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Doubled Structures of Algebroids in Gauged Double Field Theory

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Double field theory (DFT) is an effective theory of string theory. It has a manifest symmetry of T-duality. The gauge symmetry in DFT is related to some kind of algebroid structures, and they have a doubled structure. In this talk, we focus on the gauge algebra of the $O(D, D+n)$ gauged DFT and discuss an extension of the doubled structure. The gauge algebra of the $O(D, D+n)$ gauged DFT has been described by the twisted C-bracket. This bracket is related to some algebroid structures. We show that algebroids defined by the twisted C-bracket in the gauged DFT are built out of a direct sum of three (twisted) Lie algebroids. They exhibit a "triple", which we call the extended double, rather than a "double" structure. We also consider the geometrical realization of these structures in a $(2D+n)$-dimensional manifold.

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1. Introduction

T-duality provides the relationship between the different space-times. Double Field Theory (DFT) is the field theory with manifest T-duality in string theory [1]. The most fundamental DFT is defined in the $2D$-dimentional doubled space $\mathcal{M}_{2D}$. The coordinate of that space-time $X^M (M = 1, \cdots, 2D)$ is decomposed for $(x^\mu, \tilde{x}_\mu)(\mu = 1, \cdots, D)$. Here, x^μ is the Fourier dual of KK-mode, and $\tilde{x}_\mu$ is the Fourier dual of winding mode. T-duality transformation corresponds to the exchange of x^μ and $\tilde{x}_\mu$ on this space-time. The theory exhibits manifest $O(D, D)$ structure. The $O(D, D)$ DFT is the T-duality covariantized extension of the NSNS sector in type II supergravity.

The $O(D, D)$ DFT has a gauge symmetry which is described by the C-bracket. The C-bracket does not satisfy the Jacobi identity, then that is defferent from Lie bracket. This suggests the existence of more general algebraic structures. Actually, the C-bracket defines the metric algebroid which is presented by Vaisman [2]. The metric algebroid is also discussed as the pre-DFT algebroid in [3]. We call this structure as the Vaisman algebroid. The Vaisman algebroid has a doubled structure which is the direct sum of two Lie algebroids [4]. This idea is based on the Drinfel'd double for Lie algebras [5] and for Courant algebroids [6]. The notion of Drinfel'd double is important to consider the Poisson-Lie T-duality. The Poisson-Lie T-duality is well as a solution-generating technique for type II supergravities. The application for DFT is also discussed in many papers [7–9].

In this proceeding, we consider the heterotic case. It is related to the T-duality covariantized extension of the gauged supergravities called gauged DFT [10, 11]. This is the alternative formalism of the effective theory with T-duality. Gauged DFT includes non-Abelian gauge symmetries which are introduced by gauging a duality group $O(D, D + n)$. This theory is defined on the $(2D + n)$-dimentional space $\mathcal{M}_{2D+n}$. Gauged DFT also has a gauge symmetry which is described by the modified C-bracket. We call this bracket the twisted C-bracket $[\cdot,\cdot]_F$ because this includes the structure constant $F^M{}_{NK}$ of the gauge group.

$$[\Xi_1, \Xi_2]_F = \Xi_1^M \partial_K \Xi_2^M - \Xi_2^K \partial_K \Xi_1^M \partial_M - \frac{1}{2}\eta^{MN}\eta_{KL}(\Xi_1^K \partial_N \Xi_2^L - \Xi_2^K \partial_N \Xi_1^L)\partial_M + \Xi_2^N \Xi_1^K F^M{}_{NK}\partial_M. \tag{1}$$

Here, $\Xi_i (i = 1, 2)$ is the $(2D + n)$-vector on $\mathcal{M}_{2D+n}$ and η is the $O(D, D + n)$ invariant metric. We show that the twisted C-bracket (14) in gauged DFT is rewritten by the geometric quantities. We will see that the twisted C-bracket also defines the Vaisman algebroid. Then, we also see that the Vaisman algebroid with the twisted C-bracket has a tripled structure which is Drinfel'd double-like structure. The contents of this proceeding is based on [12].

2. Gauged double field theory and twisted C-bracket

First, we give a brief ntruduction about the gauged doubled field theory and the gauge symmetries. The $O(D, D + n)$ gauged DFT acton is given by

$$S_0 = \int \mathrm{d}^{2D+n}\mathbb{X}\, e^{-2d}\bigg(\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\partial_L\mathcal{H}_{MK} - 2\partial_M d\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d\partial_N d\bigg), \tag{2}$$

