

## EXACT SCATTERING IN THE $SU(n)$ SUPERSYMMETRIC

### PRINCIPAL CHIRAL MODEL

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### ABSTRACT

The complete spectrum of states in the supersymmetric principal chiral model based on  $SU(n)$  is conjectured, and an exact factorizable S-matrix is proposed to describe scattering amongst these states. The  $SU(n)_L \times SU(n)_R$  symmetry of the lagrangian is manifest in the S-matrix construction. The supersymmetries, on the other hand, are incorporated in the guise of spin-1/2 charges acting on a set of RSOS kinks associated with  $su(n)$  at level  $n$ . To test the proposed S-matrix, calculations of the change in the ground-state energy in the presence of a coupling to a background charge are carried out. The results derived from the lagrangian using perturbation theory and from the S-matrix using the TBA are found to be in complete agreement for a variety of background charges which pick out, in turn, the highest weight states in each of the fundamental representations of  $SU(n)$ . In particular, these methods rule out the possibility of additional CDD factors in the S-matrix. Comparison of the expressions found for the free-energy also yields an exact result for the mass-gap in these models:  $m/\Lambda_{\overline{MS}} = (n/\pi) \sin(\pi/n)$ .

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## 1. Introduction

Sigma-models in two dimensions have played an important rôle in helping us improve our understanding of quantum field theory; in particular, they have provided illuminating examples of integrable yet highly non-trivial quantum systems in which one can investigate non-perturbative phenomena in an explicit way. It seems that a sigma-model based on a Riemannian symmetric space  $G/H$  (with  $G$  and  $H$  compact Lie groups) is always classically integrable, but that integrability may be spoiled by anomalies at the quantum level unless  $H$  is simple (see [1] and references given there). A prominent subset of the class of quantum-integrable sigma-models consists of the principal chiral models (PCMs). The target space for a PCM is some simple, compact Lie group,  $G$ —which can of course be regarded as a symmetric space  $G \times G/G$ —so the basic field in the theory is a  $G$ -valued function  $U(x, t)$  on Minkowski space. The lagrangian governing its behaviour is

$$\mathcal{L} = \frac{1}{g} \text{Tr} (\partial_\mu U^{-1} \partial^\mu U), \quad (1.1)$$

where  $g$  is a dimensionless coupling, and the theory is clearly invariant under a global symmetry group  $G_L \times G_R$  which acts by left and right multiplication:  $U \rightarrow g_L U g_R^{-1}$  for any  $g_L$  and  $g_R$  in  $G$ .

In four dimensions, lagrangians such as (1.1) have been studied for many years: they are non-renormalizable but they are nevertheless useful in phenomenological, low-energy descriptions of strong interactions in which the chiral symmetry  $G_L \times G_R$  is spontaneously broken to a diagonal flavour subgroup  $G$ . In two dimensions—which is the case of interest here—such lagrangians are, by contrast, renormalizable, and spontaneous symmetry breaking is not allowed on general grounds [2]. Furthermore, since these two-dimensional models are integrable in the sense that they possess infinitely many conserved charges, their S-matrices must factorize, and by enforcing this powerful constraint in conjunction with the usual axioms of S-matrix theory, along with the existence of the unbroken  $G_L \times G_R$  symmetry, one can hope to determine the exact expressions for all scattering amplitudes.

For the cases where  $G$  is a classical Lie group, such S-matrices were conjectured some time ago [3,4]. The spectrum is postulated to consist of particles transforming in representations  $(R, \bar{R})$  of  $G_L \times G_R$ , where  $R$  is some (possibly reducible) representation of  $G$ . In the case  $G = \text{SU}(n)$  there are a total of  $(n-1)$  multiplets of particles, each associated with the fundamental representations  $R = R_a$  of  $G$  with  $a = 1, \dots, n-1$  and with corresponding masses given by

$$m_a = m \frac{\sin(\pi a/n)}{\sin(\pi/n)}, \quad a = 1, 2, \dots, n-1, \quad (1.2)$$

where  $m$  is the mass of the lightest state.<sup>4</sup> Quite recently these S-matrices have been subjected to a highly stringent test involving the Thermodynamic Bethe Ansatz (TBA)—see [5] for the  $SU(n)$  case, and [6] for the other classical groups—using a technique which was first applied to the  $O(N)$  sigma model [7,8] (see also [9,10,11,12]) based on ideas developed in [13,14]. This work showed that the proposed S-matrices for the PCM's are correct as they stand, without the addition of CDD factors, a potential ambiguity which was never resolved in the original papers. The calculations which were performed to test the S-matrices also led, as a by-product, to an exact formula for the mass gaps of these models, and the results have since been confirmed by lattice simulations in various cases [15,16].

Our aim in this paper is to extend the rather satisfying picture of the bosonic principal chiral models sketched above to include their supersymmetric versions, at least when the group is  $G = SU(n)$ . When fermions are coupled to bosonic sigma-models in special ways—either minimally or supersymmetrically—integrability can be maintained at the quantum level, or even re-instated if it was originally broken through quantum effects in the bosonic theory [17].<sup>5</sup> This is one major motivation for studying supersymmetric integrable theories. Another point, of more direct relevance to the work we shall describe here, is that even if the original bosonic theory *is* quantum integrable, the addition of fermions can produce dramatic changes, including a radical alteration of the spectrum. Thus, one finds that typically a supersymmetric model involves something ‘genuinely new’, beyond the mere addition of superpartners for each of the original bosonic states. The original bosonic states may even disappear completely in some cases. We shall see below that it is just this kind of radical alteration in the spectrum which is needed in order to understand the supersymmetric  $SU(n)$  PCM.

It is not difficult to construct a supersymmetric counterpart of the bosonic lagrangian (1.1) for a general group  $G$ , although there are a number of slightly different ways to write the end result. We choose to supplement the original bosonic,  $G$ -valued field  $U$  with a Majorana fermion  $\psi$  taking values in the Lie algebra of  $G$  and we take as the lagrangian<sup>6</sup>

$$\mathcal{L} = \frac{1}{g} \text{Tr} \left( \partial_\mu U^{-1} \partial^\mu U - i \bar{\psi} \gamma^\mu (\partial_\mu \psi + \frac{1}{2} [U^{-1} \partial_\mu U, \psi]) - \frac{1}{16} \{ \bar{\psi}, \gamma_5 \psi \} \{ \bar{\psi}, \gamma_5 \psi \} \right) \quad (1.3)$$

(Our conventions for spinors in two dimensions are those of [18].) Note that the anti-commutators appearing in the fermion interaction terms are elements of the Lie algebra because of the Grassmann nature of the fermion fields. This model is supersymmetric, and, like its bosonic counterpart, it has a global chiral symmetry group  $G_L \times G_R$ . In the

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<sup>4</sup> We use the obvious labelling for the fundamental representations, agreeing with [4], so  $R_1$  is the ‘defining’ representation ( $n$ ) and  $R_{n-1}$  is its complex conjugate ( $\bar{n}$ ).

<sup>5</sup> An example of the latter phenomenon occurs for the sigma models on  $\mathbb{C}P^{n-1}$ , or  $SU(n)/SU(n-1) \times U(1)$ .

<sup>6</sup> This can be derived in a number of ways; perhaps the most straightforward is to consider the obvious modification of (1.1) in superspace and then to integrate out the auxiliary fields to arrive at the expression above.

way we have chosen to write the lagrangian, a general element  $(g_L, g_R)$  in  $G_L \times G_R$  acts according to

$$U \rightarrow g_L U g_R^{-1}, \quad \psi \rightarrow g_R \psi g_R^{-1} \quad (1.4)$$

which clearly leaves the theory invariant. It may seem strange at first that the fermions transform only under  $G_R$ , but this is just a matter of convention; there is an equivalent way of writing the theory in which the fermions transform only under  $G_L$ , these formulations being related by a redefinition of  $\psi$  by conjugation with  $U$ .<sup>7</sup>

From now on we shall specialize to the case  $G = \text{SU}(n)$ . In the next section we shall construct an S-matrix which we claim describes the scattering of particles in this theory, based on a certain assumption about the spectrum. We shall afterwards subject this S-matrix—and hence also our assumption about the spectrum—to the same stringent test involving the TBA that has already been successfully carried out in the bosonic case. In this way we shall be able to show that in this model too, there are no allowed CDD ambiguities in the S-matrix, and we shall be able to extract an exact expression for the mass-gap. Before embarking on the detailed technical aspects of the construction of the S-matrix and its verification, some discussion is in order about what the spectrum of the theory defined by (1.3) is likely to be.

To get some idea of how to proceed it is useful to recall the relationship of PCM's (whether bosonic or supersymmetric) to some other well-known integrable sigma-models, particularly those with  $O(N)$  symmetry for which the target spaces are the spheres  $S^{N-1}$  with their standard round metrics. Notice that the first non-trivial member of this series is the  $O(3)$  model, which has an alternative description as a sigma-model with target space  $\mathbb{C}P^1$ , reflecting the relationship between  $O(3)$  and  $\text{SU}(2)$  (the latter group acting naturally as the symmetry group of complex projective space). The next member of the sequence is the  $O(4)$  model, which is nothing but a re-writing of the  $\text{SU}(2)$  PCM, again reflecting the usual homomorphism from  $\text{SU}(2) \times \text{SU}(2)$  to  $\text{SO}(4)$ .

