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A GENERAL ANALYTICAL EXPRESSION FOR THE CHROMATICITY OF ACCELERATOR RINGS

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ABSTRACT

We generalize the analytical approach, described in a previous note, to include magnets with inclined and/or curved boundaries. Evaluating the fringe field contribution in hard edge approximation, we arrive at relatively simple expressions for the chromaticity, general enough to cover all magnet types commonly used in proton synchrotrons and storage rings. Results are given for a number of small focusing rings for which the exact treatment of the fringe fields is important.

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C O N T E N T S

1. INTRODUCTION

In a previous note[1] we discussed methods used in the design of LEAR to calculate the chromaticity, i.e. the linear dependence of betatron wave number Q on momentum. Our approach, derived in ref.[1], starts from the equation of motion in a general time independent magnetic field[2]. Expanding these equations up to the second order in the deviation of a particle from the reference orbit and applying appropriate transformations, we obtained a differential equation from which the tune shift $\Delta Q(\Delta p)$ can be calculated using standard accelerator theory. The resulting expressions have been evaluated in ref.[1] for the case of magnets with orthogonal beam entry (sector magnets) as is appropriate for LEAR.

These analytical results agree well with those obtained by using the computer programme TRANSPORT[3] to calculate the chromaticity from the second order matrix elements, a formalism which was derived by S. Peggs[4]. But for small machines where the bending radius ρ is no longer large compared to the dispersion D most other programmes and methods disagree with our results. We believe that, with the exception of TRANSPORT, these programmes use approximations for the edge field which are not appropriate for the small rings.

In the present report we generalize the analytical approach to include magnets with inclined and/or curved boundaries. We need to calculate the dipole, quadrupole and sextupole components of the fringe field that enters into the chromaticity formula. To obtain these components, we assume that the field passes smoothly from its value well inside the magnet to zero field in the straight section. The transition takes place in a short edge region which follows the border curve of the magnet. By evaluating this transition in hard edge approximation, we arrive at relatively simple analytical expressions for the chromaticity, general enough to cover all common magnet types used in synchrotrons and storage rings.

2. THE GENERAL FORMULA

We repeat here for convenience, the general derivation given already in ref.[1] (Appendix 1).

The trajectory equations of a particle in a static magnetic field[2] are (note that we use for the horizontal coordinate a sign convention opposite to the one of Steffen[2]):

$$z'' + \frac{\dot{w}}{w^2} z' = \frac{e}{p}\frac{v}{w}\left[- x'B_s + (1 + hx)B_x\right]$$

$$x'' + \frac{\dot{w}}{w^2} x' - h(1 + hx) = \frac{e}{p}\frac{v}{w}\left[z'B_s - (1 + hx)B_z\right], \tag{2.1}$$

with $h(s) = \frac{1}{\rho}$ the curvature of the reference orbit, x the horizontal co-ordinate (counted positive radially outwards), z the vertical component and s the distance along the reference orbit. A prime denotes the derivative with respect to s, v the total particle velocity and $w = \dot{s}$ the velocity of the particle projection along the reference orbit. Introducing [2]

$$\frac{v}{w} = \sqrt{x'^2 + z'^2 + (1 + hx)^2}$$

$$\frac{\dot{w}}{w^2} = -\frac{1}{2}\frac{\frac{d}{ds}\left(\frac{v^2}{w^2}\right)}{\frac{v^2}{w^2}} \tag{2.2}$$

$$\frac{e}{p} = \frac{e}{p_o\left(1 + \frac{\Delta p}{p_o}\right)}$$

and using for the magnetic field components the expressions up to second order in z and x

$$\frac{e}{p_o} B_x(x,z,s) = - kz - rzx$$

$$\frac{e}{p_o} B_z(x,z,s) = h - kx - \frac{1}{2} rx^2 - \frac{1}{2}(h'' - hk - r)z^2 \tag{2.3}$$

$$\frac{e}{p_o} B_s(x,z,s) = h'z - (hh' + k')zx$$

with $$k(s) = -\frac{1}{B_o\rho}\frac{\partial B_z(0,0,s)}{\partial x} \quad \text{and} \quad r(s) = \frac{\partial k}{\partial x} = -\frac{1}{B_o\rho}\frac{\partial^2 B_z(0,0,s)}{\partial x^2}, \tag{2.4}$$

one can express the differential equations for the vertical and horizontal motion to second order in z,x and their derivatives :

$$\begin{aligned} &z'' - (hx' + h'x)z' = -[k - k\delta + 2khx + rx + h'x']z \\ &x'' - \frac{1}{2}hx'^2 - h'xx' = h\delta - [h^2 - k + k\delta - 2h^2\delta]x - [h^3 - 2kh - \frac{1}{2}r]\,x^2. \end{aligned} \tag{2.5}$$

Herein we use $\delta \equiv \frac{\Delta p}{p_o}$ for brevity.

For the chromaticity calculation, we expand these equations around the reference orbit by inserting $x = D\delta$ into the equation for the vertical and $x = \bar{x} + D\delta$ into the equation for the horizontal motion where D(s) (also called α_p or η in accelerator theory) is the dispersion function defined by the periodic solution $[D(s + 2\pi R) = D(s)]$ of

$$D'' + (h^2 - k)D = h \tag{2.6}$$

Neglecting all terms of higher than first order in $\bar{x}$, $\bar{x}'$ and δ, we arrive at the differential equations

$$\begin{aligned} &z'' - (hD)'z'\delta + [k + (2khD - k + rD + D'h')\delta]z = 0 \\ &\bar{x}'' - (hD)'x'\delta + [h^2 - k - (2h^2 + h'D' - 2Dh^3 - k + 4khD + rD)\delta]\bar{x} = 0. \end{aligned} \tag{2.7}$$

From these equations the familiar equations of motion can be extracted by setting $\delta \equiv 0$:

$$\begin{aligned} &z'' + kz = 0 \\ &\bar{x}'' + (h^2 - k)\bar{x} = 0 \end{aligned} \tag{2.8}$$

For the following consideration it will be useful to write eq.(2.7) in a more compact form :

$$y'' + 2gy' + (k + \Delta k)y = 0 \tag{2.9}$$

where y represents either the vertical (z) or the horizontal $(\bar{x})$ component and where $g \equiv -\frac{1}{2}(hD)'\delta$ and Δk are the perturbations linear in δ.

