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The three-loop single-mass heavy flavor corrections to deep-inelastic scattering

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J. Ablinger,[a,b] **A. Behring,**[c] **J. Blümlein,**[d,e,*] **A. De Freitas,**[a,d] **A. von Manteuffel,**[f] **C. Schneider**[a] **and K. Schönwald**[g]

[a] *Johannes Kepler University, Research Institute for Symbolic Computation (RISC),*
Altenberger Straße 69, A-4040, Linz, Austria
[b] *Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Straße 69, A-4040 Linz Austria*
[c] *Theoretical Physics Department, CERN, 1211 Geneva 23, Switzerland*
[d] *Deutsches Elektronen-Synchrotron DESY, Platanenallee 6, 15738 Zeuthen, Germany*
[e] *Institut für Theoretische Physik III, IV, TU Dortmund, Otto-Hahn Straße 4,*
44227 Dortmund, Germany
[f] *Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany*
[g] *Physik-Institut, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland*

E-mail: `Johannes.Bluemlein@desy.de`

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We report on the status of the calculation of the massive Wilson coefficients and operator matrix elements for deep-inelastic scatterung to three-loop order. We discuss both the unpolarized and the polarized case, for which all the single-mass and nearly all two-mass contributions have been calculated. Numerical results on the structure function $F_2(x, Q^2)$ are presented. In the polarized case, we work in the Larin scheme and refer to parton distirbution functions in this scheme. Furthermore, results on the three-loop variable flavor number scheme are presented.

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[*] Speaker

1. Introduction

The massive operator matrix elements (OMEs) [1, 2] and massive Wilson coefficient of deep-inelastic scattering [3–6] in the asymptotic region $Q^2 \gg m^2$, with m the heavy quark mass, have been calculated in the single-mass case to three-loop order in Quantum Chromodynamics (QCD). To two-loop order these OMEs have been computed in Refs. [1, 7–20]. At three-loop order seven unpolarized massive OMEs, $A_{qq,Q}^{\rm NS}, A_{Qq}^{\rm PS}, A_{qq,Q}^{\rm PS}, A_{Qg}, A_{qg,Q}, A_{gq,Q}$ and $A_{gg,Q}$, and the corresponding polarized OMEs contribute. First a series of Mellin moments was calculated in Ref. [21]. The computation of theses functions for general values of Mellin-N followed in Refs. [15, 16, 19, 22–31]. Two-mass corrections contribute starting from two-loop order, i.e. at next-to-leading-order (NLO), cf. [32], as factorizable terms. From three-loop order onward also irreducible two-mass terms contribute, cf. Refs. [2, 33–36]. The last missing term of this class will be published soon [37]. Also, for charged current structure functions a series of heavy-flavor corrections was calculated [38–41]. The massive Wilson coefficients depend on the massless three-loop unpolarized Wilson coefficients [42, 43] and the polarized ones [43]. The evolution of the massless parton densities to three-loop order depend on the unpolarized [25, 43–56] and polarized [57–59] three-loop anomalous dimensions.

The technical aspects of the calculation of these massive OMEs consist of a series of standard steps, described e.g. in Ref. [30]. The integration-by-parts reduction has been performed using `Reduze 2` [60, 61]. We used also more special analytic methods, such as summation and guessing methods applied to a very large number of moments [62–69], special higher transcendental function treatment of different kind [70–87], differential equation methods for first-order-factorizing systems [88] and non-first-order-factorizing systems [89, 90], including (general) semi-analytic solutions [91, 92]. In our calculations the use of the packages `Sigma` [62–64], `HarmonicSums` [70–87], `OreSys` [93–95], and others [96, 97] played an important role.

The present note is organized as follows. In Section 2 we present the three-loop single-mass contributions at next-to-next-to-leading order (NNLO) to the structure function $F_2(x,Q^2)$ for the first time. It is an important ingredient for precision QCD fits of the deep-inelastic World data to determine the strong coupling constant $a_s = \alpha_s/(4\pi)$, cf. [98–101] and of the parton distribution functions (PDFs), [102]. To obtain the same results for the polarized structure functions one needs also PDFs evolved in the Larin scheme. This is discussed in Section 3. The three-loop massive OMEs also allow one to derive the matching relations in the variable flavor number scheme (VFNS) at three-loop order, which is presented in Section 4. Section 5 contains the conclusions.

