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DERIVATION AND INTERPRETATION OF PARAMETERS DESCRIBING BETATRON MISMATCH AND CHROMATICITY

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Abstract

Expressions to quantify betatron mismatch and chromatic effects are frequently used in accelerator physics, but their derivations are not given in standard text books, making their interpretation difficult. First parameters describing betatron mismatch are introduced using normalization with respect to reference Twiss parameters describing a lattice. In a second step, the derivatives of these mismatch parameters with respect to the relative momentum offset are considered and lead naturally to the Montague W functions and a phase angle as computed by standard lattice programs.

INTRODUCTION

Parameters to describe betatron mismatch and chromatic effects are regularly used, but derivations are difficult to find, hampering their interpretation. In particular, the chromatic functions W are defined in reference [1], but without explanation of their motivation and meaning.

Derivations of these quantities will be given with the aim to improve the understanding of their meaning, in particular for the Montague W functions.

MISMATCH PARAMETER

Parameters to describe the mismatch between an ellipse described by reference Twiss parameters β_r, α_r and γ_r and another set of Twiss parameters β, α and γ representing a beam, can be derived making use of transfer to a normalized phase space as sketched in Fig. 1. Using reference Twiss parameters for normalization, one obtains:

$$\xi = \frac{1}{\sqrt{\beta_r}}\, q$$
$$p_\xi = \frac{\alpha_r}{\sqrt{\beta_r}}\, q + \sqrt{\beta_r}\, p_q \quad .$$

These relations can easily be inverted to:

$$q = \sqrt{\beta_r}\, \xi$$
$$p_q = \frac{1}{\sqrt{\beta_r}}\, p_\xi - \frac{\alpha_r}{\sqrt{\beta_r}}\, \xi \quad .$$

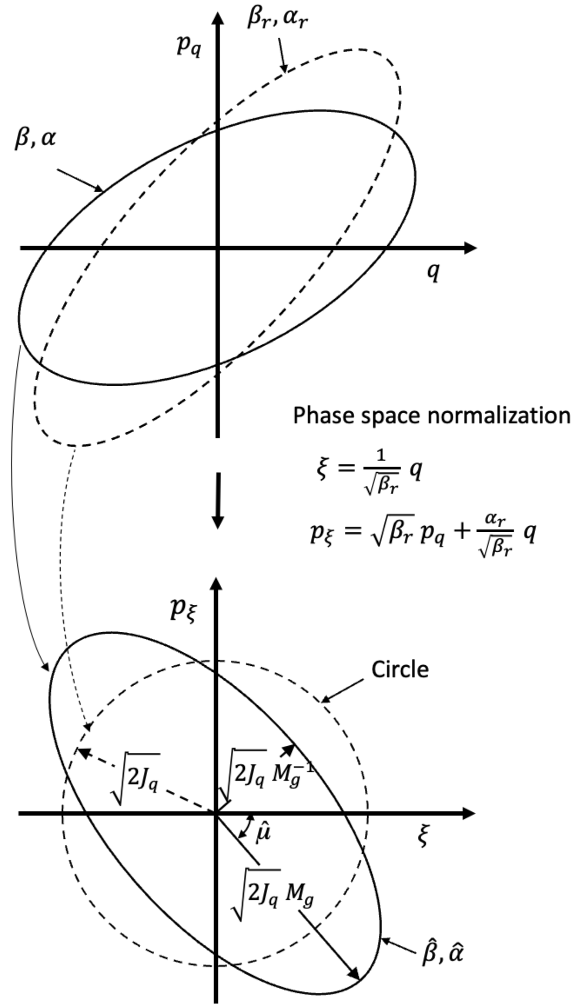

Figure 1: Transformation to normalized phase space.

The phase space ellipse for an action variable J_q in non-normalized and normalized phase space is given by:

$$\begin{aligned}
2J_q &= \gamma q^2 + 2\alpha q\, p_q + \beta p_q^2 = \\
&= \gamma \left(\sqrt{\beta_r}\,\xi\right)^2 + 2\alpha \left(\sqrt{\beta_r}\,\xi\right)\left(\frac{1}{\sqrt{\beta_r}}\, p_\xi - \frac{\alpha_r}{\sqrt{\beta_r}}\,\xi\right) \\
&\quad + \beta \left(\frac{1}{\sqrt{\beta_r}}\, p_\xi - \frac{\alpha_r}{\sqrt{\beta_r}}\,\xi\right)^2 \\
&= \underbrace{\left(\gamma\beta_r - 2\alpha\alpha_r + \frac{\beta}{\beta_r}\alpha_r^2\right)}_{=\hat{\gamma}} \xi^2 + 2\underbrace{\left(\alpha - \frac{\beta}{\beta_r}\alpha_r\right)}_{=\hat{\alpha}} \xi\, p_\xi + \underbrace{\frac{\beta}{\beta_r}}_{=\hat{\beta}} p_\xi^2
\end{aligned}$$

