mathematics

IMPACT FACTOR 2.3

CITESCORE 4.0

Article

New Analytical Formulas for the Rank of Farey Fractions and Estimates of the Local Discrepancy

Rogelio Tomás García

MDPI

https://doi.org/10.3390/math13010140

mathematics

MDPI

Article

New Analytical Formulas for the Rank of Farey Fractions and Estimates of the Local Discrepancy

Rogelio Tomás García

CERN, Esplanade des Particules 1, 1211 Meyrin, Switzerland; rogelio.tomas@cern.ch

Abstract: New analytical formulas are derived for the rank and the local discrepancy of Farey fractions. The new rank formula is applicable to all Farey fractions and involves sums of a lower order compared to the searched one. This serves to establish a new unconditional estimate for the local discrepancy of Farey fractions that decrease with the order of the Farey sequence. This estimate improves the currently known estimates. A new recursive expression for the local discrepancy of Farey fractions is also given. A second new unconditional estimate of the local discrepancy of any Farey fraction is derived from a sum of the Mertens function, again, improving the currently known estimates.

Keywords: Farey sequence; Riemann Hypothesis

MSC: 11B57

1. Introduction and Statement of Main Results

The Farey sequence F_n of order $n \in \mathbb{N}$ is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. These fractions are referred to as Farey fractions. An introduction and thorough reviews of the theory of Farey sequences can be found in [1–4], along with a few applications in [5,6]. Throughout this paper, we exclude the fraction $0/1$ from F_n. For given $n \in \mathbb{N}$ and $x \in]0,1]$, $I_n(x)$ is defined as the number of elements in F_n within $]0,x]$. We define F_n^x as a subsequence of F_n given by

$$F_n^x = F_n \cap]0,x],$$

and, therefore,

$$I_n(x) = |F_n^x|.$$

The local discrepancy $\hat{r}_n(h/k)$ of the Farey fraction h/k in F_n is defined as [7,8]

$$\hat{r}_n(h/k) = \frac{h}{k}|F_n| - I_n(h/k).$$

We also introduce the discrepancy $\hat{r}_q^{\varphi}(h/k)$, at the level of the Euler Totient function $\varphi(x)$, such that the number of Farey fractions in F_q lower than h/k and with denominators equal to q is given by

$$\frac{h}{k}\varphi(q) - \hat{r}_q^{\varphi}\left(\frac{h}{k}\right),$$

and, therefore,

$$\hat{r}_n(h/k) = \sum_{q=1}^{n} \hat{r}_q^{\varphi}\left(\frac{h}{k}\right).$$

Academic Editor: Ion Mihai

Received: 7 December 2024
Revised: 30 December 2024
Accepted: 31 December 2024
Published: 2 January 2025

Citation: Tomás García, R. New Analytical Formulas for the Rank of Farey Fractions and Estimates of the Local Discrepancy. *Mathematics* **2025**, *13*, 140. https://doi.org/10.3390/math13010140

Note that h/k is not necessarily an element of F_n. The absolute discrepancy of the Farey sequence F_n is generally defined as

$$D_n = \sup_{\alpha \in F_n} \frac{|\hat{r}_n(\alpha)|}{|F_n|}.$$

It is important to recall an equivalent formulation of the Riemann Hypothesis (RH). The Franel–Landau formulation [9,10] is expressed as

$$\frac{1}{|F_n|} \sum_{h/k \in F_n} |\hat{r}_n(h/k)| = O\left(n^{\frac{1}{2}+\epsilon}\right) \ \forall \epsilon > 0 \ \Leftrightarrow \ \text{RH}.$$

This direct connection between local discrepancies and RH shows the importance of progressing in computing estimates of $\hat{r}_n(h/k)$. The unconditional estimate of $\hat{r}_n(h/k)$ is not generally addressed in the literature, while D_n has been evaluated to be $O(1/n)$ in [7] and, therefore, using $|F_n| = O(n^2)$, we can write

$$\hat{r}_n(h/k) = O(n). \tag{1}$$

The absolute discrepancy of the Farey sequence was derived in [11] and found to be $D_n = 1/n$ by finding an upper bound of an integral of the Mertens function. This result has been qualified as "most remarkable" in [12]. Figure 1 shows an illustration of the different bounds for the local discrepancies versus the corresponding Farey fraction α for different ranges of n, as derived in [11].

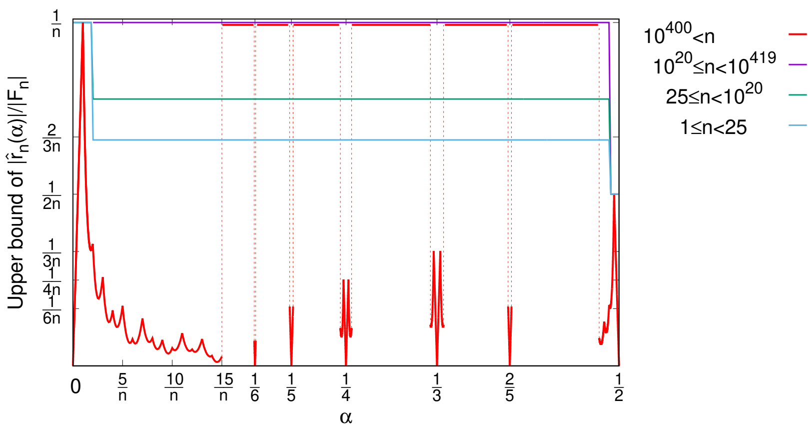

Figure 1. Illustration of the results in [11], showing the upper bounds of the local discrepancy of Farey fractions, $|\hat{r}_n(\alpha)|/|F_n|$, versus the Farey fractions α in $[0, 1/2]$ (without respecting the actual separation ratios in the horizontal axis). Note that the red curve for $\alpha \le 15/n$ has been plotted using expression (2), while in [11] (page 361) tabulated values are given.