2. The Single-Mass Heavy-Flavor Contributions to $F_2(x,Q^2)$

The current results on the three–loop massive OMEs allow us to compute the structure function $F_2(x,Q^2)$, including the massless and single-mass heavy-flavor corrections due to c- and b-quarks at large enough scales Q^2. The massive OMEs and massive asymptotic Wilson coefficients are calculated for quark masses in the on-shell scheme, $m_c = 1.59$ GeV, [104], and $m_b = 4.78$ GeV [105]. It has been shown in Ref. [1] that the criterion $Q^2 \gg m^2$, for which the asymptotic structure function represents the full structure function $F_2(x, Q^2)$ at the 1%-level is fulfilled by $Q^2/m^2 \geq 10$, i.e. for $Q^2 \geq 25$ GeV^2 in the case of charm at NLO.[1]

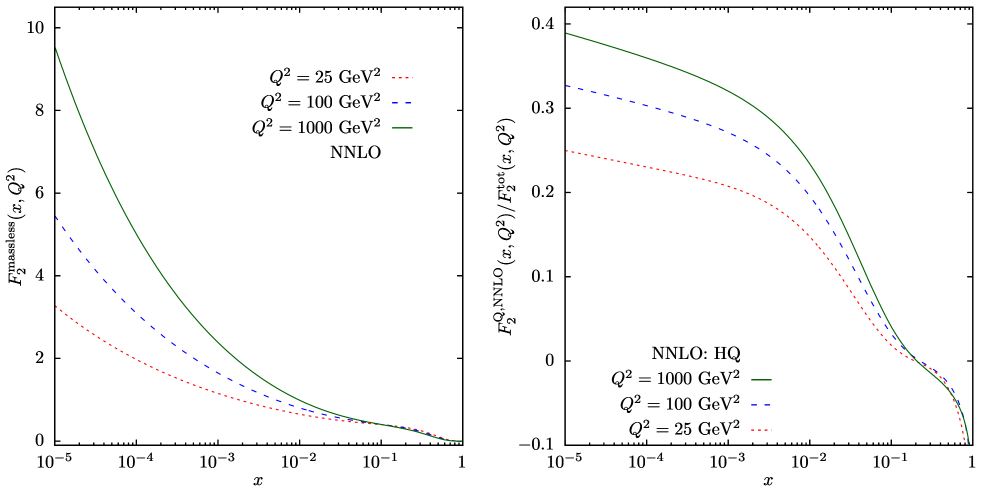

Figure 1: Left panel: the massless contributions to the structure function $F_2(x, Q^2)$ at NNLO using the PDFs of Ref. [103]. Right panel: The ratio of the NNLO single-mass charm and bottom contributions to $F_2(x, Q^2)$ to its total value. Dotted lines: $Q^2 = 25$ GeV^2; dashed lines: $Q^2 = 100$ GeV^2; full lines: $Q^2 = 10000$ GeV^2.

In Figure 1 we present both the prediction for the massless contributions to the structure function $F_2(x, Q^2)$ as well as for the single-mass c and b-quark contributions at NNLO for a wide range in the kinematic variables Bjorken x and the virtuality Q^2. The fraction of the (virtual and real) heavy quark contributions vary from $\sim 25\%$ to 40% at $x = 10^{-4}$ for Q^2 in the range between 25 GeV^2 and 10^4 GeV^2 and the contribution falls towards large values of x. Here five different massive Wilson coefficients contribute.

