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ON FOURTH ORDER RESONANCES IN MACHINES *)
WITH STRONGLY VARYING FOCUSING FUNCTIONS

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Abstract

A strong octupole component $\partial^3 B_z / \partial r^3$ was measured in the stray field of the LEAR quadrupole by M. Giesch and C. Mazeline. The present note is concerned with the resulting excitation of betatron resonances.

The observed field is explained by the edge effect i.e. the transition from quadrupole - to zero field. It is present in any quadrupole as pointed out sometime ago by Gluckstern and Regenstreif. As a function of the distance from the quad, the pseudo octupole changes from positive to negative values such that the integrated strength is zero.

Nevertheless systematic fourth order resonances can be strongly excited by this field, especially if the lattice functions vary strongly over the length of the stray field.

Non systematic resonances observed in LEAR seem only partly explained by this effect, which could be very important in machines with a low beta insertion.

Quadrupole end-fields could also be important in ACOL and AA, as recently pointed out by E.J.N. Wilson

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*) Extended version of material presented at "Journées d'études Saturne", Fontevraud, 1983

1. QUADRUPOLE STRAY FIELD AND OCTUPOLAR RESONANCES

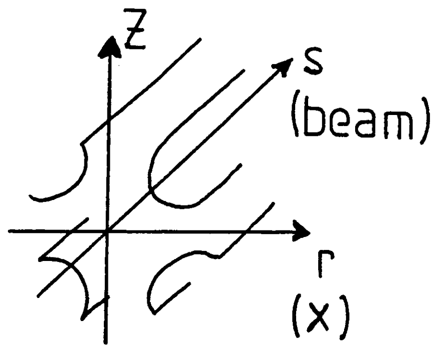
Fig. 1 gives the octupole component of the field, measured by C. Mazeline[1] and M. Giesch, on one LEAR quad. by determining $\frac{\partial^3 B_z}{\partial x^3}$ for $z = 0$, $x = 0$ along the "beam axis". Using rot $\vec{B} = 0$, div $\vec{B} = 0$, it can be shown[2][3][4][8][9] that this is the effect of the field fall-off at the entrance and exit face :

Ideal quad :

$B_z = Gx$ with $G = \frac{\partial B_z}{\partial x}$

$B_x = Gz$ $\quad G' = \frac{\partial G}{\partial s} = 0$

$B_s = 0$



Real quad : $G = G(s)$

rot $\vec{B} = 0$

$$\frac{\partial B_s}{\partial x} = \frac{\partial B_x}{\partial s} = G'z$$

$$\underline{B_s = G'xz}$$

$$\frac{\partial B_s}{\partial z} = \frac{\partial B_z}{\partial s} = G'x$$

div $\vec{B} = 0$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} + G''xz = 0$$

due to symmetry $\frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial x} = \frac{-G''}{2}\, xz$

$$B_x = \frac{-G''}{4}\left[x^2 + F_1(z)\right]z$$

or

$$B_z = \frac{-G''}{4}\left[z^2 + F_2(x)\right]x$$

Finally rot $\vec{B}/_s = 0 \rightarrow \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$

This gives :

$$\frac{-G''}{4}\left[z^2 + \left[F_2(x).\ x\right]'_x\right] = \frac{-G''}{4}\left[x^2 + \left(F_1(z)\cdot z\right)'_z\right]$$

$$\frac{d}{dz}\left[F_1(z).z\right] = z^2 \quad \rightarrow \quad F_1(z).z = \frac{1}{3}z^3 + C$$

$$\frac{d}{dx}\left[F_2(x).x\right] = x^2 \quad \rightarrow \quad F_2(x).x = \frac{1}{3}z^3 + D$$

For reasons of symmetry C = 0, D = 0. Thus the field is

$$B_z = x\left[G - \frac{G''}{4}\left(z^2 + \frac{1}{3}x^2\right)\right] + \dots$$

$$B_x = z\left[G - \frac{G''}{4}\left(x^2 + \frac{1}{3}z^2\right)\right] + \dots$$

$$B_s = G'xz + \dots$$

With the additional field, up to third order in x, z :

$$\boxed{\begin{aligned} \Delta B_z &= -\frac{G''}{4}\,x\left(z^2 + \frac{1}{3}x^2\right) \\ \Delta B_x &= -\frac{G''}{4}\,z\left(x^2 + \frac{1}{3}z^2\right) \\ \Delta B_s &= G'zx \end{aligned}} \qquad (1)$$

<u>Note</u> : The procedure can be repeated to obtain higher other terms, e.g. : rot $\vec{B} = 0$ gives $\Delta_2 B_s = -\frac{G'''}{12}\,zx\,(x^2 + z^2)$ as addition to B_s and this is a pseudo duo-decapole.

