequency shift* (change of the betatron or synchrotron fre-quency), to an increase of a small disturbance of the beam, i.e. an *instability*, or a *change of the particle distribution*, e.g. bunch lengthening. These phenomena are called *collective effects* since they are caused by a collective action of the many particles in the beam.

As an example we consider bunches in a storage ring going through a cavity; Fig. 1. Each bunch induces electromagnetic fields in this cavity which oscillate and slowly decay
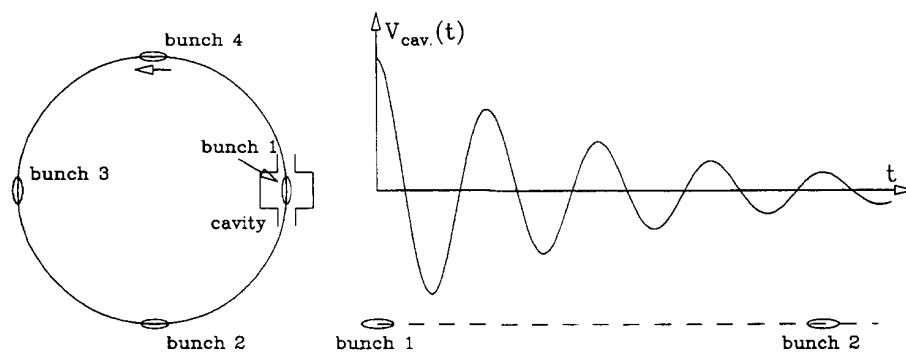

Figure 1: Field acting from one bunch to the next one

away. The next bunch, or the same bunch on the next turn, might find some field left and will be influenced by it. The phase of the field seen in the next turn can be such that a small initial synchrotron oscillation amplitude of the bunch is increased. In each turn the oscillation is amplified resulting in an exponentially growing instability.

In most cases the fields created by the beam are small compared to the guide fields and their effects can be treated as a *perturbation*. This is done in three steps:

a) First, the motion in the guide field and the stationary particle distribution are calculated.

b) A small disturbance of the bunch from its stationary motion is considered (betatron or synchrotron oscillation). The fields due to such a disturbance are determined taking the boundary condition imposed by the beam surroundings (impedance) into account.

c) The effect of these fields on the initial disturbance is investigated. If the amplitude of the disturbance is increased we have an *instability*, if it is decreased we have *damping*, or, if the frequency of the mode of oscillation is changed, we have a *frequency shift*.

For the case of small self-fields considered here the particle distribution in the bunch is given by external conditions (machine parameter, initial condition, synchrotron radiation) and is usually Gaussian in electron machines. As disturbances of the stationary distribution we consider some modes of oscillation which are orthogonal to each other such that the stability of each mode can be treated independently.

Strong self-fields, however, modify the particle distribution and also the modes of oscillation such that they are no longer independent. A self-consistent solution has to be found in this case and this can be a rather difficult task. It is usually only attempted for the case of bunch lengthening where the longitudinal particle distribution in the presence of an impedance is obtained. The beam position monitors and the transverse impedance can usually not "resolve" the local transverse particle distribution. The latter is therefore only in exceptional cases of importance for beam instabilities.

We distinguish between *single* and *multi-traversal* collective effects. For the first kind no memory of the induced field over one revolution or over the time interval between the passage of adjacent bunches is assumed. An example of a single-traversal effect is bunch lengthening. For multi-traversal effects the impedance has to have a memory such that one bunch can influence the next one or itself after one revolution. Such a memory can be provided by cavity-like objects with a relatively large quality factor Q.

Finally, we distinguish between *longitudinal* and *transverse* effects. In the first case a longitudinal impedance influences the synchrotron oscillation such that its amplitude grows or its frequency changes. The transverse impedance has a corresponding effect on the horizontal or vertical betatron oscillations.

Here, we will concentrate on longitudinal collective effects and derive the Robinson instability in some detail.

2 IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR

2.1 Cavity resonance

Impedances and wake potentials have been treated extensively in the literature. We recapitulate here some of their essential properties on the simple case of a cavity resonance.

Cavities are the most likely objects to cause coupled-bunch mode instabilities since the induced fields oscillate for a relatively long time and provide a memory over the time interval between bunch passages. Such a cavity can be of a form which resembles an RCL circuit as shown in Fig. 2, and can be treated as such. The RCL circuit has a shunt impedance R_s, an inductance L and a capacity C. In a real cavity these three parameters cannot easily be separated. For this reason we use some other related parameters which can be measured directly: The *resonance frequency* ω_r, the *quality factor* Q and the
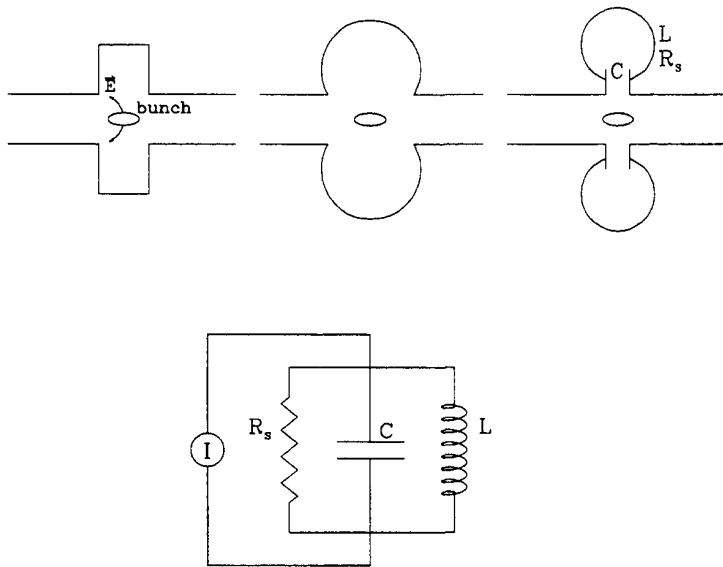
Figure 2: RLC circuit equivalent to a cavity resonance

damping rate α:

$$\omega_r = \frac{1}{\sqrt{LC}} \ , \quad Q = R_s\sqrt{\frac{C}{L}} = \frac{R_s}{L\omega_r} = R_sC\omega_r \ , \quad \alpha = \frac{\omega_r}{2Q}.$$

If this circuit is driven by a current I the voltages across each element are

$$V_R = I_R R_s \ , \quad V_C = \frac{1}{C}\int I_C dt \ , \quad V_L = L\frac{dI_L}{dt}$$

and have the relations to the currents

$$V_R = V_C = V_L = V \ , \quad I_R + I_C + I_L = I.$$

Differentiating with respect to t gives

$$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}}{R_s} + C\ddot{V} + \frac{V}{L}.$$

Using $L = R_s/(\omega_r Q)$ and $C = Q/(\omega_r R_s)$ gives the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \frac{\omega_r R_s}{Q}\dot{I}.$$

The solution of the homogeneous equation is a damped oscillation

$$V(t) = \hat{V}e^{-\alpha t}\cos\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}} + \phi\right)$$

or

$$V(t) = e^{-\alpha t}\left(A\cos\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right) + B\sin\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right)\right).$$
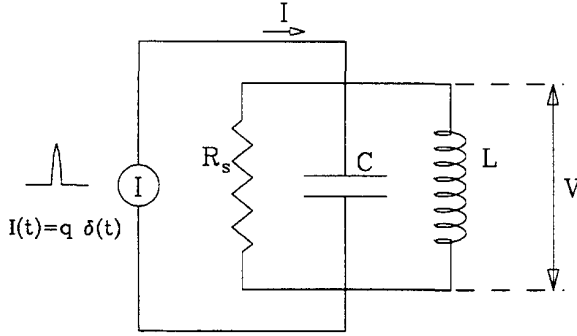
Figure 3: RLC circuit driven by a pulse

2.2 Wake potential

We now calculate the response of the RCL circuit (representing a cavity) to a delta function pulse (very short bunch), Fig. 3,

$$I(t) = q\delta(t).$$

The charge q induces a voltage in the capacity

$$V(0^+) = \frac{q}{C} = \frac{\omega_r R_s}{Q} q.$$

The energy stored in the capacitor

$$U = \frac{q^2}{C} = \frac{\omega_r R_s}{2Q} q^2 = \frac{V(0^+)}{2} q = k_{pm} q^2,$$

must be equal to the energy lost by the charge. Here we introduced the *parasitic mode loss factor* of a point charge

$$k_{pm} = \frac{U}{q^2} = \frac{\omega_r R_s}{2Q}$$

which is the energy loss normalized for the charge. The charged capacitor C will now discharge first through the resistor R_s and then also through the inductance L

$$\dot{V}(0^+) = -\frac{\dot{q}}{C} = -\frac{I_R}{C} = \frac{1}{C}\frac{V(0^+)}{R_s} = -\frac{\omega_r^2 R_s}{Q^2} q = \frac{2\omega_r k_{pm}}{Q} q.$$

The voltage in this resonance circuit has now the initial conditions

$$V(0^+) = 2k_{pm}q \quad \text{and} \quad \dot{V}(0^+) = \frac{2\omega_r k_{pm}}{Q} q.$$

