

THE LANDAU-POMERANCHUK-MIGDAL EFFECT IN QED

R. Baier ¹, Yu. L. Dokshitzer ², A. H. Mueller* ³, S. Peigné ⁴ and D. Schiff ⁴

¹*Fakultät für Physik, Universität Bielefeld, D-33501 Bielefeld, Germany*

²*Theory Division, CERN, 1211 Geneva 23, Switzerland[†]*

³*Physics Department, Columbia University, New York, NY 10027, USA*

⁴*LPTHE[‡], Université Paris-Sud, Bâtiment 211, F-91405 Orsay, France*

Abstract

The calculation of the radiative energy loss encountered by a fast charged particle which undergoes multiple scattering is being investigated. A detailed derivation of the Landau-Pomeranchuk-Migdal coherent effect in QED is given, focusing on the specific feature of the Coulomb interaction. As a result the radiation intensity per unit length in the coherent regime is shown to be proportional to $\sqrt{\omega}$ for a photon energy ω times a logarithmic enhancement which is determined exactly.

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[†]Permanent address: Petersburg Nuclear Physics Institute, Gatchina, 188350 St. Petersburg, Russia

[‡]Laboratoire associé au Centre National de la Recherche Scientifique

1 Introduction

The radiative energy loss encountered by a charged fast particle which undergoes successive scatterings in a medium has recently been the object of renewed interest. This is indeed of special importance in order to understand the energy loss mechanism for quarks and gluons, propagating in nuclear matter and in particular through the quark-gluon plasma.

Our first step [1] has been to reformulate the QED treatment given by Landau, Pomeranchuk and Migdal (LPM) [2,3,4]. The experimental observation of the LPM effect at SLAC [5] has recently triggered further studies with special emphasis on finite length effects [6].

In the present paper we focus on a detailed derivation of the radiative spectrum per unit length, assuming a very large number of scatterers. The model used which has been recently elaborated by Gyulassy and Wang [7] depicts the multiple scattering of a fast electron in the medium as due to static scattering centres with Debye screened Coulomb potentials. The assumption that the mean free path λ of the projectile is much larger than the screening radius $\lambda \gg \mu^{-1}$ allows one to treat successive scatterings as independent. Coupled to the soft photon approximation this then leads to an eikonal picture of classical propagation.

As stated in [1], the specific case of the Coulomb potential which is not screened at *short* distances requires a special treatment which was not available in the literature. Such a treatment is essential for QCD. We present here for QED a complete detailed derivation, correcting an unjustified step made in [1].

For the specific case of the Coulomb interaction, the radiation intensity per unit length in the coherent regime is proportional to $\sqrt{\omega}$ times a logarithmic enhancement. For potentials less singular than the Coulomb one at short distance, where the random walk picture is applicable, we confirm the original LPM result.

The outline of the paper is as follows :

In section 2, we show in details how the model of static Coulomb centres describes multiple scattering. The radiation spectrum induced by multiple scattering is worked out in the soft photon approximation in section 3 and studied in section 4. In this last section, we first obtain the shape of the spectrum on heuristic grounds and then carefully derive the spectrum in the coherent regime by solving an exact differential equation to leading logarithmic accuracy. Section 5 is the conclusion.

2 Model for multiple scattering

In order to work out the radiation intensity properly normalized we need first to derive the multiple scattering cross section of a fast electron propagating in a medium. We do this using the model of Gyulassy and Wang [7]. The main feature of this model consists in assuming that scattering centres are static. Thus the collisional energy loss of the charged particle vanishes, and the total energy loss will be purely radiative. The scattering centre located at \vec{x}_i creates screened Coulomb potential

$$\mathcal{V}_i(\vec{x}) = \frac{e}{4\pi} \frac{e^{-\mu|\vec{x}-\vec{x}_i|}}{|\vec{x}-\vec{x}_i|}, \quad (2.1)$$

with Fourier transform

$$\mathcal{V}_i(\vec{q}) = \frac{e}{\vec{q}^2 + \mu^2} e^{-i\vec{q}\cdot\vec{x}_i}. \quad (2.2)$$

Here μ is the Debye mass in the medium and we denote \vec{q}_i the momentum transferred to the electron during scattering on the centre \vec{x}_i .

To be able to treat successive elastic scatterings as independent, centres have to be well separated. This means that the average distance between two successive interactions (the electron mean free path λ), is large compared to the range of the potential,

$$\lambda \gg \mu^{-1}. \quad (2.3)$$

This leads to important simplifications in the study of the radiative energy loss.

We also concentrate on the high energy limit, where the transferred momenta \vec{q}_i become transverse. Moreover, their characteristic values are of order μ , and we have

$$\Theta_s = \frac{|\vec{q}_\perp|}{E} \sim \theta_1 \equiv \frac{\mu}{E} \ll 1, \quad (2.4)$$

where Θ_s is the typical electron scattering angle.

Let us consider the probability amplitude to scatter on N static centres, and call the incoming and outgoing electron 4-momenta p_0 and p_N , respectively. In the high energy limit where electron mass and spin effects are irrelevant, the S -matrix element corresponding to multiple elastic scattering reads [8]

$$S_{scatt} \propto \delta(p_N^0 - p_0^0) \sum_\sigma \int \prod_{i=1}^{N-1} \left[\frac{d^3 \vec{p}_i}{p_i^2 + i\eta} \right] \prod_{i=1}^N \left[\frac{e^{-i\vec{q}_i \cdot \vec{x}_{\sigma(i)}}}{\vec{q}_i^2 + \mu^2} \right], \quad (2.5)$$

where after the i^{th} momentum transfer \vec{q}_i , the electron momentum is

$$\vec{p}_i = \vec{p}_{i-1} + \vec{q}_i \quad ; \quad i = 1, \dots, N, \quad (2.6a)$$

and

$$p_i^0 = E \quad ; \quad i = 0, \dots, N-1. \quad (2.6b)$$

The resulting cross section may be worked out in the eikonal limit of small angle scattering (see e.g. [7]). Let us indicate the main steps of the derivation in the well known Glauber picture.

S_{scatt} contains the phase factor $\exp(i\varphi_{scatt})$ with

$$\begin{aligned} \varphi_{scatt} = & - \sum_{i=1}^N \vec{q}_i \cdot \vec{x}_{\sigma(i)} = - \sum_{i=1}^N \vec{q}_{i\perp} \cdot \vec{x}_{\sigma(i)\perp} + \sum_{i=1}^{N-1} p_{i\parallel} (z_{\sigma(i+1)} - z_{\sigma(i)}) \\ & + \left(p_{0\parallel} z_{\sigma(1)} - p_{N\parallel} z_{\sigma(N)} \right), \end{aligned} \quad (2.7)$$

where $z_{\sigma(i)}$ is the longitudinal component of $\vec{x}_{\sigma(i)}$ and we sum over all permutations σ of $\{1, \dots, N\}$.

S_{scatt} may be calculated by integrating successively over the longitudinal momenta $p_{i\parallel}$, for $i = 1, \dots, N-1$. Let us start with $\int dp_{1\parallel}$.

When $(z_{\sigma(2)} - z_{\sigma(1)}) > 0$, the integration over $p_{1\parallel}$ is performed by closing the integration contour in the upper half of the complex $p_{1\parallel}$ plane (see (2.7)), and taking the pole $p_{1\parallel} = E - p_{1\perp}^2/2E + i\eta$.

When $(z_{\sigma(2)} - z_{\sigma(1)}) < 0$, we have to close the contour in the lower half-plane. We take the pole $p_{1\parallel} = -E + p_{1\perp}^2/2E - i\eta$, but this yields a residue which is suppressed by a factor of order $(p_{1\perp}^2 + \mu^2)/E^2$ when compared to the first case, where we have chosen $\vec{p}_{0\perp} = 0$.