<u>Comparison with the field of a two dimensional octupole</u>

$$B_z = bx(x^2 - 3z^2) = 3bx\left(-z^2 + \frac{1}{3}x^2\right) \qquad \text{note the - sign !}$$

$$B_x = bz(3x^2 - z^2) = 3bz\left(x^2 - \frac{1}{3}z^2\right)$$

$$B_s = 0$$

with $b = \frac{1}{6}\frac{\partial^3 B_z}{\partial x^3}$, the octupole component

Conclude : The field (1) is a "pseudo 8-pole", a "normal" octupole follows from a longitudinal vector potential $\vec{B} = \text{rot}\,\vec{A}$:

$$\vec{A} = \begin{pmatrix} 0 \\ A_s \\ 0 \end{pmatrix} \quad \text{with} \quad A_s = \frac{b}{3}\left(\frac{x^4}{4} + \frac{z^4}{4} - \frac{3}{2}x^2z^2\right)$$

The pseudo octupole (1) can be derived from a transverse but <u>not</u> from a purely longitudinal $\vec{A}$:

$$\boxed{\vec{A} = \begin{pmatrix} A_x \\ 0 \\ A_z \end{pmatrix} \qquad \begin{aligned} A_x &= \frac{G'}{12}\left(3xz^2 + x^3\right) \\ A_z &= -\frac{G'}{12}\left(3zx^2 + z^3\right) \end{aligned}} \qquad (2)$$

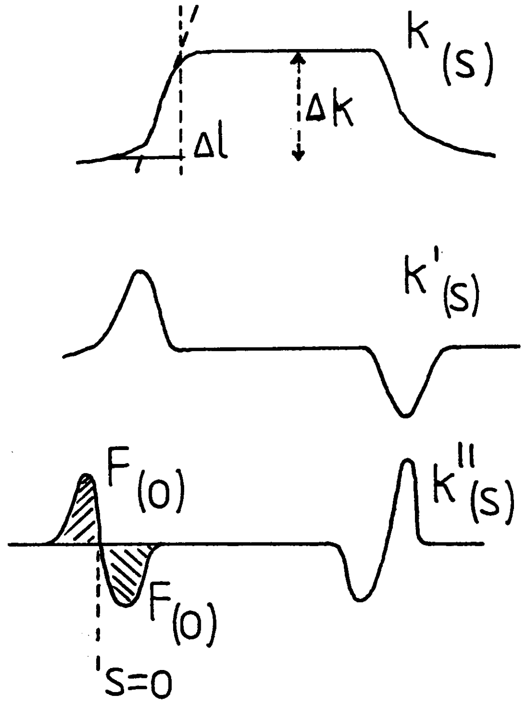
2. COMPARISON WITH MEASUREMENT

Observe from (1) : $\dfrac{\partial^3 B_z}{\partial x^3} = \dfrac{-G''}{2}$

From the definition of k : $k = \dfrac{-G}{B\rho} = \dfrac{-\partial B_z}{\partial x} \dfrac{1}{B\rho}$ and from (1) :

$\dfrac{1}{B\rho} \dfrac{\partial^3 B_z}{\partial x^3} = - \dfrac{G''}{2B\rho} = \dfrac{k''}{2}$

Here we denote $k' = \dfrac{\partial k}{\partial s}$ etc.



Define the "area" :

$F = \int_{s_1}^{s_2} k''ds = k'(s_2) - k'(s_1)$

under the octupole curve. Note that over the total stray field F = 0 because $k'(s) = 0$ for $s \gg 0$ and $s \ll 0$, i.e. positive and negative areas under the $\dfrac{\partial^3 B_z}{\partial x^3}$ curve should be equal.