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where $\mathcal{H}_{MN}(\mathbb{X})$, $(M,N=1,\ldots,2D+n)$ and $d(\mathbb{X})$ are the generalized metric and the generalized dilaton. These are the fundamental fields defined in the $(2D{+}n)$-dimensional doubled space $\mathcal{M}_{2D+n}$. The coordinate $\mathbb{X}^M$ may be decomposed into $\mathbb{X}^M=(\tilde{x}_\mu, x^\mu, \bar{x}_\alpha)$, $(\mu,\nu=1,\ldots,D, \alpha=1,\ldots,n)$. The standard parametrizations of the generalized metric and the dilaton are given by

$$\mathcal{H}_{MN}=\begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}c_{\rho\nu} & -g^{\mu\rho}A_\rho{}^\beta \\ -g^{\nu\rho}c_{\rho\mu} & g_{\mu\nu}+c_{\rho\mu}g^{\rho\sigma}c_{\sigma\nu}+\kappa^{\alpha\beta}A_{\mu\alpha}A_{\nu\beta} & c_{\rho\mu}g^{\rho\sigma}A_\sigma{}^\beta+A_\mu{}^\beta \\ -g^{\nu\rho}A_\rho{}^\alpha & c_{\rho\nu}g^{\rho\sigma}A_\sigma{}^\alpha+A_\nu{}^\alpha & \kappa^{\alpha\beta}+A_\rho{}^\alpha g^{\rho\sigma}A_\sigma{}^\beta \end{pmatrix},$$
$$e^{-2d}=\sqrt{-g}e^{-2\phi}, \tag{3}$$

where

$$c_{\mu\nu}=B_{\mu\nu}+\frac{1}{2}\kappa^{\alpha\beta}A_{\mu\alpha}A_{\nu\beta}. \tag{4}$$

$g_{\mu\nu}(\mathbb{X}), g^{\mu\nu}(\mathbb{X})$ are a symmetric $D\times D$ matrix and its inverse. An $n\times n$ symmetric constant matrix $\kappa_{\alpha\beta}$ and its inverse $\kappa^{\alpha\beta}$ and a $D\times n$ matrix $A_{\mu\alpha}(\mathbb{X})$ and a scalar quantity $\phi(\mathbb{X})$ have been introduced. $B_{\mu\nu}$ is a $D\times D$ anti-symmetric matrix. The indices $M,N,\ldots=1,\ldots,2D{+}n$ are raised and lowered by the $O(D,D+n)$ invariant metric,

$$\eta_{MN}=\begin{pmatrix} 0 & \delta^\mu{}_\nu & 0 \\ \delta_\mu{}^\nu & 0 & 0 \\ 0 & 0 & \kappa^{\alpha\beta} \end{pmatrix}, \tag{5}$$

and its inverse η^{MN}. Note that the generalized metric $\mathcal{H}_{MN}$ is an element of $O(D,D+n)$. The action (2) is manifestly invariant under the $O(D,D+n)$ transformation;

$$\mathcal{H}'^{MN}(\mathbb{X}')=O^M{}_PO^N{}_Q\mathcal{H}^{PQ}(\mathbb{X}),\quad d'(\mathbb{X}')=d(\mathbb{X}),\quad \mathbb{X}'^M=O^M{}_N\mathbb{X}^N,\quad O\in O(D,D+n). \tag{6}$$

Now we gauge a subgroup G of $O(D,D+n)$ and break it down to $O(D,D)\times G$. This is done by introducing a constant flux $F^M{}_{NK}$ such as

$$F^M{}_{NK}=\begin{cases} F_\alpha{}^{\beta\gamma} & \text{if } (M,N,K)=(\alpha,\beta,\gamma) \\ 0 & \text{else} \end{cases}. \tag{7}$$

Here $F_\alpha{}^{\beta\gamma}$ is the structure constant for the gauge group G whose dimension is $\dim G=n$. The constant $F^M{}_{NK}$ must satisfy the following relations;

$$F^{(M}{}_{PK}\eta^{N)K}=0,\qquad F_{MNK}=F_{[MNK]},\qquad F^M{}_{N[K}F^N{}_{LP]}=0. \tag{8}$$

In order to keep the gauge invariance, the action (2) is deformed such as [10]

$$S=S_0+\delta S, \tag{9}$$

where

$$\delta S=\int \mathrm{d}^{2D+n}\mathbb{X}\, e^{-2d}\Big(-\frac{1}{2}F^M{}_{NK}\mathcal{H}^{NP}\mathcal{H}^{KQ}\partial_P\mathcal{H}_{QM}-\frac{1}{12}F^M{}_{KP}F^N{}_{LQ}\mathcal{H}_{MN}\mathcal{H}^{KL}\mathcal{H}^{PQ}$$
$$-\frac{1}{4}F^M{}_{NK}F^N{}_{ML}\mathcal{H}^{KL}-\frac{1}{6}F^{MNK}F_{MNK}\Big). \tag{10}$$

