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PROGRAMS FOR SYMBOLIC SOLVING OF DIFFERENTIAL EQUATIONS

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Abstract

When one studies nonlinear resonances in storage rings by perturbation theory, one is led into a system of first order coupled differential equations. By using Lie-series, it is possible to write the solution of those as a series expansion. We will here study the possibilities to let a program do that. In other words, we study the possibilities to have a program which reads the differential equations and gives the solution as symbolic series expansions to any desired order. It is then possible to have a second program which reads those expansions and, when given starting values, it may calculate the valuesof the solutions for any chosen time.

Theory

Given a set of nonautonomous first order coupled equations

$$\frac{dx_i}{dt} = F_i\ (x_1, \ldots, x_n, t) \quad , \qquad i = 1, \ldots, n \tag{1}$$

One may define a differential operator D by

$$D = F_1(x_1(t_0), \ldots, x_n(t_0), t_0) \ . \ \frac{\partial}{\partial x_1(t_0)} + \ldots$$

$$+ F_n\ (x_1(t_0), \ldots, x_n(t_0), t_0) \ . \ \frac{\partial}{\partial x_n\ (t_0)} + \frac{\partial}{\partial t_0}$$

It can be shown that the solutions to (1) may be written as a Lie-series of the form[1].

$$x_i(t) \qquad \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \quad D^k\ x_i(t_0) = e^{(t-t_0)D} \quad . \ x_i(t_0), \ i=1\ldots,n \tag{2}$$

If one had a subroutine which makes symbolic differentiation followed by algebraic simplification, then this solution could easily be calculated to any given order.

LISP

A programming language well adapted to this kind of problems is LISP. [2]) gives an example of functions for symbolic differentiation and algebraic simplification in LISP. Those routines were taken over to the LISP on the PRIAM-VAX, and routines for the Lie-series expansion were added.

A Program to Obtain Numerical Results

When we have given the equations and the order for the expansions of the solutions, the LISP program will do these calculations and write the symbolic expansions on a file. This file is read by the second program which then reads starting values: $x_i(t_0)$ and t_0. It may now calculate $x_i(t)$ for any t. In order to improve the convergence $x_i(t)$ should be calculated in steps by first calculating $x_i(t_0 + h)$ for all i where h is the time step and then $x_i\ (t_0 + 2h)$ by using the previously calculated values as starting values, and so on until $x_i(t)$ is reached.

This program should be written in a language suitable for numerical calculations such as FORTRAN or PASCAL. By using prefix notation for the symbolic expressions the program is preferably written as a recursive program. This, of course, excludes the use of FORTRAN.

Conclusions

In the Appendix we apply the programs to some examples of differential equations. Note that a higher order differential equation may always be split into a set of first order differential equations by the following trick:

$$\ddot{x} + \omega^2 x = 0$$

define $y = \dot{x}$, then

$$y = \dot{x}$$
$$\dot{y} = - \omega^2 x$$

We see that the method works fairly well for simple equations. For more complicated equations the solutions contain a lot of terms, especially when we go to higher order for improvement of the convergence. This reduces the efficiency of the program which does the numerical calculations.

A possible means to simplify and improve the efficiency is to have a more sophisticated function for algebraic simplification. The one used here does not look for common factors in different terms etc. Another possibility is to directly generate FORTRAN code for the solutions instead of having a program reading a file with the expansions. This could easily be done by implementing this under MACSYMA, a system specially developed to solve this kind of problem.

Note that we have not taken into consideration whether the Lie-series (2) converge so that they really are solutions to the equations. For such considerations we refer to the references[1]).

References

1. F. Cap, W. Gröbner and P. Lesky, "The Astonomical n-Body Problem with Time-Dependent Forces"; Acta Physica Austriaca, vol. 15 (1962), page 212.

