

Entropy of 4D Extremal Black Holes

Clifford V. Johnson^a, Ramzi R. Khuri^{b,c} and Robert C. Myers^c

^a*Institute for Theoretical Physics, UCSB, CA 93106, USA*

^b*Theory Division, CERN, CH-1211, Geneva 23, Switzerland*

^c*Physics Department, McGill University, Montréal, PQ, H3A 2T8 Canada*

Abstract

We derive the Bekenstein–Hawking entropy formula for four–dimensional Reissner–Nordström extremal black holes in type II string theory. The derivation is performed in two separate (T–dual) weak coupling pictures. One uses a type IIB bound state problem of D5– and D1–branes, while the other uses a bound state problem of D0– and D4–branes with macroscopic fundamental type IIA strings. In both cases, the D–brane systems are also bound to a Kaluza–Klein monopole, which then yields the four–dimensional black hole at strong coupling.

CERN-TH/96-62
NSF-ITP-96-14
McGill/96-09
March, 1996

cvj@itp.ucsb.edu, khuri@nxth04.cern.ch, rcm@hep.physics.mcgill.ca

1. Introduction

Not long ago, a long-outstanding goal in theoretical physics was achieved. The Bekenstein–Hawking formula for black hole entropy was shown to have a microscopic origin in an explicit string theory computation. This computation was performed for extremal five-dimensional Reissner–Nordström black holes in ref.[1]¹, and then for the slightly non-extremal case in refs[4,5]. The computation was further generalised to five-dimensional rotating black holes in ref.[6].

String theory² has long been heralded as a complete theory of quantum gravity. Over the years, it has taught us that we should think of black holes as not merely solutions of the background field equations of string theory, but as being made of strings, in some sense. The string folklore is that a black hole solution of the background field equations is really a condensate of the graviton, described perturbatively by string theory. So the problem of computing properties of the black hole really addresses the problem of understanding how to describe a black hole in terms of its most basic constituents, and not simply treating it as a background about which to perturb. The type of calculation outlined in ref.[1] has begun to make sense of a number of these ideas.

It is a strong coupling problem to compute the entropy of a black hole. However, the insight gained over the past year into the structure of strong coupling string theory brings with it the realisation that we might (in some special cases) be able to compute the entropy in some weak coupling regime and trust that it may be successfully extrapolated. Emboldened by the discovery[9] of the relevance of D-brane technology[10] to the study of Ramond–Ramond (R-R) charged extended objects in types I and II superstring theory, the authors of ref.[1] demonstrated that the intuition that D-branes can be thought of as a weak coupling description of R-R charged black holes can be made precise.

The moral of the calculation is as follows: Extremal black holes (*i.e.*, the strong coupling description) with large multiples of the fundamental units of R-R charge, (and with some large Neveu–Schwarz–Neveu–Schwarz (NS-NS) charge corresponding to an internal momentum, in the five-dimensional example of ref.[1]) seem to correspond at weak coupling

¹ There is a qualitative proposal for a different string theory computation of the entropy in the literature. See refs.[2,3].

² We use the term cautiously. It is clear that string theory is more than a theory of strings, and that the (unknown) complete theory necessarily admits descriptions in terms of other extended objects. Overwhelming evidence for this has been accumulated over the past year. See, for example, refs.[7,8] for (pre-D-brane era) discussions of suggestive computations, with references. Precise computational power emerged with the discovery[9] that the extended objects called D-branes[10] carry the basic units of Ramond–Ramond charge. Ref.[11] contains a presentation of some of the basic techniques and applications of D-branes.

to BPS³ excitations of bound states of D-branes carrying these elementary units of R-R charges. The degeneracy of these excitations gives a result for the entropy which precisely matches the entropy which one would deduce from the horizon area of the extremal black hole using the Bekenstein–Hawking area law. The fact that the excitations are BPS is the key to complete confidence in the statement that the extrapolation of this result to strong coupling makes sense, a procedure which has borne much fruit in recent times. Additional exciting results have been presented recently showing that it is possible (in the right limits) to further compute the Bekenstein–Hawking entropy from first principles for non-extremal black holes[4,5].

All of the examples of this type of computation have been for five-dimensional black holes. It would be comforting to demonstrate that computations of the same spirit can be performed in a four-dimensional setting. Such a demonstration is presented in this paper. Our example involves the four-dimensional extremal Reissner–Nordström black hole solutions of the Einstein–Maxwell system. This solution can be embedded into string theory in many different ways[12–18]. We choose two T-dual embeddings which we study in a six-dimensional string theory. Our calculation is as follows: Looking at the five-dimensional subspace $(t, r, \theta, \phi, x^4)$ we see that the solution looks like a combination of a charged black hole and a Kaluza–Klein monopole⁴, where the fifth dimension has been fibred over the angular coordinates to give a Taub–NUT-like geometry[19]. In six dimensions, we see that the solution has a momentum (or winding number in the T-dual) in the sixth dimension x^5 . This is very similar to the situation in the pioneering examples. Once in six dimensions we see that we have two (T-dual) D-Brane bound state excitation problems. One of them is identical to that of ref.[1]. An important novelty here is of course that the D-brane composite is really bound to a Kaluza–Klein monopole, and so we are addressing a somewhat different bound state problem. In the strong coupling solution, the monopole’s magnetic charge is essential in maintaining a constant modulus for the x^4 direction, and hence ensure that the four-dimensional string coupling is constant everywhere. For the purposes of the degeneracy counting at weak coupling, the monopole does not affect the internal structure of the D-brane excitations.