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The action (9) is invariant under the following gauge transformation;

$$\delta_\Xi \mathcal{H}^{MN} = \Xi^P \partial_P \mathcal{H}^{MN} + (\partial^M \Xi_P - \partial_P \Xi^M)\mathcal{H}^{PN} + (\partial^N \Xi_P - \xi_P \Xi^N)\mathcal{H}^{MP} - 2\Xi^P F^{(M}{}_{PK}\mathcal{H}^{N)K},$$
$$\partial_\Xi d = \Xi^M \partial_M d - \frac{1}{2}\partial_M \Xi^M, \tag{11}$$

provided that the following physical conditions are satisfied;

$$\eta^{MN}\partial_M \partial_N * = 0, \qquad \eta^{MN}\partial_M * \partial_N * = 0, \qquad F^M{}_{NK}\partial_M * = 0. \tag{12}$$

Here $*$ are all the quantities in DFT including the generalized metric, the generalized dilaton and the gauge parameters Ξ^M. The first condition above is just the level matching condition of closed strings and the second is known as the strong constraint in the context of the ordinary $O(D, D)$ DFT. We call these the physical conditions. The last one is specific to the gauged version of DFT and we call it the gauge condition in the following.

The gauge transformation (11) is closed under the conditions (12), namely, for an arbitrary $O(D, D + n)$ vector V^M, we have

$$[\delta_{\Xi_1}, \delta_{\Xi_2}]V^M = \delta_{[\Xi_1,\Xi_2]_F} V^M, \tag{13}$$

where the left-hand side is the commutator of δ_{Ξ_1} and δ_{Ξ_2}. In the right-hand side, we have defined the twisted C-bracket;

$$([\Xi_1, \Xi_2]_F)^M = \Xi_1^K \partial_K \Xi_2^M - \Xi_2^K \partial_K \Xi_1^M - \frac{1}{2}\eta^{MN}\eta_{KL}(\Xi_1^K \partial_N \Xi_2^L - \Xi_2^K \partial_N \Xi_1^L) + \Xi_2^N \Xi_1^K F^M{}_{NK}. \tag{14}$$

Note that the conditions (12) are trivially solved by quantities that depend only on x^μ. In this case, the action (9) reduces to that of a gauged supergravity in D dimensions. Among other things, when $D = 10$ and $n = 496$ and G is $SO(32)$ or $E_8 \times E_8$, the theory reduces to the heterotic supergravities in ten dimensions.

3. Geometrical realization of $(2D + n)$-dimensional doubled space

In this section, we consider a geometrical realization of $(2D + n)$ space $\mathcal{M}_{2D+n}$. Then, we rewrite the twisted C-bracket geometrically.

First, we assume that a $(2D + n)$-dimentional manifold $\mathcal{M}_{2D+n}$ with the (psuedo-)Riemannian metric η_{MN}. We introduce the η_{MN} as the $O(D, D + n)$ invariant metric (5). The η_{MN} defines an endmorphism $\mathcal{P} : \mathcal{M}_{2D+n} \to \mathcal{M}_{2D+n}$ which is the integrable product structure. The endmorphism satisfies $\mathcal{P}^2 = 1$, therefore $P = \pm 1$. The $\mathcal{P}$ decomposes $T\mathcal{M}_{2D+n}$ into a rank n distoribution $\bar{L}$ and a rank $2D$ distoribution $\mathcal{D}$ on $\mathcal{M}_{2D+n}$. We can define the coordinate on $\mathcal{M}_{2D+n}$ as $(X^M, \bar{x}^\alpha)$, $M = 1, \ldots, 2D$, $\alpha = 1, \ldots, n$ because the distributions are integrable. We suppose that the $\mathcal{D}$ has a para-Hermitian structure of the $O(D, D)$ DFT (see [4]). Then, the coordinate of the base space $\mathcal{M}_{2D+n}$ is decomposed as $(x^\mu, \tilde{x}_\mu, \bar{x}_\alpha)$. Therefore, we have a decomposition

$$T\mathcal{M}_{2D+n} = L \oplus \tilde{L} \oplus \bar{L}. \tag{15}$$

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Then a vector Ξ on $\mathcal{M}_{2D+n}$ is decomposed as $\Xi = \Xi^M \partial_M = X^\mu \partial_\mu + \xi_\mu \tilde{\partial}^\mu + a_\alpha \bar{\partial}^\alpha$.

Given this decomposition, we introduce operators on the bundles $L, \tilde{L}$and $\bar{L}$. The Lie brackets in each subbundle are defined by

$$\begin{aligned} [X_1, X_2]_L &= (X_1^\nu \partial_\nu X_2^\mu - X_2^\nu \partial_\nu X_1^\mu)\partial_\mu, && X_1, X_2 \in \Gamma(L), \\ [\xi_1, \xi_2]_{\tilde{L}} &= (\xi_{1\nu} \tilde{\partial}^\nu \xi_{2\mu} - \xi_{2\nu} \tilde{\partial}^\nu \xi_{1\mu})\tilde{\partial}^\mu, && \xi_1, \xi_2 \in \Gamma(\tilde{L}), \\ [a_1, a_2]_{\bar{L}} &= (a_{1\beta} \bar{\partial}^\beta a_{2\alpha} - a_{2\beta} \bar{\partial}^\beta a_{1\alpha})\bar{\partial}^\alpha, && a_1, a_2 \in \Gamma(\bar{L}). \end{aligned} \tag{16}$$