The bosonic  $O(N)$  model has a very simple spectrum consisting of a single degenerate multiplet of particles transforming in the vector representation of the group, and the S-matrix for these states was found for  $N \geq 3$  in the original work of [19]. The super  $O(N)$  model, on the other hand, was considered in [20,21] and was found to have a much richer spectrum with additional particles transforming in each of the anti-symmetric tensor representations of  $O(N)$ . Witten and Shankar determined the S-matrix for the supermultiplet transforming in the vector representation of  $O(N)$  when  $N > 4$ , but they found no consistent solution for the cases  $N = 3, 4$ . The explanation which they proposed was based on the suggestion that the super  $O(N)$  model should actually contain yet more particles: in fact it is not difficult to see that the theory contains additional kink-like states, transforming in the spinor representations of  $O(N)$ , which arise just as in the Gross-Neveu model [22] (the Gross-Neveu model is, after all, the fermionic sector of the super  $O(N)$  theory).

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<sup>7</sup> From the superspace point of view this ambiguity corresponds to a choice in how one defines the component fields of a given group-valued superfield.

The particles in the vector or tensor representations can be regarded as bound states of these kinks, and it is now natural to suppose that in the particular cases with symmetry  $O(3)$  and  $O(4)$  the bound states simply disappear, leaving only the kinks in the spectrum (just as the ‘elementary’ particles in the sine-Gordon spectrum disappear for sufficiently large coupling, leaving only the solitons). The S-matrix for the kinks is related to the S-matrix of the kinks of the supersymmetric sine-Gordon model [23,24,25] (see also [26]). For  $N = 3$ , this picture is confirmed by the equivalence with the  $\mathbb{C}P^1$  model, in which the only states transform as doublets of  $SU(2)$ , ie. spinors of  $O(3)$ . For  $N = 4$  we are led to conclude that the only states of the  $SU(2)$  PCM should transform as spinors of  $O(4)$ , or in other words as representations  $(1/2,0)$  and  $(0,1/2)$  of  $SU(2)\times SU(2)$ .

This picture of the super  $O(N)$  model advanced by Witten and Shankar suggests a natural conjecture concerning the spectrum of the super  $SU(n)$  PCMs, based on the coincidence of these theories for  $N = 4$  and  $n = 2$ . Thus, for general  $n$ , we may expect to find states in the super  $SU(n)$  PCM transforming in the representations  $(R_a, 1)$  and  $(1, R_a)$  of  $SU(n)\times SU(n)$  (where 1 denotes the trivial representation) rather than in the ‘diagonal’ representations  $(R_a, \bar{R}_a)$  found in the bosonic theory. In the bosonic PCM we can regard as ‘fundamental’ the states transforming in a representation  $(R_1, \bar{R}_1)$ , since all other S-matrix elements can be deduced from these via the bootstrap equations. In the supersymmetric theory, we can regard as similarly fundamental the kink degrees-of-freedom transforming in representations  $(R_1, 1)$  and  $(1, R_1)$ , from which all other S-matrix elements are determined. Despite the different representations of the global symmetry group which appear in the bosonic and supersymmetric cases, we shall see that the mass relations (1.2) are unchanged.

In addition to suggesting the foregoing conjecture for the spectrum, the coincidence of the  $O(4)$  model and the  $SU(2)$  PCM also gives important insight into the way in which supersymmetry enters the construction. A closer examination of the S-matrix for the spinor particles in the  $O(4)$  model [26] shows that the supersymmetric part is identical to the soliton S-matrix in the supersymmetric sine-Gordon theory [25] at a particular choice of the coupling where the scattering is reflectionless. This S-matrix is well-known to be related to an affine  $SU(2)$  quantum group symmetry with deformation parameter  $q$  a root of unity corresponding to level 2 [23]. The supersymmetry acts on the kink-quantum numbers in a rather involved way that is intimately related to the quantum group structure, as discussed in [27]. For the  $SU(n)$  PCM it is sensible to try to implement supersymmetry in an analogous way, by association with an affine  $SU(n)$  quantum group with  $q$  a root of unity corresponding to level  $n$ . We shall have a number of comments to make below regarding various subtleties involved in this construction.

## 2. The exact S-matrix

In this section we recall some general ideas which are useful in the construction of exact S-matrices. After setting up a certain amount of technology we will be able to give a succinct statement of our conjecture for the S-matrix of the supersymmetric  $SU(n)$  PCM. The main idea in the approach we shall follow here is the notion of an S-matrix ‘block’ which is invariant under the action of a quantum group related to  $SU(n)$ .<sup>8</sup> Our S-matrix will eventually be constructed in terms of such blocks and, along the way, we will explain how the S-matrix for the bosonic PCM can be understood from the same point-of-view. There are actually two kinds of blocks, one kind associated to the vertex-type realization and another associated to the RSOS-type representations of a quantum group and we shall need both kinds.<sup>9</sup> The detailed construction of the blocks has been discussed at length in the papers cited above and so we shall concentrate here on the characteristic properties of the blocks, giving only those explicit expressions that we shall need later.

In the vertex picture, particle states transform in the fundamental representations  $R_a$ ,  $a = 1, \dots, n-1$ , of  $SU(n)$ . We denote single-particle states by vectors  $\boldsymbol{\xi}^{(a)}(\theta)$  living in the vector space  $V_a$  which carries the representation  $R_a$ , where  $\theta$  is the rapidity. The corresponding blocks are matrices of the form  $\tilde{S}_{ab}(\theta)_M^N$  with  $M = (\boldsymbol{\xi}^{(a)}, \boldsymbol{\xi}^{(b)})$  and  $N = (\boldsymbol{\eta}^{(b)}, \boldsymbol{\eta}^{(a)})$  which give the amplitude for scattering between particle states transforming under the given representations:

$$\tilde{S}_{ab}(\theta)_M^N : \boldsymbol{\xi}^{(a)}(\theta_1) + \boldsymbol{\xi}^{(b)}(\theta_2) \rightarrow \boldsymbol{\eta}^{(b)}(\theta_2) + \boldsymbol{\eta}^{(a)}(\theta_1), \quad (2.1)$$

with  $\theta = \theta_1 - \theta_2$ . The blocks enjoy the following properties. (i) They satisfy the Yang-Baxter equation. (ii) They obey the completeness or unitarity relation

$$\tilde{S}_{ab}(\theta)\tilde{S}_{ba}(-\theta) = I. \quad (2.2)$$

(iii) They are crossing symmetric. (iv) They satisfy the  $su(n)$  bootstrap equations: if  $V_c \subset V_a \otimes V_b$  (so  $c = a + b$  if  $a + b < n$  and  $c = a + b - n$  if  $a + b > n$ ) then

$$\tilde{S}_{dc}(\theta) \simeq \tilde{S}_{da}(\theta - iu_b)\tilde{S}_{db}(\theta + iu_a), \quad (2.3)$$

where  $u_a = a\pi/n$  if  $a + b < n$  and  $\pi - a\pi/n$  if  $a + b > n$ . The equality in (2.3) holds in the sense that the right-hand-side should be restricted to the subspace  $V_c \subset V_a \otimes V_b$ . In fact we shall take the blocks to be the *minimal* expressions with this property, and we

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<sup>8</sup> The relation of S-matrices to quantum groups was first developed for  $SU(2)$  in [28] and later extended to  $SU(n)$  in [29]. The extension to other algebras and to the RSOS representations appears in [30]. A more unified approach was developed in [31].

<sup>9</sup> This language is borrowed from integrable lattice models where the S-matrix elements play the rôle of Boltzmann weights

shall explain more fully below what this term means. (v) Finally, there is actually a one-parameter family of blocks meeting all these conditions, and we can distinguish between its members by labelling them with a certain real number  $\lambda$ . It turns out that these S-matrix blocks are invariant under the action of a quantum group symmetry  $U_q(su(n))^{(1)}$  where

$$q = -\exp -i\pi\lambda \quad (2.4)$$

is the deformation parameter.

Let us elaborate on the last of these properties first. The quantum group  $U_q(su(n))^{(1)}$  is defined by a set of generators  $H_i$  and  $E_i^\pm$  with  $i = 0, \dots, n-1$  which obey the ‘deformed’ commutation relations

$$[H_i, E_j^\pm] = \pm a_{ij} E_j^\pm, \quad [E_i^+, E_j^-] = \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, \quad (2.5)$$

where  $a_{ij}$  is the usual Cartan matrix for  $su(n)^{(1)}$ . We are interested here in the *centerless* extension of the finite-dimensional quantum group  $U_q(su(n))$ , in other words, we will always understand  $U_q(su(n))^{(1)}$  to mean the quantum loop group. The precise way in which this acts as a symmetry of the S-matrix blocks is rather subtle, because it incorporates the rapidity operator of the theory in a non-trivial way.