Using the transformation

$$y = ue^{-\int g ds} \tag{2.10}$$

we eliminate the y' term in eq. (2.9) and get

$$u'' + (k + \Delta k - g' - g^2)u = 0. \tag{2.11}$$

Dropping the g^2- term as it is of second order in δ, we obtain the tune shift due to the Δk and g'-term applying the Courant and Snyder theory (eq. 4.31) in ref.[5]) to eq. (2.11) above :

$$\Delta Q = \frac{1}{4\pi} \int (\Delta k - g')\beta ds. \tag{2.12}$$

Here $\beta(s)$ and Q are the betatron function and betatron wave number of the unperturbed motion. From eq. (2.12) we obtain the chromaticity defined as

$$\xi = \frac{\Delta Q/Q}{\delta}. \tag{2.13}$$

Thus the contribution to the chromaticity of any magnetic element can be expressed by

$$\begin{aligned} \xi_V &= -\frac{1}{4\pi Q_V} \{\int k\beta ds - \int rD\beta ds - 2\int khD\beta ds - \int [h'D' + \tfrac{1}{2}(hD)'']\beta ds\} \\ \xi_H &= -\frac{1}{4\pi Q_H}\{\int (h^2 - k)\beta ds + \int rD\beta ds + \int (h^2 - 2Dh^3 + 4khD)\beta ds + \int [h'D' - \tfrac{1}{2}(hD)'']\beta ds\} \end{aligned} \tag{2.14}$$

in the vertical (index V) and horizontal (index H) plane respectively. By adding up all contributions, one obtains the chromaticity of the ring. What remains to be done is to find for the different types of magnets, the quadrupole k and sextupole r components and to evaluate the integrals in eq. (2.14).

3. A SIMPLE METHOD TO DERIVE THE FIELD COMPONENTS IN THE FRINGE REGION

From eq. (2.3), the vertical field in the median plane can be written as

$$\frac{e}{p_o} B_z(x,o,s) = h - kx - \frac{1}{2} rx^2 \;\ldots\; \equiv h(s,x). \tag{3.1}$$

Here $h(s,x)$ may be interpreted as the curvature for a particle trajectory at x, s, $z = 0$ and $h(s,o) = h(s)$ is the curvature of the reference orbit as used above.

In the central part of the magnet h, k and r do not depend on s for "well designed" magnets or are slowly varying functions of s for more complicated magnets. Where necessary, we use the suffix "m" to refer to the values inside the central part.

From eq. (3.1), we find $k(s)$ and $r(s)$ as

$$\begin{aligned} k(s) &= -\left.\frac{\partial h(s,x)}{\partial x}\right|_{x=0} \\ r(s) &= -\left.\frac{\partial^2 h(s,x)}{\partial x^2}\right|_{x=0} \end{aligned} \tag{3.2}$$

We use these expressions for the fringe region. This almost solves our problem. We shall just have to express the partial derivatives with respect to x by those with respect to s in the edge region denoting the latter ones by

$$h' = \frac{\partial h(s,o)}{\partial s} = \frac{\partial h}{\partial s} \quad \text{and} \quad h'' = \frac{\partial^2 h(s,o)}{\partial s^2}$$

To express the transition from zero field to the value inside the magnet, we suggest the product

$$h(s,x) = H(\sigma,\xi)h_m(s,x) = H(A)h_m(s,x) \tag{3.3}$$

where $h_m(s,x)$ is the field well inside the magnet and is usually independent of s; $H(A)$ goes from 0 to 1 at the entrance (near $A(\sigma,\xi) = 0$). The cartesian system σ,ξ has its origin in the "hard edge point",that is the point where the reference orbit enters the idealized magnetic field border. It may be defined by

$$\int_{-s_o}^{s_m} H(\bar{s},x=0)d\bar{s} - s_m \doteq 0 \tag{3.4}$$

with s_o large enough such that $H(-s_o) \simeq 0$,

s_m large enough such that $H(s_m) \simeq 1$.

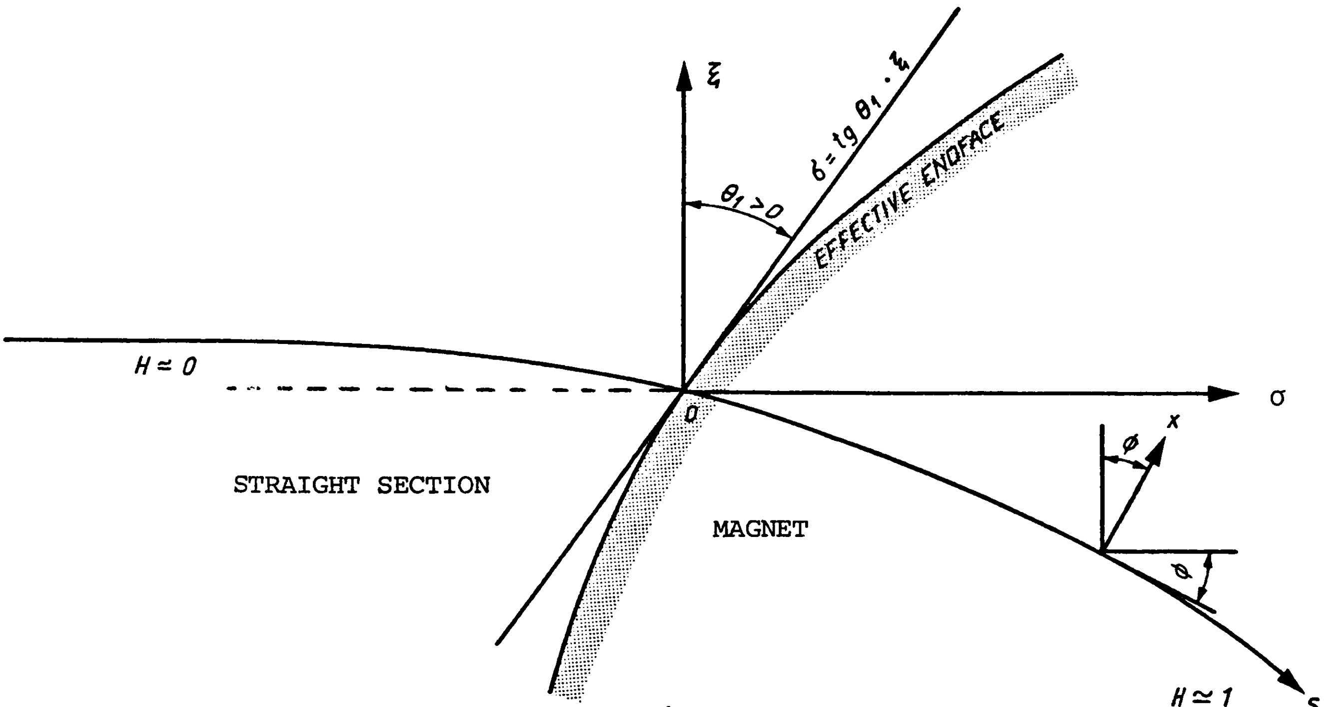

Fig. 1

From Fig. 1, one deduces :

