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TIME DEPENDENT PERTURBATION THEORY FOR ACCELERATORS

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INTRODUCTION

We start from the Hamiltonian for the linear motion of a particle in an accelerator. We transform the Hamiltonian to action-angle variables. We then add a perturbation from nonlinear fields and apply time-dependent perturbation theory.

It is found that the equations for the first order perturbation are the same as the ones obtained by B. Autin[1] through direct use of succesive approximation of the linear equations of motion. We also show how one may continue to the second order in the perturbation in the particular case of a single dipole perturbation.

1. THE LINEAR MOTION

The Hamiltonian for the linear motion is[2] :

$$\boxed{H(x,\ y,\ p_x,\ p_y;s) = (p_0 - p)\ \frac{x}{\varrho} + p_0\ \left[(\frac{1}{\varrho^2} - K_1)\ \frac{x^2}{2} + K_1\ \frac{y^2}{2}\right] + \frac{p_x^2}{2p} + \frac{p_y^2}{2p}} \tag{1}$$

The equations of motion are obtained from Hamilton's equations :

$$\begin{aligned} \frac{dx}{ds} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{p} \quad , \quad \frac{dp_x}{ds} = - \frac{\partial H}{\partial x} = -p_0\ (\frac{1}{\varrho^2} - K_1)\ x + \frac{p - p_0}{\varrho} \\ \frac{dy}{ds} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{p} \quad , \quad \frac{dp_y}{ds} = - \frac{\partial H}{\partial y} = -p_0\ K_1\ y \end{aligned} \tag{2}$$

In terms of x and y

$$\begin{aligned} x'' + \frac{p_0}{p}\ (\frac{1}{\varrho^2} - K_1)\ x &= \frac{p - p_0}{p}\ \frac{1}{\varrho} \\ y'' + \frac{p_0\ K_1}{p}\ y &= 0 \end{aligned} \tag{3}$$

Since the equation of motion is inhomogeneous, we get the general solution as a linear combination of a particular solution to the inhomogeneous equation and the general solution of the homogeneous equation. We chose as the particular solution the closed orbit $x_\varepsilon(s)$, $p_\varepsilon(s)$, $y = 0$.

We now may perform a canonical transformation generated by :

$$F_2 = (x - x_\varepsilon(s))\ (p_\beta + p_\varepsilon(s)) \tag{4}$$

The transformation is obtained from :

$$\begin{aligned} p_x &= \frac{\partial F_2}{\partial x} = p_\beta + p_\varepsilon(s) \\ x_\beta &= \frac{\partial F_2}{\partial p_\beta} = x - x_\varepsilon(s) \\ H_\beta &= H + \frac{\partial F_2}{\partial \beta} \end{aligned} \tag{5}$$

This gives :

$$\boxed{H_\beta(x_\beta,\ p_\beta,\ y,\ p_y) = p_0\ \left[(\frac{1}{\varrho^2} - K_1)\ \frac{x_\beta^2}{2} + K_1\ \frac{y^2}{2}\right] + \frac{p_\beta^2}{2\beta} + \frac{p_y^2}{2p}} \tag{6}$$

The new Hamilton's equations are :

$$\frac{dx_\beta}{ds} = \frac{\partial H_\beta}{\partial p_\beta} = \frac{p_\beta}{p} \;, \quad \frac{dp_\beta}{ds} = - \frac{\partial H_\beta}{\partial x_\beta} = - p_0 \left(\frac{1}{\varrho^2} - K_1\right) x_\beta$$

$$\frac{dy}{ds} = \frac{\partial H_\beta}{\partial p_\beta} = \frac{p_y}{p} \;, \quad \frac{dp_y}{ds} = - \frac{\partial H_\beta}{\partial_y} = - p_0 \, K_1 \, y \tag{7}$$

If we use x , p_x instead of x_β, p_β and define :

$$\boxed{K_x = \frac{p_0}{p} \left(\frac{1}{\varrho^2} - K_1\right)} \qquad \boxed{K_y = \frac{p_0}{p} K_1} \tag{8}$$

and solve for x, y. We find Hill's equation :

$$\boxed{z'' + K_z(s) \, z = 0} \;, \quad z = x \text{ or } y \tag{9}$$

Where :

K(s) + K(s + C) , C = the circumference

The periodicity of K is that of the closed orbit.