To explain how this works, let us begin by introducing the matrices  $\tilde{H}_i$  and  $\tilde{E}_i^\pm$   $i = 1, \dots, n-1$  which represent the Cartan generators and simple-root step operators for the finite-dimensional algebra  $U_q(su(n))$  in the representation  $R_a$ . By definition, these act on the vector space  $V_a$ . Let us also introduce matrices  $\tilde{E}_0^\pm$  corresponding to the step operators for the (non-simple) lowest root of  $su(n)$ , and  $\tilde{H}_0 = -\sum_{i=1}^{n-1} \tilde{H}_i$ . In terms of these matrices, the quantum loop-group generators are realized on single-particle states in  $V_a$  with definite rapidity  $\theta$  by

$$H_i \rightarrow \tilde{H}_i, \quad E_i^\pm \rightarrow e^{\pm s_i \theta} \tilde{E}_i^\pm \quad (2.6)$$

with  $i = 0, \dots, n-1$ . Notice that on a fundamental representation the deformed commutation relations (2.5) reduce to those of the Lie algebra because the eigenvalues of the  $H_i$  are only 0 or  $\pm 1$ . The set of real numbers  $s_i$  are the Lorentz spins of the symmetry generators, and they can be viewed as a choice of (non-integral) gradation of the loop algebra. For the vertex-type S-matrix blocks that we shall write down below, the appropriate choice for these spins is

$$s_i = \lambda, \quad i = 0, \dots, n-1, \quad (2.7)$$

so their values are fixed by the deformation parameter  $q$ . This can also be expressed by saying that the loop algebra  $U_q(su(n))^{(1)}$  is realized in the principal gradation, but with the standard loop parameter given by

$$t = (-q)^{-\theta/i\pi} = e^{\lambda\theta}, \quad (2.8)$$

which relates it to the rapidity and to  $q$ .

Lastly, we should note that to state precisely what is meant by a symmetry of the S-matrix, we need to extend the action of the generators introduced above to multi-particle states. For this we use the coproduct structure in the quantum group. Thus, the generators extend to two-particle states according to

$$\begin{aligned}\Delta(E_i^\pm) &= E_i^\pm \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^\pm, \\ \Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i.\end{aligned}\tag{2.9}$$

Invariance under the quantum group means that these generators commute with the action of the S-matrix blocks.

Having explained something of the symmetry properties of the vertex-type S-matrix blocks, let us return to the bootstrap condition (2.3) written above. The blocks are minimal solutions of these crucial equations in the sense that they have no poles on the physical strip for values of the parameter in the range  $0 < \lambda < 1/n$ . This means that although they satisfy the  $su(n)$  bootstrap equations, there are no simple poles to signal *dynamically* that  $V_c$  should be a bound-state of  $V_a$  and  $V_b$  if  $V_c \subset V_a \otimes V_b$ . (When we come to use the blocks to write down physically meaningful S-matrices, the necessary poles will be incorporated in additional scalar factors.) Nevertheless, the usual bootstrap procedure can still be used to build up *all* the blocks, for any given representations, out of the elementary block  $\tilde{S}_{11}(\theta)$  for the scattering of the  $n$ -dimensional defining representation  $R_1$  with itself. To complete our description of the blocks for the vertex representations, let us finally give some explicit formulas.

First of all, it is convenient to label the states by the weight vectors of the representations. In order to do this we introduce the weights  $\mathbf{e}_i$ ,  $i = 1, \dots, n$  of the representation  $R_1$ , with inner products  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} - 1/n$ . Now we can write down an expression for the scattering of states

$$\tilde{S}_{11}(\theta)_{ij}^{kl} : \mathbf{e}_i(\theta_1) + \mathbf{e}_j(\theta_2) \rightarrow \mathbf{e}_k(\theta_2) + \mathbf{e}_l(\theta_1),\tag{2.10}$$

by specifying the three non-zero components

$$\begin{aligned}\tilde{S}_{11}(\theta)_{ii}^{ii} &= f(\theta) \sin(\pi\lambda - n\lambda\theta/2i) \\ \tilde{S}_{11}(\theta)_{ij}^{ji} &= f(\theta) \sin(n\lambda\theta/2i), \quad i \neq j \\ \tilde{S}_{11}(\theta)_{ij}^{ij} &= f(\theta) e^{(2j-2i\pm n)\lambda\theta/2} \sin(\pi\lambda), \quad i > j \text{ or } i < j.\end{aligned}\tag{2.11}$$

The function  $f(\theta)$  is again *minimal* in the sense of having the least number of poles and zeros on the physical strip—in this case none—necessary in order to ensure unitarity and crossing symmetry. The explicit form of the function is [29]

$$\begin{aligned}f(\theta) &= \frac{1}{\sin(\pi\lambda - n\lambda\theta/2i)} \prod_{j=1}^{\infty} \frac{\Gamma(1 + \frac{in\lambda\theta}{2\pi} + (j-1)n\lambda)}{\Gamma(1 - \frac{in\lambda\theta}{2\pi} + (j-1)n\lambda)} \\ &\times \frac{\Gamma(\frac{in\lambda\theta}{2\pi} + jn\lambda) \Gamma(-\frac{in\lambda\theta}{2\pi} + [(j-1)n+1]\lambda) \Gamma(1 - \frac{in\lambda\theta}{2\pi} + (jn-1)\lambda)}{\Gamma(-\frac{in\lambda\theta}{2\pi} + jn\lambda) \Gamma(\frac{in\lambda\theta}{2\pi} + [(j-1)n+1]\lambda) \Gamma(1 + \frac{in\lambda\theta}{2\pi} + (jn-1)\lambda)}.\end{aligned}\tag{2.12}$$



The information we have just given now determines all the blocks, because of the bootstrap property. Of course the scattering matrix of the charge conjugate states, transforming as  $R_{n-1}$ , can also be found by using crossing symmetry. If we introduce a basis for these states denoted  $\mathbf{e}_{\bar{i}}(\theta)$  then we must have

$$\tilde{S}_{1,n-1}(\theta)_{\bar{i}\bar{j}}^{\bar{k}l} = \tilde{S}_{11}(i\pi - \theta)_{ki}^{lj}. \quad (2.13)$$

It is important that the relation (2.13) is consistent with the bootstrap equations (2.3) and also with unitarity (2.2).

We must now draw attention to some special behaviour of the quantum group symmetry for various values of  $q$ . For generic values, the only part of the conventional  $SU(n)$  symmetry which survives unscathed in the quantum group is the Cartan subalgebra. But if  $q^2 = 1$  then the quantum group invariance becomes a conventional Lie algebra invariance, and then the blocks that we have introduced above are simply  $SU(n)$ -invariant. Note in particular that the S-matrix blocks have a well-defined limit as  $\lambda \rightarrow 0$ , which means  $q \rightarrow -1$ ; this is known as the ‘rational’ limit. The Lorentz spins  $s_i$  of the generators become zero in this limit, as one would expect. The resulting  $SU(n)$ -invariant blocks have a special role to play, and we shall denote them by  $\tilde{S}_{ab}^{SU(n)}(\theta)$ .

The other significant special values for  $q$  occur when  $\lambda = 1/(n+k)$  where  $k$  is a positive integer. In these cases it is possible to ‘restrict’ the vertex-type representations that we have considered up till now and to pass to the so-called RSOS version of the S-matrix, which is realized in terms of kink-like states. We now need to describe this in some detail.

We must first specify the set of ‘kink’ states on which the RSOS S-matrices act, and this depends on the choice of the integer  $k$ . By definition, the ‘kinks’ interpolate between ‘vacua’ which are labelled by the set of integrable highest weights of the affine algebra  $su(n)^{(1)}$  at level  $k$ . This set of highest weights can be written explicitly as

$$\Lambda = \left\{ \sum_{i=1}^{n-1} m_i \mathbf{e}_i, k \geq m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0 \right\}. \quad (2.14)$$

where  $\mathbf{e}_i$  are the weight vectors introduced earlier. Now the kinks also come in a number of different ‘species’ which are labelled by the fundamental representations  $R_a$ , so we denote them  $K_{\mu\nu}^{(a)}(\theta)$ . Here  $\mu, \nu \in \Lambda$  and the kink can be thought of as carrying a topological charge  $\nu - \mu$  which is a vector of eigenvalues of the Cartan generators. The final restriction on the allowed single-kink states is that this charge must be a weight of the representation  $R_a$ ; namely  $\nu - \mu = \sum_{p=1}^a \mathbf{e}_{i_p}$  with all the  $i_p$ ’s distinct. This restriction can be expressed in terms of the representation theory of the affine algebra  $su(n)^{(1)}$  at level  $k$  by saying that there is an allowed kink state  $K_{\mu\nu}^{(a)}(\theta)$  whenever the irreducible module with highest weight  $\nu$  occurs in the decomposition of the tensor product between the module with highest weight  $\mu$  and the module associated to  $R_a$ . What we have said fixes the allowed one-kink states completely. Multi-kink states are now constructed as usual in quantum mechanics, except that there is an additional ‘adjacency’ condition which requires that

successive vacua must coincide. Thus, the allowed two-kink states, for example, are of the form  $|K_{\mu_1\mu_2}^{(a_1)}(\theta_1)K_{\mu_2\mu_3}^{(a_2)}(\theta_2)\rangle$  and similarly for any number of kinks.