2. Clask Weissman; "LISP 1.5 Primer"; Dickenson Publishing Company Inc. (1967).

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APPENDIX

We show here some examples. Note that the derivative is written with the use of an operator, where

$$Dtx$$

is the derivative of x with respect to time, etc. The different examples are :

Exponential decay	$\dot{x} + ay = 0$
Harmonic oscillator	$\ddot{x} + \omega^2 x = 0$
Pendulum equation	$x + \omega^2 \sin x = 0$

Motion of particle in sextupolar fields

$$r_2' = /k/r_2^2 \sin \psi$$

$$\theta_2' = /k/ \frac{r_2^2}{r_1} \cos \Psi \qquad \Psi = \theta_1 + 2\,\theta_2 + \theta_k + \theta e$$

$$r_2' = 2/k/r_1 r_2 \sin \Psi$$

$$\theta_2' = 2/k/r_1 \cos \Psi$$

where the prime denotes differentiation with respect to θ.

Note that D in the examples of the equations means the total derivative, whereas in the expression for the differential operator D stands for the partial derivative.

Exponential decay $\dot{x} + \alpha x = 0$

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Equations

Dtx=-ax

Differential operator

(-ax)Dx+Dt

Order of expansion 5

Expansions

x(t)=x+t(-ax)+0.5 t^2 aax+0.1666666666666667 t^3 a^2 (-ax)+0.04166666666666667 t
^4 a^2 (-a)(-ax)+0.008333333333333333 t^5 a^2 (-a)(-a)(-ax)
```

Harmonic oscillator $\ddot{x} + \omega^2 x = 0$

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Equations

Dtx=y

Dty=-xw^2

Differential operator

yDx+(-xw^2 )Dy+Dt

Order of expansion 5

Expansions

x(t)=x+ty+0.5 t^2 (-xw^2 )+0.166666666666667 t^3 y(-w^2 )+0.0416666666666667 t
^4 xw^2 w^2 +0.00833333333333333 t^5 yw^4

y(t)=y+t(-xw^2 )+0.5 t^2 y(-w^2 )+0.166666666666667 t^3 xw^2 w^2 +0.04166666666
666667 t^4 yw^4 +0.00833333333333333 t^5 w^4 (-xw^2 )
```

Pendulum equation $\ddot{x} + \omega^2 \sin x = 0$

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Equations

Dtx=y

Dty=-w^2 sinx

Differential operator

yDx+(-w^2 sinx)Dy+Dt

Order of expansion 5

Expansions

x(t)=x+ty+0.5 t^2 (-w^2 sinx)+0.1666666666666667 t^3 y(-w^2 cosx)+0.041666666666
66667 t^4 (yy(-w^2 (-sinx))+w^2 sinxw^2 cosx)+0.008333333333333333 t^5 (y(yy(-w^
2 (-cosx))+w^2 sinxw^2 (-sinx)+(w^2 cosx)^2 )+2 y(-w^2 (-sinx))(-w^2 sinx))