2. ACTION-ANGLE VARIABLES

The transformation to action-angle variables is done by a canonical transformation with the generating function :

$$F_1\ (z, \phi_z) = -\frac{z^2}{2\beta(s)}\left[\tan\phi_z - \frac{\beta'(s)}{2}\right] \qquad (10)$$

where the transformation is given by :

$$p_z = \frac{\partial F_1}{\partial z} = -\frac{z}{\beta_z}\left(\tan\phi - \frac{\beta_z'}{2}\cos\phi_z\right)$$

$$J_z = -\frac{\partial F_1}{\partial \phi_z} = \frac{z^2}{2\beta}\,\frac{1}{\cos^2\phi_z} \qquad (11)$$

$$H_1 = H + \frac{\partial F_1}{\partial s}$$

We find :

$$\boxed{z = \sqrt{2J_z\,\beta_z}\,\cos\phi_z} \quad \boxed{p_z = -\sqrt{\frac{2J_z}{\beta_z}}\left(\sin\phi_z - \frac{\beta_z'}{2}\cos\phi_z\right)} \quad z = x \text{ or } y \qquad (12)$$

The new Hamiltonian is :

$$\boxed{H_1(J_x, \phi_x J_y, \phi_y; s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}} \qquad (13)$$

Hamilton's equations give :

$$\frac{dJ_x}{ds} = -\frac{\partial H_1}{\partial \phi_x} = 0\ , \quad \frac{d\phi_x}{ds} = \frac{\partial H_1}{\partial J_x} = \frac{1}{\beta_x(s)}$$

$$\frac{dJ_y}{ds} = -\frac{\partial H_1}{\partial \phi_y} = 0\ , \quad \frac{d\phi_y}{ds} = \frac{\partial H_1}{\partial J_y} = \frac{1}{\beta_y(s)} \qquad (14)$$

These may be integrated :

J_x, J_y = constant

$$\boxed{\phi_z(s) = \phi_z(0) + \int_0^s \frac{ds'}{\beta(s')} \equiv \phi_z(0) + \mu_z(s)} \quad z = x \text{ or } y \qquad (15)$$

3. TIME DEPENDENT PERTURBATION THEORY (Variation of constants).

The unperturbed Hamiltonian in action angle variables is :

$$\boxed{H_0\ (J_x,\ \phi_x,\ J_y,\ \phi_y;\ s) = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)}} \tag{16}$$

with the solution :

$$\boxed{\begin{aligned} z &= \sqrt{2\beta_z J_z}\ \cos\phi_z \\ \phi_z &= \int_0^s \frac{ds'}{\beta(s')} + \phi_z(0) \end{aligned}} \quad z = x \text{ or } y \tag{17}$$

We now add a perturbation V from nonlinear fields given by :

$$V = \sum_{m,n\geq 0} V_{mn}\ x^m\ y^n \tag{18}$$

where :

$$\boxed{Re \sum_{n\geq 1} \frac{1}{n!}\left[B^{(n)} + i\ C^{(n)}\right]\ (x + iy)^n = \sum_{m,n\geq 0} V_{mn}\ x^m\ y^n} \tag{19}$$

$$B^{(n)} = \frac{1}{(B\varrho)_0}\left[\frac{\partial^{n-1} B_y}{\partial x^{n-1}}\right]_{x=y=0},\quad C^{(n)} = \frac{1}{(B\varrho)_0}\left[\frac{\partial^{n-1} B_x}{\partial x^{n-1}}\right]_{x=y=0} \tag{20}$$

The perturbed Hamiltonian is then :

$$H_1\ (J_x,\ \phi_x,\ J_y,\ \phi_y;\ s) = H_0 + V \tag{21}$$

or by using (17) :

$$\boxed{H_1 = \frac{J_x}{\beta_x(s)} + \frac{J_y}{\beta_y(s)} + \sum_{m,n\geq 0} 2^{(m+n)/2}\ V_{mn}\beta_x^{\,m/2}\ \beta_y^{\,n/2}\ J_x^{\,m/2}\ J_y^{\,n/2}\ \cos^m\phi_x\ \cos^n\phi_y} \tag{22}$$