Finally, whatever the sign of $(z_{\sigma(2)} - z_{\sigma(1)})$ is, the poles in $p_{1\parallel}$ corresponding to the "Coulomb denominator" $(\vec{q}_1^2 + \mu^2)$ lead to residues involving a suppression factor $\sim \exp\{-\mu|z_{\sigma(2)} - z_{\sigma(1)}|\}$, and may be neglected when $\lambda \gg \mu^{-1}$. This may be repeated for all $p_{i\parallel}$ integrations.

We deduce from the above :

- The high energy electron scatters successively on ordered scattering centres (in the z direction) : $z_{\sigma(N)} > z_{\sigma(N-1)} > \dots > z_{\sigma(1)}$. In (2.5), only the identity permutation has to be kept and backward scattering may be neglected¹.
- As a consequence of (2.3), the internal electron momenta are on-shell, which corresponds to independent elastic scatterings on static centres:

$$p_{i\parallel} \simeq E - \frac{p_{i\perp}^2}{2E}, \quad i = 1, 2, \dots, N-1; \quad p_{0\parallel} = E. \quad (2.8)$$

The phase (2.7) takes the form

$$\varphi_{scatt} = \frac{p_{N\perp}^2}{2E} z_N - \sum_{i=1}^N \vec{q}_{i\perp} \cdot \vec{x}_{i\perp} - \sum_{i=1}^{N-1} \frac{p_{i\perp}^2}{2E} (z_{i+1} - z_i). \quad (2.9)$$

Using $\vec{q}_{i\perp}$ as integration variables we obtain, up to an irrelevant phase and the energy conservation δ -factor,

$$S_{scatt} \propto \int \prod_{i=1}^N \left[d^2 \vec{q}_{i\perp} \frac{e^{-i\vec{q}_{i\perp} \cdot \vec{x}_{i\perp}}}{\vec{q}_{i\perp}^2 + \mu^2} \right] \cdot \prod_{i=1}^{N-1} \left[e^{-i\frac{p_{i\perp}^2}{2E}(z_{i+1}-z_i)} \right] \cdot \delta^2 \left(\sum_{i=1}^N \vec{q}_{i\perp} - \vec{p}_{N\perp} \right). \quad (2.10)$$

The scattering amplitude squared reads

$$|M_{scatt}|^2 \propto \int \prod_{i=1}^N \left[d^2 \vec{q}_{i\perp} d^2 \vec{q}'_{i\perp} \frac{e^{i(\vec{q}'_{i\perp} - \vec{q}_{i\perp}) \cdot \vec{x}_{i\perp}}}{(\vec{q}_{i\perp}^2 + \mu^2)(\vec{q}'_{i\perp}^2 + \mu^2)} \right] \prod_{i=1}^{N-1} \left[e^{i\frac{p_{i\perp}^2 - p_{i\perp}'^2}{2E}(z_{i+1}-z_i)} \right] \cdot \delta^2 \left(\sum_{i=1}^N \vec{q}_{i\perp} - \vec{p}_{N\perp} \right) \delta^2 \left(\sum_{i=1}^N \vec{q}'_{i\perp} - \vec{p}_{N\perp} \right). \quad (2.11)$$

The next step is to average over the coordinates \vec{x}_i . Since the centres are assumed to be uniformly distributed, averaging over $\vec{x}_{i\perp}$ leads to $\vec{q}'_{i\perp} = \vec{q}_{i\perp}$, $\vec{p}'_{i\perp} = \vec{p}_{i\perp}$, which results in

$$\frac{d\sigma_{scatt}}{d^2 \vec{p}_{N\perp}} \propto \int \prod_{i=1}^N \frac{d^2 \vec{q}_{i\perp}}{(\vec{q}_{i\perp}^2 + \mu^2)^2} \delta^2 \left(\sum_{i=1}^N \vec{q}_{i\perp} - \vec{p}_{N\perp} \right). \quad (2.12)$$

Here we have dropped an overall factor including the transverse surface $\delta^2(\vec{0})$. Integrating over $\vec{p}_{N\perp}$ we finally arrive at

$$\sigma_{scatt} \propto \int \prod_{i=1}^N \frac{d^2 \vec{q}_{i\perp}}{(\vec{q}_{i\perp}^2 + \mu^2)^2}. \quad (2.13)$$

A similar expression may be found in [9,10].

¹In other words the permutations different from identity imply the occurrence of vacuum creation of e^+e^- pairs, which process is power-suppressed when $E \rightarrow \infty$.

3 Radiation spectrum induced by multiple scattering in the soft photon approximation

Multiple scattering induces radiation. The calculation of the bremsstrahlung probability is done in the soft photon approximation

$$\omega \ll E. \quad (3.1)$$

This allows to factorize the radiation amplitude as the product of the multiple scattering amplitude times the photon emission amplitude. The ratio between the radiation cross section and the scattering cross section yields the radiation probability.

Let us consider the probability amplitude for emitting a photon of 4-momentum $k = (\omega, \vec{k})$ off the electron between centres \vec{x}_j and \vec{x}_{j+1} . The corresponding S -matrix element reads

$$S_{rad}^j \propto e\delta(p_N^0 - p_0^0) \int \prod_{i=1}^{j-1} \left[\frac{d^3 \vec{p}_i}{p_i^2 + i\varepsilon} \right] \cdot \int d^3 \vec{p}_j \frac{\varepsilon \cdot p_j}{k \cdot p_j} \left[\frac{1}{(p_j - k)^2 + i\eta} - \frac{1}{p_j^2 + i\eta} \right] \cdot \int \prod_{i=j+1}^{N-1} \left[\frac{d^3 \vec{p}_i}{(p_i - k)^2 + i\eta} \right] \cdot \prod_{i=1}^N \left[\frac{e^{-i\vec{q}_i \cdot \vec{x}_i}}{\vec{q}_i^2 + \mu^2} \right], \quad (3.2)$$

with the momentum variables p_i defined in (2.6). The two physical photon polarization states ε are chosen in the form

$$\varepsilon = (\varepsilon_0, -\varepsilon_0, \vec{\varepsilon}_\perp) \quad ; \quad \varepsilon \cdot k = 0 \implies \varepsilon_0 = \frac{\vec{\varepsilon}_\perp \cdot \vec{k}_\perp}{\omega + k_\parallel} \simeq \frac{\vec{\varepsilon}_\perp \cdot \vec{k}_\perp}{2\omega}. \quad (3.3)$$

In the same way as in the previous section, the phase φ_{rad} is given by the r.h.s. of (2.7) and the integrations over $p_{i\parallel}$ are done by closing the contour in the upper half-plane,

$$p_i^2 = 0 \implies p_{i\parallel} \simeq E - \frac{p_{i\perp}^2}{2E}, \quad (3.4a)$$

$$(p_i - k)^2 = 0 \implies p_{i\parallel} - k_\parallel \simeq E - \omega - \frac{(p_i - k)_\perp^2}{2(E - \omega)}. \quad (3.4b)$$

As far as the intermediate state j is concerned, we may have either $(p_j - k)^2 = 0$ or $p_j^2 = 0$. Since the accompanying radiation factor is invariant,

$$\frac{\varepsilon \cdot p_j}{k \cdot p_j} = \frac{\varepsilon \cdot (p_j - k)}{k \cdot (p_j - k)}; \quad [\varepsilon \cdot k = 0, \quad k^2 = 0], \quad (3.5)$$

it may be always expressed in terms of the *real* electron momentum. For example, taking $p_j^2 = 0$ we obtain

$$\varepsilon \cdot p_j = \varepsilon_0(E + p_{j\parallel}) - \vec{\varepsilon}_\perp \cdot \vec{p}_{j\perp} \simeq E \vec{\varepsilon}_\perp \cdot \left(\frac{\vec{k}_\perp}{\omega} - \frac{\vec{p}_{j\perp}}{E} \right) \equiv E \vec{\varepsilon}_\perp \cdot \vec{u}_j, \quad (3.6a)$$

$$k \cdot p_j = \frac{E\omega}{2} \left(\frac{\vec{k}_\perp}{\omega} - \frac{\vec{p}_{j\perp}}{E} \right)^2 \equiv \frac{E\omega}{2} u_j^2. \quad (3.6b)$$