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with $\hat{\beta}$, $\hat{\alpha}$ and $\hat{\gamma}$ describing the mismatched Twiss ellipse in normalized phase space. Note that the reference Twiss ellipse is mapped onto a circle with radius $\sqrt{2J_q}$. The ellipse describing the beam in normalized phase space can as well be described by the parameters introduced using Fig. 1. The quantity $M_g = \sqrt{M_\beta}$ describes the overlap of the two sets of Twiss parameters considered and $\hat{\mu}$ is a phase. Using these two parameters and the action variable, the ellipse is described by the condition

$$\begin{aligned}2J_q &= \left(\xi\cos\hat{\mu} - p_\xi\sin\hat{\mu}\right)^2\frac{1}{M_g^2} + \left(\xi\sin\hat{\mu} + p_\xi\cos\hat{\mu}\right)^2 M_g^2\\ &= \left(\frac{1}{M_g^2}\cos^2\hat{\mu} + M_g^2\sin^2\hat{\mu}\right)\xi^2 + \left(M_g^2 - \frac{1}{M_g^2}\right)\sin 2\hat{\mu}\,\xi\,p_\xi\\ &\quad + \left(\frac{1}{M_g^2}\sin^2\hat{\mu} + M_g^2\cos^2\hat{\mu}\right)p_\xi^2 \quad .\end{aligned}$$

Combining the last two expressions results in:

$$\begin{aligned}\hat{\beta} &= \frac{\beta}{\beta_r} = \frac{1}{2}\left(M_\beta + \frac{1}{M_\beta}\right) + \frac{1}{2}\left(M_\beta - \frac{1}{M_\beta}\right)\cos 2\hat{\mu} \qquad (1)\\ \hat{\alpha} &= \alpha - \frac{\beta}{\beta_r}\alpha_r = \frac{1}{2}\left(M_\beta - \frac{1}{M_\beta}\right)\sin 2\hat{\mu}\\ \hat{\gamma} &= \gamma\beta_r - 2\alpha\alpha_r + \frac{\beta}{\beta_r}\alpha_r^2\\ &= \frac{1}{2}\left(M_\beta + \frac{1}{M_\beta}\right) - \frac{1}{2}\left(M_\beta - \frac{1}{M_\beta}\right)\cos 2\hat{\mu} \quad .\end{aligned}$$

The parameters M_g and M_β are computed adding the expressions for $\hat{\beta}$ and $\hat{\gamma}$ resulting in

$$2M_{rms} = \hat{\beta} + \hat{\gamma} = M_\beta + 1/M_\beta = \beta_r\gamma - 2\alpha_r\alpha + \gamma_r\beta$$

with M_{rms} the rms mismatch factor describing the relative emittance increase if a beam is injected with a mismatch and filamentation occurs in the absence of space charge or similar effects. A derivation is given in the appendix. The last equation can easily be inverted and gives:

$$\begin{aligned}M_\beta &= M_{rms} + \sqrt{M_{rms}^2 - 1} \quad \text{and}\\ M_g &= \sqrt{M_{rms} + \sqrt{M_{rms}^2 - 1}} \quad .\end{aligned}$$

The parameter $\hat{\mu}$ can be determined from

$$\frac{\hat{\alpha}}{\hat{\beta} - \hat{\gamma}} = \frac{1}{2}\tan 2\hat{\mu} \quad .$$

Note that along a beam line, or a part of a synchrotron without focusing errors, the propagation between two locations becomes a pure rotation in normalized phase space. Thus, in the absence of focusing errors M_g and M_β are constant and the change of $\hat{\mu}$ is given by the betatron phase advance. Therefore, M_g and M_β are suitable parameters to quantify the overlap of the two Twiss ellipses and the amplitude of the betatron mismatch.

CHROMATIC FUNCTIONS

To quantify chromatic effects, the derivatives of mismatch parameters w.r.t. the relative momentum offset $\delta = \Delta P/P_0$ are considered. Developing Eq. 1 in first order in δ around $\delta = 0$ and using $W = 2\,d\,M_g/d\delta = d\,M_\beta/d\delta$ yields:

$$\begin{aligned}\hat{\beta} &= 1 + \delta\,\frac{1}{\beta}\frac{d\,\beta}{d\,\delta} = 1 + \delta\,B = 1 + \delta\,W\,\cos 2\hat{\mu}\\ \hat{\alpha} &= \delta\left(\frac{d\,\alpha}{d\,\delta} - \frac{\alpha}{\beta}\frac{d\,\beta}{d\,\delta}\right) = \delta\,A = \delta\,W\sin 2\hat{\mu}\\ \hat{\gamma} &= 1 + \delta\left(\beta\frac{d\,\gamma}{d\,\delta} - \frac{d\,\alpha^2}{d\,\delta} + \frac{\alpha^2}{\beta}\frac{d\,\beta}{d\,\delta}\right)\\ &= 1 - \delta\,B = 1 - \delta\,W\cos 2\hat{\mu}\end{aligned}$$