Due to the perturbation the old constants of motion J_x, J_y, $\phi_x(0)$, $\phi_y(0)$ are now varying with time. Their time dependence are given by Hamilton's equations :

$$\frac{dJ_x}{ds} = -\frac{\partial H_1}{d\phi_x} \quad , \quad \frac{d\phi_x}{ds} = \frac{\partial H_1}{\partial J_x}$$

$$\frac{dJ_y}{ds} = -\frac{\partial H_1}{d\phi_y} \quad , \quad \frac{d\phi_y}{ds} = \frac{\partial H_1}{\partial J_y}$$

or by (17) :

$$\frac{d\phi_x(0)}{ds} = \frac{\partial H_1}{\partial J_x} - \frac{1}{\beta_x(s)} \tag{23}$$

$$\frac{d\phi_y(0)}{ds} = \frac{\partial H_1}{\partial J_y} - \frac{1}{\beta_y(s)}$$

Until now everything is exact. However the new equations are normally not possible to solve without approximations.

Guignard[3] solved the equations by Fourier expanding the Hamiltonian, which is possible because it is periodic for a circular machine. He then only kept the dominating terms for the motion close to a single resonance and was then left with a simplified Hamiltonian for which the equations could be solved. However the fourier coefficients are calculated by using the betafunctions for the linear motion which then gives first order perturbations.

Another approach is to apply time-dependent perturbation theory[4] directly. One then gets the n-th order perturbation by after derivation using the n-1 order expressions for the variables on the right hand side in (23). In particular is the first order perturbation obtained by using the solution for the linear motion (where J_x, J_y, $\phi_x(0)$, $\phi_y(0)$ are constants) in the right hand sides of (23). Generally, we have from (22), (23) :

$$\frac{dJ_{nx}}{ds} = -\frac{\partial H_1}{\partial \phi_x} = \sum_{m,n>0} m\, 2^{(m+n)/2}\, V_{mn}\, \beta_x^{m/2}\, \beta_y^{n/2}\, J_x^{m/2}\, J_y^{n/2} \sin\phi_x \cos^{m-1}\phi_x \cos^{n}\phi_y \Big|_{n-1}$$

$$\frac{dJ_{nx}}{ds} = -\frac{\partial H_1}{\partial \phi_y} = \sum_{m,n>0} n\, 2^{(m+n)/2}\, V_{mn}\, \beta_x^{m/2}\, \beta_y^{n/2}\, J_x^{m/2}\, J_y^{n/2} \cos^{m}\phi_x \sin\phi_y \cos^{n-1}\phi_y \Big|_{n-1}$$

$$\frac{d\phi_{nx}(s)}{ds} = \frac{\partial H_1}{\partial J_x} - \frac{1}{\beta_x(s)} = \sum_{m,n>0} \frac{m}{2}\, 2^{(m+n)/2}\, V_{mn}\, \beta_x^{m/2}\, \beta_y^{n/2}\, J_x^{(m-2)/2}\, J_y^{n/2} \cos^{m}\phi_y \cos^{n}\phi_y \Big|_{n-1} \tag{24}$$

$$\frac{d\phi_{nx}(s)}{ds} = \frac{\partial H_1}{\partial J_y} - \frac{1}{\beta_y(s)} = \sum_{m,n>0} \frac{n}{2}\, 2^{(m+n)/2}\, V_{mn}\, \beta_x^{m/2}\, \beta_y^{n/2}\, J_x^{m/2} J_y^{(n-2)/2} \cos^{m}\phi_x \cos^{n}\phi_y \Big|_{n-1}$$

Where I_{n-1} means that the n-1 order expressions for J and ϕ should be used. The first order perturbation is obtained by using :

$$J_{oz} = constant$$
$$\phi_{oz} = \int_0^s \frac{1}{\beta_z(s')}\, ds' + \phi_{oz}(0) = \mu_z(s) + \phi_{oz}(0) \qquad (25)$$

from the linear motion.

We will now give explicit examples of first order perturbations due to different types of magnetic multipoles.