We take the solution of the homogeneous differential equation and its derivative

$$V(t) = e^{-\alpha t}\left(A\cos\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\, t\right) + B\sin\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\, t\right)\right)$$
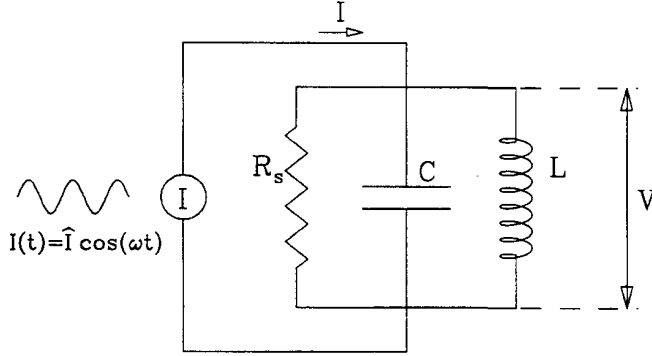
Figure 4: RLC circuit driven by a harmonic excitation

$$\dot{V}(t) \;=\; e^{-\alpha t}\left(\left(-A\alpha + B\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,\right)\cos\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right)\right.$$
$$\left. -\left(B\alpha + A\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,\right)\sin\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right)\right)$$

and satisfy the above initial conditions by

$$A = 2k_{pm}q \quad \text{and} \quad -A\alpha + B\omega_r\sqrt{1 - \frac{1}{4Q^2}} = -\frac{2\omega_r k_{pm}}{Q}q.$$

We obtain the voltage in a resonator circuit excited at the time $t = 0$ by a δ-pulse $I(t) = q\delta t$

$$V(t) = 2qk_{pm}e^{-\alpha t}\left(\cos\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right) - \frac{\sin\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right)}{2Q\sqrt{1 - \frac{1}{4Q^2}}}\right).$$

This voltage is induced by a charge q going through the cavity at the time $t = 0$. A second point charge q' going through the cavity at a later time t will gain or loose the energy $U = q'V(t)$. This energy gain/loss per unit source and unit probe charge is called the *wake potential* of a point charge or also the *Green function* $G(t)$. For our resonator (cavity resonance) we have

$$G(t) = 2k_{pm}e^{-\alpha t}\left(\cos\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right) - \frac{\sin\left(\omega_r\sqrt{1 - \frac{1}{4Q^2}}\,t\right)}{2Q\sqrt{1 - \frac{1}{4Q^2}}}\right)$$

which for a large quality factor $Q \gg 1$ simplifies to

$$G(t) \approx 2k_{pm}e^{-\alpha t}\cos\left(\omega_r t\right).$$

2.3 Impedance

We use now a *harmonic* excitation of the circuit with a current $I = \hat{I}\cos(\omega t)$, Fig. 4. This is described by the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = -\frac{\omega_r R_s}{Q}\hat{I}\omega\sin(\omega t).$$

The solution of the homogeneous equation is a damped oscillation which disappears after some time. We are left with the particular solution of the form $V(t) = A\cos(\omega t) + B\sin(\omega t)$. Inserting this into the differential equation and separating cosine and sine terms gives

$$(\omega_r^2 - \omega^2)A + \frac{\omega_r\omega}{Q}B = 0 \quad \text{and} \quad (\omega_r^2 - \omega^2)B - \frac{\omega_r\omega}{Q}A = -\frac{\omega_r\omega R_s}{Q}\hat{I}.$$

The voltage induced by the harmonic excitation of the resonator becomes

$$V(t) = \hat{I}R_s\frac{\cos(\omega t) - Q\frac{\omega_r^2-\omega^2}{\omega_r\omega}\sin(\omega t)}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}.$$

This voltage has a cosine term which is *in phase* with the exciting current. It can absorb energy and is called the *resistive* term. The sine term of the voltage is *out of phase* with the exciting current and does not absorb energy, it is called the *reactive* term. The ratio between the voltage and current is called *impedance*. It is a *function of frequency* ω and has a resistive part $Z_r(\omega)$ and a reactive part $Z_i(\omega)$

$$Z_r(\omega) = R_s\frac{1}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2} \quad , \quad Z_i(\omega) = -R_s\frac{Q\frac{\omega_r^2-\omega^2}{\omega_r\omega}}{1 + Q^2\left(\frac{\omega_r^2-\omega^2}{\omega_r\omega}\right)^2}.$$

The resonance excited by a current of the form $I(t) = \hat{I}\cos(\omega t)$ or $I(t) = \hat{I}\sin(\omega t)$ results in a voltage $V(t)$

$$I(t) = \hat{I}\cos(\omega t) \;\to\; V(t) = \hat{I}\left(Z_r(\omega)\cos(\omega t) + Z_i(\omega)\sin(\omega t)\right),$$

$$I(t) = \hat{I}\sin(\omega t) \;\to\; V(t) = \hat{I}\left(Z_r(\omega)\sin(\omega t) - Z_i(\omega)\cos(\omega t)\right).$$

2.4 Complex notation

We have used a harmonic excitation of the form

$$I(t) = \hat{I}\cos(\omega t) = \hat{I}\frac{e^{j\omega t} + e^{-j\omega t}}{2} \quad \text{with} \;\; 0 \le \omega \le \infty,$$

using positive frequencies only. A complex notation

$$I(t) = \hat{I}e^{j\omega t} \quad \text{with} \;\; -\infty \le \omega \le \infty$$

involving positive and negative frequencies leads to more compact expressions and is often convenient. We take the differential equation

$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 = \frac{\omega_r R_s}{Q}\dot{I}$$

of the resonator voltage with the excitation $I(t) = \hat{I}\exp(j\omega t)$ and seek a solution of the form $V(t) = V_0\exp(j\omega t)$, where V_0 is in general complex and get

$$-\omega^2 V_0 e^{j\omega t} + j\frac{\omega_r - \omega}{Q}V_0 e^{j\omega t} + \omega_r^2 V_0 e^{j\omega t} = j\frac{\omega_r\omega R_s}{Q}\hat{I}e^{j\omega t}.$$
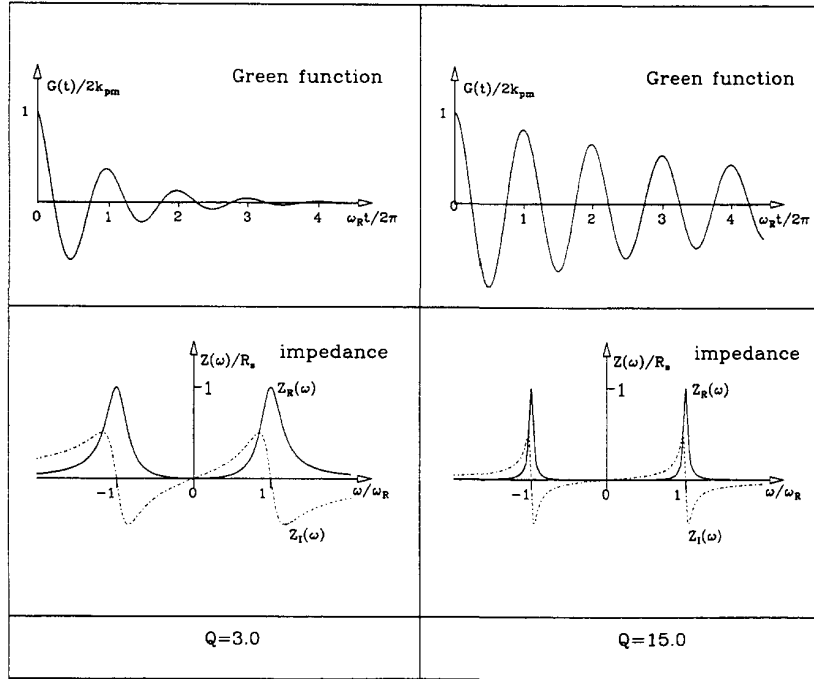
Figure 5: Green function and impedance of a resonance

The impedance, defined as the ratio V/I, is given by

$$Z(\omega) = \frac{V_0}{\hat{I}} = R_s \frac{j\frac{\omega_r \omega}{Q}}{\omega_r^2 - \omega^2 + jQ\frac{\omega_r \omega}{Q}} = R_s \frac{1 - jQ\frac{\omega^2 - \omega_r^2}{\omega \omega_r}}{1 + Q^2 \left(\frac{\omega^2 - \omega_r^2}{\omega \omega_r}\right)^2} = Z_r(\omega) + jZ_i(\omega)$$

and has a real and an imaginary part, Fig. 5. For a large quality factor the impedance is only large for $\omega \approx \omega_r$ or $|\omega - \omega_r|/\omega_r = |\Delta\omega|\omega_r \ll 1$ and can be simplified to

$$Z(\omega) \approx R_s \frac{1 - j2Q\frac{\Delta\omega}{\omega_r}}{1 + 4Q^2 \left(\frac{\Delta\omega}{\omega_r}\right)^2}.$$

