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# STOCHASTIC COOLING FOR BEGINNERS

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STOCHASTIC COOLING FOR BEGINNERS *(Lecture dedicated to Yuri Orlov)*

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ABSTRACT

These two lectures have been prepared to give a simple introduction to the principles. In Part I we try to explain stochastic cooling using the time-domain picture which starts from the pulse response of the system. In Part II the discussion is repeated, looking more closely at the frequency-domain response. An attempt is made to familiarize the beginners with some of the elementary cooling equations, from the 'single particle case' up to equations which describe the evolution of the particle distribution.

*Foreword:* This lecture has been prepared for beginners. Apologies go to those who expect a shorter and more technical presentation. Rather, I have tried hard to introduce the principles in as simple a way as possible.

In Part I we shall look at the response of the cooling system to error signals of the particles as a function of time. We start from the simplest situation where each particle interacts only with itself, and improve it step by step by including the adverse effects of other particles, of amplifier noise, of imperfect mixing of particles, etc. I hope that the repetition involved is helpful to the newcomer.

Part II restarts from the analysis of the frequency response of the system. It gives a fresh look at the cooling process from another interesting viewpoint; some of the phenomena -- such as, for example, imperfect mixing and filter techniques -- then appear more natural. Finally, we try to de-dramatize the more general cooling equations which describe the detailed evolution of the beam density distribution.

A list of references is given at the end of each Part. A more detailed bibliography is appended to Part II.

## PART I: TIME DOMAIN DESCRIPTION

### 1. INTRODUCTION

Beam cooling aims at reducing the size and the energy spread of a particle beam circulating in a storage ring. This reduction of size should not be accompanied by beam loss. Instead, the goal is to 'compress' the same number of particles into a beam of smaller size and spread, i.e. to increase the particle density.

Since the beam size varies with the focusing properties of the storage ring, it is useful to introduce normalized measures of size and density. Such quantities are the (horizontal, vertical, and longitudinal) emittances and the phase-space density. For our present purpose they may be regarded as the (square of the) horizontal and vertical beam diameter, the energy spread, and the density normalized by the focusing strength and the size of ring to make them independent of the storage ring properties.

Phase-space density is then a general figure of merit of a particle beam, and cooling improves this figure of merit.

The terms beam temperature and beam cooling have been taken over from the kinetic theory of gases. Imagine a beam of particles going around in a storage ring. Particles will oscillate around the beam centre in much the same way that particles of a hot gas bounce back and forth between the walls of a container. The larger the (mean square of the) velocity of these oscillations in a beam the larger the beam size. The mean square velocity *spread* is used to define the beam temperature in analogy to the temperature of the gas, which is determined by the kinetic energy $\frac{1}{2}\, m\, v^2_{rms}$ of the molecules.

Why beam cooling? The resultant (sometimes dramatic) increase of beam quality is very desirable for at least three reasons:

i) Accumulation of rare particles
[Example: the Antiproton Accumulator (AA) at CERN]:
Cooling to make space available so that more beam can be stacked into the same storage ring.

ii) Preservation of beam quality:
Cooling to compensate for various mechanisms leading to growth of beam size and/or loss of stored beam.

iii) Improvement of interaction rate and resolution:
Cooling to provide sharply collimated and highly mono-energetic beams for precision experiments with colliding beams or beams interacting with fixed targets.

Several cooling techniques are operative or have been discussed[1]: Electron beams have a tendency to cool 'by themselves' owing to the emission of radiation as the orbit is curved. The energy radiated decreases very strongly with increasing rest mass of the particles. For (anti-)protons and heavier particles, radiation damping is negligible at energies currently accessible in accelerators. 'Artificial' damping had therefore to be devised, and two such methods have been successfully put to work during the last decade: i) cooling of heavier particles by the use of an electron beam -- this will be the subject of Fred Mill's lecture at this School; and ii) stochastic cooling by the use of a feedback system, which will be discussed during the rest of this talk.