We can now introduce the idea of an S-matrix block associated to the scattering of these RSOS kinks, just as we did previously for the vertex representations. These RSOS blocks share many of the nice properties of the vertex blocks that we discussed earlier: (i) They satisfy the Yang-Baxter equation; (ii) they obey the unitarity condition; (iii) they are crossing symmetric; (iv) they obey the  $su(n)$  bootstrap equations. The bootstrap property means that we can, if we so wish, consider only the S-matrix elements for the kinks associated to the representation  $R_1$ , since all the others may be deduced from these. The S-matrix elements

$$\begin{aligned} \tilde{S}_{11}^{(\text{RSOS},k)}(\theta; \boldsymbol{\mu})_{ij}^{kl} &: K_{\boldsymbol{\mu}, \boldsymbol{\mu} + \mathbf{e}_i}^{(1)}(\theta_1) + K_{\boldsymbol{\mu} + \mathbf{e}_i, \boldsymbol{\mu} + \mathbf{e}_i + \mathbf{e}_j}^{(1)}(\theta_2) \\ &\rightarrow K_{\boldsymbol{\mu}, \boldsymbol{\mu} + \mathbf{e}_k}^{(1)}(\theta_2) + K_{\boldsymbol{\mu} + \mathbf{e}_k, \boldsymbol{\mu} + \mathbf{e}_k + \mathbf{e}_l}^{(1)}(\theta_1), \end{aligned} \quad (2.15)$$

are, like those in (2.11), non-zero only if charge is conserved, i.e.  $\mathbf{e}_i + \mathbf{e}_j = \mathbf{e}_k + \mathbf{e}_l$ . The explicit expressions for the non-zero elements are

$$\begin{aligned} \tilde{S}_{11}^{(\text{RSOS},k)}(\theta; \boldsymbol{\mu})_{ii}^{ii} &= f(\theta) \sin(\pi\lambda - n\lambda\theta/2i) \\ \tilde{S}_{11}^{(\text{RSOS},k)}(\theta; \boldsymbol{\mu})_{ij}^{ji} &= f(\theta) \sin(n\lambda\theta/2i) \frac{\sqrt{(s_{ij}(\boldsymbol{\mu} + \mathbf{e}_i)s_{ij}(\boldsymbol{\mu} + \mathbf{e}_j))}}{s_{ij}(\boldsymbol{\mu})}, \quad i \neq j \\ \tilde{S}_{11}^{(\text{RSOS},k)}(\theta; \boldsymbol{\mu})_{ij}^{ij} &= f(\theta) \left[ \sin(\pi\lambda - n\lambda\theta/2i) + \sin(n\lambda\theta/2i) \frac{s_{ij}(\boldsymbol{\mu} + \mathbf{e}_i)}{s_{ij}(\boldsymbol{\mu})} \right], \quad i \neq j \end{aligned} \quad (2.16)$$

(compare with (2.11)) where

$$s_{ij}(\boldsymbol{\mu}) = \sin(\pi\lambda(\mathbf{e}_i - \mathbf{e}_j) \cdot (\boldsymbol{\mu} + \boldsymbol{\rho})), \quad (2.17)$$

and  $\boldsymbol{\rho} = \sum_{j=1}^{n-1} (n-j)\mathbf{e}_j$  is the sum of the fundamental weights of  $su(n)$ .

The last point to discuss, which is central to our construction, is the issue of the symmetry properties of these RSOS blocks. We explained above that the vertex blocks are invariant under a quantum loop group  $U_q(su(n)^{(1)})$  where the generators have non-trivial Lorentz spins given by (2.7). The first step in the construction of the RSOS blocks involves starting with the vertex S-matrix and carrying out a conjugation operation which has the effect of changing the action of the  $U_q(su(n)^{(1)})$  symmetry generators so that they have modified spins, with (2.7) replaced by

$$s_0 = n\lambda, \quad s_i = 0, \quad i = 1, \dots, n-1. \quad (2.18)$$

This is effectively a change to the homogeneous gradation of the loop algebra. In a loose sense, the RSOS reduction is now achieved by ‘modding-out’ with respect to the generators which have just been rendered spin-less by this change in gradation (more properly the RSOS blocks appear as intertwiners of irreducible representations of the finite-dimensional

$U_q(su(n))$ ) so there is now just one surviving conjugate pair of symmetry charges with non-zero spin. The last step, which requires  $\lambda = 1/(n+k)$ , is to further restrict the allowed space of states (passing from SOS to RSOS in statistical mechanical terminology) according to the rules for allowed kinks which we expressed above in terms of the allowed highest weights of  $su(n)$  at level  $k$ . So we arrive finally at a set of S-matrix blocks for which most of the original quantum group symmetry has been lost, but for which there remains a pair of conserved charges with Lorentz spins

$$\pm s_0 = \pm n\lambda = \pm \frac{n}{n+k}. \quad (2.19)$$

At this stage the possible connection with supersymmetry becomes apparent: on choosing  $k = n$  the formula above tells us that the surviving charges have Lorentz spin  $\pm 1/2$ , and it is natural to suspect that they are supersymmetry generators. Unfortunately, it is apparently not known at present how to show directly that these conserved quantities really obey the correct supersymmetry algebra for all values of  $n$ . The question of how the original quantum group relations descend to the RSOS picture seems to be rather poorly understood, and there is even some considerable freedom in how these conserved charges are defined to act in the restricted theory. (See the discussions in [23,24,25,27,32].) What has been known for certain for some time is that in the simplest case,  $n = 2$ , it is possible to show explicitly that spin-1/2 conserved quantities can be defined so as to obey the supersymmetry algebra [23]. We will make the reasonable assumption that this can also be done for general values of  $n$ . We have carried out a partial check of the next simplest case based on  $SU(3)$  using brute force methods, and it appears that in this case too the residual spin half charges can be defined so as to obey the desired algebra. In the absence of a general construction for  $SU(n)$ , however, we defer a more detailed discussion of this point to another occasion.

To summarize: we will proceed on the assumption that the RSOS blocks with  $k = n$  carry a good representation of supersymmetry. To add some additional reassurance on this point, we should also emphasize that the S-matrix which we write down based on this assumption will ultimately be subjected to a very stringent test. The conclusion will be that it does indeed correspond to the original supersymmetric lagrangian (1.3).

We have now set up all the S-matrix technology that we need, but before we consider the supersymmetric model it may be helpful to explain how the S-matrix of the bosonic PCM is constructed from the point-of-view we have followed here. For the bosonic  $SU(n)$  model, the particles transform in multiplets  $(R_a, \bar{R}_a) = (R_a, R_{n-a})$  of the global  $SU(n)_L \times SU(n)_R$  symmetry. The states in the theory are consequently of the form

$$|\xi^{(a)}, \eta^{(n-a)}; \theta\rangle \quad (2.20)$$

The S-matrix acting on these states is constructed out of two  $SU(n)$ -invariant vertex blocks:

$$S_{ab}(\theta) = X_{ab}(\theta) \tilde{S}_{ab}^{SU(n)}(\theta) \otimes \tilde{S}_{n-a, n-b}^{SU(n)}(\theta), \quad (2.21)$$

where the tensor product structure corresponds to the product structure of the states in the obvious way. Notice that there are no reflection processes between the degenerate multiplets  $(R_a, R_{n-a})$  and  $(R_{n-a}, R_a)$ . The pre-factors  $X_{ab}(\theta) = X_{n-a, n-b}(\theta)$  are a set of scalar functions which obey the  $SU(n)$  bootstrap equations, the unitarity condition and crossing symmetry independently, but which introduce simple poles on the physical strip at just the right positions in order that the particle  $c$  appears as a bound-state of  $a$  and  $b$  if  $c = a + b$  or  $c = a + b - n$ . These factors make the bootstrap dynamical. In fact  $X_{ab}(\theta)$  is simply the minimal purely elastic S-matrix associated to  $su(n)$ :

$$X_{ab}(\theta) = \prod_{\substack{j=|a-b|+1 \\ \text{step } 2}}^{a+b-1} \frac{\sin(\theta/2i + \pi(j-1)/2n) \sin(\theta/2i + \pi(j+1)/2n)}{\sin(\theta/2i - \pi(j-1)/2n) \sin(\theta/2i - \pi(j+1)/2n)}. \quad (2.22)$$

(In comparing this to the result of [3] one should note that our definitions of the group  $SU(n)_R$  differ by complex conjugation.)

We are now ready to write down our conjecture for the states and scattering amplitudes in the supersymmetric  $SU(n)$  PCM. The two essential global invariances which we need to incorporate are  $SU(n)_L \times SU(n)_R$  and supersymmetry. The first of these can be built in using the vertex-type blocks introduced above, once we have decided what the allowed representations should be. Following our discussion in the first section which compared PCMs to the family of  $O(N)$  sigma models, we conjecture that the super PCM has a spectrum of particles with masses  $m_a$ , given by (1.2), which transform in reducible multiplets  $(R_a, 1) \oplus (1, R_a)$  for  $a = 1, \dots, n-1$ . (Of course, such states are always degenerate with their conjugates which transform as  $(R_{n-a}, 1) \oplus (1, R_{n-a})$ .) To incorporate supersymmetry, these states must also carry additional quantum numbers on which the super-charges act. Following our discussion above of RSOS-type blocks and their symmetry properties, we shall take these to be RSOS kinks of type  $a$  with  $k = n$ . In short then, the states in the model are of two general types

$$L : |\xi^{(a)}, 0, K_{\mu\nu}^{(a)}; \theta\rangle \quad R : |0, \xi^{(a)}, K_{\mu\nu}^{(a)}; \theta\rangle \quad (2.23)$$

where  $SU(n)_L$  acts on the first quantum number,  $SU(n)_R$  acts on the second quantum number, and supersymmetry acts on the kink degrees of freedom. We emphasize that the allowed kink states interpolate between the integrable weights of  $su(n)$  at level  $n$  according to the rules which we summarized earlier.