The following approximations are derived in [11] (page 361) for $n > 10^{400}$,

$$\begin{aligned} I_n(\alpha) &\overset{w}{=} n \sum_{j=1}^{\lfloor n\alpha \rfloor} \frac{\varphi(j)}{j} - \frac{1}{\alpha}\Phi(\lfloor n\alpha \rfloor), \\ |\hat{r}_n(\alpha)| &\overset{w}{\le} \frac{|F_n|}{n}\left(n\alpha - \frac{\pi^2}{3}\sum_{j=1}^{\lfloor n\alpha \rfloor} \frac{\varphi(j)}{j} + \frac{\pi^2}{3\alpha n}\Phi(\lfloor n\alpha \rfloor)\right), \end{aligned} \tag{2}$$

where $\Phi(x)$ is the Totient summatory function and "$\overset{w}{=}$" and "$\overset{w}{\le}$" are introduced in [11] to imply that the terms with relative influence below 10^{-100} are neglected. Neither the validity range of these approximations for α nor the estimates of the neglected terms are given in [11].

These approximations are only used for $\alpha \leq 15/n$ in [11] and, indeed, above this value of α, the quantity in parenthesis in (2) can take negative values. For example, for $\alpha = 33.6/n$, we would have $|\hat{r}_n(33.6/n)| \overset{w}{\leq} -0.001|F_n|/n$, which does not hold.

Knowing the missing terms that complete the above approximations for any n and α could lead to new bounds or estimates for the local discrepancies of Farey fractions. Partial developments in this direction are found in [13,14] for unit fractions. In Theorem 1 and Corollary 1, we develop new general expressions for $I_n(\alpha)$ and $\hat{r}_n(\alpha)$, obtaining, for $\hat{r}_n(\alpha)$,

$$\begin{aligned} \hat{r}_n(\alpha) &= |F_n|\alpha - n\sum_{j=1}^{\lfloor n\alpha \rfloor} \frac{\varphi(j)}{j} + \frac{1}{\alpha}\Phi(\lfloor n\alpha \rfloor) \\ &\quad -\hat{r}_{\lfloor n\alpha \rfloor}(\{1/\alpha\}) + \sum_{j=1}^{\lfloor n\alpha \rfloor}\sum_{d|j}\mu(d)\left\{\frac{n}{d}\right\}, \end{aligned} \tag{3}$$

where $\mu(d)$ is the Möbius funtion and $\{x\}$ represents the fractional part of x. This identity unexpectedly connects the discrepancy of α in F_n with the discrepancy of $\{1/\alpha\}$ in $F_{\lfloor n\alpha \rfloor}$. Furthermore, the new general identity (3) can be applied iteratively to $\hat{r}_{\lfloor n\alpha \rfloor}(\{1/\alpha\})$ for a finite number of steps, as $\{1/\alpha\}$ is always a Farey fraction of a lower order than α. This identity is used in Theorem 2 to derive a new unconditional estimate of $\hat{r}_n(\alpha)$ for any $\alpha \asymp O(n^{-\epsilon})$, with $\epsilon \in]0,1]$, given by

$$\hat{r}_n(\alpha) = O\Big(n\delta_A(n^{1-\epsilon})\Big), \text{ for } \alpha \asymp O(n^{-\epsilon}), \ \epsilon \in]0,1],$$

where the function $\delta_A(x)$ is a monotonic decreasing function defined as

$$\delta_A(x) = \exp\left(-A\frac{\log^{0.6} x}{(\log\log x)^{0.2}}\right), \text{ with } A > 0.$$

This new unconditional estimate of $\hat{r}_n(\alpha)$ improves the existing one, $O(n)$, from [7,11] for α values that decrease with n.

In this work, we derive another unconditional estimate of $\hat{r}_n(h/k)$. For later convenience, we define the local discrepancy with an offset as

$$r_n(h/k) = \hat{r}_n(h/k) - \frac{k-1}{2k}.$$

In Theorem 3, we demonstrate that

$$\sum_{d=1}^{n} r_{\lfloor n/d \rfloor}(h/k) = \sum_{i=1}^{b}\left\langle \frac{ih}{k}\right\rangle_k, \tag{4}$$

with $b = n \bmod k$ and $\langle\cdot\rangle_k$, the fractional part with an offset defined as

$$\langle x\rangle_k = \{x\} - \frac{k-1}{2k}.$$

Note that for h/k being a Farey fraction and for any positive integers a and p, we have

