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COMPARISON OF METHODS TO EVALUATE THE CHROMATICITY IN LEAR

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CONTENTS

1. INTRODUCTION

The aim of this note is to describe the methods which we have used to calculate the chromaticity in LEAR, i.e. the dependence of the betatron wave number on momentum

$$\xi = \frac{\Delta Q/Q_o}{\Delta p/p_o}$$

which plays an important rôle in the design considerations for the correction and extraction system.

The obvious starting point was to run computer programs like SYNCH or AGS to determine the natural chromaticity (using the FXPT-tracking routine in SYNCH or the AGS lattice output for off-momentum particles). However, to our surprise, SYNCH and AGS gave very different results for the same LEAR lattice. In fact, it had been pointed out to us by B. Autin that care is necessary in interpreting chromaticity given by AGS for small machines. This was confirmed in discussions which we had with the SATURNE Group (J.L. Laclare, G. Leleux): using their computer programs they found a chromaticity for LEAR different from both the AGS and the SYNCH values.

We have therefore tried to compare other methods and programs available at CERN and to develop an independent method based on non-linear differential equations. The following machines were used as "test cases":

a) the 25 GeV Proton Synchrotron (PS)[1], an existing combined function machine with sector magnets;

b) the Antiproton Accumulator (AA)[2], an existing separated function machine (a simplified "model" of the AA with straight parallel ended magnets was taken for the purpose of comparison);

c) the Low Energy Antiproton Ring (LEAR)[3], a designed separated function machine with sector magnets.

Our results are summarized in Table 1, page 10.

2. THE COMPUTER PROGRAMS

2.1 AGS[4)]

In AGS, the chromaticity is calculated in two different ways (see Appendix III):

i) In the normal lattice output the program gives $Q' = \frac{\Delta Q}{\Delta p/p_o} = \xi Q_o$. For simplicity, only contributions from the quadrupole components are taken into account in this calculation as explained in the AGS write-up. They are computed using the formula of Gratreau - Leleux (see 3.2). This is sufficient for machines like the PS or larger ones where the chromatic effect of the bending magnets is negligible (D/ρ small compared to unity, with D the dispersion function and ρ the bending radius).

ii) The second more general possibility is to compute the lattice properties for off-momentum particles and to determine the chromaticity from the off-momentum tune values. Sextupole lenses can be included in these computations and their strength can be adjusted by a matching routine to give the desired chromaticity in both planes.

2.2 PATRICIA[5)]

PATRICIA is a computer program which allows the tracking of particles and the determination of an appropriate set of sextupoles for chromaticity corrections provided the linear lattice is known. The calculations are restricted to machines with rectangular bending magnets (straight parallel ends). An example of the PATRICIA run for the AA is given in Appendix IV.

2.3 SYNCH[6)]

SYNCH is, like AGS, a general lattice program; computations for off-momentum particles are possible (see Appendix V).
The chromaticity estimations are carried out in a similar way to that in AGS, either by using the normal lattice calculations (CYC-routine) which compute the Q' from a simplified analytical expression or by tracking off-momentum particles (FXTP-routine) and calculating chromaticities from the corresponding phase advances $2\pi Q$.

2.4 TRANSPORT[7)]

TRANSPORT is a general purpose program for the design of beam transport systems, but it includes no direct option to calculate the chromaticity of a circular machine.

However, the first and second order matrix elements calculated by TRANSPORT (see Appendix VI) can be used as a basis for another method proposed by S. Peggs (see 3.3). It should be easy to introduce this method into TRANSPORT in order to have included in this program a standard chromaticity routine.

3. ANALYTICAL METHODS

3.1 Method of B. Autin

The formalism of B. Autin [8)] to obtain the chromaticity is based upon the determination of the trace of the transfer matrix of a period of the machine. It can be summarized as follows:

Let $[A]$ denote the transfer matrix of an element (bending magnet, lens ...) which contributes to the tune shift; at least one element of $[A]$ can be expressed as a function of the momentum. The inverse of this matrix multiplied by its derivative with respect to the momentum forms a matrix $[C]$,

$$[C] = [A]^{-1} \cdot [\Delta A] = [A]^{-1} \cdot \left[\frac{dA}{d(\frac{p}{p_o})} \right] \frac{\Delta p}{p_o}$$

If the trace of $[C]$ is zero then the tune shift due to this perturbation element is given by

$$\Delta Q_{element} = - \frac{1}{4\pi} (2C_{11}\alpha - C_{12}\gamma + C_{21}\beta),$$

where the Twiss parameters α, β and γ are to be taken at the entrance of the element.
For our purpose, we consider the C_{ij} to first order in $\Delta p/p_o$. The contributions from all structures in the ring have to be added to determine the chromaticity.

An example will demonstrate this method:

The transfer matrix for the vertical motion in a thin lens is

$$[A] = \begin{bmatrix} 1 & 0 \\ -K\ell & 1 \end{bmatrix}$$

Since the strength of the lens is a function of the momentum

$$K = \frac{K_V}{p/p_o} = K_V(1 - \frac{\Delta p}{p_0} + \ldots) \qquad \text{with } K_V = -\frac{1}{B\rho}\frac{\partial B_z}{\partial x}$$

$[\Delta A]$ is given by

$$[\Delta A] = \begin{bmatrix} 0 & 0 \\ -K_V\ell & 0 \end{bmatrix} \cdot \frac{\Delta p}{p_o}$$

and one gets

$$C_{11} = C_{12} = C_{22} = 0$$

$$C_{21} = K_V\ell \frac{\Delta p}{p_o} + \ldots \text{ higher order terms}$$

Therefore the contribution of a thin lens to the tune shift in the vertical plane is to first order

$$\Delta Q_{V\ \mathrm{lens}} = -\frac{1}{4\pi} K_V\ell\, \beta_V \frac{\Delta p}{p_o}$$

In this method, the transfer matrices of the magnetic elements play the main rôle. In an unpublished note[9] B. Autin gives the exact transfer matrices for zero gradient magnets of both sector and parallel ended type. For a sector magnet with a bending angle ϕ and a bending radius ρ the matrices lead to the following expressions for the tune shift, up to first order in $\Delta p/p_o$:

$$\Delta Q_H = -\frac{1}{4\pi}\{ 2\alpha_1 \sin\phi(D_2' \cos\phi - \sin\phi) - \gamma_1\, \rho(\cos\phi \sin\phi + D_1' - D_2' \cos^2\phi) + \beta_1 \frac{1}{\rho} \sin\phi\, (D_2' \sin\phi + \cos\phi)\ \} \frac{\Delta p}{p_o} \qquad (1)$$

$$\Delta Q_V = -\frac{1}{4\pi}\{- \gamma_1\rho(\phi + D_1' - D_2') + \frac{1}{\rho}(\beta_2 D_2' - \beta_1 D_1')\}\frac{\Delta p}{p_o}$$

where "1" denotes the entrance and "2" the exit of the magnet. Together with the well-known contributions of the quadrupoles and the equivalent horizontal focusing ($K = -\frac{1}{\rho^2}$) of the bending magnets

$$\Delta Q_H = -\frac{1}{4\pi}\ \{\int (\frac{1}{\rho^2} - K_V)\ \beta_H\ ds\ \}\ \frac{\Delta p}{p_o}$$

$$\Delta Q_V = -\frac{1}{4\pi}\ \{\int K_V \beta_V\ ds\ \}\frac{\Delta p}{p_o} \qquad (2)$$

these formulae have been used for the chromaticity calculation of LEAR included in Table 1.