Having specified the detailed structure of the states, we define the S-matrix as follows. The scattering between the L multiplets is given by

$$S_{ab}^{LL}(\theta) = X_{ab}(\theta) \tilde{S}_{ab}^{\text{SU}(n)}(\theta) \otimes I \otimes \tilde{S}_{ab}^{\text{(RSOS}, n)}(\theta), \quad (2.24)$$

and similarly between the R multiplets

$$S_{ab}^{RR}(\theta) = X_{ab}(\theta) I \otimes \tilde{S}_{ab}^{\text{SU}(n)}(\theta) \otimes \tilde{S}_{ab}^{\text{(RSOS}, n)}(\theta) \quad (2.25)$$

where  $X_{ab}(\theta)$  is the minimal elastic factor defined by (2.22). The scattering between the L and R multiplets is defined to be

$$S_{ab}^{\text{LR}}(\theta) = I \otimes I \otimes \tilde{S}_{ab}^{(\text{RSOS}, n)}(\theta). \quad (2.26)$$

As usual, the tensor products are to be understood with respect to the product structure of the states exhibited in (2.23). We emphasize that this is a conjecture for the *complete* S-matrix, describing scattering amongst *all* the states in the theory.

The following points are worthy of note. (i) The scattering between the L and R multiplets is completely diagonal in the space of global quantum numbers, as required by the form of the global symmetry. (ii) It is only the LL or RR scattering which lead to bound-states, because only these elements have poles on the physical strip provided by the scalar factors  $X_{ab}(\theta)$ . The LR scattering elements have no poles on the physical strip so no bound-states which transform non-trivially under both  $SU(n)_L$  and  $SU(n)_R$  are formed. (iii) Since the S-matrix elements are built out of the blocks, we can be assured that all the S-matrix axioms are satisfied. Furthermore, due to the existence of simple poles on the physical strip in just the right positions, provided by the factors  $X_{ab}(\theta)$ , the multiplets can all be considered as bound-states of the multiplets with  $a = 1$ . (v) There are no reflection amplitudes between any two of the degenerate multiplets  $(R_a, 1)$ ,  $(R_{n-a}, 1)$ ,  $(1, R_a)$  and  $(1, R_{n-a})$ .

Thus far, we have offered very little evidence to support our claim that this is the correct S-matrix for the super  $SU(n)$  PCM. In the following sections, however, we will carry out a very substantial test of this proposal. We shall find that our conjecture is completely consistent with the lagrangian (1.3), and in particular that it correctly reproduces the universal part of the beta-function. The nature of this test is explained more fully below. At this stage, however, it may also be helpful to draw attention to one *specific* kind of possible ambiguity in the S-matrix which our test will ultimately resolve.

As it stands, the S-matrix we have written down is the minimal<sup>10</sup> expression which satisfies all the axioms of S-matrix theory, along with the requirement that the states can all be formed as bound-states on the elementary multiplet  $(R_1, 1) \oplus (1, R_1)$ . The latter requirement was responsible for introducing the factors of  $X_{ab}(\theta)$  in (2.24) and (2.25). However, it is always possible to multiply these expressions by CDD factors which satisfy all the axioms independently, which passively respect the bootstrap equations, and, moreover, introduce no additional poles onto the physical strip (although they introduce additional zeros). These factors are of the form

$$Y_{ab}(\alpha; \theta) = \prod_{\substack{j=|a-b|+1 \\ \text{step } 2}}^{a+b-1} \frac{\sin(\theta/2i - \pi(j-1+\alpha)/2n) \sin(\theta/2i - \pi(j+1-\alpha)/2n)}{\sin(\theta/2i + \pi(j-1+\alpha)/2n) \sin(\theta/2i + \pi(j+1-\alpha)/2n)}, \quad (2.27)$$

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<sup>10</sup> ‘Minimality’ in this context means the expression with the smallest number of poles and zeros on the physical strip.

where  $\alpha$  is a parameter  $0 < \alpha < 2$ . In principle, an arbitrary number of these factors with different  $\alpha$ 's could be introduced in the S-matrix elements LL and RR, and a different set in LR scattering. One of the conclusions we shall reach in the following sections is that there are no CDD factors of this type allowed in the S-matrix for the models we are considering.

### 3. S-matrix versus lagrangian: perturbative calculation

Having arrived at a candidate S-matrix for the supersymmetric  $SU(n)$  PCM, our aim is now to test this proposal by comparing it with the original lagrangian (1.3). There is a technique for doing this which is by now well-established and which has been applied to a number of different models, following the original pioneering work of [7,8] for the  $O(N)$  sigma-model, so our explanation of the general method will be rather brief.<sup>11</sup> The technique involves modifying the Hamiltonian of the theory  $\mathcal{H} \rightarrow \mathcal{H} - hQ$  where  $Q$  is a conserved charge which generates a global symmetry of the model—in our case a generator of  $SU(n)_L \times SU(n)_R$ —and  $h$  is a parameter with the dimensions of mass which we put in by hand. The idea is to calculate the ground-state energy of this new Hamiltonian, both in perturbation theory and from the S-matrix using the TBA, and then to compare the results to test the lagrangian/S-matrix equivalence. In fact neither of these calculations can be performed exactly, but we can develop asymptotic expansions for the results which are assumed to be valid when  $h$  is very large and which are sufficient to give a highly non-trivial consistency check.

A standard perturbative analysis of the model (1.3) shows that it is asymptotically free, with the coupling constant behaving as

$$\frac{1}{g(\mu)} = \beta_1 \ln \frac{\mu}{\Lambda} + \frac{\beta_2}{\beta_1} \ln \ln \frac{\mu}{\Lambda} + \mathcal{O} \left( \ln \ln \frac{\mu}{\Lambda} / \ln \frac{\mu}{\Lambda} \right) \quad (3.1)$$

where the universal part of the beta function is given by the coefficients

$$\beta_1 = n/8\pi, \quad \beta_2 = 0. \quad (3.2)$$

The scale  $\Lambda$  in these equations is defined by the requirement that there is no constant term in the expansion for the running coupling written above; this depends on the renormalization prescription being used and we shall eventually specialize to the MS-bar scheme. For a given choice of the charge  $Q$  introduced above, it is not difficult to calculate the new ground-state energy to one-loop, or in principle to some higher order in the loop expansion, and when  $h \gg \Lambda$  we can obtain an expression for the change in the ground-state energy

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<sup>11</sup> A recent summary which attempts to collect together the known results was given in [33]; a more complete account is planned for the near future [34].

density:

$$\begin{aligned}\delta\mathcal{E}(h) &= \mathcal{E}(h) - \mathcal{E}(0) = h^2 f_1(h/\Lambda), \\ f_1(h/\Lambda) &= \frac{a_0}{g(h)} + a_1 + \mathcal{O}(g(h))\end{aligned}\tag{3.3}$$

where the numbers  $a_j$  are dimensionless quantities resulting from a calculation at a certain number of loops indicated by their subscripts. Notice that the argument of the coupling constant in this expression is  $h/\Lambda$ , which is a consequence of the fact that the final result must be a renormalization-group invariant quantity and therefore independent of  $\mu$ , the subtraction scale.

In the next section we shall see how a similar expression can be found from the S-matrix via the TBA, but rather than being a function of  $h/\Lambda$  the result is then of the form

$$\delta\mathcal{E}(h) = \mathcal{E}(h) - \mathcal{E}(0) = h^2 f_2(h/m)\tag{3.4}$$

where  $m$  is the physical mass of the particle states. Equating this to the previous expression we see that  $f_1(h/\Lambda) = f_2(h/m)$  which gives a stringent test of the S-matrix and, if we know each expression to sufficient accuracy, this equality will allow us to extract the mass-gap  $m/\Lambda$ . A simple but very important observation is that the nature of the functions  $f_1$  and  $f_2$ , and hence the values of the constants  $a_j$ , is dependent on the choice of the charge  $Q$ . It turns out that if we choose  $Q$  so that  $a_0 \neq 0$ , in other words so that there is a tree-level contribution to the ground-state energy, then it is sufficient to carry out a one-loop perturbative calculation in order to get a convincing test of the S-matrix and to extract the mass-gap. If  $a_0 = 0$  on the other hand, then one must work harder to obtain a non-trivial check, and the mass-gap can only be obtained by calculating to three-loops or beyond.