Dipole

In this case we have :

$$V_{10} = \delta_x \;,\; V_{01} = -\delta_y \qquad (26)$$

We find for the first order perturbation :

$$\frac{dJ_{1z}}{ds} = \delta_z \sqrt{2\beta_z J_{0z}}\, \sin\phi_{0z}$$
$$\frac{d\phi_{1z}(0)}{ds} = \delta_z \sqrt{\frac{\beta_z}{2J_{0z}}}\, \cos\phi_{0z} \qquad z = x \text{ or } y \qquad (27)$$

Quadrupole

For right quadrupoles we have :

$$V_{20} = \frac{1}{2} K \;,\; V_{02} = -\frac{1}{2} K \qquad (28)$$

and we find

$$\frac{dJ_{1z}}{ds} = 2\, K\, \beta_z\, J_{0z}\, \sin\phi_{0z}\, \cos\phi_{0z} = K\, \beta_z\, J_{0z}\, \sin 2\phi_{0z}$$
$$z = x \text{ or } y \qquad (29)$$
$$\frac{d\phi_{1z}(0)}{ds} = K\, \beta_z\, \cos^2\phi_{0z} = \frac{1}{2} K\, \beta_z\, (1 + \cos 2\phi_{0z})$$

Sextupole

For right sextupoles we have :

$$V_{30} = \frac{1}{6} K' \;,\; V_{12} = -\frac{1}{2} K' \qquad (30)$$

and we get :

$$\frac{dJ_{1x}}{ds} = \sqrt{2}\,K' \left[\beta_x^{3/2} J_{0x}^{3/2} \sin\phi_{0x} \cos^2\phi_{0x} - \beta_x^{1/2} \beta_y J_{0x}^{1/2} J_{0y} \sin\phi_{0x} \cos^2\phi_{0y} \right]$$

$$= \frac{K'}{4}\sqrt{2J_{0x}\beta_x} \left[\beta_x J_{0x} \sin 3\phi_{0x} + (\beta_x J_{0x} - 2\beta_y J_{0y}) \sin\phi_{0x} + \right.$$

$$\left. - \beta_y J_{0y}(\sin(\phi_{0x} + 2\phi_{0y}) + \sin(\phi_{0y} - 2\phi_{0y}) \right]$$

$$\frac{dJ_{1y}}{ds} = -2\sqrt{2}K' \beta_x^{1/2} \beta_y J_{0x}^{1/2} J_{0y} \cos\phi_x \sin\phi_{0y} \cos\phi_{0y} =$$

$$= -\frac{K'}{2} \sqrt{2\beta_x J_{0x}} \, \beta_y J_{0y} \left[\sin(\phi_{0x} + 2\phi_{0y}) - \sin(\phi_{0x} - 2\phi_{0y}) \right]$$

(31)

$$\frac{d\phi_{1x}(0)}{ds} = \frac{1}{2} K' \sqrt{2\beta_x J_{0x}} \left[\beta_x \cos^3\phi_{0x} - \beta_y J_{0x}^{-1} J_{0y} \cos\phi_{0x} \cos^2\phi_{0y} \right] =$$

$$= \frac{1}{8} K' \sqrt{2\beta_x J_{0x}} \left[\beta_x \cos^3\phi_{0x} + (3\beta_x - 2\beta_y J_{0x}^{-1} J_{0y}) \cos\phi_{0x} + \right.$$

$$\left. - \beta_y J_{0x}^{-1} J_{0y} (\cos(\phi_{0x} + 2\phi_{0y}) + \cos(\phi_{0x} - 2\phi_{0y})) \right]$$

$$\frac{d\phi_{1y}(0)}{ds} = - K' \sqrt{2\beta_x J_x} \, \beta_y \cos\phi_{0x} . \cos^2\phi_{0y} =$$

$$= -\frac{1}{4} K' \sqrt{2\beta_x J_{0x}} \, \beta_y \left[2\cos\phi_{0x} + \cos(\phi_{0x} + 2\phi_{0y}) + \cos(\phi_{0x} - 2\phi_{0y}) \right]$$

We find the same equations for the first order perturbations as B. Autin[1] by direct use of succesive approximations of the linear equations of motion. In this paper he used a thin lens approximation for the integration of these equations. This has been extended for thick lenses by symbolic integration[5]. [2] also shows how one may find the perturbed closed orbit by taking the limit when the phase μ tends to infinity.