$$\sum_{d=a}^{a+pk-1}\left\langle \frac{dh}{k}\right\rangle_k = 0. \tag{5}$$

Expression (4) can be used iteratively for an efficient calculation of $r_n(h/k)$ as done, e.g., in [15], to compute $I_n(h/k)$. Applying the Möbius inversion formula to (4) and making further developments, the following two identities are also demonstrated in Theorem 3,

$$r_n(h/k) = \sum_{d=1}^{n} \mu(d) \sum_{i=1}^{\hat{b}} \left\langle \frac{ih}{k} \right\rangle_k \tag{6}$$

$$= \sum_{d=1}^{n} M\left(\frac{n}{d}\right) \left\langle \frac{dh}{k} \right\rangle_k', \tag{7}$$

with $\hat{b} = \lfloor n/d \rfloor \bmod k$ and $M(n)$ representing the Mertens function defined as $M(n) = \sum_{d=1}^{n} \mu(d)$. Expression (6) is new and can be used to demonstrate again that $r_n(h/k) = O(n)$ as the sum over i is bounded by $k/8$, as it is shown in Lemma 3 given below. Expression (7) is not new, as it already has been given in similar forms in [3,7,11], but here, it adopts a simpler form thanks to the introduction of $\langle \cdot \rangle_k$ and the local discrepancy with an offset.

Theorem 4 establishes a second new unconditional estimate of the local discrepancy for a Farey fraction, h/k, such that $k = O(n^{1-\epsilon})$, with $1 > \epsilon > 0$, as

$$r_n(h/k) = O\Big(n \log^{0.4} n \log^{0.2} \log n \, \delta_A(n^\epsilon)\Big), \text{ for } A > 0.$$

It is important to note that this estimate includes the general case of h/k being constant. Again, this unconditional estimate improves the existing one, $O(n)$, from [7,11] for Farey fractions with denominators that can grow sublinearly with the order of the Farey sequence and complements the estimate given in Theorem 2.

2. Results

Lemma 1. *The number $\mathcal{N}_n^{h/k}(q)$ of Farey fractions in $F_n^{h/k}$ with numerators equal to q for $k \leq n$ is given by*

$$\mathcal{N}_n^{h/k}(q) = n\frac{\varphi(q)}{q} - \frac{k}{h}\varphi(q) + \hat{r}_q^{\varphi}(\{k/h\}) - \sum_{d|q} \mu(d) \left\{ \frac{n}{d} \right\},$$

Proof. Using Corollary 5 in [14], we determine the number of Farey fractions with numerators equal to q in $F_n^{1/\lfloor k/h \rfloor}$ as

$$\mathcal{N}_n^{1/\lfloor k/h \rfloor}(q) = n\frac{\varphi(q)}{q} - \lfloor k/h \rfloor \varphi(q) - \sum_{d|q} \mu(d) \left\{ \frac{n}{d} \right\}, \text{ if } \lfloor k/h \rfloor < n/q.$$

To determine $\mathcal{N}_n^{h/k}(q)$ from $\mathcal{N}_n^{1/\lfloor k/h \rfloor}(q)$, we need to compute the number of Farey fractions with numerators equal to q in $F_n \cap \left[\frac{h}{k}, \frac{1}{\lfloor k/h \rfloor}\right]$. To this end, we define F_n' as

$$F_n' = \left\{ \frac{u}{l} \in F_n \cap \left[\frac{1}{\lfloor k/h \rfloor + 1}, \frac{1}{\lfloor k/h \rfloor} \right] \; : \; u \leq q \right\},$$

and a bijective map $\tilde{M}$ between F_n' and F_q as

$$\tilde{M} \; : \; F_q \to F_n', \; \frac{t}{q'} \mapsto \frac{q'}{q'(\lfloor k/h \rfloor + 1) - t},$$

$$\tilde{M}^{-1} \; : \; F_n' \to F_q, \; \frac{h'}{k'} \mapsto \frac{h'(\lfloor k/h \rfloor + 1) - k'}{h'}.$$

This implies that the number of Farey fractions with numerators equal to q in F_n' is the same as the number of Farey fractions with denominators equal to q in F_q, that is $\varphi(q)$. Furthermore, the image of h/k under $\tilde{M}^{-1}$ is given by

$$\frac{h(\lfloor k/h \rfloor + 1) - k}{h} = 1 - \left\{ \frac{k}{h} \right\} = 1 - \frac{k \bmod h}{h},$$

and the number of Farey fractions in F_q with denominators equal to q and larger than $1 - \{k/h\}$ is given by

$$\varphi(q) \left\{ \frac{k}{h} \right\} - \hat{r}_q^{\varphi}(\{k/h\}).$$

Therefore,

$$\mathcal{N}_n^{h/k}(q) = \mathcal{N}_n^{1/\lfloor k/h \rfloor}(q) - \varphi(q) \left\{ \frac{k}{h} \right\} + \hat{r}_q^{\varphi}(\{k/h\}).$$

□

Lemma 2. *The largest numerator among the Farey fractions in F_n^{α}, with $\alpha \in [0,1]$, is equal to or below $\lfloor n\alpha \rfloor$.*

Proof. This is immediate from the fact that the largest denominator in F_n^{α} is n and $\lfloor n\alpha \rfloor$ is the largest integer that fulfills $\lfloor n\alpha \rfloor / n \leq \alpha$. □