The resonator impedance has some specific properties:

$$\omega = \omega_r \rightarrow Z_r(\omega_r) \text{ has a maximum, } Z_i(\omega_r) = 0$$
$$|\omega| < \omega_r \rightarrow Z_i(\omega) > \omega_r \text{ (inductive)}$$
$$|\omega| > \omega_r \rightarrow Z_i(\omega) < 0 \text{ (capacitive)}$$

and some properties which apply to any impedance or wake potential

$$Z_r(\omega) = Z_r(-\omega) , \quad Z_i(\omega) = -Z_i(-\omega),$$
$$Z(\omega) = \int_{-\infty}^{\infty} G(t)e^{-j\omega t}dt, \quad Z(\omega) = \text{Fourier transform of } G(t),$$
$$t < 0 \rightarrow G(t) = 0 \text{ no fields before particle arrives.}$$

Caution; sometimes one uses $I(t) = \hat{I}e^{-i\omega t}$ instead of $I(t) = \hat{I}e^{j\omega t}$, this reverses the sign $Z_i(\omega)$.
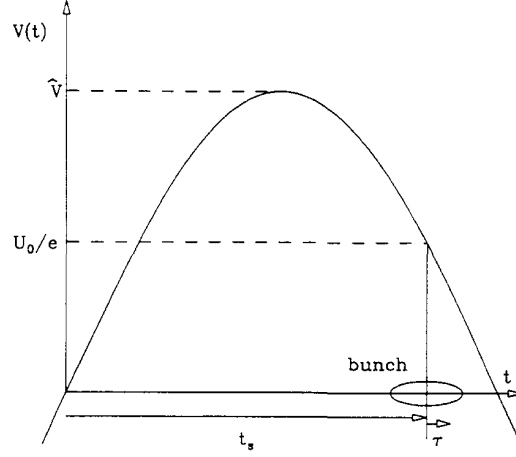
Figure 6: Longitudinal beam dynamics impedance

2.5 Review of the longitudinal dynamics

A particle with a momentum deviation Δp has a different closed orbit which is radially displaced by $\Delta x = D_x \Delta p / p$ with D_x being the dispersion. As a result the orbit length L, the revolution time T_0 and the revolution frequency ω_0 are changed

$$\frac{\Delta L}{L} = \alpha_c \frac{\Delta p}{p} \ , \quad \frac{\Delta \omega_0}{\omega_0} = -\frac{\Delta T_0}{T_0} = -\left(\alpha_c - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p} = -\eta_c \frac{\Delta p}{p}$$

with α_c being the momentum compaction and $\eta_c = \alpha_c - 1/\gamma^2$. There is a transition energy $E_T = m_0 c^2 \gamma_T$ with $\gamma_T = 1/\alpha_c^2$ for which the dependence of the revolution frequency on momentum (or energy) changes sign

$$E > E_T \ \rightarrow \ \frac{1}{\gamma^2} < \alpha_c \ \rightarrow \ \eta_c > 1 \ \rightarrow \ \omega_0 \text{ decreases with } \Delta E$$

$$E < E_T \ \rightarrow \ \frac{1}{\gamma^2} > \alpha_c \ \rightarrow \ \eta_c < 1 \ \rightarrow \ \omega_0 \text{ increases with } \Delta E \ .$$

We will assume that the particles are ultra relativistic in which case $\Delta p / p \approx \Delta E / E = \epsilon$. In the presence of an RF system and of an energy loss per turn U due to synchrotron radiation or an impedance, a circulating particle has a gain or loss δE in energy of

$$\delta E = e\hat{V} \sin(h\omega_0(t_s + \tau)) - U$$

or in relative energy $\delta E / E = \delta \epsilon$

$$\frac{\delta E}{E} = \delta \epsilon = \frac{e\hat{V} \sin(h\omega_0(t_s + \tau))}{E} - \frac{U}{E}.$$

with t_s being the synchronous arrival time of the particle in the cavity and τ the deviation from it as illustrated in Fig. 6. We introduce the synchronous phase angle $\phi_s = h\omega_0 t_s$ and assume $\tau \ll T_0$ which allows us to develop the trigonometric function

$$\delta \epsilon = \frac{e\hat{V} \sin(\phi_s)}{E} + \frac{h\omega_0 e\hat{V} \cos \phi_s}{E}\tau - \frac{U}{E}.$$

The energy gain per turn is very small $\delta E \ll E$ and we can make a smooth approximation

$$\frac{\delta E}{E} = \dot{\epsilon}T_0 = \dot{\epsilon}\frac{2\pi}{\omega_0}$$

$$\dot{\epsilon} = \frac{\omega_0 e \hat{V} \sin \phi_s}{2\pi E} + \frac{\omega_0^2 h e \hat{V} \cos \phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U}{E}. \tag{1}$$

The energy loss U suffered by a particle is in general a function of its relative energy deviation ϵ and its deviation τ from the synchronous and, to first order, can be developed as

$$U(\epsilon, \tau) \approx U_0 + \frac{\partial U}{\partial E}\Delta E + \frac{\partial U}{\partial t}\tau.$$

This leads to an expression for the time derivative of the energy loss

$$\dot{\epsilon} = \frac{\omega_0 e \hat{V} \sin \phi_s}{2\pi E} + \frac{\omega_0^2 h e \hat{V} \cos \phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{U_0}{E} - \frac{\omega_0}{2\pi}\frac{dU}{dE}\epsilon - \frac{\omega_0}{2\pi}\frac{1}{E}\frac{dU}{dt}\tau.$$

To have equilibrium for the synchronous particle $\epsilon = 0$, $\tau = 0$ we have

$$U_0 = e\hat{V} \sin \phi_s.$$

With this and using $\dot{\tau} = \omega_0 \Delta T_0/2\pi = \eta_c \epsilon$ we get a system of two first order differential equations

$$\begin{aligned} \dot{\epsilon} &= \omega_0^2 \frac{h e \hat{V} \cos \phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{dU}{dE}\epsilon - \frac{1}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\tau \\ \dot{\tau} &= \eta_c \epsilon. \end{aligned}$$

They can be combined into one second-order equation

$$\ddot{\epsilon} + \frac{\omega_0}{2\pi}\frac{dU}{dE}\dot{\epsilon} - \frac{\omega_0^2 h \eta_c e \hat{V} \cos \phi_s}{2\pi E}\epsilon - \frac{\eta_c}{E}\frac{\omega_0}{2\pi}\frac{dU}{dt}\epsilon = 0$$

which is the equation of a damped oscillation. Using

$$\omega_{s0}^2 = -\omega_0^2 \frac{h \eta_c e \hat{V} \cos \phi_s}{2\pi E} \quad , \quad \alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU}{dE} \quad ,$$

seeking a solution of the form $e^{j\omega t}$, and assuming $\alpha_s \ll \omega_{s0}$ we get

$$-\omega^2 + j\omega\alpha_s + (\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta_c}{E}\frac{dU}{dt}) = 0$$

$$\omega = j\alpha_s \pm \sqrt{(\omega_{s0}^2 + \frac{\omega_0}{2\pi}\frac{\eta_c}{E}\frac{dU}{dt}) - \alpha_s^2} \approx j\alpha_s \pm (\omega_{s0} + \frac{1}{2}\frac{\omega_0}{\omega_{s0}}\frac{\eta_c}{E}\frac{dU}{dt}).$$

Calling

$$\Delta\omega_i = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{\eta_c}{\omega_{s0}E}\frac{dU}{dt}$$

gives

$$\epsilon = A\left(e^{(-\alpha_s + j(\omega_{s0} + \Delta\omega_i)t)} + Be^{(-\alpha_s - j(\omega_{s0} + \Delta_i)t)}\right).$$

For the initial conditions $\epsilon(t) = \hat{\epsilon}$, $\dot{\epsilon}(0) = -\alpha_s\hat{\epsilon}$ we get $A = B = \hat{\epsilon}/2$ and

$$\epsilon(t) = \hat{\epsilon}\, e^{-\alpha_s t}\cos((\omega_{s0} + \Delta\omega_i)t).$$

In the absence of any energy loss U we have

$$\epsilon(t) = \hat{\epsilon}\cos(\omega_{s0}t + \phi)$$

with

$$\omega_{s0}^2 = -\omega_0^2 \frac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

In order to get a stable oscillation we need $\omega_{s0}^2 > 0$ which leads to the conditions

$$E > E_T \;\; \eta_c < 0 \;\; \to \cos\phi_s < 0 \;\; , \;\; E < E_T \;\; \eta_c > 0 \;\; \to \cos\phi_s > 0.$$

For stability in the presence of an energy loss U we need in addition

$$\alpha_s = \frac{1}{2}\frac{\omega_0}{2\pi}\frac{dU}{dE} > 0.$$

In other words the energy loss U has to increase for a positive energy deviation of the beam.