Here we introduced the dimensionless transverse vector \vec{u}_j which, in the quasi-collinear kinematics under interest, measures the angle between the photon and the electron after scattering number j ,

$$\vec{u}_j = \frac{\vec{k}_\perp}{\omega} - \frac{\vec{p}_{j\perp}}{E} = \frac{\vec{k}_\perp}{\omega} - \sum_{i=1}^j \frac{\vec{q}_{i\perp}}{E}. \quad (3.7)$$

For another pole, $(p_j - k)^2 = 0$, one has to substitute in (3.6)

$$\vec{u}_j \implies \frac{\vec{k}_\perp}{\omega} - \frac{\vec{p}_{j\perp} - \vec{k}_\perp}{E} = \vec{u}_j \left\{ 1 + \mathcal{O}\left(\frac{\omega}{E}\right) \right\} \simeq \vec{u}_j, \quad (3.8)$$

which difference, however, may be neglected in the soft approximation (3.1).

Thus, we can write the S -matrix element (3.2) as

$$S_{rad}^j \propto e\delta(p_N^0 - p_0^0) \int \prod_{i=1}^{N-1} [d^2\vec{p}_{i\perp}] \prod_{i=1}^N \left[\frac{1}{\vec{q}_{i\perp}^2 + \mu^2} \right] \cdot \frac{\varepsilon \cdot p_j}{k \cdot p_j} \left(e^{i\varphi_{rad}} \Big|_{(p_j-k)^2=0} - e^{i\varphi_{rad}} \Big|_{p_j^2=0} \right), \quad (3.9)$$

where the elementary soft radiation factor is

$$\frac{\varepsilon \cdot p_j}{k \cdot p_j} \Big|_{(p_j-k)^2=0} \simeq \frac{\varepsilon \cdot p_j}{k \cdot p_j} \Big|_{p_j^2=0} = \frac{2}{\omega} \vec{\varepsilon}_\perp \cdot \frac{\vec{u}_j}{u_j^2}, \quad (3.10)$$

with \vec{u}_j defined in (3.7).

Now we must express the phases φ_{rad} :

$$\begin{aligned} \varphi_{rad} &= -\sum_{i=1}^N \vec{q}_i \cdot \vec{x}_i = -\sum_{i=1}^N \vec{q}_{i\perp} \cdot \vec{x}_{i\perp} + \varphi_{\parallel}, \\ \varphi_{\parallel} &= \sum_{i=1}^N q_{i\parallel} z_i = p_{0\parallel} z_1 - p_{N\parallel} z_N + \sum_{i=1}^{N-1} p_{i\parallel} (z_{i+1} - z_i). \end{aligned} \quad (3.11)$$

According to (3.4), the value of the longitudinal component $p_{i\parallel}$ depends on whether $p_i^2 = 0$ or $(p_i - k)^2 = 0$. Expanding (3.4b) in ω/E and neglecting $\mathcal{O}(\omega^2/E)$ for the second case, we derive

$$p_{i\parallel} = \left(E - \frac{p_{i\perp}^2}{2E} \right) - \frac{\omega}{2} u_i^2, \quad \text{for } (p_i - k)^2 = 0. \quad (3.12)$$

Using (3.4a) and (3.12) and invoking (2.9) for the scattering phase, we represent the phase of the first term in (3.9), corresponding to $(p_j - k)^2 = 0$, as

$$\begin{aligned} \varphi_{rad} &= \varphi_{scatt} + \Phi_j, \\ \Phi_j &= -\frac{\omega}{2} \sum_{i=j}^{N-1} u_i^2 (z_{i+1} - z_i). \end{aligned} \quad (3.13)$$

Similarly, for $p_j^2 = 0$, we use (3.4a) for $i = 1, \dots, j$ and (3.12) for $i = j+1, \dots, N-1$ to obtain the expression for φ_{rad} , which differs from (3.13) simply by substituting $j+1$ for j , that is $\Phi_j \rightarrow \Phi_{j+1}$ in (3.13).

We stress that the phase difference depends only on the longitudinal separations.

Using the picture of the electron moving along a classical trajectory between scattering centres, $\vec{x}_{i+1} - \vec{x}_i = \vec{v}_i (t_{i+1} - t_i)$ with \vec{v}_i the electron velocity, one may write the phase in Lorentz-invariant form as

$$\Phi_j \simeq k \cdot (x_j - x_N). \quad (3.14)$$

After summing S_{rad}^j over j and adding the radiation off initial and final lines we get the total radiation S -matrix element

$$S_{rad} \propto e\delta(p_N^0 - p_0^0) \int \prod_{i=1}^N \left[\frac{d^2 \vec{q}_{i\perp}}{\vec{q}_{i\perp}^2 + \mu^2} \right] \cdot e^{i\varphi_{scatt}} \cdot \delta^2 \left(\sum_{i=1}^N \vec{q}_{i\perp} - \vec{p}_{N\perp} \right) \cdot \mathcal{R}, \quad (3.15a)$$

where

$$\mathcal{R} = \sum_{j=1}^{N-1} \frac{\varepsilon \cdot p_j}{k \cdot p_j} \left(e^{i\Phi_j} - e^{i\Phi_{j+1}} \right) - \frac{\varepsilon \cdot p_0}{k \cdot p_0} e^{i\Phi_1} + \frac{\varepsilon \cdot p_N}{k \cdot p_N} = \sum_{j=1}^N \left(\frac{\varepsilon \cdot p_j}{k \cdot p_j} - \frac{\varepsilon \cdot p_{j-1}}{k \cdot p_{j-1}} \right) e^{i\Phi_j}. \quad (3.15b)$$

Thus, the integrand of S_{rad} is proportional to that of S_{scatt} in (2.10). As seen from (3.13), the proportionality factor $e\mathcal{R}$ does not depend on the transverse coordinates $\vec{x}_{i\perp}$. Therefore, after squaring (3.15) to get $|M_{rad}|^2$, we may follow the same averaging procedure over $\vec{x}_{i\perp}$ as for $|M_{scatt}|^2$. The result is

$$|M_{rad}|^2 \propto e^2 \int \prod_{i=1}^N \frac{d^2 \vec{q}_{i\perp}}{(\vec{q}_{i\perp}^2 + \mu^2)^2} \delta^2 \left(\sum_{i=1}^N \vec{q}_{i\perp} - \vec{p}_{N\perp} \right) \sum_{pol} |\mathcal{R}|^2. \quad (3.16)$$

Integrating over $\vec{p}_{N\perp}$ gives

$$d\sigma_{rad} \propto e^2 \left[\int \prod_{i=1}^N \frac{d^2 \vec{q}_{i\perp}}{(\vec{q}_{i\perp}^2 + \mu^2)^2} \sum_{pol} |\mathcal{R}|^2 \right] \frac{d^3 \vec{k}}{(2\pi)^3 2\omega}. \quad (3.17)$$

Finally, using (3.10) and summing over the photon polarizations leads to the radiation intensity

$$\omega \frac{dI}{d\omega} = \frac{\alpha}{\pi^2} \int d\Omega \left\langle \left| \sum_{i=1}^N \vec{A}_i e^{i\Phi_i} \right|^2 \right\rangle, \quad (3.18a)$$

with $d\Omega$ representing the integration over the photon direction. Here we have defined the emission current

$$\vec{A}_i = \frac{\vec{u}_i}{u_i^2} - \frac{\vec{u}_{i-1}}{u_{i-1}^2}. \quad (3.18b)$$

The brackets denote the averaging over transverse momenta and longitudinal coordinates using normalized probability distributions,

$$\langle (\dots) \rangle \iff \int \prod_{\ell=1}^{N-1} \frac{d\Delta_\ell}{\lambda} \exp \left(-\frac{\Delta_\ell}{\lambda} \right) \cdot \int \prod_{i=1}^N \frac{\mu^2 d^2 \vec{q}_{i\perp}}{\pi (\vec{q}_{i\perp}^2 + \mu^2)^2} (\dots), \quad (3.19)$$

where the exponential factor reflects the survival probability of the electron over the distance $\Delta_\ell = z_{\ell+1} - z_\ell$.