It is clearly advantageous to choose a charge  $Q$  which leads to one-loop computation and there is actually a general strategy by which this can be accomplished for a wide class of (super) sigma-models [34]. In this paper, however, we shall simply mimic the choice of charges already considered in the bosonic case [5] and show that these lead to the desired classical term  $a_0 \neq 0$ , so that a one-loop calculation will suffice for a test of the supersymmetric models too. To be specific, we shall consider here a modification of the Hamiltonian

$$\mathcal{H} \rightarrow \mathcal{H} - hQ \quad \text{where} \quad Q = (\mathbf{q} \cdot \mathbf{H}, -\mathbf{q} \cdot \mathbf{H})\tag{3.5}$$

is a hermitian generator in the Lie algebra of  $SU(n)_L \times SU(n)_R$ , with  $\mathbf{H}$  denoting the vector of generators in the Cartan subalgebra of  $su(n)$  and  $\mathbf{q}$  some vector whose components are arbitrary numbers at this stage. From the action of the  $SU(n)_L \times SU(n)_R$  symmetry on the fields given in (1.4) we can deduce that the linearized action generated by this choice of  $Q$  is

$$\begin{aligned}\delta U &= i(\mathbf{q} \cdot \mathbf{H} U + U \mathbf{q} \cdot \mathbf{H}) \\ \delta \psi &= -i(\mathbf{q} \cdot \mathbf{H} \psi - \psi \mathbf{q} \cdot \mathbf{H}).\end{aligned}\tag{3.6}$$

The desired change in the Hamiltonian written in (3.5) can now be effected by making the substitution in the lagrangian (1.3):

$$\begin{aligned}\partial_0 U &\rightarrow \partial_0 U + ih(\mathbf{q} \cdot \mathbf{H} U + U \mathbf{q} \cdot \mathbf{H}), \\ \partial_0 \psi &\rightarrow \partial_0 \psi - ih(\mathbf{q} \cdot \mathbf{H} \psi - \psi \mathbf{q} \cdot \mathbf{H})\end{aligned}\tag{3.7}$$

and we must expand the resulting expression in powers of  $h$  and in powers of unconstrained fields so that we can identify the contributions which enter to various numbers of loops.

To carry out such an expansion, we first write the bosonic field in the form

$$U = \exp i \left( \mathbf{n} \cdot \mathbf{H} + \sum_{\alpha > 0} (n_\alpha E_\alpha + n_\alpha^* E_{-\alpha}) \right)\tag{3.8}$$

where  $\mathbf{n}$  (with no subscript) stands for a vector of real fields associated to our chosen Cartan subalgebra,  $n_\alpha$  are complex fields corresponding to the *positive* roots, and the sum extends over just these positive roots of the finite-dimensional Lie algebra  $su(n)$ . It is now a simple exercise to show that there is indeed a tree-level contribution for any non-zero choice of the vector  $\mathbf{q}$  and so a one-loop calculation will suffice for the purposes of making a non-trivial comparison with the S-matrix. Because of this, we need only keep terms quadratic in the fields, and it is not difficult to show that the fermions decouple completely to this order. We may further simplify the result by discarding all terms independent of  $h$ , since we are interested only in how the result changes as a function of this background parameter. This implies that the contribution to the change in the ground-state energy at one-loop is given by a lagrangian

$$\mathcal{L}_{1\text{-loop}} = \frac{4h^2 \mathbf{q}^2}{g} + \frac{1}{g} \sum_{\alpha > 0} (\partial n_\alpha \partial n_\alpha^* - h^2 (\mathbf{q} \cdot \boldsymbol{\alpha})^2 n_\alpha n_\alpha^*)\tag{3.9}$$

which corresponds to a tree-level term, plus a number of free, massive bosons (whose fields have been re-scaled to give canonical normalizations). Using standard dimensional regularization with the  $\overline{\text{MS}}$  subtraction scheme, the result for the change in the energy density as a function of the running coupling is now found to be

$$\begin{aligned}\delta \mathcal{E}(h) &= -\frac{4h^2 \mathbf{q}^2}{g(h)} - \frac{h^2}{4\pi} \sum_{\alpha > 0} (\mathbf{q} \cdot \boldsymbol{\alpha})^2 [\ln(\mathbf{q} \cdot \boldsymbol{\alpha})^2 - 1] + \mathcal{O}(g(h)) \\ &= -\frac{h^2 \mathbf{q}^2 n}{2\pi} \ln \frac{h}{\Lambda_{\overline{\text{MS}}}} - \frac{h^2}{4\pi} \sum_{\alpha > 0} (\mathbf{q} \cdot \boldsymbol{\alpha})^2 [\ln(\mathbf{q} \cdot \boldsymbol{\alpha})^2 - 1] + \mathcal{O} \left( \ln \ln \frac{h}{\Lambda_{\overline{\text{MS}}}} / \ln \frac{h}{\Lambda_{\overline{\text{MS}}}} \right)\end{aligned}\tag{3.10}$$

where we have simply substituted the two-loop expression for the running coupling given earlier in order to write the answer explicitly as a function of  $h$ .

To complete our perturbative calculation, we wish to make some convenient specific choices for the vector  $\mathbf{q}$  which defines  $Q$  in (3.5) and which has so far been left arbitrary.



We recall that the general method we are seeking to apply entails a consideration of the states with the largest charge/mass ratio, since these will make a dominant contribution to the new ground-state when  $h$  becomes sufficiently large (we will discuss this in a little more detail in the next section). We will follow standard convention and agree to normalize  $Q$  so that it has eigenvalue  $+1$  on this preferred set of states. We would like to choose  $\mathbf{q}$  so that there are as few of these preferred states as possible, since any reduction in the number leads to a significant simplification in the TBA calculation.

All states in our theory belong to either left-handed or right-handed representations  $R_a$  of  $SU(n)$ , and for each of these representations we are free to choose a basis of simultaneous eigenstates of  $\mathbf{H}$ , the eigenvalues being just the weights, of course. The action of  $Q$  on these states is then

$$\begin{aligned} Q|\boldsymbol{\xi}^{(b)}, 0, K_{\mu\nu}^{(b)}; \theta\rangle &= (\mathbf{q} \cdot \boldsymbol{\xi}^{(b)})|\boldsymbol{\xi}^{(b)}, 0, K_{\mu\nu}^{(b)}; \theta\rangle, \\ Q|0, \boldsymbol{\xi}^{(b)}, K_{\mu\nu}^{(b)}; \theta\rangle &= -(\mathbf{q} \cdot \boldsymbol{\xi}^{(b)})|0, \boldsymbol{\xi}^{(b)}, K_{\mu\nu}^{(b)}; \theta\rangle. \end{aligned} \quad (3.11)$$

The idea now is that the best we can do to select a small number of preferred states is to pick out those corresponding to the highest weight  $\boldsymbol{\omega}_a$  of the representation  $R_a$  of  $SU(n)_L$ , or the lowest weight  $-\boldsymbol{\omega}_a$  of the representation  $R_{n-a}$  of  $SU(n)_R$  by choosing  $\mathbf{q}$  proportional to  $\boldsymbol{\omega}_a$  for some fixed  $a$ . More precisely, it is clear that the correct normalization of the charge  $Q$  is achieved by taking

$$\mathbf{q} = \boldsymbol{\omega}_a / (\boldsymbol{\omega}_a^2) \quad (3.12)$$

and that then the kink-multiplets

$$|\boldsymbol{\omega}_a, 0, K_{\mu\nu}^{(a)}; \theta\rangle \quad \text{and} \quad |0, -\boldsymbol{\omega}_a, K_{\mu\nu}^{(n-a)}; \theta\rangle \quad (3.13)$$

are indeed the states in the spectrum with the largest charge/mass ratio having  $Q$  eigenvalue  $+1$ . If we substitute the choice of  $\mathbf{q}$  given by (3.12) into our general formula for the ground-state energy given above we find

$$\delta\mathcal{E}(h) = -\frac{h^2 n^2}{2\pi a(n-a)} \left[ \ln \frac{h}{\Lambda_{\overline{\text{MS}}}} + \ln \left( \frac{n}{a(n-a)} \right) - \frac{1}{2} + \mathcal{O} \left( \frac{\ln \ln(h/\Lambda_{\overline{\text{MS}}})}{\ln(h/\Lambda_{\overline{\text{MS}}})} \right) \right] \quad (3.14)$$

which we shall be able to compare with the result of the S-matrix calculation carried out in the next section.

Before concluding this section it may be useful to make some comparison with the analogous calculation for the bosonic PCM as analyzed in [6,5]. The one-loop lagrangian derived above is exactly the same, *as a function of  $\mathbf{q}$* , as that found in the bosonic case, and this means that the one-loop expression for the ground-state energy is unchanged, *as a function of the running coupling  $g$  and of  $\mathbf{q}$* . Superficially, then, it may seem that our perturbative calculation is ignorant of the presence of fermions in the theory, since they have decoupled to this order. In fact the fermions do, nevertheless, play a rôle, and the final result for the ground-state energy *as a function of  $h$* , is different from the bosonic case.

This happens for two reasons. First, the running coupling behaves quite differently in the theory with fermions and, specifically, the vanishing of the second  $\beta$ -function coefficient is characteristic of a sigma-model with supersymmetry [18]. This means that we get a different result for the ground-state energy *as a function of  $h$  and  $\mathbf{q}$* . Secondly, there is a further modification because, compared to the bosonic case, the choice of  $\mathbf{q}$  written in (3.12) involves a different normalization. This reflects the fact that in the supersymmetric PCM all states lie in either left- or right-handed representations while in the bosonic theory they live in diagonal representations (requiring an extra factor of two to ensure  $Q$  has maximum eigenvalue  $+1$ ).