Theorem 1. *The rank $I_n(h/k)$ of the Farey fraction h/k in F_n is given by*

$$\begin{aligned} I_n(h/k) &= n \sum_{j=1}^{\lfloor nh/k \rfloor} \frac{\varphi(j)}{j} - \frac{k}{h} \Phi(\lfloor nh/k \rfloor) \\ &\quad + \hat{r}_{\lfloor nh/k \rfloor}(\{k/h\}) - \sum_{j=1}^{\lfloor nh/k \rfloor} \sum_{d|j} \mu(d) \left\{ \frac{n}{d} \right\}, \end{aligned}$$

Proof. Per Lemma 2, we obtain $I_n(h/k)$ by adding $\mathcal{N}_n^{h/k}(q)$ for all $q \leq \lfloor nh/k \rfloor$,

$$I_n(h/k) = \sum_{q=1}^{\lfloor nh/k \rfloor} \mathcal{N}_n^{h/k}(q).$$

The desired result is achieved by using Lemma 1 in this relation. □

Corollary 1. *The local discrepancy of the Farey fraction h/k in F_n is given by*

$$\begin{aligned} \hat{r}_n(h/k) &= |F_n| \frac{h}{k} - n \sum_{j=1}^{\lfloor nh/k \rfloor} \frac{\varphi(j)}{j} + \frac{k}{h} \Phi(\lfloor nh/k \rfloor) \\ &\quad - \hat{r}_{\lfloor nh/k \rfloor}(\{k/h\}) + \sum_{j=1}^{\lfloor nh/k \rfloor} \sum_{d|j} \mu(d) \left\{ \frac{n}{d} \right\}. \end{aligned}$$

Proof. This follows from the definition of the local discrepancy and Theorem 1. □

Theorem 2. *The unconditional estimate of the local discrepancy of the Farey fraction h/k is given by*

$$\hat{r}_n(h/k) = \frac{3}{\pi^2} \frac{k}{h} \left\{ \frac{hn}{k} \right\}^2 + O\left(n \delta_A \left(n \frac{h}{k} \right) \right) + O\left(n \frac{h}{k} \log \left(n \frac{h}{k} \right) \right).$$

For $h/k \asymp O(n^{-\epsilon})$, with $\epsilon \in]0,1]$, the estimate simplifies to the following expression,

$$\hat{r}_n(h/k) = O\Big(n\delta_A(n^{1-\epsilon})\Big), \text{ for } h/k \asymp O(n^{-\epsilon}).$$

Proof. Recalling Theorem 1,

$$\begin{aligned} \hat{r}_n(h/k) &= |F_n|\frac{h}{k} - n\sum_{j=1}^{\lfloor nh/k \rfloor}\frac{\varphi(j)}{j} + \frac{k}{h}\Phi(\lfloor nh/k \rfloor) \\ &\quad -\hat{r}_{\lfloor nh/k \rfloor}(\{k/h\}) + \sum_{j=1}^{\lfloor nh/k \rfloor}\sum_{d|j}\mu(d)\left\{\frac{n}{d}\right\}, \end{aligned} \tag{8}$$

we proceed to provide estimates for the different terms in the right hand side of the above expression, assuming $h/k = O(n^{-\epsilon})$ with $\epsilon \in [0,1[$ and using known estimates from, e.g., [11,16,17] as follows:

$$\begin{aligned} |F_n|\frac{h}{k} &= \frac{3}{\pi^2}n^2\frac{h}{k} + E(n)\frac{h}{k}, \\ n\sum_{j=1}^{\lfloor nh/k \rfloor}\frac{\varphi(j)}{j} &= \frac{6}{\pi^2}n^2\frac{h}{k} - \frac{6}{\pi^2}n\left\{\frac{hn}{k}\right\} + nH(nh/k), \\ \frac{k}{h}\Phi(\lfloor nh/k \rfloor) &= \frac{3}{\pi^2}n^2\frac{h}{k} - \frac{6}{\pi^2}n\left\{\frac{hn}{k}\right\} + \frac{3}{\pi^2}\frac{k}{h}\left\{\frac{hn}{k}\right\}^2 + E(nh/k)\frac{k}{h}, \\ E(x) &= O(x\log^{2/3}x(\log\log x)^{4/3}), \\ E(x) &= xH(x) + O(x\delta_A(x)), \\ \hat{r}_{\lfloor nh/k \rfloor}(\{k/h\}) &= O(nh/k), \end{aligned}$$

with $A > 0$. Combining the above results we obtain the following relation,

$$\begin{aligned} \hat{r}_n(h/k) &= \frac{3}{\pi^2}\frac{k}{h}\left\{\frac{hn}{k}\right\}^2 + E(n)\frac{h}{k} \\ &\quad + O\left(n\delta_A\left(n\frac{h}{k}\right)\right) + O(nh/k) + \sum_{j=1}^{\lfloor nh/k \rfloor}\sum_{d|j}\mu(d)\left\{\frac{n}{d}\right\}. \end{aligned} \tag{9}$$