y(t)=y+t(-w^2 sinx)+0.5 t^2 y(-w^2 cosx)+0.1666666666666667 t^3 (yy(-w^2 (-sinx)
)+w^2 sinxw^2 cosx)+0.04166666666666667 t^4 (y(yy(-w^2 (-cosx))+w^2 sinxw^2 (-si
nx)+(w^2 cosx)^2 )+2 y(-w^2 (-sinx))(-w^2 sinx))+0.008333333333333333 t^5 (y(y(y
y(-w^2 sinx)+w^2 sinxw^2 (-cosx)+w^2 (-sinx)w^2 cosx+2 w^2 (-sinx)w^2 cosx)+2 y(
-w^2 (-sinx))(-w^2 cosx)+2 y(-w^2 (-cosx))(-w^2 sinx))+(y2 y(-w^2 (-cosx))+yy(-w
^2 (-cosx))+w^2 sinxw^2 (-sinx)+(w^2 cosx)^2 +2 (-w^2 (-sinx))(-w^2 sinx))(-w^2
sinx))
```

Motion in sextupolar fields

Expansion to 2:nd order


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Equations

Dtu=kw^2 sin(f+v+2 x+et)

Dtv=kw^2 1/ucos(f+v+2 x+et)

Dtw=2 kuwsin(f+v+2 x+et)

Dtx=2 kucos(f+v+2 x+et)

Differential operator

kw^2 sin(f+v+2 x+et)Du+kw^2 1/ucos(f+v+2 x+et)Dv+2 kuwsin(f+v+2 x+et)Dw+2 kucos(
f+v+2 x+et)Dx+Dt

Order of expansion 2

Expansions

u(t)=u+tkw^2 sin(f+v+2 x+et)+0.5 t^2 (kw^2 1/ucos(f+v+2 x+et)kw^2 cos(f+v+2 x+et
)+2 kuwsin(f+v+2 x+et)ksin(f+v+2 x+et)2 w+2 kucos(f+v+2 x+et)kw^2 2 cos(f+v+2 x+
et)+kw^2 ecos(f+v+2 x+et))

v(t)=v+tkw^2 1/ucos(f+v+2 x+et)+0.5 t^2 (kw^2 sin(f+v+2 x+et)kw^2 1/(ucos(f+v+2
x+et))^2 (-cos(f+v+2 x+et))+kw^2 1/ucos(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (
-u(-sin(f+v+2 x+et)))+2 kuwsin(f+v+2 x+et)k1/ucos(f+v+2 x+et)2 w+2 kucos(f+v+2 x
+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))+kw^2 1/(ucos(f+v+2 x+et
))^2 (-u(-esin(f+v+2 x+et))))

w(t)=w+t2 kuwsin(f+v+2 x+et)+0.5 t^2 (kw^2 sin(f+v+2 x+et)2 kwsin(f+v+2 x+et)+kw
^2 1/ucos(f+v+2 x+et)2 kuwcos(f+v+2 x+et)+4 kuwsin(f+v+2 x+et)kusin(f+v+2 x+et)+
4 kucos(f+v+2 x+et)kuw2 cos(f+v+2 x+et)+2 kuwecos(f+v+2 x+et))

x(t)=x+t2 kucos(f+v+2 x+et)+0.5 t^2 (kw^2 sin(f+v+2 x+et)2 kcos(f+v+2 x+et)+kw^2
 1/ucos(f+v+2 x+et)2 ku(-sin(f+v+2 x+et))+4 kucos(f+v+2 x+et)ku(-2 sin(f+v+2 x+e
t))+2 ku(-esin(f+v+2 x+et)))
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Motion in sextupolar fields

Expansion to 3:rd order.

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Equations

Dtu=kw^2 sin(f+v+2 x+et)

Dtv=kw^2 1/ucos(f+v+2 x+et)

Dtw=2 kuwsin(f+v+2 x+et)

Dtx=2 kucos(f+v+2 x+et)

Differential operator

kw^2 sin(f+v+2 x+et)Du+kw^2 1/ucos(f+v+2 x+et)Dv+2 kuwsin(f+v+2 x+et)Dw+2 kucos(
f+v+2 x+et)Dx+Dt

Order of expansion 3

Expansions

u(t)=u+tkw^2 sin(f+v+2 x+et)+0.5 t^2 (kw^2 1/ucos(f+v+2 x+et)kw^2 cos(f+v+2 x+et
)+2 kuwsin(f+v+2 x+et)ksin(f+v+2 x+et)2 w+2 kucos(f+v+2 x+et)kw^2 2 cos(f+v+2 x+
et)+kw^2 ecos(f+v+2 x+et))+0.1666666666666667 t^3 (kw^2 sin(f+v+2 x+et)(kw^2 cos
(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-cos(f+v+2 x+et))+ksin(f+v+2 x+et)2 w2