4. SECOND ORDER PERTURBATIONS

It is now straightforward to go to the next order in the perturbation. We just have to resubstitute the new solutions that has been found either by thin lens approximation or by symbolic integration in the right hand sides of (23). These new equations may also be solved by using thin lens approximation or by symbolic integration. Observe that, because the perturbations are assumed to be small we may use :

$$\frac{1}{1+\Delta} \approx 1 - \frac{1}{2}\Delta + \frac{3}{8}\Delta^2 + 0\,(\Delta^3)$$

$$1+\Delta \approx 1 + \frac{1}{2}\Delta - \frac{1}{8}\Delta^2 + 0\,(\Delta^3)$$

$$\sin(\phi + \Delta) = \sin\phi \cos\Delta + \cos\phi \sin\Delta \approx \sin\phi + \Delta\cos\phi - \frac{1}{2}\Delta^2 \sin\phi + O(\Delta^3)$$

$$\cos(\phi + \Delta) = \cos\phi \cos\Delta - \sin\phi \sin\Delta \approx \cos\phi + \Delta\sin\phi - \frac{1}{2}\Delta^2 \cos\phi + O(\Delta^3) \qquad (32)$$

Of cource one may also use direct numerical integration. The distorted closed orbit may then be found by taking the limit when μ tends to infinity.

As an example we calculate the second order perturbations due to a dipole.

Thin lens approximation

The perturbation is in this case concentrated in discrete locations s_j.

$$s = s_j + n\,C$$

where C is the circumference of the machine.

The perturbed Hamiltonian is then of the form :

$$H_1 = H_0 + \sum_{m,n>0} V_{mn}\, x^m y^n\, \delta(s-s_j-n\,C) \qquad (33)$$

The integration of the perturbed equations is then straightforward, however, it may require a lot of algebra.

Dipole

In the case of a single dipole perturbation we have from (27) :

$$\frac{dJ_{1z}}{ds} = \delta_z \sqrt{2\beta_z J_{0z}}\, \sin\phi_{0z} \;.\; \delta(s-s_0-nC)$$

$$\frac{d\phi_{1z}(0)}{ds} = \delta_z \sqrt{\frac{\beta_z}{2J_{0z}}}\, \cos\phi_{0z} \;.\; \delta(s-s_0-nC) \qquad z = x \text{ or } y \qquad (34)$$

Direct integration gives :

$$J_{1z}=J_{0z}+\delta_z\sqrt{2\beta_z J_{0z}}\sum_{n=0}^{N}\sin\phi_{0z}(s_0+nC)=J_{0z}+\delta_z\quad 2\beta_z J_{0z}\sum_{n=0}^{N}\sin(\phi_{0z}(s_0)+n2\pi Q_z)$$

$$\phi_{12}(0)=\phi_{0z}(0)+\delta_z\sqrt{\frac{\beta_z}{2J_{0z}}}\sum_{n=0}^{N}\cos\phi_{0z}(s_0+nC)=J_{0z}+\delta_z\sqrt{\frac{\beta_z}{2J_{0z}}}\sum_{n=0}^{N}\cos(\phi_{0z}(s_0)+n2\pi Q_z) \qquad (35)$$

where N is the integer part of s/C.

The sums may be evaluated, and when n tends to infinity we find :

$$\sum_{n=0}^{\infty} e^{im(\phi+n2\pi Q)}=\frac{e^{im\phi}}{1-e^{im2\pi Q}}=\frac{i}{2}\,\frac{e^{im(\phi-\pi Q)}}{\sin m\pi Q} \qquad (36)$$

or :

$$\sum_{n=0}^{\infty}\sin m(\phi+n2\pi Q)=\frac{1}{2}\,\frac{\cos m(\phi-\pi Q)}{\sin m\pi Q}\,,\quad \sum_{n=0}^{\infty}\cos m(\phi+n2\pi Q)=-\frac{1}{2}\,\frac{\sin m(\phi-\pi Q)}{\sin m\pi Q} \qquad (37)$$

so that we have :

$$J_{1z}=J_{0z}\left(1+\delta_z\sqrt{\frac{\beta_z}{2J_{0z}}}\,\frac{\cos(\phi_{0z}(s_0)-\pi Q_z)}{\sin\pi Q_z}\right.$$

$$\phi_{1z}(0)=\phi_{0z}(0)-\frac{\delta_z}{2}\sqrt{\frac{\beta_z}{2J_{0z}}}\,\frac{\sin(\phi_{0z}(s_0)-\pi Q_z}{\sin\pi Q_z} \qquad (38)$$