Also the area under the positive curve can be expressed as : $\int_{-\infty}^{0} \dfrac{1}{B\rho} \dfrac{\partial^3 B_z}{\partial x^3} = \dfrac{k'}{2}(0) = \dfrac{\Delta k}{2\Delta \ell}$

From the measured curve : $2\Delta\ell \approx 0.2$ m, $F(0) \approx 15$ m^{-3} at I = 140 A.

Compare this to LEAR theoretical $k \approx 1.2$ for I = 60 A corresponding to 300 MeV/c, $B\rho \approx 1$ Tm. We find

Theoretical : $\dfrac{\Delta k}{2\Delta\ell} \approx 6$ m^{-3}

Curve : $F(0) \dfrac{60 \text{ A}}{140 \text{ A}} \approx 5$ m^{-3}

Conclude : the field (1) agrees in shape  and value of F with the measurement (up to 20% difference, explained by the difficulty to extract more precise $\dfrac{\partial^3 B_z}{\partial x^3}$ from the measurement of $B_z(x)$).

3. EXCITATION OF 4th ORDER RESONANCES BY THIS FIELD

We use Guignard, CERN 76-06 and the formulae, CERN 77-10 but include a factor $\binom{n}{j-1} = \frac{n!}{(n-j+1)!(j-1)!}$ missing in equ. 2.2.8 of CERN 76-06. Then the formulae (p. 69) should read in our case :

$$k_z^{\prime\prime\prime} = (-1)^{(k_2+1)/2} \frac{R^2}{2.B\rho} \left[\frac{\partial^3 B_z}{\partial z^2 \partial x} - \frac{\partial^3 B_x}{\partial x^2 \partial z} \right]$$

$$F_z^{\prime\prime\prime} = \frac{R}{2B\rho} \frac{\partial^2 B_s}{\partial x \partial z}$$

(Note that all the factors $\frac{(k_2+1)!(n-k_2!)}{(n-1)!}$ etc. drop out from the formulae on p. 69 due to the correction mentioned). We then obtain for the field (1) the excitation coefficients 10).

$$\boxed{\begin{aligned} &k_z^{\prime\prime\prime} = 0 \\ &F_z^{\prime\prime\prime} = \frac{R}{2B\rho} G' = -\frac{R}{2} k' \end{aligned}}$$

$$\left(k = -\frac{\partial B_z}{\partial x} \frac{1}{B\rho} = -\frac{G}{B\rho}\right)$$

Note : This expression can be directely checked from the vector potential, equ. 2.2.7 of CERN 76-06. In our case (n = 4)

$$A_x = -\frac{B\rho}{R} \frac{1}{3!} F_z^{\prime\prime\prime} (zx^3 + 3xz^2)$$

$$A_z = +\frac{B\rho}{R} \frac{1}{3!} F_z^{\prime\prime\prime} (z^3 + 3x^2 z)$$

$$A_o = -\frac{B\rho}{R^2} \frac{1}{3!} k_z^{\prime\prime\prime} (z^3 + 3xz^2)$$

which by comparison with (2) above gives

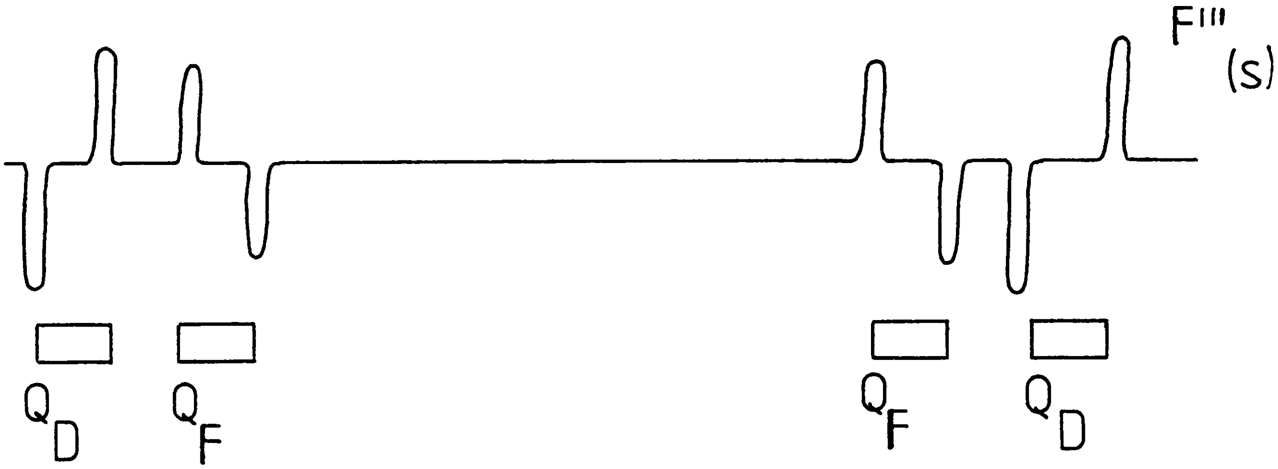
$$k_z^{\prime\prime\prime} = 0, \quad F_z^{\prime\prime\prime} . \frac{B\rho}{3!R} = \frac{G'}{12}$$