## 2. THE BASIC SET-UP

The arrangement for cooling of the horizontal beam size is sketched in Fig. 1. Assume, for the moment, that there is only one particle circulating. Unavoidably, it will have been injected with some small error in position and angle with respect to the ideal orbit (centre of the vacuum chamber). As the focusing system continuously tries to restore the resultant deviation, the particle oscillates around the ideal orbit. Details of these 'betatron

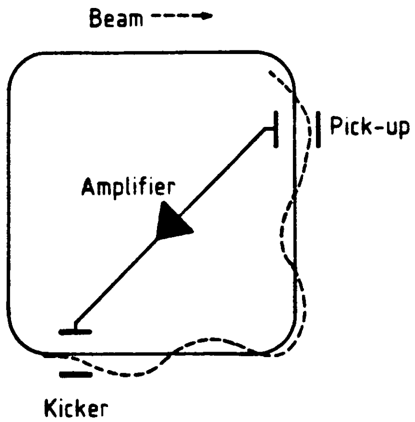


Fig. 1 The principle of (horizontal) stochastic cooling. The pick-up measures horizontal deviation; the kicker corrects angular error. They are spaced by a quarter of the betatron wavelength $\lambda_\beta$ (plus multiples of $\lambda_\beta/2$). A position error at the pick-up transforms into an error of angle at the kicker. This error of angle is corrected.

oscillations'[2] are given by the focusing structure of the storage ring, namely by the distribution of quadrupoles and gradient magnets (and higher-order 'magnetic lenses') which provide a focusing force proportional to the particle deviation (and to higher-order powers of the deviation).

For the present purpose, we can approximate the betatron oscillation by a purely sinusoidal oscillation. The cooling system is designed to damp this oscillation. A pick-up electrode senses the horizontal position of the particle on each traversal. The error signal -- ideally a short pulse with a height proportional to the particle's deviation at the pick-up -- is amplified in a broad-band amplifier and applied on a kicker which deflects the particle by an angle proportional to its error.

In the simplest case, the pick-up consists of a plate to the left of the beam and a plate to the right of it. If the particle passes to the left, the current induced on the left plate exceeds the current on the right one and vice versa. The *difference* between the two signals is a measure of the position error. The 'kicker' is, in principle, a similar arrangement of plates on which a transverse electromagnetic field is created which deflects the particle. A more rigorous account of how pick-ups and kickers work will be given in the lectures by J. Borer[3] and by C. Taylor[4].

Since the pick-up detects the position and the kicker corrects the angle their distance is chosen to correspond to a quarter of the betatron oscillation (plus an integer number of half wavelengths if more distance is necessary). A particle passing the pick-up at the crest of its oscillation will then cross the kicker with zero position error but under an angle which is proportional to its displacement at the pick-up. If the kicker corrects just this angle, the particle will, from now on, move on the central orbit. This is the most favourable situation (sketched as Case 1 in Figs. 2 and 3). A particle not crossing the pick-up at the crest of its oscillation will receive only a partial correction (Cases 2 and 3 in Figs. 2 and 3). As we shall see later, it will then take a couple of passages to eliminate the oscillation.

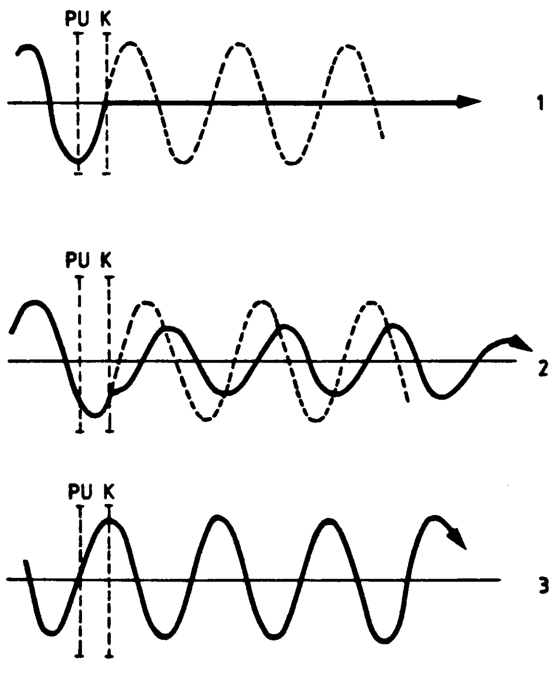


Fig. 2 The importance of betatron phase: Particle 1 crosses the pick-up with maximum elongation. Its oscillation is (ideally) completely cancelled at the kicker. Particle 2 arrives at an intermediate phase; its oscillation is only partly eliminated. Particle 3 arrives with the most unfavourable phase and is not affected by the system.