$$\sigma(s,x) = \int_o^s \cos\phi(\bar{s})d\bar{s} + x\sin\phi(s)$$
$$\xi(s,x) = -\int_o^s \sin\phi(\bar{s})d\bar{s} + x\cos\phi(s) \tag{3.5}$$

where

$$\phi(s) = \int_{-s_o}^{s} h(\bar{s})d\bar{s} \tag{3.6}$$

is the angle between the (instantaneous) s-direction and the σ-axis (which is almost the extension of the reference orbit in the straight section).

A "hard edge line" can be defined by taking eq. (3.4) for $x \neq 0$. When using the function H(A) introduced above to describe the "field rise" in the edge region, it is general enough to take

$$A = \sigma - \mathrm{tg}\theta_1 \xi - \frac{1}{2}\tau\xi^2 \tag{3.7}$$

so that A = 0 describes a tilted and curved end face whose radius of curvature is $1/(\tau\cos^3\theta_1)$. Higher powers of ξ do not contribute to the chromaticity.

The azimuthal extent of the fringe field scales with the gap height g. For a gradient magnet g is a function of the radial position and thus contributes to the derivative $\partial A/\partial x$. But this contribution goes to zero in hard edge approximation ($g \to 0$) where H(A) becomes the unit step function and dH/dA the δ function. In this approximation eqs. (3.3) and (3.7) can be considered as appropriate.

4. THE QUADRUPOLE AND THE SEXTUPOLE COMPONENT FOR A GRADIENT MAGNET WITH TILTED AND CURVED END FACES

The fact that the edge function H(A) depends on A only allows one to replace the partial derivatives for k and r (eq. (3.2)) by derivatives with respect to s. Using

$$\frac{\partial H}{\partial x} = \frac{dH}{dA}\frac{\partial A}{\partial x}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{d^2H}{dA^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{dH}{dA}\frac{\partial^2 A}{\partial x^2}$$

one gets, with A given by eq, (3.7), along with eqs. (3.1) to (3.6), the quadrupole component k

$$\begin{aligned} k &= Hk_m - h_m \frac{dH}{dA}\{\sin\phi - (\mathrm{tg}\theta_1 + \tau\xi)\cos\phi\} \\ &= Hk_m + \frac{dH}{dA}\frac{\sin(\theta-\phi)}{\cos\theta} h_m \end{aligned}$$

where we have put

$$\mathrm{tg}\theta = \mathrm{tg}\theta_1 + \tau\xi.$$

We eliminate $\frac{dH}{dA}$ using

$$h' = h_m \frac{dH}{dA} \{\cos\phi + (tg\theta_1 + \tau\xi)\sin\phi\}(1 + xh)\Big|_{x=0} \tag{4.1}$$

$$= \frac{dH}{dA} \frac{\cos(\theta - \phi)}{\cos\theta} h_m$$

and get

$$\boxed{k = Hk_m + h' tg(\theta - \phi).} \tag{4.2}$$

For the sextupole component r we obtain

$$r = Hr_m - 2\frac{dH}{dA} \frac{\sin(\theta - \phi)}{\cos\theta} k_m + \frac{dH}{dA} \cos^2\phi \, \tau h_m - \frac{d^2H}{dA^2} \frac{\sin^2(\theta - \phi)}{\cos^2\theta} h_m$$

In order to eliminate $\frac{d^2H}{dA^2}$ we need

$$h'' = \frac{d^2H}{dA^2} \frac{\cos^2(\theta - \phi)}{\cos^2\theta} h_m + \frac{dH}{dA} \left[\frac{\sin(\theta - \phi)}{\cos\theta} h - \tau \sin^2\phi \right] h_m$$

which yields, together with eq. (4.1)

$$\boxed{\begin{aligned} r &= Hr_m - 2H'k_m \, tg(\theta - \phi) - h'' tg^2(\theta - \phi) + hh' tg^3(\theta - \phi) \\ &+ \tau h' \frac{\cos\theta}{\cos(\theta - \phi)} \left[1 - \frac{\sin^2\phi}{\cos^2(\theta - \phi)}\right] \end{aligned}} \tag{4.3}$$

For most cases it is sufficient to consider k and r in hard edge approximation. Then, eqs. (4.2) and (4.3) simplify to

$$k = Hk_m + h_m \, tg\, \theta_1 \, \delta(s) \tag{4.4}$$

$$r = Hr_m + \{\tau h_m - tg\,\theta_1 \left[2k_m + h_m^2 (1 + \frac{tg^2\theta_1}{2})\right]\} \delta(s) - h_m \, tg^2\theta_1 \, \delta'(s) \tag{4.5}$$

Signs are correct for an entrance. Signs for the exit depend on sign convention for θ and τ. Adopting that of Fig. 3, eq.(4.4) remains unchanged whilst r for the exit becomes

$$r = Hr_m + \{\tau h_m - tg\ \theta_2 [2k_m + h_m^2(1 + \frac{tg^2\theta_2}{2})]\}\delta(s) + h_m\ tg^2\ \theta_2\delta'(s) \qquad (4.6)$$

A magnet with upstream-downstream symmetry has the property of doubling - never of cancelling - the contributions from the entrance at the exit. This is also true for the last term of eq. (4.6) since, when folded with another function, $-\delta'(s)$ yields the derivative at the hard edge point and that has opposite signs at entrance and exit.