Next we calculate the excitation term dp (from CERN 77-10 p. 69) . In our case :

$$dp = \frac{1}{2\pi} \int_o^{2\pi} d\theta \, \beta_x^{|n_x/2|} \beta_z^{|n_z/2|} \frac{B\rho}{R^2} RF_z^{\prime\prime\prime} \left(\left(|n_x| \frac{\alpha_x}{\beta_x} - |n_z| \frac{\alpha_z}{\beta_z} \right) - i \left(\frac{n_x}{\beta_x} - \frac{n_z}{\beta_z} \right) \right)$$

$$\exp i(n_x \mu_x + n_z \mu_z)\theta$$

4. APPLICATION TO LEAR



lattice functions at quad. entrance and exit :

Pos s(m)	β_H (m)	β_V (m)	$\mu_H/2\pi$	$\mu_V/2\pi$	α_H	α_V	Remark
Q_F 4.08 4.55	10.42 9.31	8.42 12.11	0.178 0.1857	0.105 0.1132	-1.86 4.01	-.957 -7.63	Q_F entrance exit
Q_D 4.93 4.40	6.39 5.27	18.9 21.7	0.1935 0.207	0.1172 0.121	3.30 -0.65	-8.40 4.09	Q_D entrance exit

Since the "octupoles" are short, we can use the integrated strength of each "stray octupole" and multiply it with the corresponding "central" lattice functions

$$dp = \frac{1}{2\pi}\int \dots\ F'''\ R d\theta \approx \frac{1}{2\pi}\sum \dots\ F'''\Delta\ell$$

where $F'''\Delta\ell$ is the integrated strength

$$F'''\Delta\ell = -\frac{R}{2}k'\Delta\ell = \mp\frac{R}{2}k \quad \begin{cases} -\text{ at entrance, } & k' > 0 \\ +\text{ at exit, } & k' < 0 \end{cases}$$

So far our results are for any 4th order resonance. We shall now pay special attention to the specific case $4Q_z = 11$, i.e. :

$n_x = 0$, $n_z = 4$, which has been studied in LEAR to some detail.

In this case :

- Excitation term : $dp = \frac{1}{2\pi}\frac{B\rho}{R}\sum \mp\frac{k}{2}\left[4i - 4\alpha_V\right] B_z e^{i4\mu_z}$

- Width of the resonance (formulae p. 76) : $\Delta e = \frac{R\ 4^2}{2^2 B\rho 4!} E_z|dp| = \frac{R}{6B\rho} E_z|dp|$

  (In our case N=4, n_x=0, n_z=4, E_z = vertical emittance, area/π)

We find for LEAR

Position k of quad.	Contribution to $dp.\frac{R}{B\rho}$ Re	Im	$\lvert\ \rvert$	Contribution to $\Delta e/E_z$ $[\mathrm{rad}^{-1}\mathrm{m}^{-1}]$
QF 4.08 / 4.55 -1.27	- 4.47 36.8	- 1.46 - 6.2	4.7 37.4	0.78 6.25
$\sum$	32.4	- 7.7	33.3	5.5
QD 4.93 / 5.40 1.41	81.2 39.3	- 7.4 -14.2	81.5 41.8	13.5 6.9
$\sum$	120.5	-21.6	122.4	20.4
$\sum$ QF+QD = 1 doublet :	152.9	-29.2	155.7	25.9

Conclude : Potential excitation of the resonance comes mainly from the QD where β_z and α_z are large.