Here $\kappa_{\alpha\beta}$ is an invertible matrix. We also define the exterior derivatives for function $f \in C^\infty(\mathcal{M}_{2D+n})$ as

$$\mathrm{d}f = \tilde{\partial}^\mu f \partial_\mu \in \Gamma(L), \qquad \tilde{\mathrm{d}}f = \partial_\mu f \tilde{\partial}^\mu \in \Gamma(\tilde{L}), \qquad \bar{\mathrm{d}}f = \kappa_{\alpha\beta} \bar{\partial}^\alpha f \bar{\partial}^\beta \in \Gamma(\bar{L}). \tag{17}$$

We can also define the totally anti-symmetric products for each bundle as follows,

$$\begin{aligned} X &= \frac{1}{p!} X^{\mu_1 \cdots \mu_p} \partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_p} \in \Gamma(L^{\wedge p}), \\ \xi &= \frac{1}{p!} \xi_{\mu_1 \cdots \mu_p} \tilde{\partial}^{\mu_1} \wedge \cdots \wedge \tilde{\partial}^{\mu_p} \in \Gamma(\tilde{L}^{\wedge p}), \\ a &= \frac{1}{p!} a_{\alpha_1 \cdots \alpha_p} \bar{\partial}^{\alpha_1} \wedge \cdots \wedge \bar{\partial}^{\alpha_p} \in \Gamma(\bar{L}^{\wedge p}), \end{aligned} \tag{18}$$

where $1 \leq p \leq D$ for $L, \tilde{L}$ and $1 \leq p \leq n$ for $\bar{L}$. They are defined in the subspaces in $T\mathcal{M}_{2D+n}$. The exterior derivatives $\mathrm{d}, \tilde{\mathrm{d}}, \bar{\mathrm{d}}$ act on a p-vector $\Xi^{(p)} = \Xi^{M_1 \cdots M_p} \partial_{M_1} \wedge \cdots \wedge \partial_{M_p}$ in $T\mathcal{M}_{2D+n}$ as

$$\begin{aligned} \mathrm{d}\Xi^{(p)} &= \frac{1}{p!} \partial_\mu \Xi^{M_1 \cdots M_p}(\mathbb{X}) \tilde{\partial}^\mu \wedge \partial_{M_1} \wedge \cdots \wedge \partial_{M_p}, \\ \tilde{\mathrm{d}}\Xi^{(p)} &= \frac{1}{p!} \tilde{\partial}^\mu \Xi^{M_1 \cdots M_p}(\mathbb{X}) \partial_\mu \wedge \partial_{M_1} \wedge \cdots \wedge \partial_{M_p}, \\ \bar{\mathrm{d}}\Xi^{(p)} &= \frac{1}{p!} \kappa_{\alpha\beta} \bar{\partial}^\beta \Xi^{M_1 \cdots M_p}(\mathbb{X}) \bar{\partial}^\alpha \wedge \partial_{M_1} \wedge \cdots \wedge \partial_{M_p}. \end{aligned} \tag{19}$$

These are defined by maps that increase the rank of each wedge product. For example, when $\Xi^{(p)} = X^{(p)} = X^{\mu_1 \cdots \mu_p} \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p} \in \Gamma(L^{\wedge p})$, we have

$$\begin{aligned} \mathrm{d}X &= \partial_\mu X^{\mu_1 \cdots \mu_p} \tilde{\partial}^\mu \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p}, \\ \tilde{\mathrm{d}}X &= \tilde{\partial}^\mu X^{\mu_1 \cdots \mu_p} \partial_\mu \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p}, \\ \bar{\mathrm{d}}X &= \kappa_{\alpha\beta} \bar{\partial}^\beta X^{\mu_1 \cdots \mu_p} \bar{\partial}^\alpha \wedge \partial_{\mu_1} \wedge \cdots \wedge \partial_{\mu_p}. \end{aligned} \tag{20}$$

By the definition, the $\mathrm{d}, \tilde{\mathrm{d}}$ and $\bar{\mathrm{d}}$ satisfy the properties,

$$\begin{aligned} &\mathrm{d}^2 = \tilde{\mathrm{d}}^2 = \bar{\mathrm{d}}^2 = 0, \\ &\mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d} = \mathrm{d}\bar{\mathrm{d}} + \bar{\mathrm{d}}\mathrm{d} = \tilde{\mathrm{d}}\bar{\mathrm{d}} + \bar{\mathrm{d}}\tilde{\mathrm{d}} = 0. \end{aligned} \tag{21}$$

These are a generalization of para-Dolbeault operators on the para-Hermitian geometry. We next define "inner products" by

$$\begin{aligned} \langle X, \xi \rangle &= X^\mu \xi_\mu = X(\xi) = \xi(X), \quad X \in \Gamma(L),\ \xi \in \Gamma(\tilde{L}), \\ \langle a_1, a_2 \rangle &= \kappa_{\alpha\beta} a_1^\alpha a_2^\beta = a_1(a_2) = a_2(a_1), \quad a_1, a_2 \in \Gamma(\bar{L}). \end{aligned} \tag{22}$$