In section 2 we present the two T-dual six-dimensional problems as uplifted four-dimensional black holes, calling them models ‘A’ and ‘B’ appropriately after the type II string theories they live in. We compute and display the charges, horizon area, and then the entropy in terms of these quantities. Section 3 briefly describes the weak coupling

³ States satisfying a Bogomol’nyi–Prasad–Sommerfeld bound. They are in reduced multiplets of the supersymmetry algebra. Formulae for their masses and charges are protected by supersymmetry regardless of the values of the coupling constants in the theory. So an enumeration of their number is a reliable thing to extrapolate in coupling space.

⁴ This is quite distinct from the previous examples, where at this stage one really has a five-dimensional black hole.

bound state problems to which we map the field configurations of section 2. We review the degeneracy counting arguments for model B in section 4 and then show how it translates into the same counting for the isomorphic bound state problem in model A. The irrelevance of the monopole to the counting problem is also discussed in section 4. The entropy computed by explicit enumeration of the degeneracy of BPS excitations agrees with the Bekenstein–Hawking area law, as advertised.

2. Four–Dimensional Reissner–Nordström

The electrically charged extremal Reissner–Nordström solution of the Einstein–Maxwell system:

$$I \sim \int d^4x \sqrt{-g} \left(R - \frac{1}{4} F^2 \right) \quad (2.1)$$

is given by

$$\begin{aligned} ds^2 &= -V^{-2} dt^2 + V^2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \\ A &= 2V^{-1} dt, \quad V = 1 + \frac{k}{r}. \end{aligned} \quad (2.2)$$

This solution has a degenerate horizon of non–zero area at $r = 0$. It was first considered in the context of supergravity in [20] (see also [21]). More recently, supersymmetric as well as non–supersymmetric embeddings of the extremal Reissner–Nordström black hole were found in [12–18]. The particular embedding we consider preserves 1/4 of the spacetime supersymmetries of heterotic string theory compactified on a six–torus, T^6 (see [15,18] and references therein).

This solution preserves 1/8 of the supersymmetries of $N = 8$, $D = 4$ supergravity, which arises as the low–energy limit of type II string theory compactified on T^6 . In truncating to heterotic string theory on T^6 or type II on $K3 \times T^2$, one has to make the correct sign choice for the fields in accordance with the sign choice in the truncation from $N = 8$ to $N = 4$ (or, more precisely, in the truncation from $N = 2$ to $N = 1$ in $D = 10$). In this manner, one of the four supersymmetries is preserved in the resultant $N = 4$, $D = 4$ supergravity limit of these latter compactifications. The opposite sign choice can still be seen to be supersymmetric, provided the opposite sign choice is made in truncating from $N = 8$ to $N = 4$ [18]. In any case, we choose in this paper the embedding that is supersymmetric in the appropriate theory, and which under T–duality and other duality transformations remains supersymmetric with the same amount of supersymmetry preserved [22].

2.1. Model B

The solution may be embedded into six–dimensional type IIB string theory (compactified

on $K3$ for our discussion⁵) whose action contains the terms:

$$I_{IIB} = 8V_4 \int d^6x \sqrt{-g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - (\partial\sigma)^2 - \frac{1}{12}H^2 \right) - \frac{e^{2\sigma}}{12}F_{(3)}^2 \right]. \quad (2.3)$$

As is evident from this action, we are working with the string metric g . The remaining fields σ , Φ , H and $F_{(3)}$ are the volume modulus scalar for $K3$, the six-dimensional dilaton (a linear combination of the ten-dimensional dilaton and σ), the 3-form field strength for the NS-NS 2-form B , and the 3-form field strength for the R-R 2-form $A_{(2)}$. We use units where $\alpha' = 1$. We also retain V_4 , the volume of the $K3$ surface. The six-dimensional solution is now:

$$\begin{aligned} ds^2 &= -V^{-2}dt^2 + V^2(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \\ &\quad + k^2(dx^4 - \cos\theta d\phi)^2 + L^2 \left(dx^5 + \frac{1}{L}(V^{-1} - 1)dt \right)^2; \\ A_{(2)} &= \frac{L}{\lambda\tilde{\lambda}}V^{-1}dt \wedge dx^5 - \frac{k^2}{\lambda\tilde{\lambda}}\cos\theta d\phi \wedge dx^4; \\ e^\Phi &= \lambda; \quad e^\sigma = \tilde{\lambda}; \quad B = 0. \end{aligned} \quad (2.4)$$

where $V = 1 + k/r$.