3.2 Formula of P. Gratreau - G. Leleux

A formula derived from a Hamiltonian formalism, first by P. Gratreau[10] and then given in a very concise derivation by G. Leleux[11] leads to the following expressions for the natural chromaticities:

$$\xi_H = -\frac{1}{4\pi\ Q_H}\left\{\int_o^C (\frac{1}{\rho^2} - K_V)\ \beta_H\ ds + \int_o^C \frac{2}{\rho}(D\ \beta_H\ K_V + \alpha_H\ D' - \frac{1}{2}\ \gamma_H D\ )\ ds\right\}$$

$$\xi_V = -\frac{1}{4\pi\ Q_V}\left\{\int_o^C K_V\ \beta_V\ ds - \int_o^C \frac{1}{\rho}\ D(\beta_V\ K_V + \gamma_V)\ ds\right\} \qquad (3)$$

C is the circumference of the machine, ρ the bending radius and $\alpha_{V,H}$, $\beta_{V,H}$ and $\gamma_{V,H}$ the Twiss parameters of the lattice.
The application of these relations to a machine with parallel ended magnets is not obvious, since the quadrupole effects due to the non-orthogonal beam entry and exit are not directly included in the formulae above (for the AA the corresponding result in Table 1 is denoted by (iii)). By replacing a straight magnet with a bending angle ϕ by a sector magnet enclosed between

$$\begin{bmatrix} 1 & 0 \\ \pm \frac{1}{\rho}\tan\frac{\phi}{2} & 1 \end{bmatrix}$$

(+ ... for the horizontal plane, - ... for the vertical plane) this can be overcome (for the AA the corresponding result in Table 1 is denoted by (iv)) .

A further remark is in order: In ref 11, the magnetic field $\vec{B}$ = rot $\vec{A}$ is derived from a vector potential which has only a longitudinal (s) component. Hence the contributions of the stray field B_s at the magnet edges are neglected both for straight and for sector magnets.

3.3 Method of S. Peggs

S. Peggs' method[12] needs the elements of the first and second order matrices R_{ij} and T_{ijk} of the computer program TRANSPORT. The analysis described in more detail in his thesis[13] gives the betatron wave number as a function of the momentum spread $\delta = \Delta p/p_o$ in terms of the TRANSPORT matrices for a periodical machine:

$$\cos\left[2\pi Q_H(\delta)\right] = \cos\left[2\pi Q_H(0)\right] + \frac{\delta}{2}\{(2T_{111} + T_{212})D + T_{116} + T_{226}\}$$

$$\cos\left[2\pi Q_V(\delta)\right] = \cos\left[2\pi Q_V(0)\right] + \frac{\delta}{2}\{(T_{313} + T_{414})D + T_{336} + T_{446}\}$$

The dispersion function can also be expressed as

$$D = \frac{1}{2T_{111}\delta}\{(1 - R_{11} - T_{116}\delta) - \sqrt{(1 - R_{11} - T_{116}\delta)^2 - 4T_{111}\delta(T_{166}\delta + R_{16})}\}$$

Similar relations can also be obtained for the variation of the Twiss parameters $\beta(\delta)$ and $\alpha(\delta)$ with the momentum. The machine tune for zero momentum spread can be calculated from the diagonal elements of the matrix $[R]$ for one revolution

$$\cos\left[2\pi Q_H(0)\right] = \frac{1}{2}(R_{11} + R_{22})$$

$$\cos\left[2\pi Q_V(0)\right] = \frac{1}{2}(R_{33} + R_{44})$$

These equations permit one to obtain the relevant information from the TRANSPORT output.

Peggs' method is based on the observation that the Twiss transfer matrix $[\overline{R}]$ for the motion around the off-momentum orbit can be represented in terms of the first $[R]$ and second order $[T]$ matrix for the central ray. One obtains e.g. for horizontal motion

$$\overline{R}_{11} = R_{11} + (2T_{111}\,D + T_{112}\,D' + T_{116})\delta$$

$$\overline{R}_{12} = R_{12} + (2T_{122}\,D' + T_{112}\,D + T_{126})\delta$$

$$\overline{R}_{21} = R_{21} + (2T_{211}\,D + T_{212}\,D' + T_{216})\delta$$

$$\overline{R}_{22} = R_{22} + (2T_{222}\,D' + T_{212}\,D + T_{226})\delta$$

The chromatic tune values are obtained from the transfer matrix $[\bar{R}]$ for one turn by interpreting

$$\bar{R}_{11} + \bar{R}_{22} = 2\cos[2\pi Q_H(\delta)] \qquad \text{etc.}$$

The method implies that the determinand of $[\bar{R}]$ is unity which can be used as an "accuracy test". By expansion for small ΔQ and assuming $\sin(2\pi Q) \neq 0$ one obtains the following practical formulae for ξ

$$\begin{aligned} \xi_H &= \frac{-100}{4\pi Q_H \sin(2\pi Q_H)} \{2T_{111} + T_{212}) D + T_{116} + T_{226}\} , \\ \xi_V &= \frac{-100}{4\pi Q_V \sin(2\pi Q_V)} \{(T_{313} + T_{414}) D + T_{336} + T_{446}\} \end{aligned} \tag{4}$$

The factor 100 comes in due to the convention of TRANSPORT to use δ in units of percent.

A further remark is worth mentioning: since the accuracy of this method depends only on the second order matrix elements T_{ijk}, great care should be taken in the input of the program TRANSPORT to get all second order contributions. For example, to take into account the edge effects of a magnet a pole-face rotation instruction must immediately precede the first and follow the last magnet element instruction. Even for a sector magnet where no pole-face rotations are necessary in first order, "pseudo"-rotations by an angle zero must be included as they contribute to the second order matrix elements.

3.4 A method based on non-linear differential equations

Following the Courant and Snyder [14] formalism, we derive from the 2nd order differential equations given, for example by K. Steffen[15], formulae for the chromaticity contributions of quadrupoles and sector magnets. As outlined in more detail in Appendix I, this gives the relations

$$\xi_H = -\frac{1}{4\pi Q_H}\left\{\int_0^C (h^2 - K_V)\beta_H \, ds + \int_0^C h(2K_V D\beta_H + 2D'\alpha_H - D\gamma_H)\, ds\right\} \tag{5}$$

$$\xi_V = -\frac{1}{4\pi Q_V}\left\{\underbrace{\int_0^C K_V\beta_V ds}_{\text{focusing}} - \underbrace{\int_0^C hK_V D\beta_V \, ds}_{\text{change of focusing strength}} - \underbrace{\int_0^C hD\gamma_V \, ds}_{\text{change of orbit length}} - \underbrace{\int_0^C h'D'\beta_V \, ds}_{\text{edge effect due to stray field } B_s}\right\}$$

term due to: focusing; change of focusing strength; change of orbit length; edge effect due to stray field B_s

with $h(s) = \frac{1}{\rho}$. For the horizontal plane the formula is identical to that of Gratreau-Leleux but in the vertical plane they differ by the "edge-effect" term, which using hard edge approximation[15] can be expressed as

$$\int_0^C h'D'\beta\, ds = \sum \frac{1}{\rho} (D_1'\beta_1 - D_2'\beta_2)$$

Here again "1" denotes the entrance and "2" the exit of a magnet.