kwsin(f+v+2 x+et)+kw^2 2 cos(f+v+2 x+et)2 kcos(f+v+2 x+et))+kw^2 1/ucos(f+v+2 x+
et)(kw^2 1/ucos(f+v+2 x+et)kw^2 (-sin(f+v+2 x+et))+kw^2 cos(f+v+2 x+et)kw^2 1/(u
cos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))+2 kuwsin(f+v+2 x+et)k2 wcos(f+v+2 x+et
)+ksin(f+v+2 x+et)2 w2 kuwcos(f+v+2 x+et)+2 kucos(f+v+2 x+et)kw^2 2 (-sin(f+v+2
x+et))+kw^2 2 cos(f+v+2 x+et)2 ku(-sin(f+v+2 x+et))+kw^2 e(-sin(f+v+2 x+et)))+2
kuwsin(f+v+2 x+et)(kw^2 1/ucos(f+v+2 x+et)kcos(f+v+2 x+et)2 w+kw^2 cos(f+v+2 x+e
t)k1/ucos(f+v+2 x+et)2 w+2 kuwsin(f+v+2 x+et)k2 sin(f+v+2 x+et)+ksin(f+v+2 x+et)
2 w2 kusin(f+v+2 x+et)+2 kucos(f+v+2 x+et)k4 wcos(f+v+2 x+et)+kecos(f+v+2 x+et)2
 w)+2 kucos(f+v+2 x+et)(kw^2 1/ucos(f+v+2 x+et)kw^2 (-2 sin(f+v+2 x+et))+kw^2 co
s(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))+2 kuwsin(f+v+2
 x+et)k4 wcos(f+v+2 x+et)+ksin(f+v+2 x+et)2 w2 kuw2 cos(f+v+2 x+et)+2 kucos(f+v+
2 x+et)kw^2 2 (-2 sin(f+v+2 x+et))+kw^2 2 cos(f+v+2 x+et)2 ku(-2 sin(f+v+2 x+et)
)+kw^2 e(-2 sin(f+v+2 x+et)))+kw^2 1/ucos(f+v+2 x+et)kw^2 (-esin(f+v+2 x+et))+kw
^2 cos(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-esin(f+v+2 x+et)))+2 kuwsin(f
+v+2 x+et)k2 wecos(f+v+2 x+et)+ksin(f+v+2 x+et)2 w2 kuwecos(f+v+2 x+et)+2 kucos(
f+v+2 x+et)kw^2 2 (-esin(f+v+2 x+et))+kw^2 2 cos(f+v+2 x+et)2 ku(-esin(f+v+2 x+e
t))+kw^2 e(-esin(f+v+2 x+et)))

v(t)=v+tkw^2 1/ucos(f+v+2 x+et)+0.5 t^2 (kw^2 sin(f+v+2 x+et)kw^2 1/(ucos(f+v+2
x+et))^2 (-cos(f+v+2 x+et))+kw^2 1/ucos(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (
-u(-sin(f+v+2 x+et)))+2 kuwsin(f+v+2 x+et)k1/ucos(f+v+2 x+et)2 w+2 kucos(f+v+2 x
+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))+kw^2 1/(ucos(f+v+2 x+et
))^2 (-u(-esin(f+v+2 x+et))))+0.1666666666666667 t^3 (kw^2 sin(f+v+2 x+et)(kw^2
sin(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^4 (-2 cos(f+v+2 x+et)ucos(f+v+2 x+et))(
-cos(f+v+2 x+et))+kw^2 1/ucos(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2 sin(f+v+2
x+et)+1/(ucos(f+v+2 x+et))^4 (-2 cos(f+v+2 x+et)ucos(f+v+2 x+et))(-u(-sin(f+v+2
x+et))))+kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))kw^2 1/(ucos(f+v+2 x+
et))^2 (-cos(f+v+2 x+et))+2 kuwsin(f+v+2 x+et)k2 w1/(ucos(f+v+2 x+et))^2 (-cos(f
+v+2 x+et))+k1/ucos(f+v+2 x+et)2 w2 kwsin(f+v+2 x+et)+2 kucos(f+v+2 x+et)kw^2 (1
/(ucos(f+v+2 x+et))^2 2 sin(f+v+2 x+et)+1/(ucos(f+v+2 x+et))^4 (-2 cos(f+v+2 x+e
t)ucos(f+v+2 x+et))(-u(-2 sin(f+v+2 x+et))))+kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2
sin(f+v+2 x+et)))2 kcos(f+v+2 x+et)+kw^2 (1/(ucos(f+v+2 x+et))^2 esin(f+v+2 x+et
)+1/(ucos(f+v+2 x+et))^4 (-2 cos(f+v+2 x+et)ucos(f+v+2 x+et))(-u(-esin(f+v+2 x+e