In a general case of scattering potential other than Coulomb, the averaging over transverse momenta has the form

$$\int \prod_{i=1}^N d^2 \vec{q}_{i\perp} V(q_{i\perp}^2) (\dots); \quad \int d^2 \vec{q} V(q^2) = 1, \quad (3.20)$$

with $V(q^2)$ the normalized cross section for elastic electron scattering in the medium.

For the sake of completeness and also in view of the generalization to nonabelian radiation processes, it is useful to mention the old-fashioned perturbation theory proof of (3.18).

Let us consider the radiation amplitude induced by N transfers of fixed momenta at centres \vec{x}_i and times t_i . In time-ordered perturbation theory, in the limit $E \rightarrow \infty$, M_{rad}^j is obtained by integrating the phase factor over the emission time τ_j , lying between t_j and t_{j+1} ,

$$M_{rad}^j = -ie \frac{2\varepsilon \cdot p_j}{2E} \left\{ \int_{t_j}^{t_{j+1}} d\tau_j e^{i\tau_j (|\vec{p}_j - \vec{k}| + \omega - |\vec{p}_j|)} \right\} \cdot e^{i\delta\varphi} M_{scatt}. \quad (3.21)$$

We denote by $\delta\varphi$ the phase difference between M_{rad}^j and M_{scatt} . For the interaction times t_i , $i > j$, the associated phase is $t_i (|\vec{p}_i| - |\vec{p}_{i-1}|) \simeq 0$ for M_{scatt} , but becomes $t_i (|\vec{p}_i - \vec{k}| - |\vec{p}_{i-1} - \vec{k}|) \simeq t_i \vec{k} \cdot (\vec{v}_{i-1} - \vec{v}_i)$ for M_{rad}^j . Thus

$$\delta\varphi = \sum_{i=j+1}^N t_i \vec{k} \cdot (\vec{v}_{i-1} - \vec{v}_i). \quad (3.22)$$

Using

$$\vec{x}_{i+1} - \vec{x}_i = \vec{v}_i (t_{i+1} - t_i), \quad (3.23)$$

and

$$|\vec{p}_i - \vec{k}| - |\vec{p}_i| + \omega \simeq \omega - \vec{k} \cdot \vec{v}_i = \frac{k \cdot p_i}{E}, \quad (3.24)$$

we easily get

$$M_{rad}^j = e \frac{\varepsilon \cdot p_j}{k \cdot p_j} \left(e^{ik \cdot x_j} - e^{ik \cdot x_{j+1}} \right) M_{scatt}. \quad (3.25)$$

Integrating over τ_j in (3.21) gives two contributions, proportional to $e^{ik \cdot x_j}$ and $e^{ik \cdot x_{j+1}}$, which may be respectively included in radiation amplitudes induced by the j^{th} and $(j+1)^{\text{th}}$ scatterings. Finally, the radiation amplitude induced by N scatterings is found to be identical to the previous result (3.18).

4 Explicit calculation of the radiation intensity

Here we present the analytic calculation of the energy spectrum (3.18) in the soft photon approximation. The spectrum (3.18) may be written in two equivalent forms as follows:

$$\omega \frac{dI}{d\omega} = \frac{\alpha}{\pi^2} \int d\Omega \left\langle 2 \operatorname{Re} \sum_{i=1}^N \sum_{j=i+1}^N \vec{A}_i \cdot \vec{A}_j e^{i\Phi_{ji}} + \sum_{i=1}^N |\vec{A}_i|^2 \right\rangle \quad (4.1a)$$

$$= \frac{\alpha}{\pi^2} \int d\Omega \left\langle 2 \operatorname{Re} \sum_{i=1}^N \sum_{j=i+1}^N \vec{A}_i \cdot \vec{A}_j (e^{i\Phi_{ji}} - 1) + \left| \sum_{i=1}^N \vec{A}_i \right|^2 \right\rangle, \quad (4.1b)$$

with the relative phase

$$\Phi_{ji} = \Phi_j - \Phi_i = \frac{\omega}{2} \sum_{\ell=i}^{j-1} u_\ell^2 (z_{\ell+1} - z_\ell). \quad (4.2)$$

4.1 Heuristic derivation of the LPM spectrum

The radiation pattern depends crucially on the phases. If Φ_{ji} are large, the rapid oscillations in the first term of (4.1a) wash this contribution away, and we have the Bethe-Heitler regime of independent radiation. Dividing by the size of the medium, $N\lambda$, for the differential energy spectrum *per unit length* (the radiation density) we obtain

$$\begin{aligned} \omega \frac{dI}{d\omega dz} &= \frac{1}{\lambda} \frac{\alpha}{\pi^2} \int d\Omega \frac{1}{N} \left\langle \sum_{i=1}^N |\vec{A}_i|^2 \right\rangle = \frac{\alpha}{\pi\lambda} \int \frac{d^2\vec{u}_1}{\pi} \left\langle |\vec{A}_1|^2 \right\rangle \\ &= \frac{\alpha}{\pi\lambda} \int \frac{d^2\vec{u}}{\pi} \left\langle \frac{\Theta_s^2}{u^2(\vec{u} - \vec{\Theta}_s)^2} \right\rangle \simeq \frac{2\alpha}{\pi\lambda} \left\langle \ln \frac{\Theta_s^2}{u_{min}^2} \right\rangle, \end{aligned} \quad (4.3)$$

with $u_{min}^2 \ll \Theta_s^2$ the value of the emission angle below which the phase between neighbouring centres becomes small:

$$\Phi_{i,i+1} = \frac{1}{2}\omega u_i^2 (z_{i+1} - z_i) \sim \frac{1}{2}\omega\lambda u_i^2 < 1 \Leftrightarrow u_i^2 < u_{min}^2; \quad u_{min}^{-2} = \frac{1}{2}\omega\lambda. \quad (4.4)$$

Small emission angles $u^2 < u_{min}^2$ or $(\vec{u} - \vec{\Theta}_s)^2 < u_{min}^2$ correspond to photon formation time *larger* than the distance to the neighbouring centres,

$$t_{form} = \frac{2\omega}{k_\perp^2} = \frac{2}{\omega u^2} > \lambda; \quad (4.5)$$

so that destructive interference with the radiation due to the nearest neighbour screens the logarithmic divergence. Introducing the characteristic dimensionless parameter κ , representing the typical phase difference between neighbouring centres,

$$\kappa = \frac{\lambda\mu^2}{2} \frac{\omega}{E^2}, \quad (4.6)$$

we may obtain (4.3) as

$$\left(\omega \frac{dI}{d\omega dz} \right)^{(BH)} = \frac{2\alpha}{\pi\lambda} \left\langle \ln \frac{\kappa q^2}{\mu^2} \right\rangle \simeq \frac{2\alpha}{\pi\lambda} \ln \kappa. \quad (4.7)$$

The Bethe-Heitler limit corresponds to $\kappa > 1$.