3   A STATIONARY BUNCH INTERACTING WITH AN IMPEDANCE

3.1   Spectrum of a stationary bunch

We consider now a bunch which represents in a single traversal the current $I(t)$. Its Fourier transform is

$$\tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t)e^{-j\omega t}dt. \tag{2}$$

We assume the form of the bunch to be symmetric

$$I(-t) = I(t)$$

which leads to a Fourier transform having only a real part and being symmetric in ω

$$\tilde{I}(-\omega) = \tilde{I}(\omega).$$

This assumption is used for convenience to reduce the number of terms which have to be carried along in some calculation. Since, in most practical applications, the bunches are to a good approximation symmetric, this represents a minor restriction which could easily be removed. The currents of a bunch with Gaussian distribution in time and frequency domain are illustrated in Fig. 7 and given by the expressions

$$I(t) = \frac{q}{\sqrt{2\pi}\sigma_t}e^{-\frac{t^2}{2\sigma_t^2}} \;\; ; \;\; \tilde{I}(\omega) = qe^{-\frac{\omega^2}{2\sigma_\omega^2}} \tag{3}$$

where $q = N_b e$ is the total charge of the N_b particles in a bunch. The rms width of the bunch and its spectrum are σ_t and σ_ω which are related by

$$\sigma_t = \frac{1}{\sigma_\omega}.$$

Next we investigate the case of a circulating bunch having repetitive passages at a given location. For a stationary bunch having no synchrotron oscillations the observed current is of the form

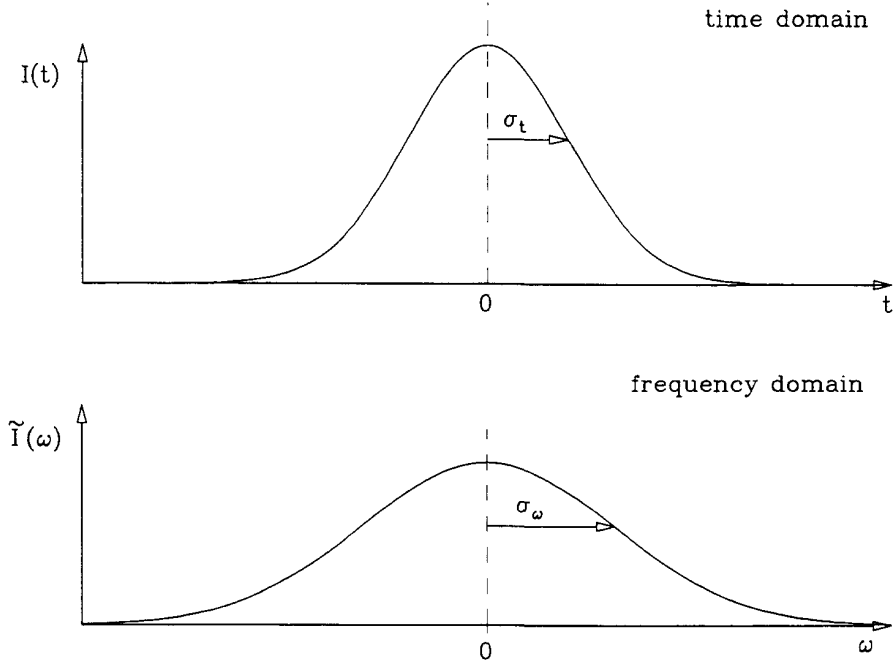$$I_k(t) = \sum_{k=-\infty}^{\infty} I(t - kT_0).$$

Figure 7: Single passage of a bunch in time and frequency domain

This is not a very useful expression for applications. Since the current is periodic it is natural to express it in a Fourier series

$$I_k(t) = I_0 + 2 \sum_1^\infty I_p \cos(p\omega_0 t) \tag{4}$$

with

$$I_p = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} I(t) \cos(p\omega_0 t) dt \text{ and } I_0 = \langle I \rangle = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} I(t) dt = \frac{q}{T_0}. \tag{5}$$

Comparing the Fourier transform Eq. (2) with the terms of the Fourier series Eq. (5) we find the relation

$$I_p = \frac{\omega_0}{\sqrt{2\pi}} \tilde{I}(p\omega_0).$$

For a Gaussian bunch Eq. (3) we get

$$I_p = \frac{q}{T_0} e^{-\frac{p^2 \omega_0^2}{2\sigma_\omega^2}}.$$

At low frequencies we have $I_p \approx I_0$. The current in time and frequency domain of such a circulating bunch is shown in Fig. 8.

It should be noted that for the frequency component I_p of the current we use a value adapted to positive and negative frequencies. If only positive frequencies are used the current component at $p\omega_0$ is $2I_p$.
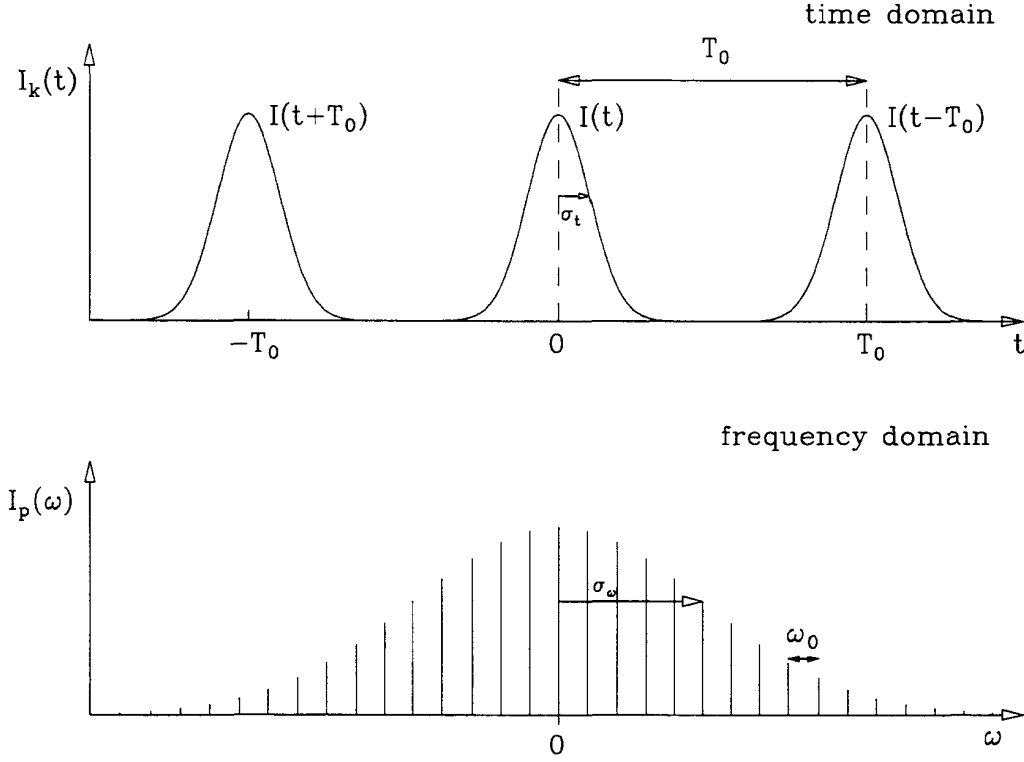
time domain

frequency domain

Figure 8: Multiple passage of a bunch in time and frequency domain

3.2 Voltage induced by the stationary bunch

In the presence of a cavity resonance or any *general impedance* $Z(\omega)$ the circulating stationary bunch induces a voltage which is in frequency domain

$$\tilde{V}_k(\omega) = \tilde{I}_k(\omega)Z(\omega) = \omega_0 \sum_{p=-\infty}^{\infty} \tilde{I}(\omega)\delta(\omega - p\omega_0)Z(\omega)$$

and in time domain

$$\begin{aligned}
V_k(t) &= \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{I}(\omega)\delta(\omega - p\omega_0)Z(\omega)e^{j\omega t}d\omega \\
&= \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \tilde{I}(p\omega_0)Z(p\omega_0)e^{jp\omega_0 t} = \sum_{p=-\infty}^{\infty} I_p Z(p\omega_0)e^{jp\omega_0 t}.
\end{aligned}$$

By combining positive and negative frequencies and observing the symmetry conditions $Z_r(-\omega) = Z_r(\omega)$, $Z_i(-\omega) = -Z_i(\omega)$ and the fact that $Z(0) = 0$ we get a real expression

$$V_k(t) = 2 \sum_{p=1}^{\infty} I_p \left(Z_r(p\omega_0) \cos(p\omega_0 t) - Z_i(p\omega_0) \sin(p\omega_0 t) \right). \tag{6}$$

We calculate the average induced voltage $\langle V \rangle$ seen by the particle in the bunch

$$\langle V \rangle = \frac{1}{I_0 T_0} \int_{-T_0/2}^{T_0/2} I_k(t) V_k(t) dt.$$

With the expressions Eq. (4) for the current and Eq. (6) for the voltage we get

$$\langle V \rangle = \frac{4}{I_0 T_0} \sum_{p=1}^{\infty} \sum_{p'=1}^{\infty} I_p' I_p$$

$$\left(Z_r(p\omega_0) \int_{-T_0/2}^{T_0/2} \cos(p'\omega_0 t) \cos(p\omega_0 t) dt - Z_r(p\omega_0) \int_{-T_0/2}^{T_0/2} \cos(p'\omega_0 t) \sin(p\omega_0 t) dt \right).$$

The first integral vanishes except for $p' = p$ in which case it has the value $T_0/2$, and the second integral always vanishes. This leads to

$$\langle V \rangle = \frac{1}{I_0} \sum_{p=-\infty}^{\infty} |I_p|^2 Z_r(p\omega_0) = \frac{2}{I_0} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0). \tag{7}$$