This solution can be seen to be supersymmetric as follows: in [18] it was shown that the truncation of the type IIB theory to an $N = 1$, $D = 10$ theory consisting of its NS-NS fields contained the extremal Reissner-Nordström black hole as a supersymmetric solution which preserved a quarter of the remaining supersymmetries. In $D = 10$, this NS-NS solution is related to the ten-dimensional uplift of the solution (2.4) by interchanging the R-R and NS-NS two-forms. However, once the 4-form potential generating the self-dual 5-form field strength of IIB is set to zero, the R-R and NS-NS 3-form field strengths appear in an identical manner in the supersymmetry transformations in $D = 10$ [23]. It then follows that the ten-dimensional R-R solution is supersymmetric, since the NS-NS version was already shown to be so [18]. In compactifying back to six dimensions, the solution (2.4) is then seen to be supersymmetric and preserves 1/4 of the spacetime supersymmetries of the $N = 2$ theory.

The metric here is written in the standard form for Kaluza-Klein reduction. The first line of the line element ds^2 yields the four-dimensional Reissner-Nordström in the (t, r, θ, ϕ) subspace. We have chosen the coordinate x^5 to have periodicity 2π , while the 4π periodicity of x^4 is fixed by the fact that the (r, θ, ϕ, x^4) subspace forms a Euclidean Taub-NUT-like space[24]. The periodicity is chosen to avoid singularities on $\theta = 0$ and $\theta = \pi$. Consequently, the (θ, ϕ, x^4) subspace has the topology of a 3-sphere[25]. The five-dimensional subspace $(t, r, \theta, \phi, x^4)$ thus forms essentially a magnetic Kaluza-Klein monopole [19], first

⁵ We could also consider a compactification on T^4 .

found to be a solution of string theory in [13]⁶. In four dimensions, g_{44} and g_{55} will play the role of scalar fields, which in this case are simply the constants k^2 and L^2 . Similarly, $g_{5\mu}/g_{55}$ and $g_{4\mu}/g_{44}$ play the role of four-dimensional gauge fields.

The solution carries the following six-dimensional conserved $F_{(3)}$ charges:

$$\begin{aligned} Q_1 &= \frac{V_4}{2\pi^{5/2}} \int_{S^3} e^{2\sigma} {}^*F_{(3)} = \frac{2^3}{\sqrt{\pi}} V_4 \frac{\tilde{\lambda}}{\lambda} k^2; \\ Q_5 &= 2^3 \pi^{3/2} \int_{S^3} F_{(3)} = 2^7 \pi^{7/2} \frac{k^2}{\tilde{\lambda}\lambda}. \end{aligned} \tag{2.5}$$

The off-diagonal metric component g_{5t} indicates that the solution also carries momentum P^5 in the x^5 direction. Calculating the total ADM momentum yields

$$P^5 = \frac{N}{L}, \quad \text{where } N = 2^8 \pi^3 V_4 \frac{k^2 L^2}{\lambda^2}. \tag{2.6}$$

Calculating the total ADM mass yields:

$$M = 2^{10} V_4 \pi^3 \frac{k^2 L^2}{\lambda^2}. \tag{2.7}$$

The extremal black hole has a horizon at $r = 0$ which has finite area given by⁷

$$A = 2^5 \pi^3 k^3 L. \tag{2.8}$$

The Bekenstein–Hawking entropy formula[29] then states that the entropy of this extremal black hole is

$$S = \frac{A}{4G_N} = 2^{10} V_4 \pi^4 \frac{k^3 L}{\lambda^2}, \tag{2.9}$$

where G_N is the effective Newton's constant $(16\pi G_N)^{-1} = 8V_4/\lambda^2$. Note that we can write the entropy in the form $S = 2\pi\sqrt{Q_1 Q_5 N}$.

⁶ In fact, this supersymmetric embedding of the extremal Reissner–Nordström black hole represents a bound state[14,3,26,18] of two electric/magnetic dual pairs of dilaton black holes[20,27,12], each in turn representing bound states of a Kaluza–Klein black hole and an H -monopole[28]. For our purposes, however, the magnetic Kaluza–Klein monopole plays a particularly important role.

⁷ We are working with the six-dimensional area here. Of course, it can be written as the product of the four-dimensional area with the volume of the $x^{4,5}$ space. Note, however that the correct topology of the horizon is $S^3 \times S^1$. It is more natural to work in terms of six-dimensional quantities, bearing in mind that the reduction to four-dimensional results is trivial.