#### 4. S-matrix versus lagrangian: TBA calculation

We now turn to the calculation of  $\delta\mathcal{E}(h)$  from the S-matrix using the TBA. It is clear that if we make the choice of charge given in (3.12) and then increase  $h$  from zero until it exceeds the threshold value  $m_a$ , it will become energetically favourable to populate the ground-state with particles in the multiplets (3.13). We shall assume in what follows that the new ground-state effectively contains *only* particles corresponding to the preferred states (3.13). The idea behind this assumption is that the preferred states (3.13) with the largest charge/mass ratio should repel all other states and thereby dominate the new ground-state. Some support for this picture is obtained by considering which other states might also appear in the new ground-state. For it to be energetically favourable for a state, labelled by  $\boldsymbol{\xi}^{(b)}$ , to appear in the ground-state, it must have positive charge, i.e.  $\boldsymbol{\xi}^{(b)} \cdot \boldsymbol{\omega}_a > 0$ . But if this condition is satisfied then  $\boldsymbol{\xi}^{(b)} + \boldsymbol{\omega}_a$  cannot be a weight of a fundamental representation (the latter requires  $\boldsymbol{\xi}^{(b)} \cdot \boldsymbol{\omega}_a < 0$  as a necessary condition); hence the states labelled by  $\boldsymbol{\xi}^{(b)}$  cannot form bound-states with the states (3.13) from which we deduce that the forces between them cannot be attractive. This kind of argument should make our assumption seem plausible, but it does not provide a rigorous proof. In fact, our hypothesis *can* actually be proven from an analysis of the full TBA equations of the theory, but the proof is rather technical and so we have chosen not to reproduce it here (see [34] for details), relying instead on the less rigorous but more physical foregoing arguments. The reason the hypothesis is so important is that it allows us to deal directly with a much simpler set of TBA equations because we can immediately restrict to configurations containing only the states (3.13). Our final result—the agreement between the perturbative calculation and the TBA calculation—will also confirm that this working hypothesis is correct.

The idea behind the TBA is to consider the thermodynamics of a gas of the particles which interact via the exact factorizable S-matrix [35]. Since the number of particles is preserved it makes sense to consider single particle states. One then imposes periodic boundary conditions to get equations relating the densities-of-states of the various particles. From these equations conventional thermodynamic arguments lead to an expression for the

free energy at finite temperature and with chemical potentials. The ground-state energy density in the presence of a coupling to a charge corresponds to the zero temperature limit with some specific chemical potential. The difficulty facing us is that, even though we have restricted to the subspace of states (3.13), these degrees of freedom still do not interact purely elastically as far as their kink degrees-of-freedom are concerned. In finding the equations for the particle densities-of-states one has to perform a diagonalization in the kink subspace. Fortunately, however, the S-matrix elements in this subspace are proportional to RSOS Boltzmann weights of an integrable lattice model and the relevant diagonalization has already been performed<sup>12</sup> and we shall simply quote the result for the Bethe equations which relate the densities-of-states.

We denote the densities of occupied states in rapidity space of the two multiplets in (3.13) as  $\sigma_L(\theta)$  and  $\sigma_R(\theta)$ . Our normalization is such that, for example  $\int_{-\infty}^{\infty} d\theta \sigma_L(\theta)$  gives the number of occupied L-states per unit length of real space. The densities of un-occupied states, or holes, are denoted  $\tilde{\sigma}_L(\theta)$  and  $\tilde{\sigma}_R(\theta)$ , respectively, so that the total densities-of-states are therefore  $\tilde{\sigma}_L(\theta) + \sigma_L(\theta)$  and  $\tilde{\sigma}_R(\theta) + \sigma_R(\theta)$ , respectively.

The diagonalization in the kink subspace introduces additional terms which behave as if they come from particles with zero mass associated to the simple roots of  $SU(n)$  and carrying a ‘string-length’. These fictitious particles are known as magnons and we denote their densities as  $r_p^a(\theta)$  and their associated hole densities as  $\tilde{r}_p^a(\theta)$ , where  $p$ , the string-length index, and  $a$ , which labels the simple roots of the algebra  $su(n)$ , both run from 1 to  $n - 1$ .<sup>13</sup> The Bethe equations relating these densities are

$$\begin{aligned}\tilde{\sigma}_L(\theta) + B^{(a)} * \sigma_L(\theta) + C^{(a)} * \sigma_R(\theta) + a_p^{(n)} * r_p^a(\theta) &= \frac{m_a}{2\pi} \cosh \theta, \\ \tilde{\sigma}_R(\theta) + B^{(a)} * \sigma_R(\theta) + C^{(a)} * \sigma_L(\theta) + a_p^{(n)} * r_p^{n-a}(\theta) &= \frac{m_a}{2\pi} \cosh \theta,\end{aligned}\tag{4.1}$$

along with the magnon equations

$$\tilde{r}_p^b(\theta) + A_{pq}^{(n)} * K_{bc}^{(n)} * r_q^c(\theta) = \delta_{ba} a_p^{(n)} * \sigma_L(\theta) + \delta_{b, n-a} a_p^{(n)} * \sigma_R(\theta).\tag{4.2}$$

In (4.1) and (4.2) we have used the notation  $f * g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta')$  and  $a, b$  and  $p, q$  all run from 1 to  $n - 1$  (repeated indices are summed). The kernels appearing in (4.1) are

$$\begin{aligned}B^{(a)}(\theta) &= \left[ A_{nn}^{(\infty)} \right]^{-1} * A_{aa}^{(n)}(\theta), \\ C^{(a)}(\theta) &= \left[ A_{nn}^{(n+1)} \right]^{-1} * A_{n-a, a}^{(n)}(\theta) - A_{n-a, a}^{(n)}(\theta), \\ a_p^{(n)}(\theta) &= \frac{1}{2\pi} \cdot \frac{\sin(\pi p/n)}{\cosh \theta - \cos(\pi p/n)}.\end{aligned}\tag{4.3}$$

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<sup>12</sup> See [36] for a discussion in the context of S-matrix theory. In fact not all the relevant eigenvalues have been obtained, as far as the authors are aware; however, the resulting TBA equations are identical to those conjectured for  $SU(n)$  on the basis of the universality of the TBA equations [37].

<sup>13</sup> In general the string length index runs from 1 to  $k - 1$ , but in this case the level  $k = n$

If we define Fourier transforms as

$$f(\theta) = \int_0^\infty \frac{dx}{\pi} \cos(\theta x) \hat{f}(x), \quad (4.4)$$

then by definition

$$[f]^{-1}(\theta) = \int_0^\infty \frac{dx}{\pi} \cos(\theta x) \frac{1}{\hat{f}(x)}, \quad (4.5)$$

and to complete the definition of the kernels we have

$$\hat{A}_{pq}^{(k)}(x) = \frac{2 \sinh(\min(p, q)\pi x/n) \sinh((k - \max(p, q))\pi x/n) \cosh(\pi x/n)}{\sinh(k\pi x/n) \sinh(\pi x/n)}, \quad (4.6)$$

where  $1 \leq p, q \leq k - 1$  and  $\hat{K}_{pq}^{(k)}(x) = \left( \hat{A}^{(k)}(x)^{-1} \right)_{pq}$ . The kernels are related to the S-matrix elements of the states (3.13), but unfortunately not in a simple way.<sup>14</sup>

To find the TBA equations (in our case at zero temperature), in the presence of the coupling to the charge, one minimizes the value of the new Hamiltonian  $H - hQ$ , which for a macroscopic configuration is

$$\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} (m_a \cosh \theta - h) (\sigma_L(\theta) + \sigma_R(\theta)), \quad (4.7)$$

subject to the Bethe equations (4.1) and (4.2) as a constraint. The result of the variational problem can be expressed in terms of the ‘excitation energies’ for the particles  $\epsilon_{L,R}(\theta)$  and the magnons  $\xi_p^a(\theta)$ , with

$$\delta \mathcal{E}(h) = \frac{m_a}{2\pi} \int_{-\infty}^{\infty} d\theta [\epsilon_L^-(\theta) + \epsilon_R^-(\theta)] \cosh \theta, \quad (4.8)$$

which satisfy the TBA equations:

$$\begin{aligned} \epsilon_L^+(\theta) + B^{(a)} * \epsilon_L^-(\theta) + C^{(a)} * \epsilon_R^-(\theta) - a_p^{(n)} * \xi_p^{a,-}(\theta) &= m_a \cosh \theta - h, \\ \epsilon_R^+(\theta) + B^{(a)} * \epsilon_R^-(\theta) + C^{(a)} * \epsilon_L^-(\theta) - a_p^{(n)} * \xi_p^{n-a,-}(\theta) &= m_a \cosh \theta - h, \\ \xi_p^{b,+}(\theta) + A_{pq}^{(n)} * K_{bc}^{(n)} * \xi_q^{c,-}(\theta) &= -\delta_{ba} a_p^{(n)} * \epsilon_L^-(\theta) - \delta_{b,n-a} a_p^{(n)} * \epsilon_R^-(\theta). \end{aligned} \quad (4.9)$$

In the above, we have defined

$$f^\pm(\theta) = \begin{cases} f(\theta) & f(\theta) \gtrless 0 \\ 0 & \text{otherwise.} \end{cases} \quad (4.10)$$