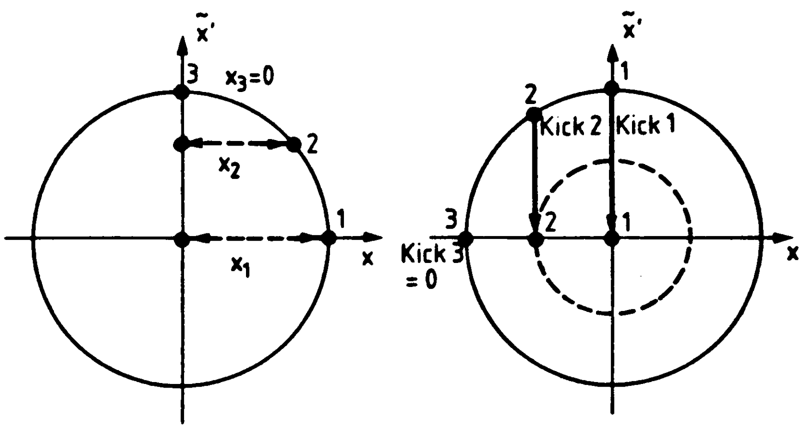


Fig. 3 Phase space representation of betatron cooling. The same as for Fig. 2 except that a 'polar diagram' $R/Q\ x' = f(x)$ is used to represent the betatron motion $x = \hat{x} \sin [Q(s/R) + \psi_0]$, $(R/Q)\ x' = \hat{x} \cos [Q(S/R) + \psi_0]$. The undisturbed motion of a particle is given by a circle with the radius equal to the betatron amplitude $\hat{x}$. Kicks correspond to a jump of $x'$. The cooling system tries to put particles onto smaller circles. Particles 1, 2, and 3 are sketched with the most favourable, the intermediate, and the least favourable initial phase, respectively. As the number of oscillations per turn is different from an integer or half-integer, particles come back with different phases on subsequent turns and all particles will be cooled progressively.

Another particularity of stochastic cooling is easily understood from the single particle model (Fig. 1) used so far: the correction signal has to arrive at the kicker at the same time as the test particle. Since the signal is delayed in the cables and the amplifier, whereas a high-energy particle moves at a speed close to the velocity of light, the cooling path has usually to take a short cut across the ring. Only at low and medium energy ($v/c \lesssim 0.5$) is a parallel path feasible.

We have thus familiarized ourselves with two constraints on the distance pick-up to kicker: taken along the beam, this distance is fixed, or rather quantized, owing to the required phase relationship of the betatron oscillation; taken along the cooling path this length is fixed by the required synchronism between particle and signal. A change of energy (particle velocity) and/or a change of the betatron wavelength will therefore require special measures. Incidentally, the first of these two conditions is due to the oscillatory nature of the betatron motion. For momentum spread cooling in a coasting beam, where the momentum deviation of a particle is constant rather than oscillatory, this constraint does not come into play and a greater freedom in the choice of pick-up to kicker distance exists.

It is now time to leave the one-particle consideration and turn our attention to a beam of particles which oscillate incoherently, i.e. with different amplitudes and with random initial phase.

By beam cooling we shall now mean a reduction with time of the amplitude of each individual particle. To understand stochasic cooling, we will next have a closer look at the response of the cooling system. This permits us to discern groups of particles -- so-called samples -- which will receive the same correcting kick during a passage through the system.

## 3. THE NOTION OF BEAM SAMPLES

To be able to analyse the response of the cooling system, let us start with an excursion into elementary pulse and filtering theory[5]. What we would like to take over is a bandwidth/pulse-length relation known as the Küpfmüller or Nyquist theorem and discussed in textbooks:

*If a signal has a Fourier decomposition of band-width $\Delta f = W$, then its 'typical' time duration will be*

$$T_s = 1/(2W) .$$

This is illustrated in Fig. 4, where we sketch the Fourier spectrum of a pulse and the resulting time-domain signal. Clearly the two representations are linked by a Fourier transformation, and this permits us to check the theorem.