2.2. Model A

We now consider a solution of type IIA string theory compactified on $K3$ to six dimensions with action:

$$I_{IIA} = 8V_4 \int d^6x \sqrt{-g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - (\partial\sigma)^2 - \frac{1}{12}H^2 \right) - \frac{e^{2\sigma}}{4}F_{(2)}^2 - \frac{e^{-2\sigma}}{4}\widehat{F}_{(2)}^2 \right], \quad (2.10)$$

which is:

$$\begin{aligned} ds^2 &= -V^{-2}dt^2 + V^2(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \\ &\quad + k^2(dx^4 - \cos\theta d\phi)^2 + L'^2(dx^5)^2; \\ e^\Phi &= \lambda'; \quad e^\sigma = \tilde{\lambda}; \\ B &= L'(V^{-1} - 1)dt \wedge dx^5; \\ A_{(1)} &= \frac{1}{\lambda'\tilde{\lambda}}V^{-1}dt; \quad \widehat{A}_{(1)} = \frac{\tilde{\lambda}}{\lambda'}V^{-1}dt, \end{aligned} \quad (2.11)$$

where again $V = 1 + k/r$. Here $F_{(2)}$ is the 2-form field strength for the (ten-dimensional) R-R vector $A_{(1)}$, while $\widehat{F}_{(2)}$ is a 2-form R-R field strength for the vector $\widehat{A}_{(1)}$. The latter field strength is the six-dimensional Hodge dual of the 4-form field strength of the (ten-dimensional) type IIA string. Finally, B is the NS-NS 2-form potential with field strength H . This solution and the previous one (2.4) are precisely dual under a T-duality transformation⁸ in the x^5 direction if $L' = 1/L$ and $\lambda' = \lambda/L$. As noted above the solutions preserve the same amount of supersymmetry.

The solution has an electric $F_{(2)}$ charge

$$Q_0 = \frac{V_4}{\pi^{7/2}} \int_{S^3 \times S^1} e^{2\sigma} *F_{(2)} = \frac{2^3}{\sqrt{\pi}} V_4 \frac{\tilde{\lambda}L'}{\lambda'} k^2. \quad (2.12)$$

There is also an electric $\widehat{F}_{(2)}$ charge:

$$Q_4 = 4\sqrt{\pi} \int_{S^3 \times S^1} e^{-2\sigma} *\widehat{F}_{(2)} = 2^7 \pi^{7/2} \frac{L'}{\tilde{\lambda}\lambda'} k^2. \quad (2.13)$$

Finally the solution also carries an ‘electric’ charge from the NS-NS 3-form H , given by

$$W = 16\pi V_4 \int_{S^3} e^{-2\Phi} *H = 2^8 \pi^3 V_4 \frac{k^2}{\lambda'^2}. \quad (2.14)$$

⁸ See for example ref.[23] for explicit details of how T-duality operates on the background fields in type II theory.

The horizon (still at $r = 0$) area is:

$$A = 2^5 \pi^3 k^3 L'. \quad (2.15)$$

The Bekenstein–Hawking entropy formula[29] then states that the entropy of this extremal black hole is

$$S = \frac{A}{4G_N} = 2^{10} \pi^4 V_4 \frac{k^3 L'}{\lambda'^2}, \quad (2.16)$$

where G_N is the effective Newton’s constant $(16\pi G_N)^{-1} = 8V_4/\lambda'^2$. Note that we can write the entropy in the form $S = 2\pi\sqrt{Q_0 Q_4 W}$.

These results are of course T–dual to the ones derived in the previous sub–section.

3. Two Bound State Problems

The next step is to identify the ingredients of the weak coupling description. This is done by recalling the result of ref.[9] that the basic carriers of R-R charge in type II string theory are p –dimensional extended objects which we shall call here ‘Dp–branes’. (We reserve the term ‘D–brane’ for the more generic extended objects, or when we have bound states.) In ten dimensions, the world volume of these objects couple to $(p + 1)$ –form R-R potentials, and they therefore carry ‘electric’ charges with respect to a $(p + 2)$ –form R-R field strength[11]. Therefore, by noting the amounts of charge that all of the R-R fields in the solutions (2.4) and (2.11) have⁹, we can determine the composition of the D–brane bound states which the solutions become at weak coupling. The NS-NS charges (associated with $g_{5\mu}$, $g_{4\mu}$ and B) then complete our picture of the underlying bound state configuration. In model A, the electric B charge indicates the presence of ‘fundamental’ strings winding around x^5 .