For the sum with the Möbius function we establish the following estimate,

$$\sum_{j=1}^{\lfloor nh/k \rfloor}\sum_{d|j}\mu(d)\left\{\frac{n}{d}\right\} = \sum_{j=1}^{\lfloor nh/k \rfloor}\mu(j)\left\lfloor\frac{nh}{kj}\right\rfloor\left\{\frac{n}{j}\right\} = O\left(n\frac{h}{k}\log\left(n\frac{h}{k}\right)\right).$$

Combining the above estimates the desired result is obtained. For the case $h/k = O(n^{-1})$, we directly evaluate identity (8), obtaining

$$\hat{r}_n(h/k) = O(n),$$

which is compatible with the formulation of the theorem and with the main result in [11]. □

Lemma 3. *For $\hat{b}$, h, and k integers fulfilling $0 \le \hat{b} \le k$, $\gcd(h,k) = 1$ and $1 \le h \le k-1$ we have*

$$\left|\sum_{i=1}^{\hat{b}}\left\langle\frac{ih}{k}\right\rangle_k\right| \equiv \left|\sum_{i=1}^{\hat{b}}\left(\left\{\frac{ih}{k}\right\} - \frac{k-1}{2k}\right)\right| \le \frac{k}{8}.$$

Proof. Since $\gcd(h,k) = 1$, the fractional part $\{ih/k\}$ takes different values for all $i \in [\![1, \hat{b}]\!]$ and, therefore, we can establish the following bounds

$$\sum_{i=1}^{\hat{b}} \frac{i}{k} \le \sum_{i=1}^{\hat{b}} \left\{ \frac{ih}{k} \right\} \le \sum_{i=1}^{\hat{b}} \frac{k-i}{k},$$

$$\frac{\hat{b}(\hat{b}+1)}{2k} \le \sum_{i=1}^{\hat{b}} \left\{ \frac{ih}{k} \right\} \le \frac{\hat{b}(2k-\hat{b}-1)}{2k}.$$

Subtracting $\hat{b}(k-1)/(2k)$ we obtain

$$-\frac{(k-2)^2}{8k} \le -\frac{\hat{b}(k-\hat{b}-2)}{2k} \le \sum_{i=1}^{\hat{b}} \left\langle \frac{ih}{k} \right\rangle_k \le \frac{\hat{b}(k-\hat{b})}{2k} \le \frac{k}{8}.$$

□

Theorem 3. *For $n \in \mathbb{N}$, h/k being a Farey fraction and b defined as $b \equiv n \mod k$, we have*

$$\sum_{d=1}^{n} r_{\lfloor n/d \rfloor}(h/k) \quad = \quad \sum_{i=1}^{b} \left\langle \frac{ih}{k} \right\rangle_k, \tag{10}$$

and, by Möbius inversion, we also have

$$r_n(h/k) \quad = \quad \sum_{d=1}^{n} \mu(d) \sum_{i=1}^{\hat{b}} \left\langle \frac{ih}{k} \right\rangle_k, \tag{11}$$

with $\hat{b} = \lfloor n/d \rfloor \mod k$. Furthermore,

$$r_n(h/k) \quad = \quad \sum_{d=1}^{n} M\left(\frac{n}{d}\right) \left\langle \frac{dh}{k} \right\rangle_k \tag{12}$$

where $M(x)$ is the Mertens function.

Proof. $I_{\lfloor n/d \rfloor}(h/k)$ represents the number of simple fractions of the form p/q and $0 \le p \le q \le n$ below or equal h/k with $\gcd(p,q) = d$. Therefore the sum over d,

$$\sum_{d=1}^{n} I_{\lfloor n/d \rfloor}(h/k) = \sum_{i=1}^{n} \left\lfloor \frac{ih}{k} \right\rfloor,$$

gives the total number of fractions below or equal h/k. This argument is commonly used, see, e.g., [15]. Developing the right hand side of the above expression, using the definition $b \equiv n \mod k$, we obtain

$$\begin{aligned}\sum_{i=1}^{n}\left\lfloor\frac{ih}{k}\right\rfloor &= \sum_{i=1}^{n}\left(\frac{ih}{k}-\left\{\frac{ih}{k}\right\}\right)\\ &= \frac{h}{2k}(n+1)n-\sum_{i=1}^{n}\left\{\frac{ih}{k}\right\}\\ &= \frac{h}{2k}(n+1)n-\left\lfloor\frac{n}{k}\right\rfloor\sum_{i=1}^{k-1}\left\{\frac{ih}{k}\right\}-\sum_{i=1}^{b}\left\{\frac{ih}{k}\right\}\\ &= \frac{h}{2k}(n+1)n-\left\lfloor\frac{n}{k}\right\rfloor\frac{k-1}{2}-\sum_{i=1}^{b}\left\{\frac{ih}{k}\right\}\\ &= \frac{h}{2k}(n+1)n-(n-b)\frac{k-1}{2k}-\sum_{i=1}^{b}\left\{\frac{ih}{k}\right\}.\end{aligned}$$

Inserting the following quantity,

$$\sum_{d=1}^{n}|F_{\lfloor n/d\rfloor}|=\sum_{d=1}^{n}I_{\lfloor n/d\rfloor}(1/1)=\sum_{i=1}^{n}\left\lfloor\frac{i\cdot 1}{1}\right\rfloor=\frac{1}{2}(n+1)n,$$

in the above derivation gives

$$\sum_{d=1}^{n}I_{\lfloor n/d\rfloor}(h/k)=\frac{h}{k}\sum_{d=1}^{n}|F_{\lfloor n/d\rfloor}|-n\frac{k-1}{2k}+b\frac{k-1}{2k}-\sum_{i=1}^{b}\left\{\frac{ih}{k}\right\}.$$