We will also need the average voltage slope

$$\langle \frac{dV}{dt} \rangle = \frac{1}{I_0 T_0} \int_{-T_0/2}^{T_0/2} I_k(t) \frac{dV_k(t)}{dt} dt.$$

With the same method we used for the average voltage we obtain

$$\langle \frac{dV}{dt} \rangle = -\frac{\omega_0}{I_0} \sum_{p=-infty}^{\infty} p|I_p|^2 Z_i(p\omega_0) = -\frac{2\omega_0}{I_0} \sum_{p=1}^{\infty} p|I_p|^2 Z_i(p\omega_0).$$

3.3 Energy loss per turn of a stationary circulating bunch

The energy W_b lost by the whole circulating stationary bunch in one turn due to the impedance $Z(\omega)$ can be obtained from the average voltage Eq. (7)

$$W_b = q\langle V \rangle = \frac{2q}{I_0} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0)$$

where $q = eN_b$ is the total charge of the bunch. The average energy loss U of a particle in the bunch is

$$U = \frac{W_b}{N_b} = \frac{2e}{I_0} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0) = \frac{2T_0}{N_b} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0).$$

We can normalize the loss W_b by the square of charge (the charge inducing the voltage and the same charge suffering an energy loss) to get the so-called parasitic mode loss factor or a bunch

$$k_{pm} = \frac{W_b}{q^2} = \frac{U}{eq} = \frac{2}{qI_0} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0) = \frac{2T_0}{q^2} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0).$$

This parameter depends on the bunch length. For a short bunch the spectrum extends to higher frequencies. The parameter k_{pm} is therefore expected to increase with decreasing bunch length.

If the impedance is broad band and does not contain resonances of bandwidth smaller than the revolution frequency, the above sum can be approximated by an integral

$$k_{pm} \approx \frac{1}{q^2} \int_{-\infty}^{\infty} |\tilde{I}(\omega)|^2 Z_r(\omega) d\omega.$$

3.4 Incoherent synchrotron frequency shift

We take now the case of a stationary bunch in the presence of an impedance $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$. As we saw before, the bunch induces an average voltage in the resistive part of the impedance

$$\langle V \rangle = \frac{2}{I_0} \sum_{p=1}^{\infty} |I_p|^2 Z_r(p\omega_0) \tag{8}$$

and an averaged voltage slope in the reactive part

$$\langle \frac{dV}{dt} \rangle = -\frac{2\omega_0}{I_0} \sum_{p=1}^{\infty} p|I_p|^2 Z_i(p\omega_0), \tag{9}$$

both being independent of the energy. We have to include these voltages in the equation of the synchrotron motion

$$\dot{\epsilon} = \frac{e\hat{V} \sin\phi_s \omega_0}{2\pi E} + \frac{\omega_0^2 he\hat{V} \cos\phi_s}{2\pi E}\tau - \frac{\omega_0}{2\pi}\frac{e\langle V \rangle}{E} - \frac{\omega_0 e}{2\pi E}\langle \frac{dV}{dt} \rangle\tau.$$

With the condition $e\hat{V} \sin\phi_s = e\langle V \rangle$ we find

$$\dot{\epsilon} = \omega_0^2 \frac{he\hat{V} \cos\phi_s}{2\pi E}\tau + \frac{\omega_0 e}{2\pi E}\langle \frac{dV}{dt} \rangle\tau$$

$$\dot{\tau} = \eta_c\epsilon,$$

or, combined into a second-order equation,

$$\ddot{\epsilon} + -\left(\frac{\omega_0^2 h\eta_c e\hat{V} \cos\phi_s}{2\pi E} + \frac{\eta_c}{E}\frac{\omega_0}{2\pi}e\langle \frac{dV}{dt} \rangle \right)\epsilon = 0.$$

The solution is an undamped oscillation with the frequency given by

$$\omega_s^2 = \omega_{s0}^2 + \frac{2\omega_0^2 \eta_c e}{2\pi E I_0} \sum_{p=1}^{\infty} p|I_p|^2 Z_i(p\omega_0)$$

which can be written as

$$\omega_s^2 = \omega_{s0}^2 \left(1 + \frac{2\omega_0^2 \eta_c e}{2\pi E \omega_{s0}^2 I_0} \sum_{p=1}^{\infty} p|I_p|^2 Z_i(p\omega_0) \right) = \omega_{s0}^2 \left(1 + \frac{2}{h\hat{V} \cos\phi_s I_0} \sum_{p=1}^{\infty} p|I_p|^2 Z_i(p\omega_0) \right).$$

There is a shift of the incoherent synchrotron frequency. For a small effect this shift can be expressed as

$$\frac{\Delta\omega_{si}}{\omega_{s0}} = \frac{1}{I_0 h\hat{V} \cos\phi_s} \sum_{p=1}^{\infty} pI_p^2 Z_i(p\omega_0). \tag{10}$$

4   INTERACTION OF AN OSCILLATING BUNCH WITH A CAVITY

4.1   Spectrum of an oscillating bunch

We consider now a bunch which executes a synchrotron oscillation with frequency $\omega_s = \omega_0 Q_s$. This results in a modulation of its passage time t_k at a cavity in successive turns k as illustrated in Fig. 9
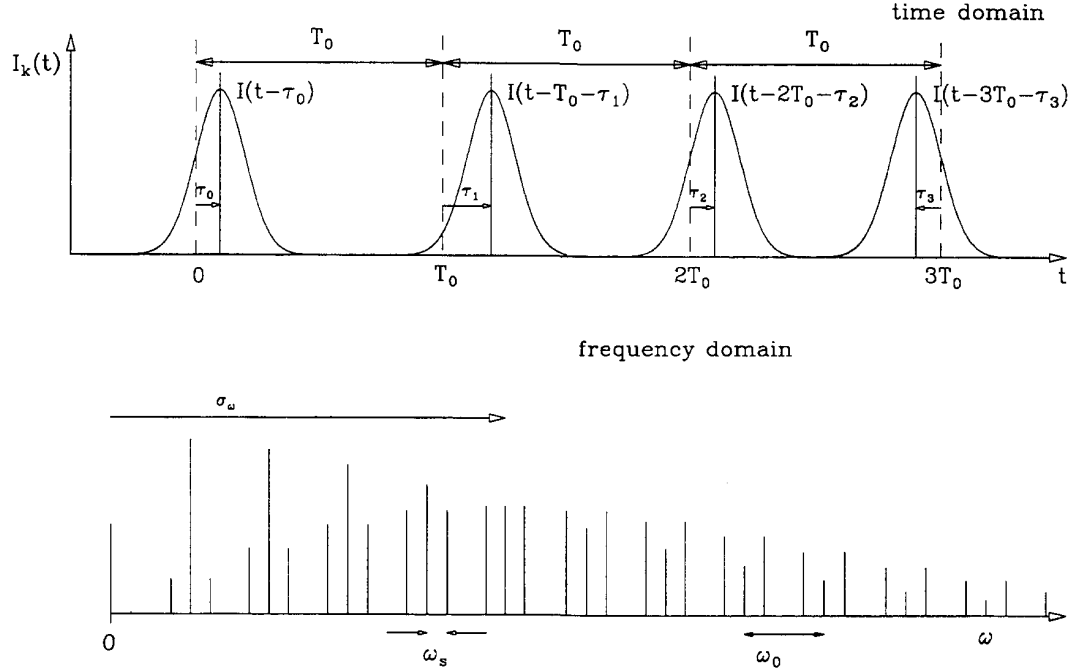
$$t_k = kT_0 + \tau_k,$$

Figure 9: Oscillating bunch in time and frequency domain

with

$$\tau_k = \hat{\tau} \cos(2\pi Q_s k),$$

where k is the revolution number and $\hat{\tau}$ the amplitude of the modulation. The current represented by this oscillating bunch is given in time domain by

$$I_k(t) = \sum_{-\infty}^{\infty} I(t - kT_0 - \tau_k).$$

To get the Fourier transform of this current we use again the shift theorem and the symmetry condition $I(t) = I(-t)$

$$\tilde{I}_k(\omega) = \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} e^{-j\omega(kT_0 + \tau_k)}.$$

We assume now that the oscillation is small $\omega\hat{\tau} \ll 1$ for all frequencies ω contained in the bunch spectrum $\tilde{I}(\omega)$ and approximate

$$\begin{aligned}
\tilde{I}_k(\omega) &= \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} e^{-j\omega(kT_0)} \left(1 - j\omega\hat{\tau} \cos(2\pi Q_s k)\right) \\
&= \tilde{I}(\omega) \sum_{k=-\infty}^{\infty} e^{-j\omega kT_0} - j\frac{\omega\hat{\tau}}{2}\tilde{I}(\omega \sum_{k=-\infty}^{\infty} \left(e^{-jk(\omega T_0 - 2\pi Q_s)} + e^{-jk(\omega T_0 + 2\pi Q_s)}\right).
\end{aligned}$$