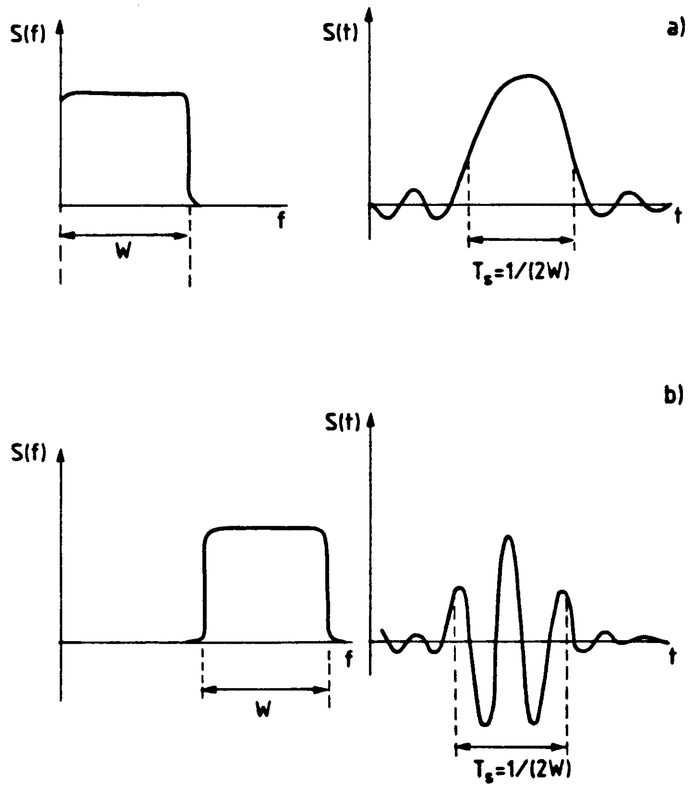


Fig. 4 Illustration of the Küpfmüller-Nyquist relation: A signal whose Fourier decomposition S(f) has a bandwidth W has a typical time duration $T_S = 1/(2W)$. Illustration for a 'low pass' [case (a)] and a 'band-pass' signal [case (b)].

For curiosity, note the difference between a pulse with a low frequency ($0 \le f \le W$) and a high frequency ($W < f_1 \le f \le f_1 + W$) spectrum (both cases are sketched in Fig. 4). In spite of the different shape of the time-domain signal, the 'typical duration' is in both cases $1/(2W)$.

A corollary to the theorem is well-known to people who design systems for transmitting short pulses:

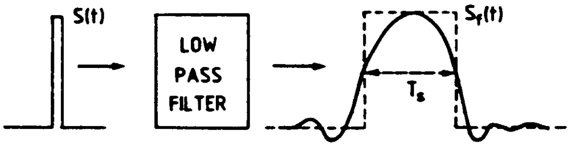


Fig. 5 Input and output signal S(t) of a band-pass system and 'rectangular' approximation to the output pulse $S_f(t)$.

*When a short pulse is filtered by a low pass or band-pass filter of band-width W, the resulting pulse has a 'typical' time width*

$$T_s = 1/(2W) \qquad \text{(Fig. 5)} . \tag{3.1}$$

In this form, the theorem is directly applicable to our cooling problem, to which we now return. Passing through the pick-up, an off-axis particle induces a short pulse with a length given by the transit time. Owing to the finite band-width (W) of the cooling system, the corresponding kicker signal is broadened into a pulse of length $T_s$. To simplify considerations, we approximate this kicker pulse by a rectangular pulse of total length $T_s$ (Fig. 6).

A test particle passing the system at $t_0$ will then be affected by the kicks due to all particles passing during the time interval $t_0 \pm T_s/2$. These particles are said to belong to the sample of the test particle. In a uniform beam of length T (revolution time), there are $\ell_s = T/T_s = 2WT$ equally spaced samples of length $T_s$ with

$$N_s = N/(2WT) \quad \text{particles per sample} . \tag{3.2}$$

The example given in Table 1 corresponds (in round numbers) to the momentum cooling system at injection in the Low-Energy Antiproton Ring (LEAR)[6]. In this example we can imagine the beam as being sliced up into 250 samples, each containing 1/250 of the total number of particles.

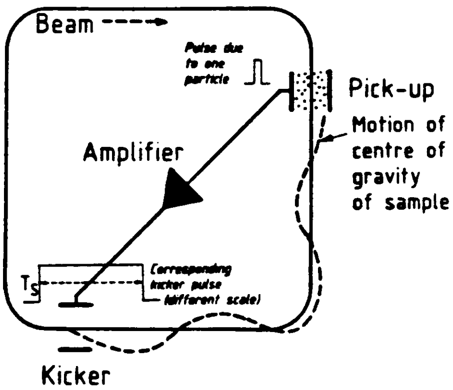


Fig. 6 Pick-up signal of a particle and corresponding kicker pulse (idealized). The test particle experiences the kicks of all other particles passing within time $-T_s/2 \le \Delta t \le T_s/2$ of its arrival at the kicker. These particles are said to belong to the sample of the test particle. Cooling may be discussed in terms of the centre-of-gravity motion of samples.