Since $r_n(h/k)$ is defined as

$$r_n(h/k)=\frac{h}{k}|F_n|-I_n(h/k)-\frac{k-1}{2k},$$

we retrieve the desired result as follows:

$$\sum_{d=1}^{n}r_{\lfloor n/d\rfloor}(h/k) = \sum_{i=1}^{b}\left\langle\frac{ih}{k}\right\rangle_k.$$

Identity (11) is directly obtained by Möbius inversion and (12) is derived as follows,

$$\begin{aligned}r_n(h/k) &= \sum_{d=1}^{n}\mu(d)\sum_{i=1}^{\hat{b}}\left\langle\frac{ih}{k}\right\rangle_k\\ &= \sum_{d=1}^{n}\left(M\left(\frac{n}{d}\right)-M\left(\frac{n}{d+1}\right)\right)\sum_{i=1}^{d \bmod k}\left\langle\frac{ih}{k}\right\rangle_k\\ &= M(n)\left\langle\frac{h}{k}\right\rangle_k+\sum_{d=2}^{n}M\left(\frac{n}{d}\right)\left\langle\frac{dh}{k}\right\rangle_k=\sum_{d=1}^{n}M\left(\frac{n}{d}\right)\left\langle\frac{dh}{k}\right\rangle_k\end{aligned}$$

with $\hat{b}=\lfloor n/d\rfloor \bmod k$. Identity (12) is very similar to Formula (1) of [11] and to its further derivations within the proof of Lemma 4 in [11]. □

Corollary 2. *For any constant $\alpha\in]0,1]$, we have*

$$\sum_{j=1}^{\lfloor n\alpha\rfloor}\sum_{d|j}\mu(d)\left\{\frac{n}{d}\right\}=O(n\log^{2/3}n(\log\log n)^{4/3}).$$

Proof. By inspecting estimate (9) for the case with constant $h/k=\alpha$, we realize that the largest growing term, the sum with the Möbius function, must have the same asymptotic behavior as the second largest term, $E(n)\alpha$, so that their sum can result in the known estimate, $\hat{r}_n(h/k) = O(n)$, on the left hand side. □

Theorem 4. *The unconditional estimate of the local discrepancy of the Farey fraction h/k is given by*

$$r_n(h/k) = O\Big(n \log^{0.4} n \log^{0.2} \log n\, \delta_A(n^{\hat{\epsilon}})\Big) + O\Big(kn^{\hat{\epsilon}}\Big),$$

for $1 > \hat{\epsilon} > 0$.

For the case $k = O(n^{1-\epsilon'})$, with $\epsilon' > \hat{\epsilon}$, the second O term can be neglected and the estimate is given by

$$r_n(h/k) = O\Big(n \log^{0.4} n \log^{0.2} \log n\, \delta_A(n^{\hat{\epsilon}})\Big).$$

Proof. Let us start from the expression (12) of the local discrepancy of the Farey fraction h/k given in Theorem 3

$$r_n(h/k) = \sum_{d \le n} M\Big(\frac{n}{d}\Big)\Big\langle \frac{dh}{k} \Big\rangle_k. \tag{13}$$

Splitting the sum in (13) in two parts at $f(n) = \lfloor n^{1-\hat{\epsilon}} \rfloor$ for any $\hat{\epsilon}$ such that $1 > \hat{\epsilon} > 0$ gives

$$\sum_{d \le n} M\Big(\frac{n}{d}\Big)\Big\langle \frac{dh}{k} \Big\rangle_k = \sum_{d=1}^{f(n)} M\Big(\frac{n}{d}\Big)\Big\langle \frac{dh}{k} \Big\rangle_k + \sum_{d=f(n)+1}^{n} M\Big(\frac{n}{d}\Big)\Big\langle \frac{dh}{k} \Big\rangle_k. \tag{14}$$

For any monotonically increasing function $g(x)$ in the range $[n/a, n]$, with $a \ge 1$, we have

$$\sum_{d=1}^{a} g(n/d) \le g(n) + \int_1^a g(n/x)\mathrm{d}x$$

for any $n > a$. Since $M(x) = O(x\delta_A(x))$ for $A > 0$, see [17], we establish the following estimate for the first sum in the right hand side of inequality (14) as

$$\sum_{d=1}^{f(n)} |M(n/d)| = O\Big(n \log^{0.4} n \log^{0.2} \log n\, \delta_A(n^{\hat{\epsilon}})\Big),$$

where we have used that

$$\int_1^{f(n)} \frac{n}{x}\delta_A(n/x)\mathrm{d}x = \int_{\frac{n}{f(n)}}^{n} \frac{n}{u}\delta_A(u)\mathrm{d}u = O\Big(n \log^{0.4} n \log^{0.2} \log n\, \delta_A(n^{\hat{\epsilon}})\Big)$$

as demonstrated in Lemma 4.