3.1. Model B

In the type IIB theory, we have a solution with magnetic charge Q_5 and electric charge Q_0 with respect to the R-R 3–form field strength. The fundamental objects which carry these charges in six dimensions are D1–branes, otherwise known as D–strings. From the ten–dimensional point of view, there are a number of ways to construct composite D–strings in compactifying to six dimensions. Type IIB string theory contains 3–, 5– and 7–form

⁹ Note that in the previous sections we have used normalisations which ensure that our charges are integer amounts of the basic units [9,11]. Thus the R-R charges simply count the number of the corresponding D–branes in the underlying bound state. The energetic reader may find it instructive to check that the analogous formulae yield also the correct integral quantisation for the momentum and winding number.

field strengths in ten dimensions, for which D1-branes, D3-branes and D5-branes carry electric charge. In compactifying from ten to six dimensions on $K3$, the last two extended objects can wrap themselves around the 2- and 4-cycles of $K3$ and appear as D-strings in six dimensions.

The case which we consider here is a compactification in which Q_5 D5-branes (which carry magnetic R-R 3-form charge) wrap around the whole of an internal $K3$, appearing as strings in six dimensions, forming a composite with Q_1 D1-branes (which carry electric R-R 3-form charge). BPS excitations of such a configuration preserve 1/4 of the spacetime supersymmetries in the $N = 2$ theory, as the problem requires. This D-string composite must also have momentum $P^5 = N/L$ in the compact x^5 direction in order to match the $g_{5\mu}$ charge of the black hole configuration (2.4). This D-brane composite is the same object as found in ref.[1]. Finally, we ascertain from the $g_{4\mu}$ charge in the solution (2.4) the presence of a magnetic Kaluza-Klein monopole. Hence, the complete configuration is comprised of the D-string composite bound to a Kaluza-Klein monopole.

3.2. Model A

In the (T-dual) type IIA theory, the situation is slightly different. We have electric NS-NS 3-form charge (winding number) W , the smallest unit of which is carried by fundamental type IIA strings, together with electric R-R charges Q_0 and Q_4 , which are carried by D0-branes in six dimensions. In ten dimensions, our type IIA compactification has a D4-brane (which carries electric 6-form charge in ten dimensions) wrapped about the $K3$, appearing as a D-particle in six dimensions. The bound state problem is one involving a D-particle composed of Q_0 D0-branes and Q_4 wrapped D4-branes, threaded by¹⁰ fundamental type IIA strings winding around the x^5 circle with total winding number W . As before, the D-particle is placed in the monopole background, which provides the extra magnetic charge. Once again, BPS excitations of this bound state will preserve 1/4 of the supersymmetries.

4. Microscopic Entropy

4.1. Model B

To evaluate the entropy of the black hole, we need to simply count the various BPS excitations of the bound states which we discussed in the previous section. The problem of R-R bound states has received much attention recently, and much of the counting techniques used in the black hole computations rely on results derived in refs[31,32].

¹⁰ Such configurations of R-R ‘beads’ on a NS-NS ‘necklace’ have been considered recently in ref.[30] in a different but not unrelated context.

In the problem presented in the type IIB theory the answer is very familiar, as it has been used several times now[1][4][5]. Let us set it up as a standard piece of elementary D-brane calculus[11]. We have Q_1 parallel D1-branes and Q_5 parallel D5-branes bound together. This configuration yields the following decomposition of the spacetime Lorentz group:

$$SO(1, 9) \supset SO(1, 1) \otimes SO(4) \otimes SO(4), \quad (4.1)$$

where the first factor acts along the D-string world sheet (t, x^5) , the third acts in the rest of the D5-brane world-volume (x^6, x^7, x^8, x^9) and the second in the rest of spacetime (x^1, x^2, x^3, x^4) . In studying the BPS excitations of the bound states which have the highest degeneracy, we study the (1,5) and (5,1) open string sector[4], *i.e.*, oriented strings stretching between the D1- and D5-branes¹¹. There are 2 bosons with NN boundary conditions¹², 4 with DD and 4 with ND. Working in the light-cone gauge, these give a zero point energy of $-1/12$. The NS sector fermions in the four ND directions will be integer moded. Accordingly, the vacuum energy of the NS fermions cancels that of the bosons, while the integer moding produces a degenerate vacuum (like the R sector does in ordinary NN string theory) forming a spinor under the world-volume $SO(4)$ mentioned above. It is a boson under the spacetime $SO(1, 5)$ Lorentz group. After the GSO projection, it is a two-dimensional representation. The R sector fermions will have half-integer moding in the four ND directions and thus produce vacuum energy $1/12$ again, giving a zero energy degenerate vacuum, which is a spinor of the $SO(1, 5)$, *i.e.*, it is a spacetime fermion. The GSO projection and a requirement that the state is left-moving (as we are interested in BPS excitations) reduces the number of states to two, matching the spacetime bosons from the previous sector. As (1,5) and (5,1) states are different (we are considering oriented strings here), we have $4Q_1Q_5$ boson-fermion ground states.

Our configuration carries momentum N in the x^5 direction around which the D-string is wrapped. The number of ways, $d(N)$, of distributing a total momentum N amongst the (1,5) and (5,1) strings is given by the partition function:

$$\sum d(N)q^N = \left(\prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n} \right)^{4Q_1Q_5}. \quad (4.2)$$

For large N , this gives $d(N) \sim \exp(2\pi\sqrt{Q_1Q_5N})$, and $S = \ln d(N)$ yields precisely the entropy (2.16) we computed for our black hole using the Bekenstein-Hawking area law, in section 2.