In the opposite limit, $\kappa \ll 1$, the phases between neighbouring centres are vanishingly small, so that a group of ν centres radiates coherently. To estimate the *coherence number* ν [11], we look for the separation between two centres, such that an accumulated phase becomes of order unity. In a first approximation one may assume that the trajectory of the electron is a random walk in the transverse momentum space,

$$u_{\ell+1}^2 \simeq u_\ell^2 + \langle \Theta_s^2 \rangle. \quad (4.8)$$

The phase $\Phi_{i,i+\nu}$ is estimated then as

$$\Phi \approx \frac{1}{2}\omega\lambda \left[\nu u_i^2 + \frac{1}{2}\nu(\nu-1) \langle \Theta_s^2 \rangle \right] \approx \kappa \left[\nu \cdot \frac{u_i^2}{\theta_1^2} + \frac{\nu^2}{2} \cdot \frac{\langle \Theta_s^2 \rangle}{\theta_1^2} \right] \sim 1. \quad (4.9)$$

As will become apparent later from (4.27a), the second term here is ν times bigger than the first one, which gives

$$\nu^{-1} \approx \sqrt{\frac{\kappa}{2} \frac{\langle \Theta_s^2 \rangle}{\theta_1^2}} \ll 1. \quad (4.10)$$

If the *coherence length* exceeds the size of the medium, $\nu\lambda > L$, all the phases are small, and the first term in (4.1b) vanishes. In the last term the emission currents (3.18b) add up into the expression

$$\sum_{i=1}^N \vec{A}_i = \frac{\vec{u}_N}{u_N^2} - \frac{\vec{u}_0}{u_0^2} \quad ; \quad \vec{u}_N = \vec{u}_0 - \frac{\vec{q}_{tot}}{E}, \quad (4.11)$$

which corresponds to the radiation induced by a single scattering with the momentum transfer $\vec{q}_{\perp tot} = \sum_{i=1}^N \vec{q}_i$. It is usually called the *factorization* limit. For a given \vec{q}_{tot} the radiation is independent of the size and the properties of the medium and reads

$$\omega \frac{dI}{d\omega} = \frac{2\alpha}{\pi} \left\langle \ln \frac{q_{tot}^2}{m^2} \right\rangle. \quad (4.12)$$

Here we have introduced a finite electron mass m to regularize the collinear divergence. It is the only place where the electron mass enters in the high-energy limit. Otherwise, the induced radiation is collinear-safe. When the size of the medium is large enough to embody a few coherence lengths, $L \gg \nu\lambda$, the LPM-suppression of the Bethe-Heitler spectrum takes place. In order to quantify it, it suffices to “slice” the medium and substitute the number of “effective radiators” N/ν for N . For the radiation density we write

$$\omega \frac{dI}{d\omega dz} \approx \frac{1}{\nu} \left(\omega \frac{dI}{d\omega dz} \right)^{(BH)}. \quad (4.13)$$

In the standard Bethe-Heitler spectrum off a point-like source (4.3) the logarithmic enhancement factor appears due to emission angles much smaller than the scattering angle. The effective centre, however, radiates at typical angles u^2 such that the formation time $2(\omega u^2)^{-1}$ is of the order of the length of the radiator $\nu\lambda$. This implies an emission angle u^2 of the order of the accumulated diffusion angle,

$$u^2 \sim \frac{1}{2}\nu \langle \Theta_s^2 \rangle. \quad (4.14)$$

Under these conditions the logarithmic enhancement is absent, and (4.13) becomes

$$\omega \frac{dI}{d\omega dz} \approx \frac{1}{\nu} \cdot \frac{2\alpha}{\pi\lambda} C = C \frac{\alpha}{\pi\lambda} \sqrt{2\kappa \frac{\langle q_{\perp}^2 \rangle}{\mu^2}} = C \frac{\alpha}{\pi} \sqrt{\omega \frac{\langle q_{\perp}^2 \rangle}{\lambda E^2}} = C \frac{\alpha}{\pi} \sqrt{\omega \frac{\langle \Theta_s^2 \rangle}{\lambda}}, \quad (4.15)$$

with C a constant of order 1. This reproduces the well-known Landau-Pomeranchuk-Migdal result ($C = 1$). Strictly speaking, the above derivation based on the random walk picture applies only to the cases when the mean squared momentum transfer in a single scattering is well defined. However, this is not true for Coulomb scattering where the integral determining $\langle q_{\perp}^2 \rangle$ formally diverges logarithmically:

$$\langle q_{\perp}^2 \rangle = \int \frac{\mu^2 dq^2}{(\mu^2 + q^2)^2} \cdot q^2 = \mu^2 \int \frac{\theta^2 d\theta^2}{(\theta_1^2 + \theta^2)^2} = \mu^2 \ln \frac{\Theta_{max}^2}{\theta_1^2}. \quad (4.16)$$

Replacing the upper limit of the angular integral by the characteristic angle of the problem u^2 from (4.14), we obtain

$$\frac{\langle q_{\perp}^2 \rangle}{\mu^2} = \frac{\langle \Theta_s^2 \rangle}{\theta_1^2} = \ln \left(\frac{1}{\sqrt{\kappa}} \sqrt{\frac{\langle \Theta_s^2 \rangle}{\theta_1^2}} \right) \simeq \frac{1}{2} \ln \frac{1}{\kappa}, \quad (4.17)$$

where we have neglected the log-factor under logarithm. Substituting this ratio into (4.15), for the Coulomb case we derive

$$\left(\omega \frac{dI}{d\omega dz} \right)_{\text{Coulomb}} \simeq \frac{\alpha}{\pi \lambda} \sqrt{\kappa \ln \frac{1}{\kappa}}. \quad (4.18)$$

This heuristic estimate coincides with the true answer derived below.

The origin of the extra logarithmic enhancement in (4.18) is readily understood. The singularity of the Coulomb potential at small distances corresponds to a long tail in the transverse momentum distribution. This enriches the contribution from “large jumps”, when the momentum transfer exceeds the inverse Debye radius, $q_{\perp}^2 \gg \mu^2$. Effective “random walk” steps become (logarithmically) larger, which reduces by the $\sqrt{\ln \kappa^{-1}}$ factor the coherence number ν and, thus, the LPM suppression.

Hereafter we shall concentrate on the coherent LPM regime

$$1 \ll \nu \ll N \quad \Longrightarrow \quad \frac{1}{N^2} \ll \kappa \ll 1. \quad (4.19a)$$

In terms of the photon energy the coherent region is limited by

$$\frac{L_{cr}^2}{L^2} \equiv \frac{\lambda E}{L^2 \mu^2} < \frac{\omega}{E} < \frac{2E}{\lambda \mu^2} \equiv \frac{E}{E_{\text{LPM}}}. \quad (4.19b)$$

4.2 General expression for the induced radiation spectrum

We turn to the original expression for the radiation spectrum (4.1). As we shall see below, a finite (though large) number of scatterings ($j-i \sim \nu$) is essential, so that for $N \gg \nu$ a given term of the double sum clearly depends only on the relative position of the two centres, $j-i = n+1$, $n \geq 0$. Moreover, the internal sum over j in (4.1b) converges and, therefore, does not depend on i in the same approximation. The sum over i then gives the total number of scatterings N .