5. EVALUATION OF THE INTEGRALS

Integrating eq.(2.14) by parts, using [5]

$$\beta' = -2\alpha, \quad \alpha' = k\beta - \gamma$$

and eq.(2.6) to express D" in terms of D one gets

$$\xi_V = -\frac{1}{4\pi Q_V}\{\int k\beta ds - \int rD\beta ds - \int h(kD\beta + D\gamma)ds - \int h'D'\beta ds - \frac{1}{2}(h'D\beta + hD'\beta + 2hD\alpha)\Big|_{entr.}^{exit}\}$$

$$\xi_H = -\frac{1}{4\pi Q_H}\{\int(h^2-k)\beta ds + \int rD\beta ds + \int h(2kD\beta + 2D'\alpha - D\gamma)ds - \frac{1}{2}(h'D\beta - hD'\beta + 2hD\alpha)\Big|_{entr.}^{exit}\} \qquad (5.1)$$

In order to simplify the integration we use the hard edge model as discussed above to describe the longitudinal field distribution h(s) over a magnet:

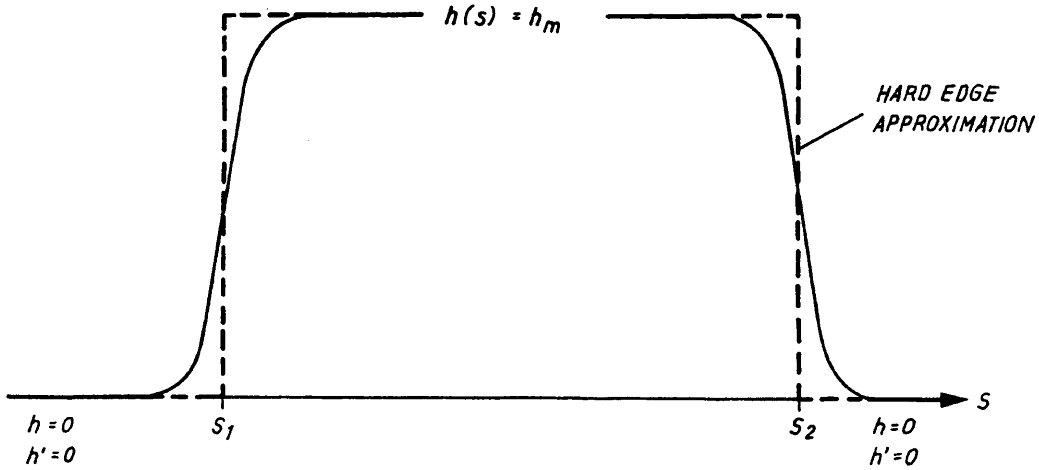

Fig. 2

Thus, the function h(s) is constant everywhere except at the points s_1 and s_2 where h'(s) becomes $\pm h_m$ times the delta function $\delta(s)$. The following integration rule[6)] holds the n-th derivative with respect to s of the delta function:

$$\int_{s-\varepsilon}^{s+\varepsilon} \delta^{(n)}(s-t)\, f(t)dt = (-1)^n\, f^{(n)}(s) \qquad (5.2)$$

Since in hard edge approximation we shall need only the r.h.s. of eqs. (4.4) and (4.5) we omit the suffix "m" from hereon. Thus all h, k and r refer to the central part.

Assuming that the central part of the magnet goes from s_1 to s_2 we evaluate the contributions to the chromaticity in three steps:

i) entrance (suffix "1")
ii) central part
iii) exit (suffix "2")

Central part

The contribution can be found from eq.(5.1). Using the hard edge approximation one gets

$$\Delta\xi_V = -\frac{1}{4\pi Q_V}\left\{\int_{s_1}^{s_2} k\beta ds - \int_{s_1}^{s_2} rD\beta ds - \int_{s_1}^{s_2} h(kD\beta + D\gamma)ds - \frac{1}{2}(hD'\beta + 2hD\alpha)\Big|_{s_1}^{s_2}\right\}$$

$$\Delta\xi_H = -\frac{1}{4\pi Q_V}\left\{\int_{s_1}^{s_2} (h^2-k)\beta ds + \int_{s_1}^{s_2} rD\beta ds + \int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma)ds + \frac{1}{2}(hD'\beta - 2hD\alpha)\Big|_{s_1}^{s_2}\right\} \qquad (5.3)$$

The various integrals can be expressed analytically by the values at s_1. With ℓ defined by $\ell = s_2 - s_1$ and introducing the following abbreviations

$$C = \begin{cases}\cos(\sqrt{K}\,\ell) \\ \cosh(\sqrt{|K|}\,\ell)\end{cases} \qquad S = \begin{cases}\dfrac{\sin(\sqrt{K}\,\ell)}{\sqrt{K}} \\ \dfrac{\sinh(\sqrt{|K|}\,\ell)}{\sqrt{|K|}}\end{cases} \qquad \text{for } K = h^2-k \begin{cases}> 0 \\ < 0\end{cases}$$

$$C2 = \begin{cases}\cos(2\sqrt{K_\beta}\,\ell) \\ \cosh(2\sqrt{|K_\beta|}\,\ell)\end{cases} \qquad S2 = \begin{cases}\dfrac{\sin(2\sqrt{K_\beta}\,\ell)}{\sqrt{K_\beta}} \\ \dfrac{\sinh(2\sqrt{|K_\beta|}\,\ell)}{\sqrt{|K_\beta|}}\end{cases} \qquad \text{for } K_\beta \begin{cases}> 0 \\ < 0\end{cases}$$

$$\text{where} \quad K_\beta = \begin{cases}k & \text{vertically} \\ K & \text{horizontally}\end{cases}$$

one obtains

$$\int_{s_1}^{s_2} K_\beta \beta ds = \frac{1}{2}\{(K_\beta\beta + \gamma)\ell + \alpha(C2-1) + \frac{1}{2}(K_\beta\beta - \gamma)S2\} \qquad (5.4)$$

$$\begin{aligned}\int_{s_1}^{s_2} D\beta ds = &\frac{1}{2K_\beta}(K_\beta\beta+\gamma)\left[(D-\frac{h}{K})S - D'\frac{1}{K}(C-1) + \frac{h}{K}\ell\right] \\ &+ \frac{h}{2KK_\beta}\;\frac{1}{2}(K_\beta\beta-\gamma)S2 + \alpha(C2-1) \\ &+ \frac{1}{4K_\beta-K}\;\{(D-\frac{h}{K})\alpha(K\cdot S\cdot S2 + 2C\cdot C2 - 2) + D'\alpha(2S\cdot C2 - C\cdot S2) \\ &+ \frac{1}{2K_\beta}(K_\beta\beta-\gamma)\left[(D-\frac{h}{K})(2K_\beta\cdot C\cdot S2 - K\cdot S\cdot C2) + D'(2K_\beta\cdot S\cdot S2 + C\cdot C2 - 1)\right]\}\end{aligned} \qquad (5.5)$$