This is the effect of 1 doublet. For the whole machine with 8 strictly equal doublets (position and gradient error = 0), the effect for the 11th harmonic is zero due to the phase factor in dp.

Effect of errors

Let us assume that gradient errors are such that :

$$\sqrt{\sum(\Delta k_D)^2} = 10^{-2} \quad \text{and} \quad \sqrt{\sum(\Delta k_F)^2} = 10^{-2}$$

corresponding to an r.m.s. error of

$$\Delta k_{rms} = \sqrt{\frac{1}{8}\sum(\Delta k)^2} = 10^{-2}/\sqrt{8} \approx 3.5\ 10^{-3}$$

for the k_F and the k_D. Then :

$$\Delta e/E_z/_{rms} = \sqrt{\sum(\Delta e_F/E_z)^2 + \sum(\Delta e_D/E_z)^2} = \sqrt{(5.5^2 + 20.4^2)}\cdot 10^{-4}$$
$$\approx 21 \times 10^{-2}\ [\mathrm{rad}^{-1}\mathrm{m}^{-1}]$$

where the numbers 5.5, 20.4 come from the table above and we have used square addition due to the phase factor in dp.

Taking $E_z = 40 \times 10^{-6}$ (π m rad) we expect :

$$\boxed{\Delta e_{rms} = 8.5 \times 10^{-6} \text{ (for gradient error of 1\%)}}$$

Crossing speed to have amplitude blow up g < 1.25 (CERN 76-06 p.79), as observed in LEAR :

$$\frac{1 - g_y}{2} = \frac{\Delta e \pi}{4} \sqrt{\frac{\beta c}{2\pi R} / (de/dt)}$$

$$\beta c / 2\pi R = f_o = 1.2 \text{ MHz}$$

$$\Delta e = 8.5 \times 10^{-6}$$

requires $\boxed{de/dt > 1.6 \times 10^{-3}}$ per sec.

In the LEAR MD-s the resonance was crossed with $\Delta Q \approx 0.02$ in 500 ms. $\Delta\dot{Q} \approx 0.04 \text{ s}^{-1}$ or $\Delta\dot{e} = 4\Delta\dot{Q} \approx 0.15 \text{ s}^{-1}$ which is two order of magnitude faster than estimated above. Thus one needs an gradient error of ≈ 10% to explain the supposed strength of the resonance. This seems a rather large error.

5. EFFECT OF SEXTUPOLE LENSES CLOSE TO THE DOUBLET

The deformation of the quadrupole stray field due to the close by sextupole lenses will be studied in a different note. From the above consideration one concludes that to explain the observed width of $4Q_V = 11$ in LEAR one needs a perturbation changing the stray octupole by several percent from one doublet to another. Although the sextupole shield has a strong effect we do not expect that it suffices to explain the observed excitation.

6. LOW BETA INSERTION

Lattice parameters for the LEAR low beta insertion[5)] quads for $Q_h \simeq 3.2$, $Q_v \simeq 2.75$ are given in the table below :

Lattice functions entrance and exit of low beta quads

Position from center of LS4		β_H (m)	β_V (m)	$\mu_H/2\pi$	$\mu_V/2\pi$	α_H	α_V	Remark
QI2 k = 2.5	1.5	4	22.5	0.18292	0.2394	2.3	14.9	entrance QI2
	2	10.8	22.6	0.19629	0.2425	14.1	-14.8	exit QI2
QI1 k = -2.4	2.2	17.2	17.1	0.19862	0.24416	17.8	-12.9	entrance QI1
	2.7	24.2	13.8	0.20214	0.24987	-6.8	5.1	exit QI1

One notes the large values of $\alpha = -\frac{1}{2}\beta'$ which will lead to a strong excitation.

Further since the insertion has the periodicity 1 all resonances will now very strongly be excited. In fact for $4\,Q_V = 11$ one finds in LEAR with the above parameters : $\Delta e/E_z = 247.5 \; [\text{rad}^{-1}\text{m}^{-1}]$.

(combined effect of all 4 insertion quads).

This is an order of magnitude larger than the effect of a normal LEAR doublet.

Hence with low beta the resonance $4\ Q_V = 11$ as well as other fourth order resonances will be very large and seriously restrict the choice of working point and admissible tune spreads.