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The other combinations vanish. They are maps that decrease the rank in each wedge products. They act, for example, like

$$\begin{aligned}\tilde{\iota}_\xi X^{(p)} &= \frac{1}{(p-1)!}\xi_\mu X^{\mu\mu_2\cdots\mu_p}\partial_{\mu_2}\wedge\cdots\partial_{\mu_p},\\ \iota_X\xi^{(p)} &= \frac{1}{(p-1)!}X^\mu\xi_{\mu\mu_2\cdots\mu_p}\tilde{\partial}^{\mu_2}\wedge\cdots\wedge\tilde{\partial}^{\mu_p},\\ \bar{\iota}_{a_1}a_2^{(p)} &= \frac{1}{(p-1)!}\kappa^{\alpha\alpha_1}a_{1\alpha}a_{2\alpha_1\alpha_2\cdots\alpha_p}\bar{\partial}^{\alpha_2}\wedge\cdots\wedge\bar{\partial}^{\alpha_p}.\end{aligned} \tag{23}$$

Note that all the interior products anti-commute with all the exterior derivatives. Then, we have the following properties;

$$\begin{aligned}&\tilde{\iota}^2 = \iota^2 = \bar{\iota}^2 = 0,\\ &\tilde{\iota}\,\iota + \iota\,\tilde{\iota} = \iota\,\bar{\iota} + \bar{\iota}\,\iota = \bar{\iota}\,\tilde{\iota} + \tilde{\iota}\,\bar{\iota} = 0.\end{aligned} \tag{24}$$

These are economically written as

$$\mathbf{i}_{\partial_M}\partial_N = \eta_{MN} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & \kappa^{\alpha\beta}\end{pmatrix}, \tag{25}$$

where $\mathbf{i} = (\tilde{\iota}, \iota, \bar{\iota})$. These inner products determine the relevance of each bundle. L and $\tilde{L}$ are dual vector spaces each other and $\bar{L}$ is seld-dual. Then, we can introduce the inner products $\iota_\xi, \iota_X, \iota_a$ as the anti-derivatives of the d, $\tilde{\mathrm{d}}$ and $\bar{\mathrm{d}}$. Using these operators, we can define the Lie derivatives,

$$\begin{aligned}&\mathcal{L}_X = \iota_X\mathrm{d} + \mathrm{d}\iota_X, \qquad \tilde{\mathcal{L}}_\xi = \tilde{\iota}_\xi\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\tilde{\iota}_\xi, \qquad \bar{\mathcal{L}}_a = \bar{\iota}_a\bar{\mathrm{d}} + \bar{\mathrm{d}}\bar{\iota}_a,\\ &X \in \Gamma(L), \quad \xi \in \Gamma(\tilde{L}), \quad a \in \Gamma(\bar{L}).\end{aligned} \tag{26}$$

We can also "Lie-like derivatives" as follows,

$$\bar{\mathfrak{L}}_X = \iota_X\bar{\mathrm{d}} + \bar{\mathrm{d}}\iota_X, \qquad \bar{\mathfrak{L}}_\xi = \tilde{\iota}_\xi\bar{\mathrm{d}} + \bar{\mathrm{d}}\tilde{\iota}_\xi, \qquad \mathfrak{L}_a = \bar{\iota}_a\mathrm{d} + \mathrm{d}\bar{\iota}_a, \qquad \tilde{\mathfrak{L}}_a = \bar{\iota}_a\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\bar{\iota}_a. \tag{27}$$

Using these operators and Lie(-like) derivatives, we can rewrite the twisted C-bracket (14) geometrically as follows,