```

```
t)))))+kw^2 1/ucos(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2
sin(f+v+2 x+et)+1/(ucos(f+v+2 x+et))^4 (-2 u(-sin(f+v+2 x+et))ucos(f+v+2 x+et))(
-cos(f+v+2 x+et)))+kw^2 1/(ucos(f+v+2 x+et))^2 (-cos(f+v+2 x+et))kw^2 cos(f+v+2
x+et)+kw^2 1/ucos(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-cos(f+v+2 x+et)))
+1/(ucos(f+v+2 x+et))^4 (-2 u(-sin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-sin(f+v+2 x
+et))))+(kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et))))^2 +2 kuwsin(f+v+2 x
+et)k2 w1/(ucos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))+k1/ucos(f+v+2 x+et)2 w2 ku
wcos(f+v+2 x+et)+2 kucos(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-2 cos(f+v+
2 x+et)))+1/(ucos(f+v+2 x+et))^4 (-2 u(-sin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-2
sin(f+v+2 x+et))))+kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))2 ku(-sin
(f+v+2 x+et))+kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-ecos(f+v+2 x+et)))+1/(ucos(f+v+2
 x+et))^4 (-2 u(-sin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-esin(f+v+2 x+et)))))+2 ku
wsin(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)k1/(ucos(f+v+2 x+et))^2 (-cos(f+v+2 x+et))2
 w+kw^2 1/(ucos(f+v+2 x+et))^2 (-cos(f+v+2 x+et))ksin(f+v+2 x+et)2 w+kw^2 1/ucos
(f+v+2 x+et)k1/(ucos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))2 w+kw^2 1/(ucos(f+v+2
 x+et))^2 (-u(-sin(f+v+2 x+et)))k1/ucos(f+v+2 x+et)2 w+2 kuwsin(f+v+2 x+et)k2 1/
ucos(f+v+2 x+et)+k1/ucos(f+v+2 x+et)2 w2 kusin(f+v+2 x+et)+2 kucos(f+v+2 x+et)k1
/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))2 w+k1/(ucos(f+v+2 x+et))^2 (-u(-e
sin(f+v+2 x+et)))2 w)+2 kucos(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)kw^2 (1/(ucos(f+v+
2 x+et))^2 2 sin(f+v+2 x+et)+1/(ucos(f+v+2 x+et))^4 (-2 u(-2 sin(f+v+2 x+et))uco
s(f+v+2 x+et))(-cos(f+v+2 x+et)))+kw^2 1/(ucos(f+v+2 x+et))^2 (-cos(f+v+2 x+et))
kw^2 2 cos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-
2 cos(f+v+2 x+et)))+1/(ucos(f+v+2 x+et))^4 (-2 u(-2 sin(f+v+2 x+et))ucos(f+v+2 x
+et))(-u(-sin(f+v+2 x+et))))+kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))k
w^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))+2 kuwsin(f+v+2 x+et)k2 w1/(u
cos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))+k1/ucos(f+v+2 x+et)2 w2 kuw2 cos(f+v
+2 x+et)+2 kucos(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-4 cos(f+v+2 x+et))
)+1/(ucos(f+v+2 x+et))^4 (-2 u(-2 sin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-2 sin(f+
v+2 x+et))))+kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)))2 ku(-2 sin(f+v