The advantage of the form (5) for the vertical chromaticity ξ_V is that it exhibits the origin of the different contributions and can therefore be helpful for a physical interpretation of these contributions. For practical calculations the following equivalent relation (see Appendix II) may be useful:

$$\xi_V = -\frac{1}{4\pi Q_V}\left[\int_0^C K_V\beta_V ds - \int_0^C h(D\gamma_V + 2D'\alpha_V - h\beta_V + h^2 D\beta_V)\, ds\right] \quad (5a)$$

4. COMPARISON OF THE RESULTS AND THE CHROMATICITY IN LEAR

Results are summarized in Table 1 below.

For the PS, all methods applied led to chromaticities which agree within a few percent error with each other and with the measurements. Since the PS bending radius is about 70 m and the dispersion function $D \lesssim 3$ m, the contributions of the bendings are negligible. For this machine where $D_{max} << \rho$ all the approaches considered are satisfactory.

For the AA, which has a bending radius of about 7 m and a maximum dispersion function of about 11 m, the tracking for off-momentum particles in SYNCH gives the correct chromaticity. The other big computer programs (AGS, PATRICIA) are unsatisfactory. Since the "simplified AA" considered has parallel-ended bending magnets, the formulae of Gratreau-Leleux as we interpret them cannot be used as mentioned in section 3.2. Replacing the straight magnets by sector magnets enclosed between the edge-rotation matrices the application of the same formulae give an acceptable value only in the horizontal plane but not in the vertical since there the edge effect contribution of the bending magnets in the form

$$\Delta\xi_V = -\frac{1}{4\pi Q_V}\left\{\sum (1 + \tan^2\frac{\phi}{2}) \left(-\frac{1}{\rho}\beta_1 D_1' + \frac{1}{\rho}\beta_2 D_2'\right)\right\} \quad (6)$$

is missing, The other two analytical possibilities, namely Autin's approach which was developed for the AA, and Pegg's method, lead to results which agree very well with the measured chromaticities.

For LEAR (bending radius and D_{max} both about 4 m) the off-momentum tracking calculations of both AGS and SYNCH failed. The same values for the horizontal chromaticity in LEAR are obtained using the formulae of Gratreau-Leleux, Peggs and Autin. In the vertical plane only the results of Autin's and Pegg's methods agree with each other. The edge effect neglected by Gratreau-Leleux changes the result by about 20%.

Since also the approach based on non-linear differential equations gives chromaticities which agree exactly with the results obtained by other analytical methods we believe that the natural chromaticity of LEAR is close to the values

$$\xi_H = -1.29$$

$$\xi_V = -2.76.$$

TABLE 1 : CHROMATICITIES EVALUATED BY DIFFERENT METHODS

	P S ρ = 70.08 m (Q_H = 6.25 ; Q_V = 6.25)		A A ρ = 7.3 m ; ρ = 6.49 m (Q_H = 2.2811 ; Q_V = 2.2834)		L E A R ρ = 4.29 m ; ρ = 3.99 m (Q_H = 2.33 ; Q_V = 2.75)	
	ξ_H	ξ_V	ξ_H	ξ_V	ξ_H	ξ_V
AGS (i) (ii)	-1.04 -0.93	-1.04 -0.97	-1.21 -0.89	-0.55 -0.55	-1.17 -0.60	-2.11 -2.18
PATRICIA	-	-	-0.89	-2.95	-	-
SYNCH (i) (ii)	-1.03 -0.89	-1.04 -0.98	-0.90 -0.65	-0.76 -1.49	-0.90 -2.32	-2.11 -2.77
TRANSPORT	-	-	-	-	-	-
Method of B. Autin	-	-	-0.65	-1.49	-1.29	-2.76
Formula of P. Gratreau-G. Leleux	-0.85	-1.09	(iii) -0.98 (iv) -0.68	-0.74 -0.90	-1.29	-2.19
Method of S. Peggs	-0.85	-0.98	-0.65	-1.50	-1.29	-2.76
Non-linear different equations	-0.85	-1.09	(iv) -0.68	-1.48	-1.29	-2.76
Measurements	-1.00	-1.00	-0.62	-1.56		

(i) Normal lattice output
(ii) Tracking routine
(iii) Pure application of the formulae (3) to a machine with rectangular magnets
(iv) Rectangular magnets replaced by sector magnets enclosed between "short lenses" with $K\ell = \frac{1}{\rho} \tan \frac{\phi}{2}$

5. CONCLUSION

The problem of estimating the natural chromaticity of LEAR has led us to compare various computer programs available at CERN and analytical methods.

Whereas all programs work in a satisfactory way for large machines, care has to be taken for small synchrotrons and storage rings where the dispersion function is not negligible compared to the bending radius. In this situation the tracking routine of SYNCH gives the correct chromaticity for parallel ended magnets. The other programs fail in all situations.

From the non-linear differential equations we have derived analytical formulae to calculate the natural chromaticity which for the horizontal plane turn out to be identical with the ones of Gratreau-Leleux but differ in the vertical plane by the edge effect. The formulae of Gratreau-Leleux are also applicable for machines with parallel ended magnets if the magnets are replaced by sector magnets enclosed between "short lenses". Also in this case the edge effect contributions should be taken into account to get acceptable results for the vertical plane.

The approach of Peggs' and Autin's method - both based on matrix formalism - seem to work in a satisfactory manner for both types of magnets. Thus these two methods or the Gratreau-Leleux formulae augmented by the edge effect term (eq.(5)) can be used to calculate the natural chromaticity of small machines.

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REFERENCES

1. E. Regenstreif, CERN 59-29 (1959).

2. H. Koziol, AA parameter list, PS/AA/Note 80-1 (1980).

3. W. Hardt, P. Lefèvre, D. Möhl and G. Plass, CERN/PS/DL 80-13 (1980).

4. E. Keil, Y. Marti, B.W. Montague and A. Sudboe, CERN 75-13 (1975).

5. H. Wiedemann, PTM-230 (1980), SLAC.

6. A.A. Garren and J.W. Eusebio, UCID-10153 (1965), LBL.

7. K.L. Brown, D.C. Carey, Ch. Iselin and F. Rothaker, CERN 80-04 (1980).

8. B. Autin, XIth Int. Conf. on High Energy Accelerators, CERN, July 1980, pp. 661.

9. B. Autin, The exact equations of the betatron oscillations, to be published.
   We thank B. Autin for making this note available to us prior to publication.

10. P. Gratreau, NT/12-65, Orsay (1965).

11. G. Leleux, Unpublished manuscript. Thanks are due to the author for communicating this work to us.

12. S. Peggs, CBN 76-22, Cornell University (1976).

13. S. Peggs, Thesis, Cornell University (1981).

14. E.D. Courant and H.S. Snyder, Ann. Phys. 3,1 (1958).

15. K. Steffen, High energy beam optics, Interscience Publishers, New York, 1965.

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APPENDIX 1

An independent derivation of the chromaticity

1. An approach based on differential equations

We start from the trajectory equations of a particle in a static magnetic field, as given by K. Steffen[15], but with opposite sign convention for the x-coordinate:

$$z'' + \frac{\dot{w}}{w^2} z' = \frac{e}{p}\frac{v}{w}\left[-x' B_s + (1 + hx) B_x\right]$$

$$x'' + \frac{\dot{w}}{w^2} x' - h(1 + hx) = \frac{e}{p}\frac{v}{w}\left[z' B_s - (1 + hx) B_z\right] \tag{A1}$$

with

$h(s) = \frac{1}{\rho}$	: curvature of the reference orbit
$w = \dot{s}$	: velocity of particle projection along the reference orbit
v	: total particle velocity
s	: distance along reference orbit (' indicates the derivative with respect to s)
x, z	: radial and vertical deviation from the reference orbit.