$$\begin{aligned}&[\Xi_1, \Xi_2]_F\\ &= [X_1, X_2]_L + \big(\tilde{\mathcal{L}}_{\xi_1}X_2 - \tilde{\mathcal{L}}_{\xi_2}X_1\big) + \big(\bar{\mathcal{L}}_{a_1}X_2 - \bar{\mathcal{L}}_{a_2}X_1\big) + \frac{1}{2}\big(\tilde{\mathfrak{L}}_{a_1}a_2 - \tilde{\mathfrak{L}}_{a_2}a_1\big) + \frac{1}{2}\tilde{\mathrm{d}}\big(\iota_{X_1}\xi_2 - \iota_{X_2}\xi_1\big)\\ &+ [\xi_1, \xi_2]_{\tilde{L}} + \big(\mathcal{L}_{X_1}\xi_2 - \mathcal{L}_{X_2}\xi_1\big) + \big(\bar{\mathcal{L}}_{a_1}\xi_2 - \bar{\mathcal{L}}_{a_2}\xi_1\big) + \frac{1}{2}\big(\mathfrak{L}_{a_1}a_2 - \mathfrak{L}_{a_2}a_1\big) - \frac{1}{2}\mathrm{d}\big(\iota_{X_1}\xi_2 - \iota_{X_2}\xi_1\big)\\ &+ \frac{1}{2}[a_1, a_2]_{\bar{L}} + \frac{1}{2}\big(\bar{\mathcal{L}}_{a_1}a_2 - \bar{\mathcal{L}}_{a_2}a_1\big) + \big(\mathcal{L}_{X_1}a_2 - \mathcal{L}_{X_2}a_1\big) + \big(\tilde{\mathcal{L}}_{\xi_1}a_2 - \tilde{\mathcal{L}}_{\xi_2}a_1\big)\\ &+ \frac{1}{2}\big(\bar{\mathfrak{L}}_{X_1}\xi_2 - \bar{\mathfrak{L}}_{X_2}\xi_1\big) + \frac{1}{2}\big(\bar{\mathfrak{L}}_{\xi_1}X_2 - \bar{\mathfrak{L}}_{\xi_2}X_1\big) + \mathbf{i}_{\Xi_2}\mathbf{i}_{\Xi_1}F.\end{aligned} \tag{28}$$

This bracket includes the C-bracket that appeared in $O(D,D)$ DFT. We expect that the algebroid defined by the twisted C-bracket has the extension of the doubled structure. The first line of the right-hand side is in $\Gamma(L)$. Similarly, the second line is in $\Gamma(\tilde{L})$, and the third and fourth line is in $\Gamma(\bar{L})$. This strongly suggests the tripled structure. We discuss algebroid structures related to the twisted C-bracket in the next section.

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4. Vaisman algebroid by the twisted C-bracket

In this section, we introduce some algebroid structures. Then, we discuss the algebroid structure defined by the twisted C-bracket and the Drinfel'd double-like structures.

Lie algebroid

A Lie algebroid is a most fundamental algebroid structure. This is defined by a vector bundle E on the manifold M, an anchor map $\rho : E \to TM$, and a Lie algebroid bracket $[\cdot,\cdot] : \Gamma(E)\times\Gamma(E) \to \Gamma(E)$satisfying the Jacobi identity. We can say that this is defined as a generalization of Lie algebras. The structures ρ,$[\cdot,\cdot]$ and E are satisfies following properties.

$$[X_1, fX_2] = \rho(X_1)[f]X_2 + f[X_1, X_2], \quad X_i\ (i = 1,2) \in \Gamma(E), \tag{29}$$

$$\rho([X_1, X_2]) = [\rho(X_1), \rho(X_2)]. \tag{30}$$

If we assume a bundle E as the L (or $\tilde{L}, \bar{L}$), the Lie bracket $[\cdot,\cdot]_L$ (or $[\cdot,\cdot]_{\tilde{L}}, [\cdot,\cdot]_{\bar{L}}$) satisfies the Jacobi identity. The anchor $\rho : L \to T\mathcal{M}_{2D+n}$ (or $\tilde{\rho} : \tilde{L} \to T\mathcal{M}_{2D+n}, \bar{\rho} : \bar{L} \to T\mathcal{M}_{2D+n}$) is defined with the exterior derivatives (17) as

$$\begin{aligned} &\mathrm{d}f(X) = \rho(X)\cdot f, \qquad \tilde{\mathrm{d}}f(\xi) = \tilde{\rho}(\xi)\cdot f, \qquad \bar{\mathrm{d}}f(a) = \bar{\rho}(a)f, \\ &X \in \Gamma(L), \quad \xi \in \Gamma(\tilde{L}), \quad a \in \Gamma(\bar{L}). \end{aligned} \tag{31}$$

Then, the set $(L, \rho_L, [\cdot,\cdot]_L)$ becomes the Lie algebroid. The same applies to $\tilde{L}$ and $\bar{L}$. The term includes the structure constant F can be interpreted as a twisted term.

Vaisman algebroid

A Vaisman algebroid is defined by a vector bundle $\mathcal{V}$ on a manifold M, an anchor map $\boldsymbol{\rho}$: $\mathcal{V} \to TM$, a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ and a Vaisman bracket $[\cdot,\cdot]_{\mathrm{V}} : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$. If the $(\mathcal{V}, \boldsymbol{\rho}, (\cdot,\cdot), [\cdot,\cdot]_{\mathrm{V}})$ satisfies the following two axioms, this quadruple becomes the Vaisman algebroid.