We use again the relations

$$\sum_{k=-\infty}^{\infty} e^{-jkx} = 2\pi \sum_{p=-\infty}^{\infty} \delta(x - 2\pi p) \quad \text{and} \quad \delta(ax) = \frac{1}{a}\delta(x)$$

and get

$$\tilde{I}_k(\omega) = \omega_0 \tilde{I}(\omega) \left[\sum_{p=-\infty}^{\infty} \delta(\omega - p\omega_0) - j\frac{\omega\hat{\tau}}{2} \sum_{p=-\infty}^{\infty} (\delta(\omega - p\omega_0 - \omega_s) + \delta(\omega - p\omega_0 + \omega_s)) \right].$$

(11)

This is the Fourier-transformed current of a bunch executing a synchrotron oscillation. It is represented by a line spectrum having spectral lines at harmonics of the revolution frequency $p\omega_0$ due to the stationary bunch motion and side bands caused by the bunch oscillation. This is not astonishing since the stationary motion is periodic which leads to a simple line spectrum shown in an earlier section. The modulation of the passage time with the synchrotron frequency is expected to lead to side bands spaced by ω_s from the revolution harmonics. One would think that the above spectrum could be obtained directly using the theory of phase modulation without the relatively lengthy derivation given above. However there is a subtlety concerning the size of the spectral lines which will be discussed below. We prefer to get this current in time domain and apply the inverse Fourier transform

$$I_k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{I}_k(\omega) e^{j\omega t} d\omega$$

(12)

to the above expression, so obtaining

$$I_k(t) = \frac{\omega_0}{\sqrt{2\pi}} \sum_{p=-\infty}^{\infty} \left[\tilde{I}(p\omega_0) e^{jp\omega_0 t} - \right.$$

$$\left. j\frac{\omega_0}{\sqrt{2\pi}} \frac{\omega_0\hat{\tau}}{2} \left((p - Q_s)\tilde{I}((p - Q_s)\omega_0) e^{j(p-Q_s)\omega_0 t} + (p + Q_s)\tilde{I}((p + Q_s)\omega_0) e^{j(p+Q_s)\omega_0 t} \right) \right].$$

This is a relatively complicated way to express the current of an oscillating bunch but is, however, of some use for certain calculations. We can simplify it with some approximations. The bunch length is usually much shorter than the circumference of the machine. As a consequence its spectrum is much larger than the revolution frequency ω_0. We can therefore neglect the very small difference in the bunch spectrum between the harmonics of the revolution frequency and its side band

$$\tilde{I}(p\omega_0) \approx \tilde{I}((p + Q_s)\omega_0) \approx \tilde{I}((p - Q_s)\omega_0) = \frac{\sqrt{2\pi}}{2\omega_0} I_p.$$

We use now the current component I_p at the frequency $p\omega_0$ instead of the Fourier transform of $\tilde{I}(p\omega_0)$, and represent the current of an oscillating bunch as a standard Fourier series

$$I_k(t) = \sum_{p=-\infty}^{\infty} I_p \left[e^{jp\omega_0 t} - j\frac{\omega_0\hat{\tau}}{2} \left((p - Q_s) e^{j(p-Q_s)\omega_0 t} + (p + Q_s) e^{j(p+Q_s)\omega_0 t} \right) \right].$$

By combining terms with positive and negative values of the summing parameter p and using the fact that for a symmetric bunch we have $\tilde{I}(\omega) = \tilde{I}(-\omega)$ we can express
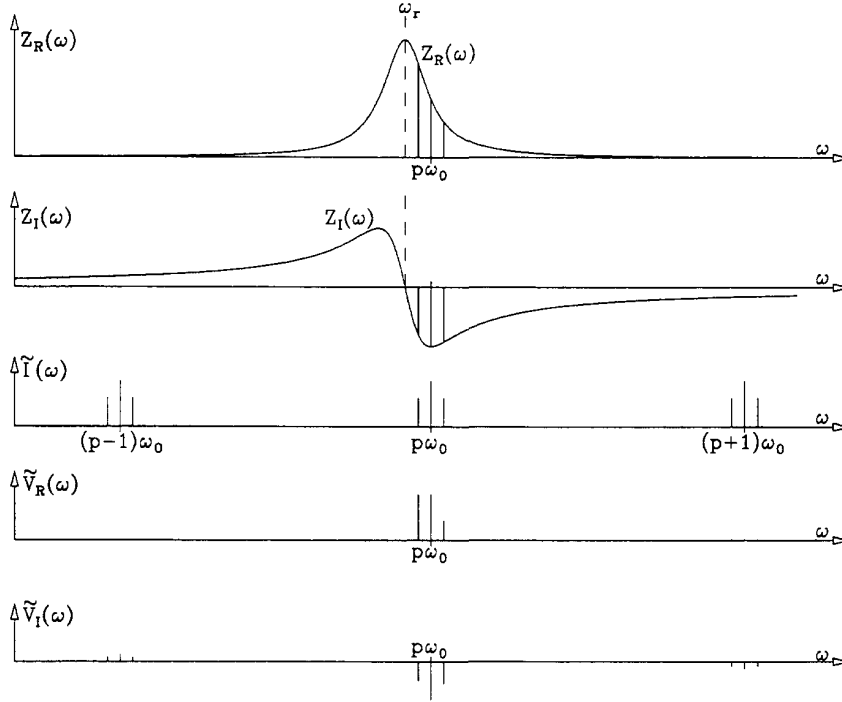
Figure 10: Voltage induced by an oscillating bunch in a narrow band impedance

the current of the oscillating bunch with real functions

$$I_k(t) = I_0 + 2 \sum_{p=1}^{\infty} I_p \left[\cos(p\omega_0 t) \right. \tag{13}$$

$$+ \left. \frac{\omega_0 \hat{\tau}}{2} \left((p - Q_s)\sin((p - Q_s)\omega_0 t) + (p + Q_s)\sin((p + Q_s)\omega_0 t)\right)\right].$$

The line spectrum of the current represented by an oscillating bunch is shown in Fig. 9. We can split the trigonometric function in the expression Eq. (14) for the current

$$I_k(t) = I_0 + 2 \sum_{p=1}^{\infty} I_p \left[\cos(p\omega_0 t) \right. \tag{14}$$

$$+ \frac{\omega_0 \hat{\tau}}{2} \left((p - Q_s)(\sin(p\omega_0 t)\cos(Q_s\omega_0 t) + \cos(p\omega_0 t)\sin(Q_s\omega_0 t)) \right.$$

$$\left. \left. + (p + Q_s)(\sin(p\omega_0 t)\cos(Q_s\omega_0 t) - \cos(p\omega_0 t)\sin(Q_s\omega_0 t))\right)\right].$$

4.2 Voltage induced by an oscillating bunch

We calculate the voltage induced by the current $I_k(t)$ in an impedance $Z(\omega)$. The Fourier transform of this voltage is given by

$$\tilde{V}_k(\omega) = \tilde{I}_k(\omega)Z(\omega).$$

To get the voltage in time domain we use the expression Eq. (11) for the current and apply the inverse Fourier transform Eq. (12)

$$V_k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{I}_k(\omega) Z(\omega) e^{j\omega t}$$

which, with our approximation, can be written as

$$V_k(t) = 2 \sum_{p=1}^{\infty} \left[I_p Z(p\omega_0) e^{p\omega_0 t} + \right.$$
$$\left. \frac{\omega_0 \hat{\tau}}{2} I_p \left((p+Q_s) Z((p+Q_s)\omega_0) e^{j((p+Q_s)\omega_0 t)} + (p-Q_s) Z((p-Q_s)\omega_0) e^{j((p-Q_s)\omega_0 t)} \right) \right].$$

We consider now a narrow-band impedance $\delta\omega < \omega_0$ and calculate the voltage which is induced in it by an oscillating bunch. Due to the small bandwidth we have to consider only one revolution harmonic p and its two side bands to the induced voltage as indicated in Fig. 10.