Within a small range of S the co-ordinates z, x, s can be viewed as a cyclindrical system z, r, θ with $r = \rho + x$ and $\theta = s/\rho$

The terms v^2/w^2 and $\dot{w}/w^2$ are given by[15]

$$\frac{v^2}{w^2} = x'^2 + z'^2 + (1 + hx)^2$$

$$\frac{\dot{w}}{w_2} = -\frac{1}{2}\frac{d/ds\,(v^2/w^2)}{(v^2/w^2)} \tag{A2}$$

These equations so far include no approximation.

To calculate the chromaticity we expand them around the reference orbit characterized by $p = p_o$, $h = h_o = e\frac{B_o}{p_o}$, $s = s_o$. We retain terms up to second order in x, z and their derivatives and insert $x = D\delta$ with $\delta = \Delta p/p_o$ into the vertical and $x = \bar{x} + D\delta$, $z = 0$, $z' = 0$ into the horizontal equation, i.e. we define the chromaticity in one plane by referring to particles of zero betatron amplitude in the other plane. In the same spirit we neglect terms with $\bar{x}^2$, $\bar{x}'^2$, δ^2 ... thus referring to small amplitude oscillations around an off-momentum orbit for small Δp but mixed terms like $\bar{x}\,D\delta$, $\bar{x}'D'\delta$ ... have to be retained as they give a linear contribution to the chromatic frequency shift.

We then arrive at differential equations of the type

$$y'' + (k + \Delta k)y + 2g\, y' = 0 \tag{A3}$$

where g and Δk are the perturbations linear in δ. We elimintate the y' term by the transformation

$$y = u \exp(-\textstyle\int g\, ds)$$

which yields

$$u'' + (k + \Delta k - g' - g^2)u = 0 \tag{A4}$$

Dropping the second order g^2 term, we obtain the frequency shift from the Courant and Snyder[14)] theory as

$$\Delta Q = \frac{1}{4\pi}\int(\Delta k - g')\ \beta\ ds \tag{A5}$$

What remains to be done is to find field representations for the magnetic elements. This will be carried out below for sector magnets.

2. Application to sector magnets

With the field expansion for a sector magnet

$$\frac{e}{p_o} B_z = h - Kx$$

$$\frac{e}{p_o} B_x = - Kz \qquad \text{with } K = K_V = - \frac{1}{\rho_o B_o} \frac{\partial B_z}{\partial x}$$

$$\frac{e}{p_o} B_s = h'z$$

we obtain the following differential equations for z and x up to second order:

$$z'' - (hx' + h'x)z' = - \left[K - K\delta - 2Khx + x'h'\right] z$$

$$x'' - \frac{1}{2} hx'^2 - h'xx' = h\delta - \left[h^2 - K + K\delta - 2h^2\delta\right]x - \left[h^3 - 2Kh\right] x^2$$

Putting $x = D$ in the equation for the vertical plane and substituting $x = \overline{x} + D\delta$ (where $\overline{x}$ presents the deviation from the off-momentum orbit $D\delta$) in the equation for the horizontal plane, we get

$$z'' - (hD)'\delta z' + \left[K + (h'D' + 2KhD - K)\delta\right] \; z = 0$$

$$\overline{x}'' - (hD)'\delta\overline{x}' + \left[h^2 - K - (h'D' + 2h^2 - 2Dh^3 + 4Kh - K)\delta\right] \overline{x} = 0$$

By virtue of (A3) - (A5) these equations yield the tune shifts

$$\frac{\Delta Q_V}{\delta} = \frac{1}{4\pi} \int \left[h'D' + \frac{1}{2}(hD)'' + 2KhD - K\right] \beta ds$$

$$\frac{\Delta Q_H}{\delta} = - \frac{1}{4\pi} \int \left[h'D' - \frac{1}{2}(hD)'' + 2h^2 - 2Dh^3 + 4KhD - K\right] \beta ds$$

Integrating by parts, using the identities

$$\beta' - 2\alpha \quad , \quad \alpha' = K\beta - \gamma$$

and the differential equation for the dispersion function

$$D'' + (h^2 - K)D = h$$

and taking into account that $h = 0$, $h' = 0$ far enough outside the magnet, one finds the following expressions for the chromaticity in the vertical and horizontal plane respectively:

$$\xi_V = -\frac{1}{4\pi Q_V}\{\int K\beta_V ds - \int hD(K\beta_V + \gamma_V)\,ds - \int h'D'\beta_V ds\}$$

$$\xi_H = -\frac{1}{4\pi Q_H}\{\int (h^2 - K)\beta_H\,ds + \int h(2KD\beta_H + 2D'\alpha_H - D\gamma_H)\,ds\}$$

3. A further derivation for sector dipole magnets

Since our formula for the vertical chromaticity is different from the one of Gratreau-Leleux we give yet another derivation of the vertical contribution for the special case of dipole sector magnets:

For $\delta = 0$ the magnet simply acts as a drift space with length $\ell = \int_M ds$ in the vertical plane. For $\delta \neq 0$ the drift length is changed to

$$\ell(\delta) = \int_M (1 + hD\delta)\,ds$$

to first order in δ. In addition the off-momentum orbit enters and leaves the magnet with an edge angle $D_1'\delta$ and $D_2'\delta$ respectively. We represent the corresponding edge focusing by a thin lens with

$$K\ell = \frac{1}{\rho_o}\tan(D'\delta) \approx \frac{1}{\rho_o}D'\delta$$

We calculate the contributions of the changed orbit length and the non-orthogonal beam entry and exit to the tune shift Q_V following the method of Courant and Snyder (eq. 4.24 - 4.30). In our case, the matrices are given by:

$$m_o = \begin{bmatrix} 1 & ds_o \\ 0 & 1 \end{bmatrix} \quad , \quad m = \begin{bmatrix} 1 & (1+hD\delta)ds_o \\ -h'D'\delta ds_o & 1 \end{bmatrix}$$

We need the trace of

$$[M] = mm_o^{-1}M_o = \begin{bmatrix} 1 & hD\delta ds_o \\ -h'D'\delta ds_o & 1 \end{bmatrix} \begin{bmatrix} \cos\mu_o + \alpha\sin\mu_o & \beta\sin\mu_o \\ -\gamma\sin\mu_o & \cos\mu_o - \alpha\sin\mu_o \end{bmatrix},$$

$$\mathrm{Tr}[M] = 2\cos\mu = 2\cos\mu_o - \gamma hD\delta ds_o \sin\mu_o - \beta h'D'\delta ds_o \sin\mu_o$$

and get for $\sin\mu \neq 0$ with $\mu = \mu_o + \Delta\mu$

$$2\Delta\mu = 4\pi\Delta Q = \gamma h D \delta ds_o + \beta h' D' \delta ds_o$$

Adding the contributions over the whole circumference we obtain for the chromaticity contribution of sector dipole magnets in the vertical plane

$$\xi_V = \frac{1}{4\pi Q_V}\{\int hD\gamma_V ds + \int h'D'\beta_V ds\}$$

This agrees exactly with the formula derived from the non-linear differential equation putting K = 0.
This derivation helps in interpreting the contributions physically: the first integral is due to the changed orbit length and the second one is due to the edge focusing.