Dividing by the size of the medium, $N\lambda$, and neglecting the factorization contribution in the $N \rightarrow \infty$ limit, we obtain the following expression for the differential energy spectrum per unit length:

$$\omega \frac{dI}{d\omega dz} = \frac{\alpha}{\pi\lambda} \int \frac{d^2\vec{U}_1}{\pi} \left\langle 2 \operatorname{Re} \sum_{n=0}^{\infty} \vec{J}_1 \cdot \vec{J}_{n+2} \left[\exp \left\{ i\kappa \sum_{\ell=1}^{n+1} U_\ell^2 \frac{z_{\ell+1} - z_\ell}{\lambda} \right\} - 1 \right] \right\rangle \left(1 + \mathcal{O}\left(\frac{1}{N}\right) \right) \quad (4.20)$$

Here we have introduced new variables \vec{U}_ℓ to represent \vec{u}_ℓ in units of the typical scattering angle $\theta_1 = \mu/E$:

$$\vec{U}_\ell = \theta_1^{-1} \cdot \vec{u}_\ell; \quad d\Omega = \theta_1^2 d^2\vec{U}, \quad (4.21)$$

and rescaled the emission currents correspondingly:

$$\vec{J}_i = \theta_1 \cdot \vec{A}_i = \frac{\vec{U}_i}{U_i^2} - \frac{\vec{U}_{i-1}}{U_{i-1}^2}. \quad (4.22)$$

It is also convenient to express the transferred momenta in units of μ ,

$$\vec{Q}_\ell = \frac{\vec{q}_\ell}{\mu}, \quad \vec{U}_\ell = \vec{U}_{\ell-1} - \vec{Q}_\ell, \quad (4.23)$$

so that

$$|\vec{Q}_\ell| = |\vec{U}_\ell - \vec{U}_{\ell-1}| \sim 1.$$

Performing in (4.20) the averaging over longitudinal separations with use of (3.19), we obtain

$$\omega \frac{dI}{d\omega dz} = \frac{\alpha}{\pi\lambda} \int \frac{d^2\vec{U}}{\pi} 2 \operatorname{Re} \sum_{n=0}^{\infty} \int \prod_{\ell=1}^{n+2} d^2\vec{Q}_\ell V(Q_\ell^2) \vec{J}_1 \cdot \vec{J}_{n+2} \left[\prod_{m=1}^{n+1} \psi(U_m^2) - 1 \right], \quad (4.24a)$$

where

$$\psi(U^2) = (1 - i\kappa U^2)^{-1}. \quad (4.24b)$$

This holds in general for arbitrary interactions. For the particular case of Coulomb scattering, V is given by

$$V(Q_\ell^2) = \frac{1}{\pi(Q_\ell^2 + 1)^2}. \quad (4.25)$$

In (4.24), the dependence on \vec{Q}_1 and \vec{Q}_{n+2} is contained only in the product of currents

$$\vec{J}_1 \cdot \vec{J}_{n+2} = \left(\frac{\vec{U}_1}{U_1^2} - \frac{\vec{U}_1 + \vec{Q}_1}{(\vec{U}_1 + \vec{Q}_1)^2} \right) \cdot \left(\frac{\vec{U}_{n+1} - \vec{Q}_{n+2}}{(\vec{U}_{n+1} - \vec{Q}_{n+2})^2} - \frac{\vec{U}_{n+1}}{U_{n+1}^2} \right). \quad (4.26)$$

Keeping $\vec{U}_1, \vec{U}_2, \dots, \vec{U}_{n+1}$ fixed, we integrate first over \vec{Q}_1 (which is equivalent to integrating over \vec{U}_0 , the direction of the incoming electron with respect to the photon) and \vec{Q}_{n+2} (outgoing electron). To this end we define

$$\vec{f}_0(\vec{U}_1) \equiv \int d^2\vec{Q}_1 V(Q_1^2) \vec{J}_1 = \pi \frac{\vec{U}_1}{U_1^2} \int dQ_1^2 V(Q_1^2) \Theta(Q_1^2 - U_1^2) = \pi \frac{\vec{U}_1}{U_1^2} \int_{U_1^2}^{\infty} dQ^2 V(Q^2). \quad (4.27a)$$

In the same way we have

$$\vec{f}_0(\vec{U}_{n+1}) \equiv - \int d^2 \vec{Q}_{n+2} V(Q_{n+2}^2) \vec{J}_{n+2} = \pi \frac{\vec{U}_{n+1}}{U_{n+1}^2} \int_{U_{n+1}^2}^{\infty} dQ^2 V(Q^2). \quad (4.27b)$$

The spectrum can be rewritten as

$$\omega \frac{dI}{d\omega dz} = \frac{2\alpha}{\pi\lambda} \operatorname{Re} \int \frac{d^2 \vec{U}_1}{\pi} \vec{f}_0(\vec{U}_1) \cdot \vec{f}(\vec{U}_1) \Big|_{\kappa}^{\kappa=0}, \quad (4.28)$$

where $\vec{f}(\vec{U}_1)$ is given by

$$\vec{f}(\vec{U}_1) = \psi(U_1^2) \vec{f}_0(\vec{U}_1) + \psi(U_1^2) \sum_{n=1}^{\infty} \prod_{\ell=2}^{n+1} \left[\int d^2 \vec{Q}_{\ell} V(Q_{\ell}^2) \psi(U_{\ell}^2) \right] \vec{f}_0(\vec{U}_{n+1}). \quad (4.29)$$

The κ -dependence of \vec{f} comes from the κ -dependence of ψ (see (4.24b)).

Master equation. By expanding the sum over n , it is straightforward to see that $\vec{f}(\vec{U})$ satisfies the following integral equation

$$(1 - i\kappa U^2) \vec{f}(\vec{U}) = \vec{f}_0(\vec{U}) + \int d^2 \vec{Q} V(Q^2) \vec{f}(\vec{U} - \vec{Q}). \quad (4.30)$$

For $\kappa = 0$ it has a trivial potential-independent solution

$$\vec{f}(\vec{U}) = \frac{\vec{U}}{U^2}. \quad (4.31)$$

Indeed, substituting (4.31) under the integral, we have

$$\int d^2 \vec{Q} V(Q^2) \frac{\vec{U} - \vec{Q}}{(\vec{U} - \vec{Q})^2} = \pi \frac{\vec{U}}{U^2} \int_0^{U^2} dQ^2 V(Q^2).$$

Taken together with (4.27) for \vec{f}_0 , this gives for the r.h.s.

$$\pi \frac{\vec{U}}{U^2} \left\{ \int_{U^2}^{\infty} dQ^2 V(Q^2) + \int_0^{U^2} dQ^2 V(Q^2) \right\} = \frac{\vec{U}}{U^2} \cdot 1$$

(due to the normalization of the scattering cross section (3.20)), which is identical to the l.h.s. ($\kappa = 0$).

In order to solve the equation (4.30) for $\kappa \neq 0$ we adopt a method different from the derivation advocated in [1], which relied on an incorrect approximation. Going to the impact parameter space by defining the Fourier transform

$$\widetilde{\vec{f}}(\vec{B}) = \int d^2 \vec{U} e^{-i\vec{B} \cdot \vec{U}} \vec{f}(\vec{U}), \quad (4.32a)$$

$$\widetilde{V}(B^2) = \int d^2 \vec{Q} e^{-i\vec{B} \cdot \vec{Q}} V(Q^2), \quad (4.32b)$$

we first derive the B -space image of the function (4.27):

$$\widetilde{f}_0(\vec{B}) = -2\pi i \frac{\vec{B}}{B^2} (1 - \widetilde{V}(B^2)). \quad (4.33)$$

In these terms (4.30) converts into the differential equation

$$(1 + i\kappa \vec{\nabla}_B^2) \widetilde{f}(\vec{B}) = \widetilde{f}_0(\vec{B}) + \widetilde{V}(B^2) \widetilde{f}(\vec{B}). \quad (4.34)$$

Now we introduce the scalar function \widetilde{g}

$$\widetilde{f}(\vec{B}) = \frac{\vec{B}}{B^2} \widetilde{g}(B^2) \quad (4.35a)$$

to obtain

$$\vec{\nabla}_B^2 \widetilde{f}(\vec{B}) = 4 \vec{B} \widetilde{g}''(B^2), \quad \widetilde{g}'' = \frac{d^2 \widetilde{g}}{d(B^2)^2}. \quad (4.35b)$$