The above TBA equations can be drastically simplified because  $a_p^{(n)}(\theta)$  is a positive kernel for all  $\theta$ ; hence the solution for the magnon terms is simply

$$\xi_p^b(\theta) = -\delta_{ba} a_p^{(n)} * \epsilon_L^- - \delta_{b,n-a} a_p^{(n)} * \epsilon_R^-(\theta), \quad (4.11)$$

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<sup>14</sup> The reason for this is that the macro-states are actually full of magnon  $n$  strings and so the kernels  $B^{(a)}(\theta)$  and  $C^{(a)}(\theta)$  are the derivatives of the phases shifts of the scattering of the particles (3.13), but in the background of an  $n$  string magnon ‘sea’ [36].

with the consequence that  $\xi_p^{b,-}(\theta) = 0$ . After taking this into account, it is easy to see that the remaining equations are symmetric in L and R; hence the solution to (4.9) clearly has  $\epsilon_L(\theta) = \epsilon_R(\theta) \equiv \epsilon(\theta)$ . So finally we are left with a single integral equation

$$\epsilon^+(\theta) + R * \epsilon^-(\theta) = m_a \cosh \theta - h, \quad (4.12)$$

and the change in the ground-state energy density is

$$\delta\mathcal{E}(h) = \frac{m_a}{\pi} \int_{-\infty}^{\infty} d\theta \epsilon^-(\theta) \cosh \theta. \quad (4.13)$$

The Fourier transform of the kernel  $R(\theta) = B^{(a)}(\theta) + C^{(a)}(\theta)$  is explicitly

$$\hat{R}(x) = \frac{2e^{\pi x/2} \sinh(\pi a x/n) \sinh(\pi(n-a)x/n) \sinh(\pi x/2)}{\sinh^2(\pi x)}. \quad (4.14)$$

From the single integral equation we can use the results of [9,5] (based on the original work of [38]) to develop an expansion for  $\delta\mathcal{E}(h)$  in the asymptotic regime  $h \gg m$ . The nature of the solution depends upon whether or not  $\hat{R}(x)$  vanishes at the origin. In the present case we see that  $\hat{R}(0) = 0$  which implies that  $\delta\mathcal{E}(h)$  for this model is therefore of the type encountered in the bosonic sigma models, like the principal chiral models [5], rather than the fermionic models. In other words it has an expansion which precisely matches the perturbative result with a classical term of  $\mathcal{O}(1/g)$  present, so that  $a_0 \neq 0$ .

To find the first few terms in the expansion of the solution one has to write the Fourier transform of the kernel in the form  $1/(G_+(x)G_-(x))$  where  $G_{\pm}(x)$  are analytic in the upper (lower) half planes with  $G_-(x) = G_+(-x)$ . This determines uniquely

$$G_+(i\xi) = \frac{n}{\sqrt{\pi a(n-a)\xi}} \frac{\Gamma(1+a\xi/n)\Gamma(1+(n-a)\xi/n)\Gamma(1+\xi/2)}{\Gamma^2(1+\xi)} \times \exp\left(\frac{\xi}{2} \ln \xi + \xi \left(-\frac{1}{2} - \frac{a}{n} \ln \frac{a}{n} - \frac{n-a}{n} \ln \frac{n-a}{n} - \frac{1}{2} \ln \frac{1}{2}\right)\right). \quad (4.15)$$

Following the discussion in [5], if  $G_+(i\xi)$  has an expansion for small  $\xi$  like

$$G_+(i\xi) = \frac{k}{\sqrt{\xi}} e^{-a\xi \ln \xi} (1 - b\xi + \mathcal{O}(\xi^2)), \quad (4.16)$$

then the first few terms of the ground-state energy for  $h \gg m$  are given by

$$\delta\mathcal{E} = -\frac{h^2 k^2}{2} \left[ \ln \frac{h}{m_a} + \ln \left( \frac{\sqrt{2\pi} k e^{-b}}{G_+(i)} \right) - 1 + a(\gamma_E - 1 + \ln 8) + (a + \frac{1}{2}) \ln \ln \frac{h}{m_a} + \mathcal{O}\left(\frac{\ln \ln(h/m_a)}{\ln(h/m_a)}\right) \right]. \quad (4.17)$$

Our kernel does indeed have an expansion of the form (4.16) with

$$k = \frac{n}{\sqrt{\pi a(n-a)}}, \quad a = -\frac{1}{2}, \quad \frac{\sqrt{2\pi} k e^{-b}}{G_+(i)} = \frac{2^{3/2} n^2 e^{\gamma_E/2}}{\pi a(n-a)} \sin\left(\frac{\pi a}{n}\right), \quad (4.18)$$

and so the first few terms in the ground-state energy are

$$\delta\mathcal{E}(h) = -\frac{h^2 n^2}{2\pi a(n-a)} \left[ \ln \frac{h}{m} + \ln \left( \frac{n^2 \sin(\pi/n)}{\pi a(n-a)} \right) - \frac{1}{2} + \mathcal{O}\left(\frac{\ln \ln(h/m)}{\ln(h/m)}\right) \right], \quad (4.19)$$

where we have used the mass formula (1.2) to relate  $m_a$  to the mass of the lightest multiplet, namely  $m$ .

## 5. Comparison and Conclusions

Comparing (4.19) with (3.14), we find that the results from the TBA calculation and the perturbative calculation are in complete agreement for each of the charges defined by (3.12). It is important to realize that the results of the calculations for these different charges are logically independent; each of them probes the S-matrix in a slightly different way, and we have shown that they all correctly reproduce the universal beta-function coefficients written in (3.2). We take this as very strong evidence that our conjectured S-matrix does indeed provide the correct description of the supersymmetric  $SU(n)$  principal chiral sigma model. The comparison also leads to the exact result for the mass-gap:

$$\frac{m}{\Lambda_{\overline{\text{MS}}}} = \frac{n}{\pi} \cdot \sin\left(\frac{\pi}{n}\right). \quad (5.1)$$

Once again, it is non-trivial that the same result is obtained for each of the charges in (3.12).

The question of CDD ambiguities deserves special attention. As we have said previously, in writing down any S-matrix proposed on the basis of symmetries and general axioms, we always have the freedom to multiply by CDD factors which respect all the basic principles automatically, which do not introduce any new poles onto the physical strip, and which satisfy the bootstrap equations. In our case this corresponds to multiplying the S-matrix elements  $S_{ab}(\theta)$  by products of factors of the form (2.27). More precisely, suppose we introduce such factors with parameters  $\alpha_j$  for LL and RR scattering and parameters  $\beta_k$  for LR scattering. The effect of these is to modify the kernels  $B^{(a)}(\theta)$  and  $C^{(a)}(\theta)$ , and hence the kernel  $R(\theta)$ :

$$\hat{R}(x) \rightarrow \hat{R}(x) - \frac{\hat{A}_{aa}^{(n)}(x)}{\cosh(\pi x/n)} \sum_j \cosh(\pi(1-\alpha_j)x/n) - \frac{\hat{A}_{a,n-a}^{(n)}(x)}{\cosh(\pi x/n)} \sum_k \cosh(\pi(1-\beta_k)x/n). \quad (5.2)$$

But from (4.6), we see that the modified kernel no longer has  $\hat{R}(0) = 0$  and the agreement with perturbation theory is therefore destroyed, by virtue of the remarks made in section 3. Modification by CDD factors in particular is therefore ruled out if we are to maintain consistency with the lagrangian. Of course one could always introduce additional CDD factors which give extra poles on the physical strip, but in that case the resulting S-matrix would require the existence of new states.

For the case of  $SU(2)$ , our result provides the solution of the supersymmetric  $O(4)$  sigma model. As suspected in [21], the vector particle of the theory is no longer stable and the spectrum just consists of the kinks transforming in the spinor and anti-spinor representations (that is  $(1/2, 0) \oplus (0, 1/2)$  of  $SU(2) \times SU(2)$ ). It is interesting that the analogous spectrum which we have proposed for the general super  $SU(n)$  model—and which we have checked by our calculations—is markedly different from the bosonic case, with particles transforming non-trivially under the left-handed or right-handed symmetry groups, but not under both. It would be interesting to find some semi-classical understanding of this.

Finally, we must return to an important assumption which we made and whose validity we have not demonstrated directly, but which is certainly borne out by our final results. We have not shown explicitly that our S-matrix for the super  $SU(n)$  PCM is invariant under  $N = 1$  supersymmetry. However, we do know that it commutes with conserved spin-1/2 charges, it is only their algebra which has not been directly established for general  $n$ , and in fact this algebra *has* been checked explicitly for  $n = 2$  in [23] and partially by us for  $n = 3$ . It would be interesting and worthwhile to show explicitly how  $N = 1$  supersymmetry is realized on the  $SU(n)$  RSOS kink S-matrices at level  $n$ ; in a sense this would relate the construction to the general scheme for supersymmetric scattering laid out in [39]. There are similar—although perhaps not identical—issues concerning the  $N = 2$  S-matrices proposed in [32] and to our knowledge these points are still unresolved. We believe that the final agreement of our S-matrix with a *supersymmetric* lagrangian provides compelling evidence that our assumption about supersymmetry is correct; we also hope to examine this question in more detail in the future.

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