APPENDIX II

Equivalence between different formulae

1. Vertical plane: equivalence of the equations (5) and (5a) with (1), (2)

Using the differential equation for the dispersion function $D'' + D(h^2 - K) = h$ and the identity $\beta' = -2\alpha$, taking into account that $h = 0$ far enough outside the magnet, and integrating by parts, one finds that the magnet contribution to (5) may be expressed as

$$- \int h(KD\beta + D\gamma)ds - \int h'D'\beta ds = - \int h(KD\beta + D\gamma)ds + \int h(D''\beta + D'\beta')ds$$

$$= - \int h(KD\beta + D\gamma - h\beta + h^2D\beta - KD\beta - D'\beta')ds$$

$$= - \int h(D\gamma + 2D'\alpha - h\beta + h^2D\beta)ds$$

In order to prove that the corresponding term in Autin's formula

$$- \gamma_1\rho(\phi + D_1' - D_2') + \frac{1}{\rho}(\beta_2D_2' - \beta_1D_1')$$

is equal to the above expression with $K = 0$ we divide the magnet into infinitesimal slices of length $d\phi$. Applying the formula to the i-th slice gives

$$- \gamma_i\rho_i(d\phi + D_i' - D_{i+1}') + \frac{1}{\rho_i}(\beta_{i+1}\, D_{i+1}' - \beta_iD_i')$$

Adding over all slices and putting $ds = \rho_i\, d\phi$ leads to

$$- \int \gamma\, ds + \Sigma\gamma_i\rho_i(D_{i+1}' - D_i') + \Sigma\, \frac{1}{\rho_i}(\beta_{i+1}\, D_{i+1}' - \beta_iD_i')$$

The sums can be expressed as

$$\Sigma\, \gamma_i\rho_i\, \frac{D_{i+1}' - D_i'}{\Delta s}\, \Delta s \quad \text{and} \quad \Sigma\, \frac{1}{\rho_i}\, \frac{\beta_{i+1}D_{i+1}' - \beta_iD_i'}{\Delta s}\, \Delta s$$

which for infinitesimal Δs becomes $\int \gamma\rho\, D''\, ds$ and $\int \frac{1}{\rho}(\beta D')'\, ds$.
Substitution of $D'' = h - Dh^2$ gives then

$$- \int \gamma ds + \int \gamma ds - \int \gamma hDds + \int h\beta'D'ds + \int h^2\beta ds - \int h^3D\beta ds$$

and finally

$$- \int h(D\gamma + 2D'\alpha - h\beta + h^2D\beta)ds$$

2. <u>Horizontal plane</u>: equivalence of the equations (5) and (1), (2)

Neglecting terms with $d\phi^2$, Autin's formula

$$2\alpha_1 \sin\phi(D_2' \cos\phi - \sin\phi) - \gamma_1 \rho(\cos\phi \sin\phi + D_1' - D_2' \cos^2\phi)$$

$$+ \beta_1 \frac{1}{\rho} \sin\phi\,(D_2' \sin\phi + \cos\phi)$$

gives for the i-th slice

$$2\alpha_i D_{i+1}'\, d\phi - \gamma_i \rho_i (d\phi + D_i' - D_{i+1}') + \beta_i \frac{1}{\rho_i}\, d\phi.$$

Under the same conditions as used above for the vertical plane, this leads to

$$2 \int hD'\alpha ds - \int \gamma\, ds + \int \gamma\, \rho\, D''ds + \int h^2\beta ds$$

which finally can be written as

$$\int h^2\beta ds + \int h(2D'\alpha - D\gamma)ds$$

This agrees with equation (5), derived from the non-linear differential equation, putting K = 0.

AGS - Output for LEAR

*** L E A R *** Q(H) = 2.33 Q(V) = 2.75 AGS VERSION 75.03 09/07/81 10.05.08

CIRCUMFERENCE = 78.53982 4 HALF SUPERPERIODS WEDGE MAGNETS ALL VALUES AT EXIT OF ELEMENTS

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

NO	ELEM	L(M)	DIST(M)	ANG(MR)	K(1/M2)	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PI	MUH/2PI	ALPHAP(M)	ALPHAP'
INITIAL			0.0000			5.07830	1.98809	0.00000	0.00000	0.00000	0.00000	3.51841	-.000000
1	S3	4.0645	4.0645	0.00000	0.0000000	8.33139	10.29766	-.80037	-2.04443	.10742	.17760	3.51841	-.000000
2	QF	.5000	4.5645	0.00000	-1.2839286	12.34197	9.07820	-8.06120	4.21664	.11570	.18540	2.96868	-2.139787
3	S2	.3500	4.9145	0.00000	0.0000000	18.63972	6.37996	-9.93237	3.49259	.11937	.19272	2.21976	-2.139787
4	QD	.5000	5.4145	0.00000	1.4371403	21.60985	5.15100	4.72117	-.74711	.12311	.20747	1.49548	-.843536
5	S1	1.1259	6.5404	0.00000	0.0000000	12.34477	7.21685	3.50774	-1.08770	.13409	.23703	.54573	-.843536
6	BH1	1.0921	7.6325	254.73150	0.0000000	5.96873	9.38586	2.33080	-.85531	.15439	.25798	-.24480	-.596407
7	BH35	2.0650	9.6975	517.16700	0.0000000	.93817	10.74591	.10530	.25649	.32319	.28947	-.86802	.006321
8	FRB	.0539	9.7514	13.49900	.4864190	.92863	10.73176	.07154	.00616	.33238	.29027	-.86785	.000000
9	GAP	.0661	9.8175	0.00000	0.0000000	.92390	10.73135	.00000	-.00000	.34375	.29125	-.86785	.000000

DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
0.0000	-.86863	2.3300	-2.7193	10.7459	-.38268	2.7500	-5.8119	21.6099	3.5184	-5.5345

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

NO	ELEM	DL/DX	D2L/DX2	DK/DX	D2K/DX2	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PU	MUH/2PI	ALPHAP	ALPHAP'
INITIAL						5.03118	1.98453	0.00000	0.00000	0.00000	0.00000	3.51423	-.000000
1	S3	0.000	0.000	0.000000	0.00000	8.31473	10.30899	-.80786	-2.04809	.10815	.17771	3.51423	-.000000
2	QF	0.000	0.000	0.000000	0.00000	12.33258	9.08635	-8.07057	4.22569	.11644	.18550	2.96462	-2.139263
3	S2	0.000	0.000	0.000000	0.00000	18.63889	6.38258	-9.94746	3.49935	.12011	.19282	2.21587	-2.139263
4	QD	0.000	0.000	0.000000	0.00000	21.61721	5.15001	4.72264	-.74554	.12384	.20757	1.49151	-.844495
5	S1	0.000	0.000	0.000000	0.00000	12.34918	7.21181	3.50890	-1.08568	.13482	.23714	.54068	-.844495
6	BH1	0.000	0.000	0.000000	0.00000	5.97111	9.37633	2.33170	-.85339	.15512	.25811	-.25069	-.597039
7	BH35	0.000	0.000	0.000000	0.00000	.93794	10.73209	.10532	.25654	.32393	.28964	-.87448	.006498
8	FRB	0.000	0.000	0.000000	0.00000	.92840	10.71794	.07156	.00617	.33313	.29044	-.87430	.000000
9	GAP	0.000	0.000	0.000000	0.00000	.92367	10.71753	.00000	.00000	.34450	.29142	-.87430	.000000

DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
-.0010	-.86755	2.3314	-2.7225	10.7321	-.37395	2.7560	-5.8196	21.6172	-.0035	-5.4723