The equations for the second order perturbations are obtained by using (38) in (24).

$$\frac{dJ_{2z}}{ds}=\delta_z\sqrt{2\beta_z J_{1z}}\,\sin\phi_{1z}$$

$$\frac{d\phi_{2z}}{ds}=\delta_z\sqrt{\frac{\beta_z}{2J_{1z}}}\,\cos\phi_{1z} \qquad (39)$$

The equations are integrated by using the thin lens approximation :

$$J_{2z}=J_{0z}+\delta_z\sqrt{2\beta_z J_{1z}}\sum_{n=0}^{N}\sin(\phi_{1z}(s_0)+n2\pi Q_z)$$

$$Q_{2z}(0)=\phi_{0z}(0)+\delta_z\sqrt{\frac{\beta_z}{2J_{1z}}}\sum_{n=0}^{N}\cos(\phi_{1z}(s_0)+n\,1\pi Q_z) \qquad (40)$$

where N is the integer part of s/C.

The sums are evaluated by (37) when N tends to infinity.

$$
\begin{aligned}
J_{1z} &= J_{0z} + \delta_z \sqrt{\frac{\beta_z J_{1z}}{2}} \; \frac{\cos(\phi_{1z}(s_0) - \pi Q_z)}{\sin \pi Q_z} \\
\phi_{2z} &= \phi_{0z}(0) - \delta_z \sqrt{\frac{\beta_z J_{1z}}{2}} \; \frac{\sin(\phi_{1z}(s_0) - \pi Q_z)}{\sin \pi Q_z}
\end{aligned}
\qquad (41)
$$

By using the expansions (32) to second order in δ_z this may be simplified to :

$$
J_{2z} = J_{0z} \left[1 + \delta_z \sqrt{\frac{\beta_z}{2J_{0z}}} \; \frac{\cos(\phi_{0z}(s_0) - \pi Q_z)}{\sin \pi Q_z} + \delta_z^2 \frac{\beta_z}{2J_{0z}} \; \frac{1}{\sin^2 \pi Q_z} \right]
\qquad (42)
$$

$$
\phi_{2z}(0) = \phi_{0z}(0) - \delta_z \frac{1}{2} \sqrt{\frac{\beta_z}{2J_{0z}}} \; \frac{\sin(\phi_{0z}(s_0) - \pi Q_z)}{\sin \pi Q_z} + \delta_z^2 \frac{1}{2} \frac{\beta_z}{2J_{0z}} \; \frac{\sin(2\phi_{0z}(s_0) - 2\pi Q_z^2)}{\sin^2 \pi Q_z}
$$

The motion to second order is given by putting (42) in (17) :

$$
z = \sqrt{2\beta_z J_{2z}} \cos\phi_{2z}(s) \; , \; \phi_{2z}(s) = \mu_z(s) + \phi_{2z}(0) \qquad (43)
$$

or if we expand by using (32) :

$$
\boxed{
\begin{aligned}
z(s) = \sqrt{2\beta_z J_{0z}} \Big[& \left(1 + \delta_z^2 \frac{1}{2} \frac{\beta_z}{2J_{0z}} \frac{1}{\sin^2 \pi Q_z}\right) \cos\phi_z(s) \\
& + \delta_z \frac{1}{2} \; \frac{\sin(\phi_z(s) - \phi_{0z}(s_0) + \pi Q_z)}{\sin \pi Q_z} \\
& - \delta_z^2 \frac{1}{4} \frac{\beta_z}{2J_{0z}} \; \frac{\cos(\phi_z(s) - 2\phi_{0z}(s_0) + 2\pi Q_z)}{\sin^2 \pi Q_z} \\
& + \delta_z^2 \frac{1}{4} \frac{\beta_z}{2J_{0z}} \; \frac{\cos(\phi_z(s) + 2\phi_{0z}(s_0) - 2\pi Q_z)}{\sin^2 \pi Q_z}
\end{aligned}
}
\qquad (44)
$$

This may be continued for higher multipoles. However due to the amount of algebra this is preferably done by symbolic computation.

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6. REFERENCES

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