For $K_\beta = 0$: $$\int_{s_1}^{s_2} D\beta ds = D\beta\ell + (D'\beta - 2D\alpha)\frac{\ell^2}{2} + (D\gamma - 2D'\alpha + \frac{1}{2}h\beta)\frac{\ell^3}{3} + (D'\gamma - h\alpha)\frac{\ell^4}{4} + h\gamma\frac{\ell^5}{10}$$

$$\int_{s_1}^{s_2} hD(k_\beta+\gamma)ds = h(k\beta+\gamma)\{(D-\frac{h}{K})S + \frac{1}{K}[D'(1-C) + h\ell]\}$$

$$\text{For } K = 0 : \quad \int_{s_1}^{s_2} hD(k\beta+\gamma)ds = h(k\beta+\gamma)\{D\ell + \frac{1}{2}D'\ell^2 + \frac{1}{6}h\ell^3\} \tag{5.6}$$

Since horizontally one has $C2 = C^2 - 1$, $S2 = 2SC$ and $K = K_\beta$ the corresponding integral can be written in terms of S, C and K only:

$$\begin{aligned}\int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma)ds = {} & h(K\beta+\gamma)(\frac{k}{K} - \frac{1}{2})\ \{(D-\frac{h}{K})S + \frac{1}{K}[D'(1-C) + \ell h]\} \\ & + h^2\frac{1}{K}(\frac{k}{K} + \frac{1}{2})\{(K\beta-\gamma)SC - 2\alpha KS^2\} \\ & + \frac{1}{3}h\{2(C^3-1)\left[2\alpha(D-\frac{h}{K})(\frac{k}{K}+\frac{1}{2}) - \frac{1}{K}D'(K\beta-\gamma)\right] \\ & -2KS^3\left[2\alpha D'(\frac{k}{K}+\frac{1}{2}) + (D-\frac{h}{K})(K\beta-\gamma)\right] \\ & +S(1+2C^2)\ \left[2\alpha D' + (D-\frac{h}{K})(K\beta-\gamma)(\frac{k}{K}+\frac{1}{2})\right] \\ & -(3C-1-2C^3)\left[2\alpha(D-\frac{h}{K}) - \frac{1}{K}D'(K\beta-\gamma)(\frac{k}{K}+\frac{1}{2})\right]\}\end{aligned} \tag{5.7}$$

For $K = 0$: $$\int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma)ds = h\{(2kD\beta + 2D'\alpha - D\gamma)\ell + (kD'\beta - 2kD\alpha + h\alpha - \frac{3}{2}D'\gamma)\ell^2 + (hk\beta - 4kD'\alpha + 2kD\gamma - \frac{5}{2}h\gamma)\frac{\ell^3}{3} + k(D'\gamma - h\alpha)\frac{\ell^4}{2} + hk\gamma\frac{\ell^5}{5}\}$$

<u>Entrance</u>

In order to carry out the integration of eq. (2.14) over the fringe region, we insert k and r as given in chapter 4, eq. (4.4) and eq. (4.5) respectively. We must be careful with the functions α and D' because of their discontinuity at the edges:

$$\text{vertical: } \alpha = \alpha_1 + \beta h\ tg\theta_1\ H(s)\ ;\ \text{horizontal: } \alpha = \alpha_1 - \beta h\ tg\theta_1\ H(s), \quad D' = D_1' + Dh\ tg\theta_1\ H(s) \tag{5.8}$$

But their average values $<\alpha>$ and $<D'>$, needed for the integration, are simply obtained by letting $<H(s)> = \frac{1}{2}$ in eq. (5.8).

Inserting the eqs. (4.4) and (4.5) into the eq. (2.14), using eq. (2.6) and the hard edge approximation (where H becomes the step function and H' the delta function), one finds for the contribution of the entrance edge applying the relation (5.2)

$$\Delta\xi_{V_1} = -\frac{1}{4\pi Q_V}\{h\beta_1 tg\theta_1 - \tau h D_1\beta_1 + tg\theta_1 [2k + h^2(1 + tg\theta_1)]D_1\beta_1$$

$$- \frac{1}{2} h^2 D_1\beta_1 \; tg^3\theta_1 - h tg^2\theta_1 < \frac{d}{ds}(D\beta) > - h^2 D_1\beta_1 \; tg\theta_1 - 2h\beta_1 <D'>$$

$$+ \frac{1}{2} h < \frac{d}{ds}(D\beta) > - \frac{1}{4} h^2 D_1\beta_1 \; tg\theta_1\}$$

$$\Delta\xi_{H_1} = -\frac{1}{4\pi Q_H}\{-h\beta_1 tg\theta_1 + \tau h D_1\beta_1 - tg\theta_1 [2k + h^2(1 + tg^2\theta_1)]D_1\beta_1$$

$$+ \frac{1}{2} h^2 D_1\beta_1 tg^3\theta_1 + h \; tg^2\theta_1 < \frac{d}{ds}(D\beta) > + 2h^2 D\beta \; tg\theta_1 + \frac{1}{2} h < \frac{d}{ds}(D\beta) >$$

$$- \frac{1}{4} h^2 D_1\beta_1 \; tg\theta_1\}$$

Carrying out the derivatives with respect to s and substituting the proper averages one obtains

$$\Delta\xi_{V_1} = -\frac{1}{4\pi Q_V}\{-\tau_1 h \beta_1 D_1 + \beta_1 \; tg \; \theta_1(h + 2kD_1) - h^2\beta_1 D_1 \; tg \; \theta_1(\frac{3}{2} - tg^2 \; \theta_1)$$

$$- h\alpha_1 D_1 \; (1 - 2 \; tg^2\theta_1) - h\beta_1 \; D_1' \; (\frac{3}{2} + tg^2 \; \theta_1)\} \tag{5.9}$$

$$\Delta\xi_{H_1} = -\frac{1}{4\pi Q_H}\{\tau_1 h\beta_1 \; D_1 - \beta_1 \; tg \; \theta_1(h + 2kD_1) + h^2\beta_1 D_1 \; tg \; \theta_1(\frac{3}{2} + tg^2 \; \theta_1)$$

$$- h\alpha_1 D_1(1 + 2 \; tg^2 \; \theta_1) + h\beta_1 D_1' \; (\frac{1}{2} + tg^2 \; \theta_1)\}$$