The second sum in the r.h.s of (14) can be bounded as

$$\left|\sum_{d=f(n)+1}^{n} M\Big(\frac{n}{d}\Big)\Big\langle \frac{dh}{k} \Big\rangle_k\right| \le \sum_{d \in J}\left|\Big(M\Big(\frac{n}{d}\Big) - M\Big(\frac{n}{d+\alpha_d}\Big)\Big)\Big\langle \frac{dh}{k} \Big\rangle_k\right| + \sum_{d=n-k+1}^{n}\left|M\Big(\frac{n}{d}\Big)\Big\langle \frac{dh}{k} \Big\rangle_k\right| \tag{15}$$

where we have used the fact that for every $d < n - k + 1$ there exists one $\alpha_d < k$ such that

$$\Big\langle \frac{dh}{k} \Big\rangle_k = -\Big\langle \frac{(d+\alpha_d)h}{k} \Big\rangle_k.$$

The set J is a subset of $[\![f(n)+1, n-k]\!]$ such that the map A

$$A \,:\, J \to [\![f(n)+1, n-k]\!] - J,\; d \mapsto d + \alpha_d$$

is bijective. The second sum in the r.h.s of Expression (15) includes the elements that cannot be paired when $d + \alpha_d > n$ and accepts the following bound,

$$\sum_{d=n-k+1}^{n} \left| M\left(\frac{n}{d}\right) \left\langle \frac{dh}{k} \right\rangle_k \right| \leq \frac{kn}{2(n-k+1)},$$

where we have used that $|M(x)| \leq x$, for all x.

To derive a bound for the first sum in the r.h.s of Expression (15) we use the fact that

$$\left| M\left(\frac{n}{d}\right) - M\left(\frac{n}{d+\alpha_d}\right) \right| \leq \frac{\alpha_d n}{d^2} \leq \frac{kn}{d^2},$$

where we have used that $|M(x) - M(y)| \leq |x - y|$. Furthermore,

$$\sum_{d \in J} \frac{kn}{d^2} \left| \left\langle \frac{dh}{k} \right\rangle_k \right| \leq \frac{1}{2} \sum_{d=f(n)+1}^{n-k+1} \frac{kn}{d^2} \leq \frac{kn}{2(f(n)+1)} - \frac{kn}{2(n-k+1)}.$$

Combining the above bounds and recalling that $f(n) = \lfloor n^{1-\hat{\epsilon}} \rfloor$, we conclude that

$$\left| \sum_{d=f(n)+1}^{n} M\left(\frac{n}{d}\right) \left\langle \frac{dh}{k} \right\rangle_k \right| \leq \frac{kn^{\hat{\epsilon}}}{2} = O(kn^{\hat{\epsilon}}).$$

Inserting the above estimates into inequality (14), we obtain the wanted result. □

Lemma 4. *For any $a > x$, we have*

$$\int_x^a \frac{\delta_A(u)}{u} \mathrm{d}u = O\Big(\log^{0.4} x \log^{0.2} \log x\, \delta_A(x)\Big).$$

Proof. This is demonstrated using the following derivative,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{5}{3A} \frac{\log^{0.4} x \log^{0.2} \log x}{\left(1 - \frac{1}{3 \log \log x}\right)} \delta_A(x) \right) = \\
&\frac{\delta_A(x)}{x} \left(-1 + \frac{\log^{0.2} \log x}{A \log^{0.6} x} \frac{(6 \log^2 \log x + \log \log x - 6)}{(3 \log \log x - 1)^2} \right) = \\
&\frac{\delta_A(x)}{x} \left(-1 + O\left(\frac{\log^{0.2} \log x}{\log^{0.6} x} \right) \right),
\end{aligned}$$

and therefore,

$$\int_x^a \frac{\delta_A(u)}{u} \mathrm{d}u = O\left(\frac{\log^{0.4} u \log^{0.2} \log u}{\left(1 - \frac{1}{3 \log \log u}\right)} \delta_A(u) \Bigg|_{u=a}^{u=x} \right).$$

The factor in the denominator inside the O term can be neglected for large x. □

3. Discussion

We have developed new exact formulas for the rank and discrepancy of Farey fractions using an interesting technique based on a bijection between Farey subsequences. These formulas complete an approximation presented in the classical paper [11]. It is remarkable that this formula, see Corollary 1, connects the local discrepancy of two different Farey fractions, namely α and $\{1/\alpha\}$. As a curiosity, the largest solution of the equation $\alpha = \{1/\alpha\}$ is the fractional part of the Golden ratio, $\left(\sqrt{5}-1\right)/2$, which is not a Farey fraction. These new formulas are used to compute a new estimate of the local discrepancy in Theorem 2 that improves the currently known estimates.