+2 x+et))+kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-e2 cos(f+v+2 x+et)))+1/(ucos(f+v+2 x
+et))^4 (-2 u(-2 sin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-esin(f+v+2 x+et)))))+kw^2
 sin(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))^2 esin(f+v+2 x+et)+1/(ucos(f+v+2 x+et
))^4 (-2 u(-esin(f+v+2 x+et))ucos(f+v+2 x+et))(-cos(f+v+2 x+et)))+kw^2 1/(ucos(f
+v+2 x+et))^2 (-cos(f+v+2 x+et))kw^2 ecos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+et)kw^
2 (1/(ucos(f+v+2 x+et))^2 (-u(-ecos(f+v+2 x+et)))+1/(ucos(f+v+2 x+et))^4 (-2 u(-
esin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-sin(f+v+2 x+et))))+kw^2 1/(ucos(f+v+2 x+e
t))^2 (-u(-sin(f+v+2 x+et)))kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-esin(f+v+2 x+et)))+
2 kuwsin(f+v+2 x+et)k2 w1/(ucos(f+v+2 x+et))^2 (-u(-esin(f+v+2 x+et)))+k1/ucos(f
+v+2 x+et)2 w2 kuwecos(f+v+2 x+et)+2 kucos(f+v+2 x+et)kw^2 (1/(ucos(f+v+2 x+et))
^2 (-u(-2 ecos(f+v+2 x+et)))+1/(ucos(f+v+2 x+et))^4 (-2 u(-esin(f+v+2 x+et))ucos
(f+v+2 x+et))(-u(-2 sin(f+v+2 x+et))))+kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+
v+2 x+et)))2 ku(-esin(f+v+2 x+et))+kw^2 (1/(ucos(f+v+2 x+et))^2 (-u(-eecos(f+v+2
 x+et)))+1/(ucos(f+v+2 x+et))^4 (-2 u(-esin(f+v+2 x+et))ucos(f+v+2 x+et))(-u(-es
in(f+v+2 x+et)))))

w(t)=w+t2 kuwsin(f+v+2 x+et)+0.5 t^2 (kw^2 sin(f+v+2 x+et)2 kwsin(f+v+2 x+et)+kw
^2 1/ucos(f+v+2 x+et)2 kuwcos(f+v+2 x+et)+4 kuwsin(f+v+2 x+et)kusin(f+v+2 x+et)+
4 kucos(f+v+2 x+et)kuw2 cos(f+v+2 x+et)+2 kuwecos(f+v+2 x+et))+0.166666666666666
7 t^3 (kw^2 sin(f+v+2 x+et)(kw^2 1/ucos(f+v+2 x+et)2 kwcos(f+v+2 x+et)+2 kuwcos(
f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-cos(f+v+2 x+et))+4 (kuwsin(f+v+2 x+et)k
sin(f+v+2 x+et)+kusin(f+v+2 x+et)kwsin(f+v+2 x+et))+4 (kucos(f+v+2 x+et)kw2 cos(
f+v+2 x+et)+kuw2 cos(f+v+2 x+et)kcos(f+v+2 x+et))+2 kwecos(f+v+2 x+et))+kw^2 1/u
```

```
cos(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)2 kwcos(f+v+2 x+et)+2 kwsin(f+v+2 x+et)kw^2
cos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+et)2 kuw(-sin(f+v+2 x+et))+2 kuwcos(f+v+2 x+
et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))+4 (kuwsin(f+v+2 x+et)kucos
(f+v+2 x+et)+kusin(f+v+2 x+et)kuwcos(f+v+2 x+et))+4 (kucos(f+v+2 x+et)kuw2 (-sin
```



```
(f+v+2 x+et))+kuw2 cos(f+v+2 x+et)ku(-sin(f+v+2 x+et)))+2 kuwe(-sin(f+v+2 x+et))
)+2 kuwsin(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)2 ksin(f+v+2 x+et)+2 kwsin(f+v+2 x+et
)ksin(f+v+2 x+et)2 w+kw^2 1/ucos(f+v+2 x+et)2 kucos(f+v+2 x+et)+2 kuwcos(f+v+2 x
+et)k1/ucos(f+v+2 x+et)2 w+4 (kusin(f+v+2 x+et))^2 +4 kucos(f+v+2 x+et)ku2 cos(f