Finally, defining

$$\widetilde{h}(B^2) \equiv \widetilde{g}(B^2) + 2\pi i = \vec{B} \cdot \widetilde{f}(\vec{B}) + 2\pi i, \quad (4.36)$$

we represent (4.34) in the form

$$4i\kappa \widetilde{h}''(B^2) + \frac{1 - \widetilde{V}(B^2)}{B^2} \widetilde{h}(B^2) = 0. \quad (4.37a)$$

This is a linear second order differential equation. The corresponding boundary conditions are

$$\widetilde{h}(0) = 2\pi i, \quad (4.37b)$$

$$\widetilde{h}(\infty) = 0. \quad (4.37c)$$

Indeed, (4.37b) follows from the convergence at $B^2 = 0$ of the integral term in

$$\widetilde{h}(B^2) \equiv 2\pi i + \vec{B} \cdot \int d^2 \vec{U} e^{-i\vec{B} \cdot \vec{U}} \vec{f}(\vec{U}). \quad (4.38)$$

This in turn proceeds from the behaviour of $\vec{f}(\vec{U})$ when $\vec{U} \rightarrow \vec{0}$ or $\vec{U} \rightarrow \infty$ which may be inferred using (4.30) and (4.27a) namely

$$\vec{f}(\vec{U}) \underset{U \rightarrow 0}{\sim} \vec{f}_0(\vec{U}) \underset{U \rightarrow 0}{\sim} \frac{\vec{U}}{U^2}; \quad (4.39a)$$

$$\vec{f}(\vec{U}) \underset{U \rightarrow \infty}{\sim} \frac{i\vec{f}_0(\vec{U})}{\kappa U^2} \underset{U \rightarrow \infty}{\sim} \frac{i\pi \vec{U}}{\kappa U^4} \int_{U^2}^{\infty} dQ^2 V(Q^2). \quad (4.39b)$$

The second boundary condition follows from the fact that in the $B \rightarrow \infty$ limit, $\widetilde{V}(B^2)$ vanishes, and $\widetilde{f}(\vec{B})$ tends to $\widetilde{f}_0(\vec{B})$. Plugged into (4.33) and (4.36), this results in (4.37c).

The spectrum. Going to B -space in the expression for the radiation density (4.28), we obtain

$$\omega \frac{dI}{d\omega dz} = \frac{2\alpha}{\pi^2 \lambda} \operatorname{Re} \int \frac{d^2 \vec{B}}{(2\pi)^2} \widetilde{f}_0(\vec{B}) \cdot \widetilde{f}(-\vec{B}) \Big|_{\kappa}^{\kappa=0}. \quad (4.40)$$

The subtraction term ($\kappa = 0$) given by the B -image of the trivial solution (4.31) corresponds to $\widetilde{h}(B^2) = 0$. Bearing this in mind, we substitute (4.33) and make use of (4.36) to arrive at

$$\omega \frac{dI}{d\omega dz} = \frac{2\alpha}{\lambda\pi} \operatorname{Re} \int \frac{dB^2}{2\pi i} \frac{1 - \widetilde{V}(B^2)}{B^2} \widetilde{h}(B^2). \quad (4.41)$$

Now, invoking the differential equation (4.37) leads to a surprisingly simple result:

$$\omega \frac{dI}{d\omega dz} = \frac{4\alpha}{\lambda\pi^2} \operatorname{Re} \left\{ \kappa \widetilde{h}'(0) \right\}. \quad (4.42)$$

Thus the determination of the spectrum is equivalent to the following mathematical problem: calculate $\widetilde{h}'(0)$ for \widetilde{h} the solution of (4.37).

4.3 Solution in the small κ limit

The previous discussion is valid for general V 's and therefore the differential equation (4.37) is easily treated in the limit $\kappa \ll 1$ by applying the WKB method. The (appropriately normalized) WKB-solution reads

$$\widetilde{h}(B^2) \simeq 2\pi i \left[\frac{-\widetilde{V}'(B^2) B^2}{1 - \widetilde{V}(B^2)} \right]^{\frac{1}{4}} \exp \left\{ -\sqrt{\frac{i}{4\kappa}} \int_0^{B^2} dB'^2 \sqrt{\frac{1 - \widetilde{V}(B'^2)}{B'^2}} \right\}. \quad (4.43)$$

As long as $(1 - \widetilde{V}(B^2))/B^2 \approx -\widetilde{V}'(B^2)$ has a finite $B^2 \rightarrow 0$ limit, this solution can be applied down to $B^2 = 0$. This corresponds to the case when the mean squared transverse momentum $\langle q_{\perp}^2 \rangle$ is well defined:

$$\widetilde{V}(0) = \int d^2 Q V(Q^2) \equiv 1; \quad (4.44a)$$

$$-4\widetilde{V}'(0) = -4 \frac{d}{dB^2} \int d^2 Q V(Q^2) \frac{(i\vec{B}\vec{Q})^2}{2!} = \int d^2 \vec{Q} Q^2 V(Q^2) \equiv \frac{\langle q_{\perp}^2 \rangle}{\mu^2} < \infty. \quad (4.44b)$$

Evaluating $\widetilde{h}'(0)$ and substituting into (4.42) yields the radiation density

$$\omega \frac{dI}{d\omega dz} = \frac{\alpha}{\lambda\pi} \sqrt{2\kappa(-4\widetilde{V}'(0))} = \frac{\alpha}{\pi} \sqrt{\frac{\langle q_{\perp}^2 \rangle}{\lambda E^2}} \omega; \quad \kappa \ll 1, \quad (4.45)$$

which coincides with the original Migdal's result [3,4]. In Appendix A we make a closer contact with the derivation of Migdal, based on the random walk picture.

In the case of interest here of Coulomb interactions, the WKB approximation (4.43) is not suitable, since $\widetilde{V}'(B^2)$ develops the logarithmic singularity near $B^2 = 0$. This case is dealt with in Appendix B, the result of which is corroborated by a more rigorous calculation of $\widetilde{h}'(0)$ due to Chadan, Martin and Stubbe [12]. The final answer reads

$$\omega \frac{dI}{d\omega dz} = \frac{\alpha}{\lambda\pi} \sqrt{\kappa \ln \frac{1}{\kappa}}; \quad \kappa \ll 1. \quad (4.46)$$

5 Conclusion

The aim of this paper has been to revisit the LPM effect in QED. The suppression of the radiation spectrum due to destructive interferences between radiation amplitudes induced by multiple scattering on static Coulomb centres has been studied. In the soft photon approximation and in the limit of large electron energy and infinite medium, the radiation spectrum has been shown to depend on the single parameter $\kappa = \frac{\lambda \mu^2 \omega}{2 E^2}$ which characterizes the coherent nature of the effect: the LPM suppression appears when $\kappa < 1$.

The present approach corrects the derivation given in [1] for QED. The result for the spectrum is only slightly different from [1] within a minor change in the logarithmic factor.

The dominant interference terms are due to centres separated by a distance of the order of the coherence length $\nu\lambda \equiv \frac{1}{\sqrt{\kappa}}\lambda$ much larger than the mean free path λ , which corresponds to large formation times of the radiated photon. The spectrum is determined by photon angles which become as large as $u^2 = U^2 \frac{\mu^2}{E^2} \sim \frac{1}{\sqrt{\kappa}} \frac{\mu^2}{E^2}$, compared to a typical scattering angle of order $\frac{\mu^2}{E^2}$. The fact that U^2 becomes as large as $\frac{1}{\sqrt{\kappa}}$ means that the region $B^2 \sim \sqrt{\kappa} \ll 1$ is the dominant region in “impact parameter” space. In case the Fourier transform of the normalized scattering cross section $\tilde{V}(B^2)$ has a finite derivative $\tilde{V}'(0)$, corresponding to a scattering potential decreasing faster than $1/Q^2$ at large Q , the radiation spectrum is proportional to $\sqrt{\langle q_{\perp}^2 \rangle} = \sqrt{-4\tilde{V}'(0)\mu^2}$. For a potential which is Coulombic at short distances $\tilde{V}'(B^2) \simeq -\frac{1}{4} \ln \frac{1}{B^2}$ and the logarithm gives an enhancement factor $\sqrt{\ln \frac{1}{\sqrt{\kappa}}}$ in the radiation spectrum. For a general potential (4.42) gives the radiation spectrum in terms of the solution of a “Schrödinger” equation (4.37).