Axiom V1. $[\Xi_1, f\Xi_2]_V = f[\Xi_1, \Xi_2]_V + (\boldsymbol{\rho}(\Xi_1)\cdot f)\Xi_2 - (\Xi_1, \Xi_2)\mathcal{D}f.$

Axiom V2. $\rho_V(\Xi_1)\cdot(\Xi_2, \Xi_3) = ([\Xi_1, \Xi_2]_V + \mathcal{D}(\Xi_1, \Xi_2), \Xi_3) + (\Xi_2, [\Xi_1, \Xi_3]_V + \mathcal{D}(\Xi_1, \Xi_3)).$

In the following, we check the twisted C-bracket (28) defines a Vaisman algebroid in $\mathcal{M}_{2D+n}$. Since we have seen the tripled structure in the twisted C-bracket, we deduce the bilinear form as

$$\boldsymbol{(}\Xi_1, \Xi_2\boldsymbol{)} = \frac{1}{2}\big(\tilde{\iota}_{\xi_1}X_2 + \iota_{X_1}\xi_2 + \bar{\iota}_{a_1}a_2\big). \tag{32}$$

and the anchor map as $\boldsymbol{\rho} = \rho + \tilde{\rho} + \bar{\rho}$ and the derivation as $\mathcal{D} = \mathrm{d} + \tilde{\mathrm{d}} + \bar{\mathrm{d}}$. Then, we check the axioms with the quadraple $(L \oplus \tilde{L} \oplus \bar{L}, \boldsymbol{(}\cdot,\cdot\boldsymbol{)}, [\cdot,\cdot]_F, \boldsymbol{\rho})$.

We first check the Axiom V1.

$$[\Xi_1, f\Xi_2]_F = f[\Xi_1, \Xi_2]_F + \big(\boldsymbol{\rho}(\Xi_1)\cdot f\big)\Xi_2 - \boldsymbol{(}\Xi_1, \Xi_2\boldsymbol{)}\mathcal{D}f. \tag{33}$$

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The $(2D+n)$-dim vectors Ξ_i $(i=1,2)$ are given by $\Xi_i = X_i + \xi_i + a_i$ $(i=1,2)$. The left hand side in (33) is decomposed as

$$\begin{aligned}[\Xi_1, f\Xi_2]_F &= [X_1, fX_2]_F + [X_1, f\xi_2]_F + [X_1, fa_2]_F \\ &\quad + [\xi_1, fX_2]_F + [\xi_1, f\xi_2]_F + [\xi_1, fa_2]_F \\ &\quad + [a_1, fX_2]_F + [a_1, f\xi_2]_F + [a_1, fa_2]_F. \end{aligned} \tag{34}$$

For example, the gauge part $[a_1, fa_2]_F$ contains following terms. It is not only the Lie bracket because of Lie-like derivatives.

$$\begin{aligned}[a_1, fa_2]_F &= \frac{1}{2}[a_1, fa_2]_{\bar{L}} + \frac{1}{2}\left(\tilde{\mathfrak{L}}_{a_1}(fa_2) - \tilde{\mathfrak{L}}_{fa_2}a_1\right) + \frac{1}{2}\left(\mathfrak{L}_{a_1}(fa_2) - \mathfrak{L}_{fa_2}a_1\right) \\ &\quad + \frac{1}{2}\left(\bar{\mathcal{L}}_{a_1}(fa_2) - \bar{\mathcal{L}}_{fa_2}a_1\right) + \iota_{a_2}\iota_{a_1}F. \end{aligned} \tag{35}$$

Then we have

$$\begin{aligned}[a_1, fa_2]_F &= \frac{1}{2}f\Big([a_1, a_2]_{\bar{L}\text{-}F} + (\tilde{\mathfrak{L}}_{a_1}a_2 - \tilde{\mathfrak{L}}_{a_2}a_1) + (\mathfrak{L}_{a_1}a_2 - \mathfrak{L}_{a_2}a_1) + (\bar{\mathcal{L}}_{a_1}a_2 - \bar{\mathcal{L}}_{a_2}a_1)\Big) \\ &\quad + \frac{1}{2}\left(\left(\bar{\rho}(a_1)\cdot f\right)a_2 - \bar{\iota}_{a_2}a_1\tilde{\mathrm{d}}f - \bar{\iota}_{a_2}a_1\mathrm{d}f + \left(\bar{\rho}(a_1)\cdot f\right)a_2 - \bar{\iota}_{a_2}a_1\bar{\mathrm{d}}f\right) \\ &= f[a_1, a_2]_F + \left(\bar{\rho}(a_1)\cdot f\right)a_2 - \frac{1}{2}\bar{\iota}_{a_2}a_1\mathcal{D}f, \end{aligned} \tag{36}$$

If we repeat a similar calculation for the other eight parts in (34), we can show the relation (33). Therefore, the quadraple $(L\oplus\tilde{L}\oplus\bar{L}, (\cdot,\cdot), [\cdot,\cdot]_F, \boldsymbol{\rho})$ satisfies the Axiom V1.