Exit

The contribution of the exit edge reads, with the sign convention as defined in Fig. 3,

$$\Delta\xi_{V_2} = -\frac{1}{4\pi Q_V}\{-\tau_2 h\beta_2 D_2 + \beta_2 \operatorname{tg}\theta_2(h + 2kD_2) - h^2\beta_2 D_2 \operatorname{tg}\theta_2(\frac{3}{2} - \operatorname{tg}^2\theta_2)$$

$$+ h\alpha_2 D_2 (1 - 2\operatorname{tg}^2\theta_2) + h\beta_2 D_2'(\frac{3}{2} + \operatorname{tg}^2\theta_2)\}$$

(5.10)

$$\Delta\xi_{H_2} = -\frac{1}{4\pi Q_H}\{\tau_2 h\beta_2 D_2 - \beta_2 \operatorname{tg}\theta_2(h + 2kD_2) + h^2\beta_2 D_2 \operatorname{tg}\theta_2(\frac{3}{2} + \operatorname{tg}^2\theta_2)$$

$$+ h\alpha_2 D_2 (1 + 2\operatorname{tg}^2\theta_2) - h\beta_2 D_2'(\frac{1}{2} + \operatorname{tg}^2\theta_2)\}$$

6. A CHROMATICITY FORMULA FOR PRACTICAL CALCULATION

Combining the three formulae for the contributions of the entrance, eq. (5.8), the central part, eq. (5.3), and the exit, eq. (5.9), using eq. (5.7) and the equivalent relations for the end field region one arrives at a practical formula to calculate the chromaticity contribution of a magnet element in terms of the lattice functions at the reference orbit:

$$\Delta\xi_V = -\frac{1}{4\pi Q_V}\{\int_{s_1}^{s_2} k\beta ds - \int_{s_1}^{s_2} rD\beta ds - \int_{s_1}^{s_2} hD(k+\gamma)ds$$

$$+\left[\operatorname{tg}\theta(h\beta + 2Dk\beta) - h\operatorname{tg}^2\theta(\beta D' - 2\alpha D - hD\beta\operatorname{tg}\theta) - \beta hD' - \tau h\beta D\right]\Big|_{"1"}$$

$$+\left[\operatorname{tg}\theta(h\beta + 2Dk\beta) + h\operatorname{tg}^2\theta(\beta D' - 2\alpha D + hD\beta\operatorname{tg}\theta) + \beta hD' - \tau h\beta D\right]\Big|_{"2"}\}$$

(6.1)

$$\Delta\xi_H = -\frac{1}{4\pi Q_H}\{\int_{s_1}^{s_2} (h^2-k)\beta ds + \int_{s_1}^{s_2} rD\beta ds + \int_{s_1}^{s_2} h(2kD\beta + 2D'\alpha - D\gamma)ds$$

$$+\left[-\operatorname{tg}\theta(h\beta + 2Dk\beta) + h\operatorname{tg}^2\theta(\beta D' - 2\alpha D + hD\beta\operatorname{tg}\theta) + \tau h\beta D\right]\Big|_{"1"}$$

$$+\left[-\operatorname{tg}\theta(h\beta + 2Dk\beta) - h\operatorname{tg}^2\theta(\beta D' - 2\alpha D - hD\beta\operatorname{tg}\theta) + \tau h\beta D\right]\Big|_{"2"}\}$$

where s_1 = beginning of the central part (H = 1)

s_2 = end " " " " "

"1" = entrance of the fringe region (H = 0)

"2" = exit " " " "

θ = entrance or exit angle of the trajectory

$\frac{1}{\tau\cos^3\theta}$ = radius of curvature of the end faces

$h = \frac{1}{\rho}$ = curvature of the reference orbit

$k = -\frac{1}{B\rho}\frac{\partial B_z(0,0,S)}{\partial x}$ is the quadrupole component } of the magnet profile

$r = -\frac{1}{B\rho}\frac{\partial^2 B_z(0,0,S)}{\partial x^2}$ is the sextupole component } of the magnet profile

D, D' are the dispersion function and its derivative

α, β, γ are the Twiss functions.

The various integrals were already evaluated, see eqs. (5.4) to (5.7).

The sign convention for θ, τ and ρ are all positive as shown in the following figure:

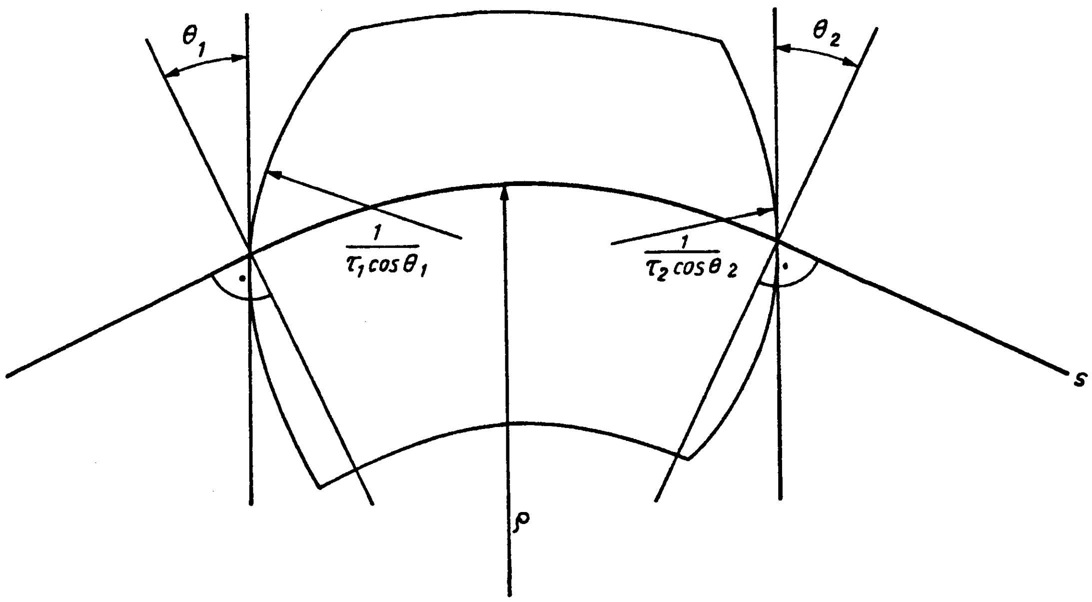

Fig. 3

Summing the contributions of all magnetic elements of the circumference one obtains the total chromaticity.

7. EXAMPLES

Eqs. (6.1) were evaluated to obtain the natural chromaticity of a number of small focusing rings. The results, summarized in Table 1, agree with the chromaticity obtained by the method of Peggs[4] which uses the program TRANSPORT[3]. All other computer programs known to us gave significantly different results for our small rings where D/ρ is not negligible. Details about the selected machines may be seen from Table 1 and from the computer outputs attached.