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

NO	ELEM	DL/DX	D2L/DX2	DK/DX	D2K/DX2	BETAV(M)	BETAH(M)	ALPHAV	ALPHAH	MUV/2PU	MUH/2PI	ALPHAP	ALPHAP'
INITIAL						5.12584	1.99163	0.00000	0.00000	0.00000	0.00000	3.52259	-.000000
1	S3	0.000	0.000	0.000000	0.00000	8.34876	10.28641	-.79294	-2.04079	.10670	.17749	3.52259	-.000000
2	QF	0.000	0.000	0.000000	0.00000	12.35243	9.07012	-8.05253	4.20764	.11497	.18529	2.97274	-2.140305
3	S2	0.000	0.000	0.000000	0.00000	18.64217	6.37738	-9.91816	3.48588	.11864	.19262	2.22364	-2.140305
4	QD	0.000	0.000	0.000000	0.00000	21.60436	5.15200	4.72014	-.74868	.12237	.20737	1.49945	-.842577
5	S1	0.000	0.000	0.000000	0.00000	12.34139	7.22190	3.50691	-1.08972	.13335	.23692	.55078	-.842577
6	BH1	0.000	0.000	0.000000	0.00000	5.96676	9.39538	2.33012	-.85723	.15366	.25785	-.23891	-.595775
7	BH35	0.000	0.000	0.000000	0.00000	.93832	10.75971	.10529	.25644	.32245	.28930	-.86156	.006146
8	FRB	0.000	0.000	0.000000	0.00000	.92879	10.74556	.07153	.00615	.33164	.29010	-.86139	.000000
9	GAP	0.000	0.000	0.000000	0.00000	.92406	10.74516	.00000	-.00000	.34300	.29108	-.86139	.000000

DP/P	COSMU(H)	Q(H)	QPRIME(H)	BETAMAX(H)	COSMU(V)	Q(V)	QPRIME(V)	BETAMAX(V)	XMAX(H)	GAMMA TR.
.0010	-.86970	2.3286	-2.7163	10.7597	-.39135	2.7440	-5.8047	21.6044	.0035	-5.5988

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

TIME = .041 SECONDS

PATRICIA - Output for the AA

```
INPUT-STREAM

CHROMATICITY CALCULATIONS WITH  "PATRICIA" FOR THE  *A A*
     9    1    2    1    0    0    0    0    1
0.100D+01 0.200D+01 0.300D+01 0.400D+01 0.500D+00
0.100D+01 0.200D+01 0.300D+01 0.400D+01
0.220D+02 0.100D+02 0.200D+01 0.0       0.0       0.100D+01 0.120D+01
   0    0    0    0    0    0    0    0    0    0    0    0    0    0
D0     1    0           0.0               0.0              0.0  0.0   0.0
D1     1    0           0.1923500D+01     0.0              0.0  0.0   0.0
B2     2    0           0.4970000D+01    -0.7301951D+01    0.0  0.0   0.0
B1     2    0           0.2890000D+01    -0.6493215D+01    0.0  0.0   0.0
QD1    3    0           0.3750000D+00     0.1558955D+00    0.0  0.0   0.0
QF1    3    0           0.3750000D+00    -0.2138321D+00    0.0  0.0   0.0
QD2    3    0           0.3750000D+00     0.1558955D+00    0.0  0.0   0.0
QF2    3    0           0.3750000D+00    -0.2138321D+00    0.0  0.0   0.0
QD3    3    0           0.3750000D+00     0.1558955D+00    0.0  0.0   0.0
QF3    3    0           0.3750000D+00    -0.2138321D+00    0.0  0.0   0.0
QD4    3    0           0.3750000D+00     0.1558955D+00    0.0  0.0   0.0
SD     4    0           0.0              -0.5000000D-01    0.0  0.0   0.0
SF     4    0           0.0               0.5000000D-01    0.0  0.0   0.0
END    0    0           0.0               0.0              0.0  0.0   0.0
D0  QD1 D1  QF1 QF1 B2  SD  QD2 QD2 D1  QF2 QF2 D1  QD3 QD3 SF  B1  QF3 QF3 SF
   1   2   3   3   1   4   5   5   1   3   3   1   3   3   1   6   6   7   1   8
B1  SD  QD4 D0
   8   6   6   2    0    0    0    0    0    0    0    0    0    0    0    0
0.0       0.0       0.192D+01 0.400D+00 0.200D+00 0.189D+00 0.250D+01 0.125D+01



                 CHROMATICITY CALCULATIONS WITH  "PATRICIA" FOR THE  *A A*
                      **** MULTIPOLE-STRUCTURE IN ONE HALF-SUPERPERIOD ****

                 J  MULTIPOLE     BETX(M)   BETY(M)    PHIX      PHIY     ETAX(M)  SM(1/M**2)

END OF CELL STRUCTURE
 1. MULTIPOLE AT: J =    7  SD          7.470    34.266    0.168     0.189     1.769     0.43284
 2. MULTIPOLE AT: J =   16  SF          6.966    21.473    0.399     0.318     6.288    -0.18571
 3. MULTIPOLE AT: J =   20  SF         15.437     5.318    0.497     0.466     8.990    -0.18571
 4. MULTIPOLE AT: J =   22  SD          8.044    10.646    0.559     0.559     6.267     0.43284

TOTAL NUMBER OF MULTIPOLES IN STORAGE RING:    16

                     WITHOUT SEXTUPOLES      WITH SPEC.SEXTUPOLES     WITH ALL SEXTUPOLES     CHROMATICITY WANTE

 CHROMATICITY IN X   =      -2.026       *        -2.026       *          0.000        *          0.0
*
 CHROMATICITY IN Y   =      -6.738       *        -6.738       *         -0.000        *          0.0
*
```