where for the computation of $\hat{\gamma}$ the relation

$$\begin{aligned}&\beta\frac{d\,\gamma}{d\,\delta} - \frac{d\,\alpha^2}{d\,\delta} + \frac{\alpha^2}{\beta}\frac{d\,\beta}{d\,\delta} =\\ &= \frac{d}{d\,\delta}\left(\beta\gamma - \alpha^2\right) - \left(\gamma - \frac{\alpha^2}{\beta}\right)\frac{d\,\beta}{d\,\delta} = -\frac{1}{\beta}\frac{d\,\beta}{d\,\delta}\end{aligned}$$

has been used. The coefficients found:

$$\begin{aligned}B &= \frac{1}{\beta}\frac{d\,\beta}{d\,\delta} = W\cos 2\hat{\mu} \quad \text{and}\\ A &= \frac{d\,\alpha}{d\,\delta} - \frac{\alpha}{\beta}\frac{d\,\beta}{d\,\delta} = W\sin 2\hat{\mu}\end{aligned}$$

are identical to the ones given in [1,2]. From these expressions one obtains:

$$\begin{aligned}W &= \sqrt{A^2 + B^2} \qquad (2)\\ \hat{\mu} &= \frac{1}{2}\arctan\frac{A}{B}\end{aligned}$$

where W is identical to the corresponding quantity in [2] and $\hat{\mu} = \Phi/2$ with Φ defined in [2]. The quantity W defined here is a factor 2 larger than the corresponding quantity in reference [1].

Similarly to the betatron mismatch parameters, the Montague chromatic functions W are invariant for momentum independent transfer matrices. This is the case for a drift space if the trajectory of a particle is described by a coordinate q and its derivative q' with respect to the longitudinal position. This led to the conclusion in reference [1] that the chromatic functions are constant within a drift space. With this formalism, the chromatic functions change quickly inside strong quadrupoles for beams with large β.

However, if a (normalized) transverse momentum p_q is used instead of the slope q', as for example by the MAD program [2], the transfer matrix of a drift becomes momentum dependent. As a consequence, the chromatic functions W vary along drift spaces, but there are no fast changes within strong quadrupoles.

SUMMARY

Derivations of betatron mismatch parameters and chromatic functions have been given.

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The chromatic W functions defined here, as used by reference [2], can be interpreted as twice the derivative of the geometric mismatch factor M_g w.r.t. the relative momentum offset δ (around $\delta = 0$). For modern colliders, as for example the muon collider lattice presented in [3], the product of the chromatic functions W and the typical relative momentum spread around the final focus reach large values with respect to unity. Thus, higher order chromatic effects are expected to become relevant so that designing a chromatic compensation section allowing a sufficiently small vales of W to be reached may not be sufficient to obtain a working lattice.

APPENDIX: DERIVATION OF THE RMS MISMATCH FACTOR M_{rms}

Another approach to betatron mismatch is to compute the rms emittance of a beam with Twiss parameters β and α injected into a ring with Twiss parameter β_r and α_r after filamentation without space charge. The phase space coordinates of a particle with phase μ and action variable J, relative to the Twiss parameters of the injected beam, are given by:

$$\begin{aligned} q &= \sqrt{2J\beta}\cos\mu \\ p_q &= \frac{2J}{\sqrt{\beta}}\left(\sin\mu - \alpha\cos\mu\right) \quad . \end{aligned}$$

The action variable relative to the Twiss parameter of the ring is given by

$$\begin{aligned} J_r &= \frac{1}{2}\left[\left(\frac{q}{\sqrt{\beta_r}}\right)^2 + \left(\sqrt{\beta_r}p_q + \frac{\alpha_r}{\sqrt{\beta_r}}\right)^2\right] \\ &= J\left[\left(\sqrt{\frac{\beta}{\beta_r}}\cos\mu\right)^2\right. \\ &\quad \left. + \left(\sqrt{\frac{\beta_r}{\beta}}\left(\sin\mu - \alpha\cos\mu\right) + \sqrt{\frac{\beta}{\beta_r}}\alpha_r\cos\mu\right)^2\right] \end{aligned}$$

The rms emittance ε_r of the beam after filamentation is obtained by averaging over phases μ and action variables J:

$$\varepsilon_r = \langle J_r\rangle = \frac{\langle J\rangle}{2}\left(\beta\frac{1+\alpha_r^2}{\beta_r} - 2\alpha\alpha_r + \frac{1+\alpha^2}{\beta_r}\beta_r\right) = M_{rms}\varepsilon$$

with ε the initial emittance of the injected beam.

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