The new notation introduced in this work, namely the discrepancy with an offset $r_n(h/k) = \hat{r}_n(h/k) - (k-1)/(2k)$ and the fractional part with the same offset $\langle\cdot\rangle_k$, simplifies the known formula and has helped in the development of the new formulas for $r_n(h/k)$ in Theorem 3. These are the basis for the development of the second new estimate of $r_n(h/k)$ in Theorem 4. The new and previous estimates of $r_n(h/k)$, or equivalently $\hat{r}_n(h/k)$, are put together in the following expression:

$$\hat{r}_n(h/k) = \begin{cases} O\big(n\delta_A(n^{1-\epsilon})\big), & \text{for } h/k \asymp O(n^{-\epsilon})\text{ , Theorem 2} \\ O\Big(n\log^{0.4} n \log^{0.2}\log n\, \delta_A(n^{\hat{\epsilon}})\Big), & \text{for } k = O(n^{1-\hat{\epsilon}})\text{ , Theorem 4} \\ O(n), & \text{otherwise,} \end{cases}$$

with $\epsilon \in]0,1]$, $\hat{\epsilon} \in]0,1[$, and $A > 0$. The cases where $k \asymp O(n)$ or $h/k = O(n^{-1})$ remain with the known estimate $\hat{r}_n(h/k) = O(n)$ from [7]. For the cases $h/k \asymp O(n^{-\epsilon})$ or $k = O(n^{1-\hat{\epsilon}})$, the new unconditional estimates of $\hat{r}_n(h/k)$ are sublinear in n. Theorem 4 applies to h/k being constant.

The Fanel–Landau formulation of the Riemann Hypothesis is expressed as

$$\frac{1}{|F_n|}\sum_{h/k\in F_n} |\hat{r}_n(h/k)| = O\left(n^{\frac{1}{2}+\epsilon}\right) \ \forall\epsilon > 0 \ \Leftrightarrow \ \text{RH}.$$

The known estimate $\hat{r}_n(h/k) = O(n)$, for all h/k in F_n, implies that

$$\frac{1}{|F_n|}\sum_{h/k\in F_n} |\hat{r}_n(h/k)| = O(n)\ ,$$

which is far from $O\left(n^{\frac{1}{2}+\epsilon}\right)$. For the RH to be true, $\hat{r}_n(h/k)$ would need to be $\hat{r}_n(h/k) = O\left(n^{\frac{1}{2}+\epsilon}\right)$ for most of the Farey fractions in F_n. The new sublinear estimates of $\hat{r}_n(h/k)$ in Theorems 2 and 4 go in the direction of the RH, but further developments would be needed for a significant improvement.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Cobeli, C.; Zaharescu, A. The Haros-Farey sequence at two hundred years. A survey. *Acta Univ. Apulensis. Math. Inform.* **2003**, *5*, 1–38.
2. Hardy, G.H.; Wright, E.M. *An Introduction to the Theory of Numbers*, 5th ed.; Oxford Science Publications: Docklands, Australia , 1996.
3. Kanemitsu, S.; Yoshimoto, M. Farey series and the Riemann hypothesis. *Acta Arith.* **1996**, 75, 351–374. [CrossRef]
4. Matveev, A.O. *Farey Sequences: Duality and Maps Between Subsequences*, 1st ed.; De Gruyter: Berlin, Germany; Boston, MA, USA, 2017; ISBN 978-3110546620.
5. Khoshnoudirad, D. Farey lines defining Farey dia-grams and application to some discrete structures. *Appl. Anal. Discrete Math.* **2015**, 9, 73–84. [CrossRef]
6. Tomás, R. From Farey sequences to resonance dia-grams. *Phys. Rev. ST Accel. Beams* **2014**, *17*, 014001. [CrossRef]
7. Niederreiter, H. The Distribution of Farey Points. *Math. Ann.* **1973**, *201*, 341–345. [CrossRef]

8. Codecà, P. *Alcune Proprietà della Discrepanza Locale delle Sequenze di Farey*; Atti della Accademia delle Scienze dell'Istituto di Bologna. Classe di Scienze Fisiche. Anno 269 rendiconti serie XIII, tomo VIII, 1980–1981, fasc.I/II; Tipografia Compositori: Bologna, Italy, 1980.
9. Franel, J. Les suites de Farey et le problème des nom-bres premiers. *Göttinger Nachrichten* **1924**, *1924*, 198–201.
10. Landau, E. Bemerkungen zu der vorstehenden Abhandlung von Herrn Franel. *Göttinger Nachrichten* **1924**, *1924*, 202–206.
11. Dress, F. Discrépance des suites de Farey. *J. Théorie Des Nr. Bordx.* **1999**, *11*, 345–367. [CrossRef]
12. Ledoan, A.H. The discrepancy of Farey series. *Acta Math. Hung.* **2018**, *156*, 465–480. [CrossRef]
13. Tomás, R. Partial Franel Sums. *J. Integer Seq.* **2022**, *25*, 22.1.5.
14. Tomás, R. Number of Farey fractions with equal numerators and the rank of unit fractions. *Integers* **2024**, *26*, 014001.
15. Pawlewicz, J.; Pătraşcu, M. Order Statistics in the Farey Sequences in Sublinear Time and Counting Primitive Lattice Points in Polygons. *Algorithmica* **2009**, *55*, 271–282. [CrossRef]
16. Kanemitsu, S.; Kuzumaki, T.; Yoshimoto, M. Some sums involving Farey fractions II. *J. Math. Soc. Jpn.* **2000**, *52*, 125–142. [CrossRef]
17. Walfisz, A. *Weylsche Exponentialsummen in der Neueren Zahlentheorie, Mathematische Foschungsberichte*; VEB Deutscher Verlag der Wissenschaften: Berlin, Germany, 1963; Volume 15, p. 191