Already in 1990 the massive OME A_{Qg} has been investigated in its ultimate small x limit to any order in a_s, using methods of $k_\perp$-factorization [106]. In 1995 the respective expansion term of $O(a_s^2)$ has been confirmed by expanding the complete NLO result in Ref. [1]. After 34 years we have now confirmed also the $O(a_s^3)$ term for the first time in Refs. [30, 31]. One obtains

$$a_{Qg}^{(3),x\to 0}(x) = \frac{64}{243} C_A^2 T_F \left[1312 + 135\zeta_2 - 189\zeta_3\right] \frac{\ln(x)}{x} \tag{1}$$

for the constant part of the unrenormalized three-loop OME $A_{Qg}^{(3)}$. Here $C_A = N_c, T_F = 1/2, C_F = (N_c^2 - 1)/(2N_c)$ denote the color factors, with $N_c = 3$ for QCD, and ζ_k are the values of Riemann's ζ-function at integer argument, $k \geq 2$. However, this term does not describe the small x behaviour, neither of the massive OME nor of the structure function, due to very large sub-leading small x corrections, as the numerical analysis in Ref. [31] shows. This is a quite common observation for a long list of BFKL predictions.[2] Already in Ref. [25] we have computed the pure-singlet OME $A_{Qq}^{(3),PS}$ and derived the corresponding quantity $a_{Qq}^{(3),PS,x\to 0}(x)$ and its leading small x limit. It is related to (1) through rescaling by the factor C_F/C_A, as has been found by an explicit analytic calculation now.

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[1] This criterion may be different in the case of other structure functions.

[2] For a survey, see Ref. [107].

3. Polarized Parton Distributions in the Larin Scheme

The three-loop massless [43] and massive Wilson coefficients in the polarized case were calculated in the Larin scheme [108, 109]. Currently it is not possible to construct the transformation into the $\overline{\rm MS}$ scheme at three-loop order for them. However, the polarized structure function $g_1(x, Q^2)$, as an observable, can be expressed in terms of the Wilson coefficients and parton distributions [110] in the Larin scheme. The scaling violations of the polarized massless parton densities are described in this scheme as well by using the corresponding anomalous dimensions [57, 59]. In Ref. [110] the polarized parton distribution functions have been provided in the Larin scheme up to NNLO recently.

Starting at NLO the scale evolution is different in the Larin and the $\overline{\rm MS}$ scheme. We illustrate this in Figure 2 for the ratio $r = f^{\rm Larin}/f^{\overline{\rm MS}} - 1$ at NNLO. The effects are larger for the quarkonic distributions than for the gluon distribution. In the latter case they are caused by mixing effects with the quarkonic anomalous dimensions only, since $\Delta P_{gg}^{(1,2)}$ is the same in both schemes. In the large x limit the anomalous dimensions in both schemes approach each other.

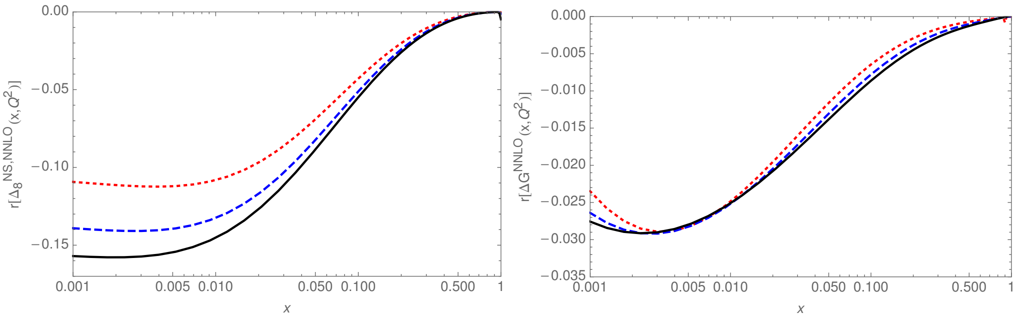

Figure 2: The relative change of the polarized parton distribution functions $\Delta_8(x, Q^2) = \Delta u(x, Q^2) + \Delta d(x, Q^2)$ and $\Delta G(x, Q^2)$ comparing the evolution in the $\overline{\rm MS}$ scheme and the Larin scheme. Dotted line: $Q^2 = 100$ GeV; dashed line: $Q^2 = 1000$ GeV; full line: $Q^2 = 10000$ GeV; from Ref. [110].