8. REFERENCES

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TABLE 1: CHARACTERISTICS AND CHROMATICITIES FOR SOME SMALL RINGS

	Type of bending magnets	Circumference/2π R (m)	Maximum bending radius ρ (m)	Maximum dispersion D (m)	Nominal working point Q_H ; Q_V	Natural chromaticity ξ_H ; ξ_V
LEAR	sector	12.5	4.64	3.63	2.27 ; 2.78	-1.42; -2.69
TARN	sector	5.06	1.32	1.52	2.42 ; 2.26	-0.68; -0.63
AA	parallel ended	24.91	7.30	10.01	2.28 ; 2.28	-0.65; -1.50
PSR	parallel ended	14.36	4.06	2.41	3.25 ; 2.25	-0.82; -1.30
SIS12A	parallel ended	31.25	10.0	2.71	4.2 ; 3.4	-1.51; 1.43
BPS-BOOSTER	parallel ended	22.35	9.87	1.74	4.73 ; 4.80	-1.10; -1.47

APPENDIX

Outputs of the Twiss functions for machines used as test cases

- LEAR, TARN
- AA, PSR, SIS12A
- BPSB

********** CERN: L E A R (QH = 2.27, QV = 2.78) **LOW ENERGY ANTIPROTON RING (running-in)**

CIRCUMFERENCE = 78.53982 4 HALF SUPERPERIODS WEDGE MAGNETS ALL VALUES AT EXIT OF ELEMENTS

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NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M2)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP'
INITIAL			0.0000			4.83034	1.87603	0.00000	0.00000	0.00000	0.00000	3.63396	-.000000
1	SL	4.0545	4.0545	0.00000	0.0000000	8.23361	10.63865	-.83938	-2.16121	.11114	.18103	3.63396	-.000000
2	QF	.5200	4.5745	0.00000	-1.2244380	12.41537	9.40058	-8.07098	4.27336	.11979	.18886	3.04879	-2.188189
3	SSS	.3300	4.9045	0.00000	0.0000000	18.32235	6.80330	-9.82899	3.59719	.12327	.19543	2.32669	-2.188189
4	QD	.5200	5.4245	0.00000	1.3874410	21.38597	5.56860	4.69331	-.93301	.12719	.20975	1.56660	-.826065
5	SS	1.0519	6.4764	0.00000	0.0000000	12.70390	7.90302	3.56073	-1.28633	.13736	.23509	.69770	-.826065
6	EDGE*	.0000	6.4764	0.00000	**********	12.70390	7.90302	3.52807	-1.26601	.13736	.23509	.69770	-.827858
7	BH1	1.1168	7.5932	240.90640	0.0000000	6.14363	10.57578	2.34588	-1.08067	.15753	.25439	-.10424	-.601274
8	EDGE*	.0000	7.5932	0.00000	**********	6.14363	10.57578	2.36576	-1.11489	.15753	.25439	-.10424	-.601612
9	SSS	.0158	7.6089	0.00000	0.0000000	6.06937	10.61095	2.34885	-1.11823	.15794	.25463	-.11371	-.601612
10	BH35	2.1355	9.7445	544.49180	0.0000000	.93420	12.52578	.05578	.31199	.33501	.28282	-.75232	.018389
11	EDGE*	.0000	9.7445	0.00000	**********	.93420	12.52578	.07861	.00583	.33501	.28282	-.75232	.000000
12	SSS	.0730	9.8175	0.00000	0.0000000	.92846	12.52536	.00000	-.00000	.34750	.28375	-.75232	.000000

DP/P	CUSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	CUSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
0.0000	-.91140	2.2700	-2.7987	12.5258	-.33874	2.7800	-5.7822	21.3860	3.6340	-6.5412

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* edge focusing to account for stray fields at end of ma gnet blocks, $k\ell$ = -0.00257, 0.00324, 0.02444 for the elements 6, 8, 11 respectively.

********** TOKYO: T A R N (QH = 2.42147, QV = 2.25941) **TEST ACCUMULATOR RING FOR NUMATRON (operating)**

CIRCUMFERENCE = 31.79488 8 SUPERPERIODS WEDGE MAGNETS ALL VALUES AT EXIT OF ELEMENTS

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NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M2)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP'
INITIAL			0.0000			2.57005	1.97908	-1.22183	1.07393	0.00000	0.00000	1.21651	-.351232
1	LLS	.8685	.8685	0.00000	0.0000000	5.42402	.93436	-2.06425	.12896	.03736	.11026	.91146	-.351232
2	QDH	.2630	1.1315	0.00000	2.8867942	5.42083	1.13686	2.07556	-.94948	.04483	.15274	.90850	.328380
3	LS	.4969	1.6284	0.00000	0.0000000	3.59990	2.49344	1.58901	-1.78060	.06277	.20042	1.07168	.328380
4	B	1.0450	2.6734	785.39820	0.0000000	1.34821	5.09620	.56581	-.17565	.14143	.24293	1.45642	.369747
5	LF	.1695	2.8429	0.00000	0.0000000	1.18453	5.16156	.39983	-.20994	.16284	.24819	1.51909	.369747
6	QFH	.2630	3.1059	0.00000	-1.7845642	1.17937	4.66521	-.37941	2.01891	.19930	.25655	1.52155	-.351232
7	LLS	.8685	3.9744	0.00000	0.0000000	2.57005	1.97908	-1.22183	1.07393	.28243	.30268	1.21651	-.351232

DP/P	CUSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	CUSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
0.0000	-.32501	2.4215	-2.5552	5.1616	-.20233	2.2594	-2.3667	5.4240	1.5216	2.0037

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********** CERN: A A (QH = 2.2811, QV = 2.2834) ANTIPROTON ACCUMULATOR * (operating)