+v+2 x+et)+2 kuecos(f+v+2 x+et))+2 kucos(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)2 kw2 c
os(f+v+2 x+et)+2 kwsin(f+v+2 x+et)kw^2 2 cos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+et)
2 kuw(-2 sin(f+v+2 x+et))+2 kuwcos(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2
 sin(f+v+2 x+et)))+4 (kuwsin(f+v+2 x+et)ku2 cos(f+v+2 x+et)+kusin(f+v+2 x+et)kuw
2 cos(f+v+2 x+et))+4 (kucos(f+v+2 x+et)kuw2 (-2 sin(f+v+2 x+et))+kuw2 cos(f+v+2
x+et)ku(-2 sin(f+v+2 x+et)))+2 kuwe(-2 sin(f+v+2 x+et)))+kw^2 sin(f+v+2 x+et)2 k
wecos(f+v+2 x+et)+2 kwsin(f+v+2 x+et)kw^2 ecos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+e
t)2 kuw(-esin(f+v+2 x+et))+2 kuwcos(f+v+2 x+et)kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-
esin(f+v+2 x+et)))+4 (kuwsin(f+v+2 x+et)kuecos(f+v+2 x+et)+kusin(f+v+2 x+et)kuwe
cos(f+v+2 x+et))+4 (kucos(f+v+2 x+et)kuw2 (-esin(f+v+2 x+et))+kuw2 cos(f+v+2 x+e
t)ku(-esin(f+v+2 x+et)))+2 kuwe(-esin(f+v+2 x+et)))
```

```
x(t)=x+t2 kucos(f+v+2 x+et)+0.5 t^2 (kw^2 sin(f+v+2 x+et)2 kcos(f+v+2 x+et)+kw^2
 1/ucos(f+v+2 x+et)2 ku(-sin(f+v+2 x+et))+4 kucos(f+v+2 x+et)ku(-2 sin(f+v+2 x+e
t))+2 ku(-esin(f+v+2 x+et)))+0.166666666666667 t^3 (kw^2 sin(f+v+2 x+et)(kw^2 1
/ucos(f+v+2 x+et)2 k(-sin(f+v+2 x+et))+2 ku(-sin(f+v+2 x+et))kw^2 1/(ucos(f+v+2
x+et))^2 (-cos(f+v+2 x+et))+4 (kucos(f+v+2 x+et)k(-2 sin(f+v+2 x+et))+ku(-2 sin(
f+v+2 x+et))kcos(f+v+2 x+et))+2 k(-esin(f+v+2 x+et)))+kw^2 1/ucos(f+v+2 x+et)(kw
^2 sin(f+v+2 x+et)2 k(-sin(f+v+2 x+et))+2 kcos(f+v+2 x+et)kw^2 cos(f+v+2 x+et)+k
w^2 1/ucos(f+v+2 x+et)2 ku(-cos(f+v+2 x+et))+2 ku(-sin(f+v+2 x+et))kw^2 1/(ucos(
f+v+2 x+et))^2 (-u(-sin(f+v+2 x+et)))+4 (kucos(f+v+2 x+et)ku(-2 cos(f+v+2 x+et))
+ku(-2 sin(f+v+2 x+et))ku(-sin(f+v+2 x+et)))+2 ku(-ecos(f+v+2 x+et)))+2 kuwsin(f
+v+2 x+et)(2 kcos(f+v+2 x+et)ksin(f+v+2 x+et)2 w+2 ku(-sin(f+v+2 x+et))k1/ucos(f
+v+2 x+et)2 w)+2 kucos(f+v+2 x+et)(kw^2 sin(f+v+2 x+et)2 k(-2 sin(f+v+2 x+et))+2
 kcos(f+v+2 x+et)kw^2 2 cos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+et)2 ku(-2 cos(f+v+2
 x+et))+2 ku(-sin(f+v+2 x+et))kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-2 sin(f+v+2 x+et)
))+4 (kucos(f+v+2 x+et)ku(-4 cos(f+v+2 x+et))+(ku(-2 sin(f+v+2 x+et)))^2 )+2 ku(
-e2 cos(f+v+2 x+et)))+kw^2 sin(f+v+2 x+et)2 k(-esin(f+v+2 x+et))+2 kcos(f+v+2 x+
et)kw^2 ecos(f+v+2 x+et)+kw^2 1/ucos(f+v+2 x+et)2 ku(-ecos(f+v+2 x+et))+2 ku(-si
n(f+v+2 x+et))kw^2 1/(ucos(f+v+2 x+et))^2 (-u(-esin(f+v+2 x+et)))+4 (kucos(f+v+2
 x+et)ku(-2 ecos(f+v+2 x+et))+ku(-2 sin(f+v+2 x+et))ku(-esin(f+v+2 x+et)))+2 ku(
-eecos(f+v+2 x+et)))
```

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