4.2. Model A

The counting problem as presented above readily lends itself to adaptation to other problems. One such is the T-dual type IIA configuration discussed as the second bound state

¹¹ This is true for large Q_1 and Q_5

¹² Here, ‘N’ means Neumann, while ‘D’ means Dirichlet.

problem in the previous section. Here, we have first to consider the D–particle as a bound state of Q_0 D0–branes and Q_4 D4–branes, the latter wrapping around the $K3$ as before. Clearly we do not need to study this configuration’s vacua in any more detail, as we will arrive at exactly the same conclusions as for the problem above, because they are T–dual: T–duality exchanges Neumann with Dirichlet boundary conditions for the x^5 direction and so the modings will not change.

We therefore have a bound state D–particle with $4Q_0Q_4$ microstates in boson–fermion pairs. However, now we do not have momentum in the x^5 direction. It is the winding number that plays the important role here. The electric NS–NS 3–form charge of the black hole solution tells us that there is a large x^5 winding number W in the problem, which is of course carried by the fundamental type IIA strings. The BPS excitations of the bound state which we want to consider are therefore those where we have distributed this winding number W amongst the $4Q_0Q_4$ boson–fermion pairs which we counted as $(0, 4)$ and $(4, 0)$ fundamental type IIA strings. We simply ask that we give these states winding number in as many ways as is possible to make total winding number W . So the configurations of strings which we considered connecting the constituents of the bound state are allowed to wind w times around the x^5 direction before connecting between two D–branes. The configurations thus look like NS–NS ‘necklaces’ made of fundamental winding IIA strings, with a R–R ‘bead’ (the D–particle composite) somewhere along its length. The number of ways, $d(W)$ of distributing the winding W is given by the partition function:

$$\sum d(W)q^W = \left(\prod_{w=1}^{\infty} \frac{1+q^w}{1-q^w} \right)^{4Q_0Q_4}. \quad (4.3)$$

This yields the same entropy previously computed for model B, which is in agreement with the Bekenstein–Hawking area law.

Once again, we stress that the bound state exists in a Kaluza–Klein monopole background, whose role at weak coupling was simply to contribute to the measured charges, but at strong coupling returns us to the configurations of section 2.

4.3. The Monopole

The above discussions (which follow that presented in ref.[4]) are complementary to the one presented in ref.[1], where the same behaviour in the large N (or W) limit arises from the elliptic genus of a sigma model whose target is a symmetric product[32] of $K3$. The $K3$ is assumed small compared to the circle x^5 and the D1–brane components of the D–string bound state are free to explore the $K3$, about which the (relatively frozen) D5–brane is wrapped.

It is clear from this point of view that only relative motions of the D–branes in the x^5, x^6, x^7, x^8 and x^9 directions are relevant to the degeneracy counting problem. Exci-

tations in the transverse directions corresponding to separating the bound state are irrelevant. The monopole has no structure in the x^5, x^6, x^7, x^8 and x^9 directions and hence it only produces a transverse potential. Thus it will not affect the counting discussions presented above.

Further note that in isolation the monopole is a soliton and hence has no intrinsic entropy. Hence the entropy of the total bound state system arises entirely from the contribution of the D-branes computed above.

5. Conclusions

We have demonstrated that the entropy of four-dimensional extremal Reissner–Nordström black holes can be computed in essentially the same way as pioneered in ref.[1] for five-dimensional black holes: Computing in the black hole is a strong coupling string theory problem, but we can compute the entropy in the weak coupling limit using D-brane calculus, and extrapolate our results back to the strong coupling regime, secure in the knowledge that as we have computed the degeneracy of BPS excitations, our data is protected by supersymmetry. A crucial difference between our situation and that of ref.[1] is that our bound states live in a monopole background. This monopole has no effect on the weak coupling counting problem, however the extra magnetic charge is essential in producing the non-singular four-dimensional field configuration.

It is interesting to note that in the five-dimensional cases like ref.[1], there are three charges parametrising the black hole, two R-R and one NS-NS. The precise arrangement of the charges is essential in producing a nonsingular black hole with a finite area horizon. One may observe[33]¹³ that all solutions related to the latter under the five-dimensional U-duality group $E_{6(6)}$ are similarly nonsingular because the horizon area is related to cubic invariant of $E_{6(6)}$. In the present problem, the addition of the monopole charge is essential in producing a finite area horizon. The combined arrangement of four charges must combine in the quartic invariant of the four dimensional U-duality group, $E_{7(7)}$, (see refs.[34,35]) which is conjectured to determine the area of the black hole horizon [34]. It would be of some interest to verify explicitly that the area in the present solutions is given by this $E_{7(7)}$ invariant.