CIRCUMFERENCE = 156.49240 2 HALF SUPERPERIODS STRAIGHT MAGNETS ALL VALUES AT EXIT OF ELEMENTS

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NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M2)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP'
INITIAL			0.0000			9.26265	7.35956	0.00000	0.00000	0.00000	0.00000	.00205	.000000
1	OD1H	.3750	.3750	0.00000	.1569347	9.07481	7.54242	.49724	-.49122	.00649	.00804	.00207	.000121
2	S3	5.7705	6.1455	0.00000	0.0000000	7.91279	18.69176	-.29586	-1.44090	.12571	.08882	.00276	.000121
3	QF1	.7500	6.8955	0.00000	-.2142088	9.46592	18.60935	-1.85748	1.54633	.13980	.09509	.00269	-.000322
4	S4	.4003	7.2958	0.00000	0.0000000	11.02835	17.40055	-2.04568	1.47339	.14604	.09863	.00256	-.000322
5	B2	4.9700	12.2658	680.64000	0.0000000	33.39899	7.70763	-1.37063	.63604	.18773	.16357	1.62817	.707872
6	S4	.4003	12.6661	0.00000	0.0000000	34.51013	7.22762	-1.40513	.56309	.18960	.17211	1.91154	.707872
7	QD2	.7500	13.4161	0.00000	.1569347	33.58521	7.09496	2.60185	-.38104	.19306	.18905	2.53528	.967661
8	S3	5.7705	19.1866	0.00000	0.0000000	11.26061	16.86727	1.26690	-1.31245	.24100	.27749	8.11917	.967661
9	QF2	.7500	19.9366	0.00000	-.2142088	10.75152	16.81674	-.56107	1.37711	.25208	.28444	8.34617	-.368414
10	S3	5.7705	25.7071	0.00000	0.0000000	21.29897	6.65871	-1.26675	.38323	.31435	.37623	6.22024	-.368414
11	QD3	.7500	26.4571	0.00000	.1569347	21.33051	6.75450	1.22594	-.51469	.31987	.39433	6.21642	.358152
12	S1	.3775	26.8346	0.00000	0.0000000	20.42164	7.16978	1.18164	-.58539	.32275	.40297	6.35162	.358152
13	B1	2.8900	29.7246	445.08000	0.0000000	11.35323	11.90621	1.64018	-1.10890	.35314	.45189	7.98544	.810728
14	S2	2.5030	32.2276	0.00000	0.0000000	5.17883	18.63062	.82662	-1.57764	.40606	.47877	10.01469	.810728
15	QF3	.7500	32.9776	0.00000	-.2142088	4.67664	18.75657	-.13036	1.41651	.43091	.48503	10.01329	-.814436
16	S2	2.5030	35.4806	0.00000	0.0000000	6.69162	12.66973	-.67467	1.01530	.50474	.51098	7.97475	-.814436
17	B1	2.8900	38.3706	445.08000	0.0000000	10.73469	8.24578	-.58342	.56721	.56012	.55507	6.33057	-.361860
18	S1	.3775	38.7481	0.00000	0.0000000	11.19296	7.84039	-.63055	.50670	.56560	.56255	6.19397	-.361860
19	QD4H	.3750	39.1231	0.00000	.1569347	11.43117	7.65176	-.00000	-.00000	.57086	.57029	6.12624	-.000000

DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
0.0000	.63466	2.2811	-2.7690	18.7566	.62908	2.2834	-1.2581	34.5101	10.0147	2.4287

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* Note: a simplified model of the AA with straight parallel ended magnets is used in the present calculation.

********** LOS ALAMOS SL: P S R (QH = 3.25, QV = 2.25) PROTON STORAGE RING (under construction)

CIRCUMFERENCE = 90.22400 10 SUPERPERIODS STRAIGHT MAGNETS ALL VALUES AT EXIT OF ELEMENTS

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NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M2)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP'
INITIAL			0.0000			6.62673	5.02783	-1.12290	1.27794	0.00000	0.00000	1.82525	-.254888
1	SL	2.2865	2.2865	0.00000	0.0000000	13.54534	1.92182	-1.90300	.08050	.03879	.13154	1.24246	-.254888
2	QD	.5000	2.7865	0.00000	.4510538	13.91974	2.19604	1.18255	-.64940	.04448	.17142	1.18332	.016108
3	SS	.4500	3.2365	0.00000	0.0000000	12.89034	2.91159	1.10501	-.94073	.04983	.19990	1.19057	.016108
4	BEND	2.5495	5.7859	628.31850	0.0000000	4.80021	11.08146	1.42041	-2.48478	.10236	.26886	2.00392	.665948
5	SS	.4500	6.2359	0.00000	0.0000000	3.64914	13.44886	1.13753	-2.77611	.11951	.27473	2.30360	.665948
6	QF	.5000	6.7359	0.00000	-.7691866	3.27544	13.60967	-.34281	2.47539	.14336	.28043	2.40804	-.254888
7	SL	2.2865	9.0224	0.00000	0.0000000	6.62673	5.02783	-1.12290	1.27794	.22500	.32500	1.82525	-.254888

DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
0.0000	-.45399	3.2500	-4.5653	13.6097	.15643	2.2500	-1.4887	13.9197	2.4080	3.1370

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********** DARMSTADT: S I S 12A (QH = 4.2, QV = 3.4) SCHWERIONEN INJEKTOR SYNCHROTRON (planned)

CIRCUMFERENCE = 196.34999 12 SUPERPERIODS STRAIGHT MAGNETS ALL VALUES AT EXIT OF ELEMENTS

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NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M2)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP'
INITIAL			0.0000			18.20022	5.47556	.80180	.67284	0.00000	0.00000	1.23869	-.249882
1	BEND	2.6180	2.6180	261.79940	0.0000000	13.53241	3.76990	.91972	-.01383	.02667	.09646	.93269	.013423
2	SS	.6500	3.2680	0.00000	0.0000000	12.39441	3.89997	.83106	-.18628	.03467	.12357	.94142	.013423
3	BEND	2.6180	5.8860	261.79940	0.0000000	8.28822	6.64146	.68333	-.87295	.07625	.20848	1.31690	.276728
4	LS	6.2565	12.1425	0.00000	0.0000000	6.66581	27.95001	-.42402	-2.53287	.23548	.28442	3.04825	.276728
5	QXF	.9900	13.1325	0.00000	-.4091095	10.99608	22.16806	-4.51956	7.57092	.25519	.29033	2.71329	-.930653
6	DRF	.5100	13.6425	0.00000	0.0000000	16.11285	15.12999	-5.51331	6.22923	.26129	.29477	2.23866	-.930653
7	QYF	.9900	14.6325	0.00000	.3621130	21.24461	8.59764	.95796	1.13183	.26931	.30943	1.67099	-.249882
8	DRF	1.7300	16.3625	0.00000	0.0000000	18.20022	5.47556	.80180	.67284	.28333	.35000	1.23869	-.249882

DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
0.0000	-.58779	4.2000	-6.7841	27.9500	-.20791	3.4000	-3.6080	21.2446	3.0483	5.4555

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