SYNCH - Output for LEAR

```
          SYNCH RUN BEGIN
==========================================================================================================================

            CHROMATICITY-ESTIMATION FOR  L E A R (4 SUPERPERIODS)
***  QPL     =              //  0.50
***  B*RH    =              //  1.00

***  S1      DRF            //   1.125917
***  S2      DRF            //   0.35
***  S3      DRF            //   4.0645
***  GAP     DRF            //   0.0661

***  KF      =              //   1.2839286
***  KD      =              //  -1.4371403

***  BH1     MAG            //  1.09206    0.0         B*RH       0.23325779
***  BH35    MAG            //  2.065      0.0         B*RH       0.25044436
***  FRB     MAG            //  0.0539    -0.486419    B*RH       0.25044527
***  QD      MAG            //  QPL        KD          B*RH
***  QF      MAG            //  QPL        KF          B*RH

***  BML     BML            //  S3    QF    S2    QD    S1    BH1   BH35 FRB   GAP
***          CYC         -4 //  BML
--------------------------------------------------------------------------------------------------------------------------
POS            S       QX       QY          BX          BY        AX        AY         X         DX         Y        DY        DS

 0           0.0000  0.00000  0.00000    1.98809     5.07830   0.00000   0.00000   3.51841  -.00000   0.00000   0.00000   0.00000
 1  S3       4.0645   .17760   .10742   10.29766     8.33139  -2.04443   -.80037   3.51841  -.00000   0.00000   0.00000   0.00000
 2  QF       4.5645   .18540   .11570    9.07820    12.34197   4.21664  -8.06120   2.96868 -2.13979   0.00000   0.00000   0.00000
 3  S2       4.9145   .19272   .11937    6.37990    18.63972   3.49259  -9.93237   2.21976 -2.13979   0.00000   0.00000   0.00000
 4  QD       5.4145   .20747   .12311    5.15100    21.60985   -.74711   4.72117   1.49548  -.84354   0.00000   0.00000   0.00000

 5  S1       6.5404   .23703   .13409    7.21685    12.34477  -1.08770   3.50774    .54573  -.84354   0.00000   0.00000    .03259
 6  BH1      7.6325   .25798   .15439    9.38586     5.96873   -.85531   2.33080   -.24480  -.59641   0.00000   0.00000   -.30904
 7  BH35     9.6975   .28947   .32319   10.74591      .93817    .25649    .10530   -.86802   .00632   0.00000   0.00000   -.32076
 8  FRB      9.7514   .29027   .33238   10.73176      .92863    .00616    .07154   -.86785   .00000   0.00000   0.00000   -.32076
 9  GAP      9.8175   .29125   .34375   10.73135      .92390   -.00000    .00000   -.86785   .00000   0.00000   0.00000   -.64152

10  REFL    19.6350   .58250   .68750    1.98809     5.07830    .00000    .00000   3.51841  -.00000   0.00000   0.00000   -.64152
R=  12.50000                                 THETX=  6.28318484     THETY=  0.00000000     QX=      2.33000     QY=      2.75000
              TGAM=(    0.00000,     5.53235)
MAXIMA       BETAX(   7)=  10.74591      XEQ=(    1)=    3.51841      BETAY(    4)=  21.60985      YEQ(  10)=   0.00000
MINIMA       BETAX(  10)=   1.98809      XEQ=(    6)=    -.24480      BETAY(    9)=    .92390      YEQ(  10)=   0.00000
CONTRIBUTIONS TO CHROMATICITY (DNU/(DP/P)) FROM MAGNETS EXPLICITLY IN         CHRX =   -2.0872, CHRY =   -5.8119
--------------------------------------------------------------------------------------------------------------------------

***  GO      SUB      0    0 //
..........................................................................................................................
***  P       PVEC             //  -0.0177                                                   -0.001
***          FXPT      0  -4  //  P    BML
*                             //
***          INCR      6      //  P               0.001
***          END       0    0 //
..........................................................................................................................

***          CALL      3      //  GO
..........................................................................................................................
```

```
CALCULATION OF THE EQUILIBRIUM ORBIT AND BETATRON FUNCTIONS OF
INITIAL REFERENCE RAY DEFINED BY P
 X = -.01770000   DX = 0.00000000   Y = 0.00000000   DY = 0.00000000    DS = 0.00000000   DP/P = -.00100000    1.00000000
  3
7X7 MATRIX FOR
              -.86441045   -1.01013808    0.00000000    0.00000000    0.00000000    6.56199582     .00000186
               .25025745    -.86441045    0.00000000    0.00000000    0.00000000    -.88080837    -.00000025
              0.00000000    0.00000000    -.37140855   -4.65919325    0.00000000    0.00000000    0.00000000
              0.00000000    0.00000000     .18502252    -.37140855    0.00000000    0.00000000    0.00000000
               .88080837   -6.56199582    0.00000000    0.00000000    1.00000000   -2.42654966     .00002005
              0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    1.00000000    0.00000000
              0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    1.00000000
EIGENVALUES OF THE 4X4 SUBMATRIX
X... LMD1   = (  -.86441045   .50278681    ); C(1) =   1.00000000, MU(1) =   2.61477295 RAD,    Q(1) =   .66461616
     1/LMD1 = (  -.86441045  -.50278681    ); C(2) =   1.00000000, MU(2) =  -2.61477295 RAD,    Q(2) =   .33538384
Y... LMD3   = (  -.37140855   .92846954    ); C(3) =   1.00000000, MU(3) =   1.95132195 RAD,    Q(3) =   .24225014
     1/LMD3 = (  -.37140855  -.92846954    ); C(4) =   1.00000000, MU(4) =  -1.95132195 RAD,    Q(4) =   .75774986
                   X            DX            Y             DY            DS           DP/P
 EQ ORBIT      -.00351861    0.00000000    0.00000000    0.00000000    0.00000000    -.00100000    1.00000000
 ETA ORBIT     3.51960901    -.00000000    0.00000000    0.00000000    0.00000000    1.00000000    0.00000000
BETATRON FUNCTIONS OF

 R=  12.50042                              THETX=  6.28947432    THETY=  0.00000000    QX=    2.33538    QY=    2.75775
                TGAM=(   0.00000,    5.39929)
MAXIMA       BETAX(   7)=  10.60490     XEQ=(  10)=   3.51961     BETAY(   4)=  21.61613    YEQ(  10)=   0.00000
MINIMA       BETAX(   1)=   2.09908     XEQ=(   6)=   -.25736     BETAY(   9)=    .92339    YEQ(  10)=   0.00000
CONTRIBUTIONS TO CHROMATICITY (DNU/(DP/P)) FROM MAGNETS EXPLICITLY IN       . CHRX =.   -2.0803, CHRY =   -5.8240

CALCULATION OF THE EQUILIBRIUM ORBIT AND BETATRON FUNCTIONS OF
INITIAL REFERENCE RAY DEFINED BY P
 X = -.01770000   DX = 0.00000000   Y = 0.00000000   DY = 0.00000000    DS = 0.00000000   DP/P =  .00100000    1.00000000
  3
7X7 MATRIX FOR
              -.87281467    -.95970996    0.00000000    0.00000000    0.00000000    6.58721863     .00000178
               .24817362    -.87281467    0.00000000    0.00000000    0.00000000    -.87289677    -.00000024
              0.00000000    0.00000000    -.39352935   -4.72280106    0.00000000    0.00000000    0.00000000
              0.00000000    0.00000000     .17894776    -.39352935    0.00000000    0.00000000    0.00000000
               .87289677   -6.58721863    0.00000000    0.00000000    1.00000000   -2.46045533     .00001988
              0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    1.00000000    0.00000000
              0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    0.00000000    1.00000000
EIGENVALUES OF THE 4X4 SUBMATRIX
X... LMD1   = (  -.87281467   .48805179    ); C(1) =   1.00000000, MU(1) =   2.63173640 RAD,    Q(1) =   .67541543
     1/LMD1 = (  -.87281467  -.48805179    ); C(2) =   1.00000000, MU(2) =  -2.63173640 RAD,    Q(2) =   .32458457
Y... LMD3   = (  -.39352935   .91931205    ); C(3) =   1.00000000, MU(3) =   1.97526390 RAD,    Q(3) =   .25749205
     1/LMD3 = (  -.39352935  -.91931205    ); C(4) =   1.00000000, MU(4) =  -1.97526390 RAD,    Q(4) =   .74250795
                   X            DX            Y             DY            DS           DP/P
 EQ ORBIT       .00351823    0.00000000    0.00000000    0.00000000    0.00000000     .00100000    1.00000000
 ETA ORBIT     3.51728269    -.00000000    0.00000000    0.00000000    0.00000000    1.00000000    0.00000000
BETATRON FUNCTIONS OF

 R=  12.49960                              THETX=  6.27690794    THETY=  0.00000000    QX=    2.32458    QY=    2.74251
                TGAM=(   0.00000,    5.67447)
MAXIMA       BETAX(   7)=  10.89284     XEQ=(   1)=   3.51728     BETAY(   4)=  21.60568    YEQ(  10)=   0.00000
MINIMA       BETAX(  10)=   1.96057     XEQ=(   6)=   -.23233     BETAY(   9)=    .92433    YEQ(  10)=   0.00000
CONTRIBUTIONS TO CHROMATICITY (DNU/(DP/P)) FROM MAGNETS EXPLICITLY IN       . CHRX =.   -2.0948, CHRY =   -5.8008
*************************************************************************************************************************
***          INCR      6      //   P          0.000000      VALUE =         -.001000
 ***         FIN       0    0  // CORE USE SUMMARY                     MAXIMUM              USED          UNUSED
                                  STORE   (ELEMENT STORAGE)             9600 (LMAX)          1404            8196
                                  INFF    (ELEMENT DEFINITIONS)          400 (MAX)             44             356

=========================================================================================================================
END OF SYNCH RUN BEGIN
```