One interesting aspect of the computation was to study how it took on different forms in a T-dual picture. Of course, the entropy should be independent of T-duality (and more generally, U-duality[35]), and it is pleasing to see how this happens in detail so cleanly. In particular with the substitution $L' = 1/L$ and $\lambda' = \lambda/L$ in the model A results, we have $Q_0 = Q_1$, $Q_4 = Q_5$ and $W = N$. One should expect that this result will work more generally. We displayed only a subset of the class of embeddings of the four-dimensional

¹³ We thank Joe Polchinski for bringing this matter to our attention.

Reissner–Nordström solution into type II string theory. There are other embeddings which are T–dual to those discussed here, and others which are not [17,18]. One example of the former is to T–dualise model A along the x^4 direction. The resulting configuration involves D1–branes wrapping the x^4 direction, fundamental strings wrapping the x^5 direction and D5–branes wrapping the $K3$ and the x^4 direction. There is also a NS-NS solitonic 5–brane wrapping the $K3$ and x^5 , replacing the role of the monopole. Some of the latter embeddings can be obtained, for example, by using string/string duality[35][36] to go from type IIA on $K3$ to heterotic string theory on T^4 , performing an $O(22,6)$ rotation and then using string/string duality to return to type IIA. The class of backgrounds thus generated contains a rich family of interesting (and presumably equivalent) bound state problems involving varying amounts of R-R and NS-NS charged extended objects in diverse backgrounds. It would certainly be interesting to study more of these, as they would shed more light on the problem of bound states in string theory, extremal black holes and perhaps non–extremal black holes.

Acknowledgments:

We would like to thank Gary Horowitz, Joe Polchinski and Andy Strominger for discussions. CVJ was supported in part by the National Science Foundation under Grant No. PHY94–07194. RCM’s research was supported by NSERC of Canada, Fonds FCAR du Québec. RCM would also like to thank the ITP at Santa Barbara for hospitality while this research was carried out. RRK was supported by a World Laboratory Fellowship.

References

- [1] A. Strominger and C. Vafa, ‘*Microscopic Origin of the Bekenstein–Hawking Entropy*’, hep-th/9601029.
- [2] F. Larsen and F. Wilczek, ‘*Internal Structure of Black Holes*’, hep-th/9511064
- [3] M. Cvetič and A. A. Tseytlin, ‘*Solitonic Strings and BPS Saturated Dyonic Black Holes*’, hep-th/9512031.
- [4] C. G. Callan Jr. and J. M. Maldacena, ‘*D–Brane Approach to Black Hole Quantum Mechanics*’, hep-th/9602043.
- [5] G. Horowitz and A. Strominger, ‘*Counting States of Near–Extremal Black Holes*’, hep-th/9602051.
- [6] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, ‘*D–Branes and Spinning Black Holes*’, hep-th/9602065;
J. C. Breckenridge, R. C. Myers, A. W. Peet, A. Strominger and C. Vafa, in preparation.
- [7] P. K. Townsend, ‘*P-brane Democracy*’, hep-th/9507048.
- [8] C. V. Johnson, N. Kaloper, R. R. Khuri and R. C. Myers, ‘*Is String Theory a Theory of Strings?*’, Phys. Lett. **B368** (1996) 71, hep-th/9509070
- [9] J. Polchinski, ‘*Dirichlet Branes and Ramond–Ramond Charges in String Theory.*’, Phys. Rev. Lett. **75** (1995) 4724, hep-th/9510017.
- [10] J. Dai, R. G. Leigh and J. Polchinski, Mod. Phys. Lett. **A4** (1989) 2073;
P. Hořava, Phys. Lett. **B231** (1989) 251;
R. G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767;
J. Polchinski, Phys. Rev. D50 (1994) 6041, hep-th/9407031.
- [11] J. Polchinski, S. Chaudhuri and C. V. Johnson, ‘*Notes on D–Branes*’, hep-th/9602052.
- [12] R. Kallosh, A. Linde, T. Ortin, A. Peet and A. Van Proeyen, Phys. Rev. **D46** (1992) 5278, hep-th/9205027.
- [13] M.J. Duff, R.R. Khuri, R. Minasian and J. Rahmfeld, ‘*New Black Hole, String and Membrane Solutions of the Four–Dimensional Heterotic String*’, Nucl. Phys. **B418** (1994) 195.
- [14] M.J. Duff and J. Rahmfeld, ‘*Massive String States as Extreme Black Holes*’, Phys. Lett. **B345** (1995) 441.
- [15] M. Cvetič and D. Youm, ‘*Four–Dimensional Supersymmetric Dyonic Black Holes in Eleven–Dimensional Supergravity*’, Nucl. Phys. **B453** (1995) 259; ‘*Dyonic BPS Saturated Black Holes of Heterotic String on a Six Torus*’, hep-th/9509070;