We start with the resistive part Z_r of the impedance and get for the voltage

$$
\begin{aligned}
V_{kr}(t) &= 2 I_p Z_r(p\omega_0) \cos(p\omega_0 t) \\
&+ \omega_0 \hat{\tau} I_p \left((p+Q_s) Z_r((p+Q_s)\omega_0) \sin((p+Q_s)\omega_0 t) \right. \\
&\quad \left. + (p-Q_s) Z_r((p-Q_s)\omega_0) \sin((p-Q_s)\omega_0 t) \right).
\end{aligned}
$$

We split the trigonometric functions into two parts

$$
\begin{aligned}
V_{kr}(t) &= 2 I_p Z_r(p\omega_0) \cos(p\omega_0 t) \\
&+ \omega_0 \hat{\tau} I_p \left[(p+Q_s) Z_r((p+Q_s)\omega_0) \left(\sin(p\omega_0 t) \cos(\omega_s t) + \cos(p\omega_0 t) \sin(\omega_s t) \right) \right. \\
&\quad \left. (p-Q_s) Z_r((p-Q_s)\omega_0) \left(\sin(p\omega_0 t) \cos(\omega_s t) - \cos(p\omega_0 t) \sin(\omega_s t) \right) \right].
\end{aligned}
$$

The synchrotron motion consists of a modulation of the arrival time each revolution k. We approximate it now as a modulation in time

$$\tau_k = \hat{\tau} \cos(2\pi Q_s k) \quad \rightarrow \quad \tau = \hat{\tau} \cos(\omega_s t) \ , \quad \dot{\tau} = -\omega_s \hat{\tau} \sin(\omega_s t).$$

Since the synchrotron phase advance $2\pi Q_s$ per turn is very small in all practical cases this is a very good approximation. Using these expressions we get for the voltage induced in the resistive impedance

$$
\begin{aligned}
V_{kr}(t) &= 2 I_p \left[Z_r(p\omega_0) \cos(p\omega_0 t) \right. \\
&+ \frac{\omega_0}{2} \left((p+Q_s) Z_r((p+Q_s)\omega_0) \left(\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \right) \right. \\
&\quad \left. \left. + (p-Q_s) Z_r((p-Q_s)\omega_0) \left(\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \right) \right) \right].
\end{aligned}
$$

We can also express the current Eq. (14) of the oscillating bunch with τ and $\dot{\tau}$

$$
\begin{aligned}
I_k(t) &= I_0 + 2 \sum_{p=1}^{\infty} I_p \left[\cos(p\omega_0 t) \right. \tag{15} \\
&+ \frac{\omega_0}{2} \left((p+Q_s) \left(\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \right) \right. \\
&\quad \left. \left. + (p-Q_s) \left(\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \right) \right) \right]. \tag{16}
\end{aligned}
$$

Next we calculate the average voltage seen by the particles in one revolution

$$\langle V_r \rangle = \frac{\int I_k(t) V_k(t) dt}{\int I_k(t) dt} = \frac{1}{I_0 T_0} \int_0^{T_0} I_k(t) V_k(t) dt.$$

In carrying out this integration we approximate for small oscillation amplitudes and take only linear terms in τ and $\dot{\tau}$. We also neglect the change of τ and $\dot{\tau}$ within one turn. This leads to

$$\langle V_r \rangle = \frac{2 I_p^2}{I_0} \Big[Z_r(p\omega_0) -$$

$$\frac{\dot{\tau}\omega_0}{2\omega_s} \big(2 Q_s Z_r(p\omega_0) + (p + Q_s) Z_r((p + Q_s)\omega_0) - (p - Q_s) Z_r((p - Q_s)\omega_0) \big) \Big].$$

For the voltage induced in the reactive (imaginary) part of the impedance we have

$$
\begin{aligned}
V_{ki}(t) &= 2 I_p \Big[-Z_i(p\omega_0) \sin(p\omega_0) \\
&+ \frac{\omega_0 \dot{\tau}}{2} \big((p + Q_s) Z_i((p + Q_s)\omega_0) \cos((p + Q_s)\omega_0 t) \\
&+ (p - Q_s) Z_i((p - Q_s)\omega_0) \cos((p - Q_s)\omega_0 t) \big) \Big].
\end{aligned}
$$

Splitting the trigonometric functions and using τ and $\dot{\tau}$ leads to

$$
\begin{aligned}
V_{ki}(t) &= 2 I_p \Big[-Z_i(p\omega_0) \sin(p\omega_0) \\
&+ \frac{\omega_0 \dot{\tau}}{2} \Big((p + Q_s) Z_i((p + Q_s)\omega_0) \Big(\cos(p\omega_0 t)\tau + \sin(p\omega_0)\frac{\dot{\tau}}{\omega_s} \Big) \\
&+ (p - Q_s) Z_r((p - Q_s)\omega_0) \Big(\cos(p\omega_0 t)\tau - \sin(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \Big) \Big) \Big].
\end{aligned}
$$

For the average voltage per turn we get within linear approximation in τ and $\dot{\tau}$

$$\langle V_i \rangle = \frac{I_p^2 \omega_0 \tau}{I_0} \big[-2 p Z_i(p\omega_0) + (p + Q_s) Z_i((p + Q_s)\omega_0) + (p - Q_s) Z_i((p - Q_s)\omega_0) \big].$$

We derived the voltage induced by an oscillating bunch in a narrow-band impedance. We assumed that the oscillation frequency is small compared to the revolution frequency $Q_s \ll 1$. We will find later when we treat the synchrotron oscillation that only the average voltage seen by the particles in the bunch is of importance for the coherent dipole oscillation. We found that this voltage has two components. The first one, V_r, is induced in the resistive impedance and is proportional to the derivative $\dot{\tau}$ of the bunch excursion. The second component, V_i, is induced in the reactive impedance and is proportional to the bunch excursion τ itself. We can collect the two parts and get the average voltage per turn seen by the particles in the bunch. To make the expressions more compact we introduce

$$Z_{pr}^+ = Z_r(p\omega_0 + \omega_s) \ , \quad Z_{pr}^- = Z_r(p\omega_0 - \omega_s) \ , \quad Z_{pr}^0 = Z_r(p\omega_0)$$

$$Z_{pi}^+ = Z_i(p\omega_0 + \omega_s) \ , \quad Z_{pi}^- = Z_i(p\omega_0 - \omega_s) \ , \quad Z_{pi}^0 = Z_i(p\omega_0)$$

and get

$$\langle V \rangle = \frac{2I_p^2}{I_0} \left[Z_{pr}^0 - \frac{p\omega_0 \dot{\tau}}{2\omega_s} \left(Z_{pr}^+ - Z_{pr}^- - Q_s(2Z_{pr}^0 + Z_{pr}^+ + Z_{pr}^-) \right) \right.$$
$$\left. - \frac{p\omega_0 \tau}{2} \left(-2Z_{pr}^0 + Z_{pi}^+ + Z_{pi}^- + Q_s(Z_{pi}^+ - Z_{pi}^-) \right) \right].$$

Within the approximation $Q_s \ll 1$ this gives

$$\langle V \rangle = \frac{2I_p^2}{I_0} \left[Z_{pr}^0 + \frac{p\omega_0 \dot{\tau}}{2\omega_s}(Z_{pr}^+ - Z_{pr}^-) + \frac{p\omega_0 \tau}{2}(-2Z_{pr}^0 + Z_{pi}^+ + Z_{pi}^-) \right]. \tag{17}$$

The part of the voltage shown in the first line of the above equation does not depend on τ or $\dot{\tau}$ and is therefore present for a stationary bunch as we calculated before. Its resistive part leads to an energy loss and the reactive part to a change of the incoherent synchrotron frequency as we will see later.

We took here the case of a narrow-band impedance and in which only one sideband pair of the bunch spectrum leads to an induced voltage. This case is easier to understand in detail and we will use it also for the most simple case of a longitudinal instability. However, a generalization to a broadband impedance is straightforward: Many sidebands of the bunch spectrum induce a voltage in the impedance at the corresponding frequencies $\omega_0(p \pm Q_s)$. The total voltage is obtained by a sum over the parameter p. This leads to the expression

5   ROBINSON INSTABILITY

5.1   Qualitative treatment

The most important effect of the interaction between a longitudinally-oscillating bunch and a cavity is the so-called Robinson instability [1] which is treated here in some detail since it can be generalized to describe all multi-turn instabilities in storage rings. We start with a qualitative treatment by considering a single bunch circulating in a storage ring and exciting a cavity resonance with resonance frequency ω_r and impedance $Z(\omega)$ of which we consider only the resistive part Z_r.

The revolution frequency ω_0 of the circulating bunch depends on its energy deviation ΔE

$$\frac{\Delta \omega_0}{\omega_0} = -\eta_c \frac{\Delta E}{E} \quad \text{or} \quad \omega_0 = p\omega_0 \left(1 - \eta_c \frac{\Delta E}{E} \right).$$

While the bunch is executing a coherent dipole mode oscillation $\epsilon(t) = \hat{\epsilon}\cos(\omega_s t)$ its revolution frequency is modulated. *Above transition* the revolution frequency ω_0 is *small* when the *energy is high* and ω_0 is *large* when the *energy is small*. If the cavity is tuned to a resonant frequency slightly smaller than the RF frequency $\omega_r < p\omega_0$, Fig. 11 left, the bunch sees a higher impedance and *loses more energy* when it has an *energy excess* and it *loses less energy* when it has a *lack of energy*. This leads to a *damping* of the oscillation. If $\omega_r > p\omega_0$ this is reversed, Fig. 11 right, and leads to an *instability*. Below transition energy the dependence of the revolution frequency is reversed which changes the stability criterion.