TRANSPORT - Output for LEAR

```
"CHROMATICITY FOR L E A R     "
  0
  1.000000     "BEAM"    3.00000   9.00000   2.00000   4.00000   0.00000   0.00000   2.00000;
 17.           "2ND " ;
  3.0          "SL  "    3.59450;
  3.0          "XF  "     .38000;
  3.0          "GAP "     .09000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "SS  "    1.12592;
  2.0          "PFR "    0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  2.0          "PFR "    0.00000;
  3.0          "GAP "     .06610;
  3.0          "GAP "     .06610;
  2.0          "PFR "    0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  2.0          "PFR "    0.00000;
  3.0          "SS  "    1.12592;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "GAP "     .09000;
  3.0          "XF  "     .38000;
  3.0          "SL  "    3.59450;
  3.0          "SL  "    3.59450;
  3.0          "XF  "     .38000;
  3.0          "GAP "     .09000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "SS  "    1.12592;
  2.0          "PFR "    0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  2.0          "PFR "    0.00000;
  3.0          "GAP "     .06610;
  3.0          "GAP "     .06610;
  2.0          "PFR "    0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  2.0          "PFR "    0.00000;
  3.0          "SS  "    1.12592;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "GAP "     .09000;
  3.0          "XF  "     .38000;
  3.0          "SL  "    3.59450;
  3.0          "SL  "    3.59450;
  3.0          "XF  "     .38000;
  3.0          "GAP "     .09000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "SS  "    1.12592;
  2.0          "PFR "    0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  2.0          "PFR "    0.00000;
  3.0          "GAP "     .06610;
  3.0          "GAP "     .06610;
  2.0          "PFR "    0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  2.0          "PFR "    0.00000;
  3.0          "SS  "    1.12592;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "GAP "     .09000;
  3.0          "XF  "     .38000;
  3.0          "SL  "    3.59450;
  3.0          "SL  "    3.59450;
  3.0          "XF  "     .38000;
  3.0          "GAP "     .09000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "SS  "    1.12592;
  2.0          "PFR "    0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  2.0          "PFR "    0.00000;
  3.0          "GAP "     .06610;
  3.0          "GAP "     .06610;
  2.0          "PFR "    0.00000;
  4.000        "FRBQ"     .05390  16.70766   7.75510;
  4.000        "BH35"    2.06500  16.70766   0.00000;
  4.000        "BH1 "    1.09206  15.56121   0.00000;
  2.0          "PFR "    0.00000;
  3.0          "SS  "    1.12592;
  5.00         "QD  "     .50000  -6.99892   7.30000;
  3.0          "XD  "     .35000;
  5.00         "QF  "     .50000   6.25278   7.30000;
  3.0          "GAP "     .09000;
  3.0          "XF  "     .38000;
  3.0          "SL  "    3.59450;
 13.                     4.00000;
```

```
*TRANSFORM 1*
          -.48165     .17422    0.00000    0.00000    0.00000    5.21298
         -4.40834    -.48165    0.00000    0.00000    0.00000   15.51018
          0.00000    0.00000    -.00000    -.50783    0.00000    0.00000
          0.00000    0.00000    1.96916    -.00000    0.00000    0.00000
         -1.55102    -.52130    0.00000    0.00000    1.00000    8.02307
          0.00000    0.00000    0.00000    0.00000    0.00000    1.00000
*2ND ORDER TRANSFORM*
  1 11 -5.606E-04
  1 12 -1.363E-04    1 22 -2.290E-05
  1 13  0.           1 23  0.           1 33  1.680E-03
  1 14  0.           1 24  0.           1 34 -6.095E-03   1 44 -2.670E-03
  1 15  0.           1 25  0.           1 35  0.          1 45  0.          1 55  0.
  1 16  1.690E-01    1 26  1.239E-02    1 36  0.          1 46  0.          1 56  0.          1 66 -5.906E-01

  2 11 -9.038E-05
  2 12 -6.075E-05    2 22 -1.305E-04
  2 13  0.           2 23  0.           2 33 -6.408E-02
  2 14  0.           2 24  0.           2 34 -1.813E-02   2 44  9.869E-03
  2 15  0.           2 25  0.           2 35  0.          2 45  0.          2 55  0.
  2 16 -6.108E-01    2 26  1.653E-01    2 36  0.          2 46  0.          2 56  0.          2 66  2.142E+00

  3 11  0.
  3 12  0.           3 22  0.
  3 13 -3.887E-02    3 23  1.052E-02    3 33  0.
  3 14  9.209E-03    3 24  3.095E-03    3 34  0.          3 44  0.
  3 15  0.           3 25  0.           3 35  0.          3 45  0.          3 55  0.
  3 16  0.           3 26  0.           3 36 -3.396E-01   3 46 -9.175E-02   3 56  0.          3 66  0.

  4 11  0.
  4 12  0.           4 22  0.
  4 13  3.571E-02    4 23  1.200E-02    4 33  0.
  4 14  6.508E-02    4 24 -1.706E-03    4 34  0.          4 44  0.
  4 15  0.           4 25  0.           4 35  0.          4 45  0.          4 55  0.
  4 16  0.           4 26  0.           4 36 -3.558E-01   4 46 -7.054E-01   4 56  0.          4 66  0.

*LENGTH*        78.53982 M
```

TABLE 1 : CHROMATICITIES EVALUATED BY DIFFERENT METHODS

	P S $\rho = 70.08$ m ($Q_H = 6.25$; $Q_V = 6.25$)		A A $\rho = 7.3$ m ; $\rho = 6.49$ m ($Q_H = 2.2811$; $Q_V = 2.2834$)		L E A R $\rho = 4.29$ m ; $\rho = 3.99$ m ($Q_H = 2.33$; $Q_V = 2.75$)	
	ξ_H	ξ_V	ξ_H	ξ_V	ξ_H	ξ_V
AGS (i)	-1.04	-1.04	-1.21	-0.55	-1.17	-2.11
AGS (ii)	-0.93	-0.97	-0.89	-0.55	-0.60	-2.18
PATRICIA	-	-	-0.89	-2.95	-	-
SYNCH (i)	-1.03	-1.04	-0.90	-0.76	-0.90	-2.11
SYNCH (ii)	-0.89	-0.98	-0.65	-1.49	-2.32	-2.77
TRANSPORT	-	-	-	-	-	-
Method of B. Autin	-	-	-0.65	-1.49	-1.29	-2.76
Formula of P. Gratreau-G. Leleux	-0.85	-1.09	(iii) -0.98 (iv) -0.68	-0.74 -0.90	-1.29	-2.19
Method of S. Peggs	-0.85	-0.98	-0.65	-1.50	-1.29	-2.76
Non-linear different equations	-0.85	-1.09	(iv) -0.68	-1.48	-1.29	-2.76
Measurements	-1.00	-1.00	-0.62	-1.56		

(i) Normal lattice output
(ii) Tracking routine
(iii) Pure application of the formulae (3) to a machine with rectangular magnets
(iv) Rectangular magnets replaced by sector magnets enclosed between "short lenses" with $K\ell = \frac{1}{\rho} \tan \frac{\phi}{2}$