Next, we check the axiom V2. We need to consider the extension of Lemma 3.2 in [6]. We introduce a scalar T_F as

$$\begin{aligned}T_F(\Xi_1, \Xi_2, \Xi_3) = &\ \frac{1}{3}\big(([\Xi_1, \Xi_2]_F, \Xi_3) + \text{c.p.}\big) \\ &= T(e_1, e_2, e_3) \\ &\quad + \frac{1}{4}\Big\{\left(\iota_{\xi_2}\iota_{a_3}\bar{\mathrm{d}}X_1 + \iota_{X_2}\iota_{a_3}\bar{\mathrm{d}}\xi_1 + \iota_{a_2}\iota_{X_3}\mathrm{d}a_1 + \iota_{a_2}\iota_{\xi_3}\tilde{\mathrm{d}}a_1 + \iota_{a_2}\iota_{a_3}\bar{\mathrm{d}}a_1\right) \\ &\qquad - \left(\iota_{\xi_3}\iota_{a_2}\bar{\mathrm{d}}X_1 + \iota_{X_3}\iota_{a_2}\bar{\mathrm{d}}\xi_1 + \iota_{a_3}\iota_{X_2}\mathrm{d}a_1 + \iota_{a_3}\iota_{\xi_2}\tilde{\mathrm{d}}a_1\right) + \text{c.p.}\Big\} \\ &\quad + \frac{1}{2}\iota_{\Xi_3}\iota_{\Xi_2}\iota_{\Xi_1}F. \end{aligned} \tag{37}$$

After some calculations, we obtain the following relation.

$$\big([\Xi_1, \Xi_2]_F, \Xi_3\big) = T_F(\Xi_1, \Xi_2, \Xi_3) + \frac{1}{2}\boldsymbol{\rho}(\Xi_1)\cdot\big(\Xi_3, \Xi_2\big) - \frac{1}{2}\boldsymbol{\rho}(\Xi_2)\cdot\big(\Xi_1, \Xi_3\big) + \frac{1}{2}\mathbf{i}_{\Xi_3}\mathbf{i}_{\Xi_2}\mathbf{i}_{\Xi_1}F. \tag{38}$$

By summing up the equation (38) after the label of 2 and 3 are replaced, we obtain

$$\begin{aligned}\boldsymbol{\rho}(\Xi_1)\cdot\big(\Xi_2, \Xi_3\big) &= \big([\Xi_1, \Xi_2]_F, \Xi_3\big) + \big([\Xi_1, \Xi_3]_F, \Xi_2\big) \\ &\quad + \frac{1}{2}\boldsymbol{\rho}(\Xi_2)\cdot\big(\Xi_1, \Xi_3\big) + \frac{1}{2}\boldsymbol{\rho}(\Xi_3)\cdot\big(\Xi_1, \Xi_2\big). \end{aligned} \tag{39}$$

Since we have $\boldsymbol{\rho} = \rho_L + \rho_{\tilde{L}} + \rho_{\bar{L}}$, finally we obtain the following relation.

$$\boldsymbol{\rho}(\Xi_1)\cdot\big(\Xi_2, \Xi_3\big) = \big([\Xi_1, \Xi_2]_F + \mathcal{D}(\Xi_1, \Xi_2), \Xi_3\big) + \big([\Xi_1, \Xi_3]_F + \mathcal{D}(\Xi_1, \Xi_3), \Xi_2\big). \tag{40}$$

This is just the Axiom V2. The quadraple $(L\oplus\tilde{L}\oplus\bar{L}, (\cdot,\cdot), [\cdot,\cdot]_F, \boldsymbol{\rho})$ satisfies the Axiom V2. Therefore, the $(L\oplus\tilde{L}\oplus\bar{L}, (\cdot,\cdot), [\cdot,\cdot]_F, \boldsymbol{\rho})$ defines the Vaisman algebroid with the tripled structure.

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5. Conclusion

In this proceeding, we discussed the extended doubled structure of algebroids. This is related to the gauge symmetry in the gauged DFT which is defined by the twisted C-bracket (14). First, we consider the geometrical realization of $(2D + n)$ space and we rewrite the twisted C-bracket as geometrical language (28). This gives explicit expression of the twisted C-bracket in the Drinfel'd double-like (tripled) form that is different from the one for the C-bracket in the ordinary DFT. There are not only ordinary operators and Lie derivatives but also "Lie-like derivatives". Next, we check the definition of the Vaisman algebroid with the twisted C-bracket (please see our paper [12] for details of the proof). Finally, we can show that the twisted C-bracket also defines the Vaisman algebroid. It has the tripled structure $L \oplus \tilde{L} \oplus \bar{L}$ which is an extension of the doubled structure with the C-bracket.

Based on this result, we can consider the heterotic case of the Poisson-Lie T-duality. In general, the Drinfel'd double structure is needed to treat this duality. We discussed the tripled structure on the gauged DFT as a generalization of the Drinfel'd double. I expect that this will be useful to consider the heterotic Poisson-Lie T-duality

We can also consider other algebroid structures with the twisted C-bracket for example Courant algebroid. Partially discussed in [13] and tripred case is in progress. We can also discuss the finite gauge transformation in gauged DFT. This is to consider the "integration" of the Vaisman algebroid with the twisted C-bracket. The relationship with the pre-Rackoid structure is recently discussed in [14].

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