5.2   Quantitative treatment

We consider a narrow-band cavity with a circulating bunch as before. The bunch executes a synchrotron oscillation which is approximately described as $\tau = \hat{\tau}\cos(\omega_s t)$ and produces sidebands to the revolution frequency harmonics of the bunch. The current of this oscillating bunch is obtained from Eq. (16) in the approximation of $Q_s \ll 1$

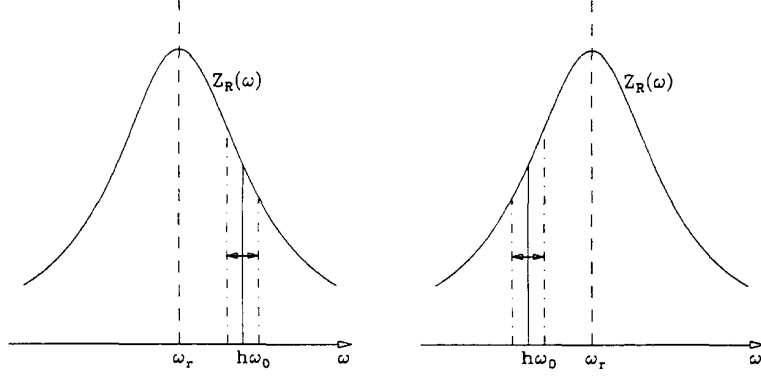$$I_k(t) = I_0 + 2\sum_{p=1}^{\infty} I_p \left[\cos(p\omega_0 t) \right.$$

Figure 11: Qualitative treatment of the Robinson instability

$$+ \quad p\omega_0 \left(\sin(p\omega_0 t)\tau - \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \right) + p\omega_0 \left(\sin(p\omega_0 t)\tau + \cos(p\omega_0 t)\frac{\dot{\tau}}{\omega_s} \right) \right].$$

Since we assumed a narrow-band impedance we consider only one revolution harmonic p with its side bands and get for the averaged induced voltage Eq. (17) or the energy loss U of the particle in the bunch

$$U = e\langle V \rangle = e\frac{2I_p^2}{I_0} \left[Z_{pr}^0 + \frac{p\omega_0 \dot{\tau}}{2\omega_s}(Z_{pr}^+ - Z_{pr}^-) + \frac{p\omega_0 \tau}{2}(-2Z_{pr}^0 + Z_{pi}^+ + Z_{pi}^-) \right].$$

We include this induced voltage in the equation Eq. (1) for the energy gain and loss.

$$\dot{\epsilon} = \frac{\omega_0 e\hat{V}\sin\phi_s}{2\pi E} + \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E}\tau - \frac{\omega_0 e}{2\pi E}\langle V \rangle.$$

Using the equilibrium condition

$$e\hat{V}\sin\phi_s = \frac{2I_p^2 Z_{pr}^0}{I_0}$$

we get

$$\dot{\epsilon} = \frac{\omega_0^2 he\hat{V}\cos\phi_s}{2\pi E} + \frac{I_p^2}{I_0}\frac{ep\omega_0^2}{2\pi E\omega_s}(Z_{pr}^+ - Z_{pr}^-)\dot{\tau} + \frac{I_p^2}{I_0}\frac{ep\omega_0^2}{2\pi E}(-2Z_{pi}^0 + Z_{pi}^+ + Z_{pi}^-)\tau.$$

Using the synchrotron frequency in the absence of an impedance

$$\omega_{s0}^2 = -\omega_0^2\frac{h\eta_c e\hat{V}\cos\phi_s}{2\pi E}$$

and the relation $\dot{\tau} = \eta_c\epsilon$ we get the second-order equation

$$\ddot{\tau} + \frac{pI_p^2\omega_{s0}}{2I_0 h\hat{V}\cos\phi_s}(Z_{pr}^+ - Z_{pr}^-)\dot{\tau} + \omega_{s0}^2 \left(1 - \frac{pI_p^2}{2I_0 h\hat{V}\cos\phi_s}(-2Z_{pi}^0 + Z_{pi}^+ + Z_{pi}^-) \right)\tau = 0.$$
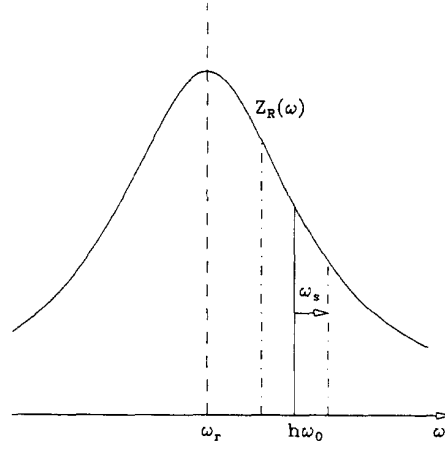
Figure 12: Quantitative treatment of the Robinson instability

This is the oscillation equation with the solution

$$\epsilon = \hat{\epsilon}e^{-\alpha_s t}\cos(\omega_s t + \phi)$$

and the damping or growth rate

$$\alpha_s = \frac{\omega_{s0}pI_p^2(Z_{pr}^+ - Z_{pr}^-)}{2I_0 h\hat{V}\cos\phi_s}.$$

The growth rate of the Robinson instability is given by the difference of the resistive impedance at the upper and lower synchrotron sideband, Fig. 12. Above transition energy we have $\cos\phi_s < 0$ and $\alpha_s > 0$, i.e. stability if $Z_{pr}^- > Z_{pr}^+$ as we found already from qualitative arguments.

The RF cavity itself has a narrow-band impedance around $h\omega_0$ which can drive an instability. Since the bunch length is usually much shorter than the RF wavelength we have $I_p = I_h \approx I_0$ so that

$$\alpha_s \approx \frac{\omega_{s0}I_0(Z_{pr}^+ - Z_{pr}^-)}{2I_0\hat{V}\cos\phi_s}.$$

There is also a frequency shift due to the reactive part of the impedance

$$\omega_s^2 = \omega_{s0}^2\left(1 + \frac{pI_p^2 Z_{pi}^0}{I_0 h\hat{V}\cos\phi_s} - \frac{pI_p^2(Z_{pi}^+ + Z_{pi}^-)}{I_0 h\hat{V}\cos\phi_s}\right).$$

The second term in the parenthesis only depends on the impedance at the revolution harmonic $p\omega_0$ and not on the one at the sidebands. It is present also in the absence of a coherent motion and produces a change of the incoherent synchrotron frequency which we calculated before Eq. (10)

$$\omega_{si}^2 = \omega_{s0}^2\left(1 + \frac{2pI_p^2 Z_{pi}^0}{I_0 h\hat{V}\cos\phi_s}\right).$$

The coherent synchrotron motion produces a further shift compared to ω_{si}

$$\omega_s^2 = \left(\omega_{si}^2 - \frac{pI_p^2(Z_{pi}^+ + Z_{pi}^-)}{I_0 h \hat{V} \cos \phi_s}\omega_{s0}^2\right).$$

For a small effect, the shift of the coherent frequency with respect to the incoherent one is given by

$$\frac{\Delta \omega_s}{\omega_{s0}} \approx -\frac{pI_p^2(Z_{pi}^+ + Z_{pi}^-)}{2I_0 h \hat{V} \cos \phi_s}.$$

5.3 Discussion and generalization of the Robinson instability

We have derived the Robinson instability for the case of a single bunch and a single, narrow band and relatively weak resonance. We will here discuss the way this instability can be extended to cover more general cases.

A more general impedance will cover not just a single revolution harmonic with the two synchrotron oscillation sidebands but many such frequency lines. In this case the voltage induced in the impedance by each such line contained in the spectrum of the bunch current has to be considered. The growth rate will no longer be given by the difference between the impedance at the upper and lower synchrotron sideband but between the sums of the impedance times the spectral power taken at each upper and each lower synchrotron sideband contained in the spectrum of the oscillating bunch.

This can be extended to the case of many bunches [3, 4]. With M equidistant bunches in the machine we have M different modes of oscillation each having a different phase between the oscillations executed by adjacent bunches. The spectrum of each such coupled bunch mode has synchrotron sidebands at distinct revolution harmonics. In calculating the stability of a certain coupled-bunch mode we have to sum over these sidebands.

So far, we considered only dipole oscillations where the bunch makes a rigid oscillation around the nominal phase without changing the form. There are higher modes of oscillation, called bunch-shape oscillations, which can be classified as quadrupole ($m = 2$), sextupole ($m = 3$), octupole ($m = 4$), etc. modes. Each mode has a spectrum with side bands at $m\omega_s$ from the revolution harmonics. Again, to calculate the stability of these modes we have to sum over these sidebands.

We have assumed that the effect of the impedance is relatively weak such that the changes of the synchrotron frequency and the growth rate of the instability are small compared to the synchrotron frequency itself. For very narrow-band cavities with high shunt impedance, e.g. superconducting cavities, this might no longer be true. In this case we have to evaluate the impedance not at the unperturbed sideband ω_{s0} but at the shifted synchrotron frequency ω_s. Furthermore, if we are interested in the growth rate we have to consider the cavity impedance for a growing oscillation which is different as soon as the growth time of the oscillation becomes comparable to the filling time of the cavity. Taking all this into account one arrives at a 4th-order equation for the shifted synchrotron frequency and the growth rate for which a more general stability criterion can be derived, often called the second Robinson instability [1].

In the above paragraph we have considered stability only for the case of infinitesimally small oscillations and we have calculated their growth or damping time. If, however, the oscillation amplitude becomes large, some non-linear effects should be included. The modulation index of the phase oscillation will become large leading to sidebands at twice the synchrotron frequency. They have to be included when summing over the impedance contribution. This can lead to a situation where the beam is unstable for small oscillation amplitudes but becomes stable again at large amplitudes. In practice, such cases have bunches oscillating with finite but more or less constant amplitudes, [5, 6].

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