Similar effects due to the stray field of the low beta quadrupoles are presumably important in other machines which have a low beta insertion such as $\bar{p}p$ or ep colliders.

We believe that in the SPS with its low beta instertion[6)7)] the excitation (stopband width per emittance $\Delta e/E$) is as large as in LEAR, thus restricting the permissible excursions of the working point. Quadrupole endfields will also be important in ACOL, as pointed out in ref. 9, where the tune shifts due to the pseudo octupoles are estimated

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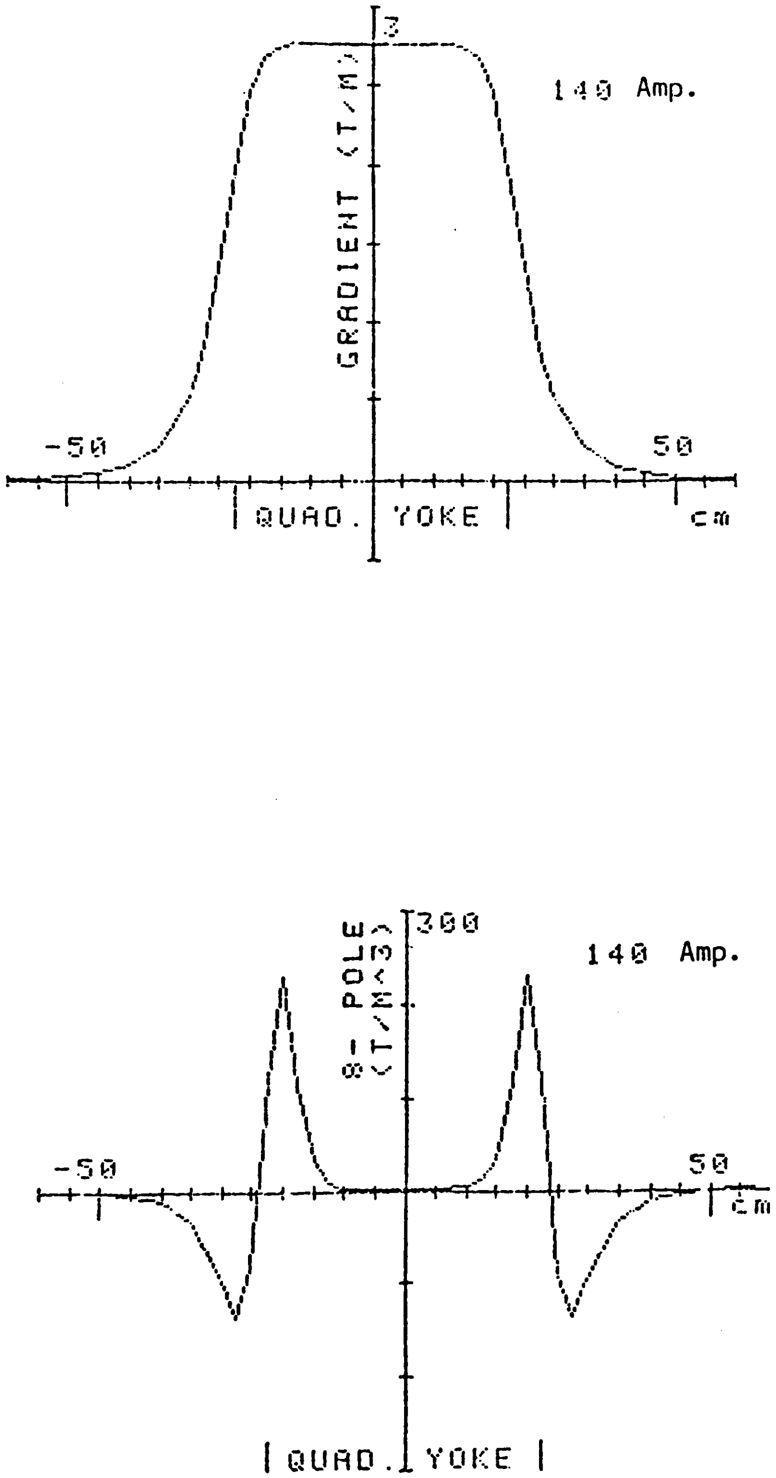

Fig. 1: Gradient and Octupole Component obtained from field measurement on LEAR quadrupole.