Finally, the present procedure can be naturally generalized to high temperature QCD in the spirit of [1]. The result of [1] for QCD should be modified accordingly, which gives for the radiated gluon spectrum

$$\omega \frac{dI^{(\text{QCD})}}{d\omega dz} = \frac{3\alpha_s C_R}{2\pi \lambda_g} \sqrt{\kappa_{\text{QCD}} \ln \frac{1}{\kappa_{\text{QCD}}}}, \quad (5.1)$$

where $C_R = C_F (N_c)$ for a fast propagating quark (gluon). This result is valid for $\kappa_{\text{QCD}} = \lambda_g \mu^2 / \omega \ll 1$, where λ_g is the mean free path of the gluon.

The derivation given in the present work may be extended to finite length media [13], which is important for future phenomenology. Last, since the method of derivation of the radiative energy loss induced by multiple scattering is independent of the detailed form of the scattering potential, provided it satisfies some requirements at large momentum transfer, a generalization to cold nuclear matter seems to be possible [9,13].

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A Gaussian interaction in the limit $\kappa \ll 1$

For comparison it is instructive to relate our derivation of the radiation spectrum to the one originally performed by Migdal [3,4]. Instead of averaging over transverse momenta and longitudinal coordinates as described in (3.19) a Gaussian probability density [14] is used as a solution of an underlying Fokker-Planck equation [3,4], which reflects the assumed random walk nature of multiple scattering. When fixed time steps equal to λ are considered the normalized density reads

$$\prod_l \frac{d^2 \vec{q}_{l\perp}}{\pi \mu^2} \exp\left(-\frac{\vec{q}_{l\perp}^2}{\mu^2}\right), \quad (\text{A.1})$$

where μ^2 is identified as the squared average scattering transverse momentum, $\mu^2 \equiv \langle \vec{q}_{l\perp}^2 \rangle$.

Starting from (4.20), and averaging with (A.1), one gets (4.24) where V is now the Gaussian interaction

$$V(Q^2) = \frac{1}{\pi} \exp(-Q^2), \quad (\text{A.2})$$

and the function $\psi(U^2)$ is just given by the phase $\psi(U^2) = \exp(i\kappa U^2) \simeq 1 + i\kappa U^2$ for $\kappa \ll 1$ which indeed coincides with (4.24b). In order to obtain the soft radiation spectrum we follow the steps of section 4.3 noting that in B -space $\tilde{V}'(0) = -1/4$. The result is already stated in (4.45). The absence of a logarithmic dependence on κ is qualitatively understood since scattering at large transferred momentum is exponentially suppressed.

The averaging prescription of a random walk (A.1) is also used by Blankenbecler and Drell [6], but expressed in terms of the transverse electric field off which the charged particle scatters.

B The spectrum for the Coulomb case in the limit $\kappa \ll 1$

Here we derive the value of $\tilde{h}'(0)$ in the Coulomb case. The Fourier transform of $V(Q^2)$ is

$$\begin{aligned} \tilde{V}(B^2) &= \int \frac{d^2 \vec{Q}}{\pi} \frac{e^{-i\vec{B}\cdot\vec{Q}}}{(Q^2 + 1)^2} = B K_1(B) \\ &\underset{B^2 \ll 1}{\simeq} 1 - \frac{B^2}{4} \ln \frac{1}{B^2} + O(B^2), \end{aligned} \quad (\text{B.1})$$

with the modified Bessel function K_1 [15].

The differential equation (4.37) becomes

$$4i\kappa \tilde{h}''(B^2) + \frac{1 - BK_1(B)}{B^2} \tilde{h}(B^2) = 0. \quad (\text{B.2})$$

For $B^2 \ll 1$, \tilde{h} satisfies the approximate equation

$$\tilde{h}''(B^2) \underset{B^2 \ll 1}{\simeq} -\frac{1}{16i\kappa} \ln\left(\frac{1}{B^2}\right) \tilde{h}(B^2). \quad (\text{B.3})$$

The approximate solution

$$\tilde{h}(B^2) \simeq C \exp\left[\pm \sqrt{\frac{i \ln(1/B^2)}{16\kappa}} B^2\right] \quad (\text{B.4})$$

may be seen, by implementation in (B.3), to be valid in the restricted region

$$\sqrt{\frac{\kappa}{\ell_\kappa^3}} \ll B^2 \ll 1, \quad \ell_\kappa \equiv \ln \frac{1}{\sqrt{\kappa}} \gg 1. \quad (\text{B.5})$$

The constant C and the sign of the exponent in (B.4) are fixed by continuity when looking at the asymptotic forms of the solution \tilde{h} of (B.2). For $B^2 \gg 1$ [15]

$$\tilde{h}(B^2) \underset{B^2 \gg 1}{\sim} C' B H_1^{(1)} \left[\frac{1+i}{4\sqrt{2}} \sqrt{\frac{B^2}{\kappa}} \right] \underset{B^2 \gg 1}{\sim} C'' \sqrt{B} \exp \left[\frac{-1+i}{4\sqrt{2}} \sqrt{\frac{B^2}{\kappa}} \right], \quad (\text{B.6})$$

which satisfies $\tilde{h}(\infty) = 0$. The constants C' and C'' are of order unity.

Extending the forms (B.4) and (B.6) to their limit of validity $B^2 \sim 1$, it is clear that when $\kappa \ll 1$, we have to choose the minus sign in (B.4).

For $B^2 \rightarrow 0$, the solution of (B.3) may be written as

$$\tilde{h}(B^2) \underset{B^2 \rightarrow 0}{\simeq} 2i\pi + \tilde{h}'(0)B^2 - \frac{\pi}{16} \frac{B^4}{\kappa} \ln \left(\frac{1}{B^2} \right) + O(B^4). \quad (\text{B.7})$$

This expansion is valid for

$$B^2 \ll \left| \frac{\kappa \tilde{h}'(0)}{\ln(\kappa |\tilde{h}'(0)|)} \right|. \quad (\text{B.8})$$

Assuming that $\kappa |\tilde{h}'(0)| \underset{\kappa \ll 1}{\sim} \sqrt{\kappa \ell_\kappa}$ (this will be checked a posteriori in (B.11)), (B.7) is valid for

$$B^2 \ll \sqrt{\frac{\kappa}{\ell_\kappa}}. \quad (\text{B.9})$$

Using (B.5) and (B.9), it appears that the forms (B.4) and (B.7) must coincide in the region

$$\sqrt{\frac{\kappa}{\ell_\kappa^3}} \ll B^2 \ll \sqrt{\frac{\kappa}{\ell_\kappa}}. \quad (\text{B.10})$$

Thus we have

$$\begin{aligned} C &= 2i\pi, \\ \tilde{h}'(0) &= (1-i) \frac{\pi}{4} \sqrt{\frac{2\ell_\kappa}{\kappa}}. \end{aligned} \quad (\text{B.11})$$

The final result for the radiation spectrum is obtained from (4.42)

$$\omega \frac{dI}{d\omega dz} \Big|_{\kappa \ll 1} = \frac{\alpha}{\lambda\pi} \sqrt{\kappa \ln \frac{1}{\kappa}}. \quad (\text{B.12})$$

The dominant impact parameter region which determines this spectrum is given by

$$B^2 \sim \sqrt{\kappa}, \quad (\text{B.13})$$

corresponding to

$$U^2 \sim \frac{1}{\sqrt{\kappa}} \gg 1. \quad (\text{B.14})$$

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