For the quark distributions the relative change in the small x region, $x \sim 0.001$, may reach 10–15%, while for the gluon distribution the corresponding effect amounts to $O(3\%)$. High precision QCD fits in the polarized case therefore require to use Larin-scheme PDFs.

4. The Single-Mass Variable Flavor Number Scheme

The single-mass three-loop massive OMEs allow one to construct the corresponding matching relations in the VFNS to three-loop order. The principal structure of the matching relations has been given in Ref. [10] and was corrected in Ref. [21]. The VFNS relates the massless parton densities with N_F massless quark flavors to the ones of $N_F + 1$ massless quark flavors in the region $Q^2 \gg m_Q^2$, where m_Q is the mass of the heavy quark becoming effectively massless. In course of this, one also obtains massive quark distributions, $f_Q(x, Q^2) + f_{\bar{Q}}(x, Q^2)$. The relations are derived from the structure functions at high scales Q^2 in the fixed flavor number scheme and are determined by the process-independent massive OMEs. It has been shown in Refs. [1, 9, 16–18] at two-loop order for the cases in which the complete heavy-quark-mass dependence is known analytically that the effective massless approach in the case of charm and bottom quarks only applies at high scales Q^2 and neither at the scales m_c^2 nor m_b^2. The new PDFs obtained in the VFNS are inserted into the massless representation of the corresponding structure functions. If one analytically expands the resulting expressions in the coupling constant a_s one obtains again the structure functions in the fixed flavor number scheme to the order one worked in. The differences in the VFNS to the result in the direct calculation are of higher order in a_s. Depending on the matching scale chosen, different size pile-up effects due to these terms are obtained.

An implementation of the three-loop matching relations will be given in Ref. [111]. In Figure 3 we illustrate the charm distribution $f_c(x, Q^2) + f_{\bar{c}}(x, Q^2)$ normalized to the singlet distribution for $N_F = 3$. The effects grow with Q^2 due to the logarithmic terms $\ln(m_Q^2/Q^2)$ in the matching relations.

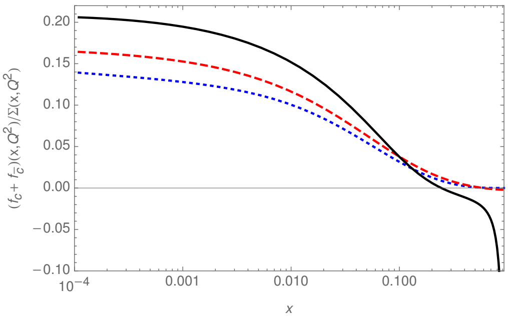

Figure 3: The distribution $f_c(x, Q^2) + f_{\bar{c}}(x, Q^2)$ normalized to $\Sigma^{\mathrm{NF}=3}(x, Q^2)$. Dotted lines: $Q^2 = 30$ GeV^2; dashed lines: $Q^2 = 100$ GeV^2; full lines: $Q^2 = 10000$ GeV^2; from Ref. [111].

5. Conclusions

We finished the calculation of all single-mass OMEs and asymptotic inclusive heavy-flavor Wilson coefficients to three-loop order and made numerical predictions for the structure function $F_2(x, Q^2)$. Very soon, also the two-mass corrections will also be finished both in the unpolarized and polarized cases. In the polarized case we worked in the Larin scheme and provided a first set of NNLO parton densities. Furthermore, the single-mass matching relations in the VFNS are now available both in the unpolarized and polarized cases.

The present results are of special importance for phenomenological predictions of the precision physics at future facilities such as the EIC [112] and LHeC [113, 114], but also for re-analysis of the HERA [115] and other World deep-inelastic data, as well as for inclusive measurements at the LHC in its high luminosity phase at CERN, and its future successor, the FCC [116]. In the flavor non-singlet case, the relations for a QCD-fit in the unpolarized and polarized cases were given in [117] in the scheme-invariant representation, which allows a direct fit of $a_s(M_Z)$ using measured input distributions at the starting scale Q_0^2.

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