- M. Cvetič and A. A. Tseytlin, ‘General Class of BPS Saturated Dyonically Black Holes as Exact Superstring Solutions’, hep-th/9510097.
- [16] M. Cvetič and D. Youm, ‘All the Static Spherically Symmetric Black Holes of Heterotic String on a Six Torus’, hep-th/9512127; ‘Singular BPS Saturated States and Enhanced Symmetries of Four-Dimensional $N=4$ Supersymmetric String Vacua’, Phys. Lett. **359B** (1995) 87-92.
- [17] H. Lu and C.N. Pope, ‘ P -brane Solitons in Maximal Supergravities’, hep-th/9512012; ‘Multi Scalar p -brane Solitons’, hep-th/9512153; ‘An Approach to the Classification of p -brane Solitons’, hep-th/9601089.
- [18] R. R. Khuri and T. Ortín, ‘Supersymmetric Black Holes in $N = 8$ Supergravity’, hep-th/9512177; ‘A Non-Supersymmetric Dyonically Extreme Reissner-Nordström Black Hole’, hep-th/9512178.
- [19] D. J. Gross and M. J. Perry, Nucl. Phys. **B226** (1983) 29;
R. D. Sorkin, Phys. Rev. Lett. **51** (1983) 87.
- [20] G.W. Gibbons, Nucl. Phys. **B207**, (1982) 337; Nucl. Phys. **B298**, (1988) 741.
- [21] M.J. Duff and C.N. Pope, Nucl. Phys. **B255** (1985) 355.
- [22] I. Bakas, ‘Space-Time Interpretations of S Duality and Supersymmetry Violations of T Duality’, Phys. Lett. **343B** (1995) 103;
E. Bergshoeff, R. Kallosh and T. Ortín, ‘Duality Versus Supersymmetry and Compactification’, Phys. Rev. **D51** (1995) 3009;
I. Bakas and K. Sfetsos, ‘ T Duality and World-Sheet Supersymmetry’, Phys. Lett. **349B** (1995) 448;
S.F. Hassan, ‘ T -duality And Non-Local Supersymmetry’, hep-th/9504148;
E. Alvarez, L. Alvarez-Gaume and I. Bakas, ‘ T Duality and Space-Time Supersymmetry’, hep-th/9510028.
- [23] E. Bergshoeff, C.M. Hull and T. Ortín, Nucl. Phys. **B451** (1995) 547, hep-th/9504081;
K. Behrndt, E. Bergshoeff, B. Janssen, ‘Type II Duality symmetries in Six Dimensions’, hep-th/9512152.
- [24] A. H. Taub, Ann. Math. **53** (1951) 472;
E. Newmann, L. Tamburino and T. Unti, J. Math. Phys. **4** (1963) 915.
- [25] C. W. Misner, J. Math. Phys. **4** (1963) 924.
- [26] J. Rahmfeld, ‘Extremal Black Holes as Bound States’, hep-th/9512089.
- [27] D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. **D43** (1991) 3140; erratum **D45** (1992) 3888.

- [28] R. R. Khuri, Phys. Lett. **B259** (1991) 261; Nucl. Phys. **B387** (1992) 315, hep-th/9205081.
- [29] J. Bekenstein, Phys. Rev. **D7** (1973) 2333; Phys. Rev. **D9** (1974) 3292; S. W. Hawking, Comm. Math. Phys. **43** (1975) 199; Phys. Rev. **D13** (1976) 191.
- [30] G. Horowitz, A. Strominger and J. Maldacena, to appear.
- [31] E. Witten, ‘*Bound States of Strings and P–Branes*’, hep-th/9510135; M. Li, ‘*Boundary States of D–Branes and Dy–Strings*’, hep-th/9510161; M. Douglas, ‘*Branes Within Branes*’, hep-th/9512077; C. Vafa, ‘*Gas of D–Branes and Hagedorn Density of BPS States*’, hep-th/9511088; M. Bershadsky, C. Vafa and V. Sadov, ‘*D–Strings on D–Manifolds*’, hep-th/9510225; A. Sen, ‘*U–Duality and Intersecting D–Branes*’, hep-th/9511026.
- [32] C. Vafa, ‘*Instantons on D–Branes*’, hep-th/9512078.
- [33] S. Ferrara and R. Kallosh, ‘*Supersymmetry and Attractors*’, hep-th/9602136; J. Polchinski, unpublished.
- [34] R. Kallosh and B. Kol, ‘*E(7) Symmetric Area of the Black Hole Horizon*’, hep-th/9602014.
- [35] C. M. Hull and P. K. Townsend, ‘*Unity of Superstring Dualities*’, Nucl. Phys. **B438** (1995) 109, hep-th/9410167.
- [36] E. Witten, ‘*String Theory Dynamics in Various Dimensions*’, Nucl. Phys. **B443** (1995) 85, hep-th/9503124.