Table 1

An example of samples

| | | |
|---|---|---|
| No. of particles in the beam | N | $10^9$ |
| Revolution time | T | 0.5 μs |
| Transit time in one-pick-up unit | $T_t$ | 0.1 ns |
| Cooling system bandwidth | W | 250 MHz |
| Sample length | $T_s$ | 2 ns |
| No. of samples per turn | $\ell_s$ | 250 |
| No. of particles per sample | $N_s$ | $4 \times 10^6$ |

## 4. THE 'TEST PARTICLE PICTURE' AND THE 'SAMPLING PICTURE'

The model of samples has permitted us to subdivide the beam into a large number of slices which are treated independently of each other by the cooling system. If the bandwidth can be made large enough so that there are no other particles in the sample of the test particle, then the single-particle analysis is still valid. However, to account for the reality of some million particles per sample, we have to go a step further and do some simple algebra. This will permit us to discern two slightly different pictures of the cooling process: In the test particle picture we shall view cooling as the competition between: i) the 'coherent effect' of the test particle upon itself via the cooling loop; and ii) the 'incoherent effect', i.e. the disturbance of the test particle by the other sample members (see Fig. 7a). In the 'sampling picture' we shall understand stochastic cooling as a process where samples are taken from the beam at a rate $\ell_s$ per turn. By measuring and reducing the average sample error, the error of each individual particle will (on the average) slowly decrease.

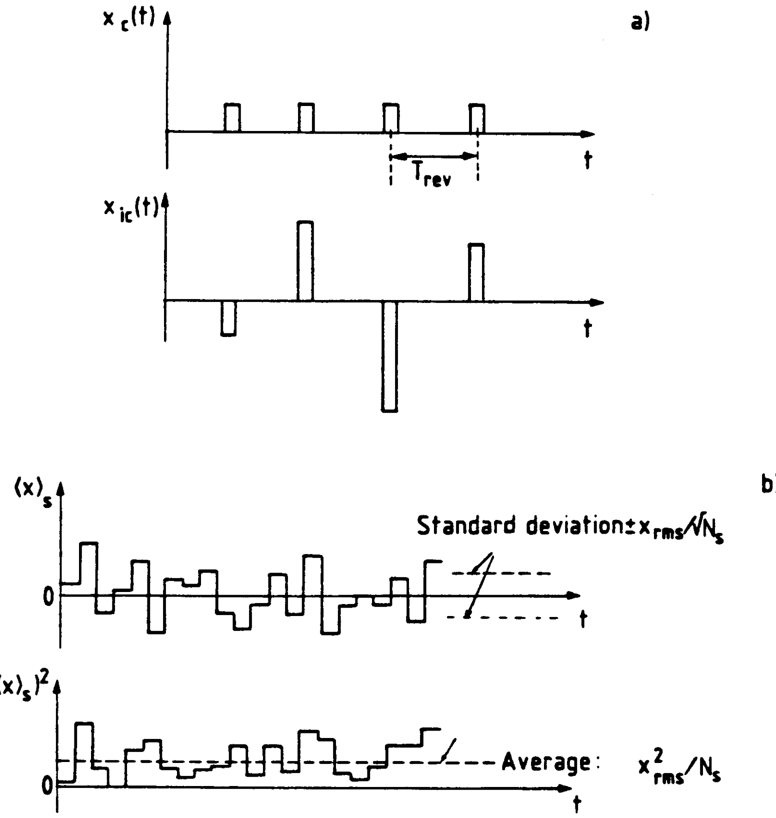


Fig. 7 Cooling system signals in a) the test particle picture and b) the sampling picture. Under (a), signals at the instant of passage of the test particle are sketched. The upper trace gives the coherent correction signal due to the test-particle itself. The lower trace sketches the incoherent signal due to the other particles in the sample. The kick experienced is the sum of coherent and incoherent effect. If the amplification is not too strong and the sample population is small, the coherent effect which is systematic will predominate over the random heating by the incoherent signals. Figure 7b sketches the total cooling signal as a function of time. This signal and the correction proportional to it is given by the centre of gravity $\langle x\rangle_s(t)$ of successive beam samples. Owing to the finite sample population ($N_s$) the deviation $x_{rms}$ within the beam leads to a fluctuation of the sample signal of $\pm x_{rms}/\sqrt{N_s}$. By correcting the sample error $\langle x\rangle_s(t) \to 0$ for each sample, the deviation in the beam is slowly reduced.

A few simple equations will illustrate these pictures. Let us denote by x the error of the test particle and assume that the corresponding correction at the kicker is proportional to x, say $\lambda x$. With no other particles present, the error would be changed from x to a corrected

$$x_c = x - \lambda x \,, \tag{4.1}$$

i.e. the test particle receives a correcting kick,

$$\Delta x = -\lambda x \,.$$

In reality the kicks $-\lambda x_i$ of the other sample members have to be added, and the corrected error after one turn and the corresponding kick are

$$x_c = x - \underbrace{\lambda x}_{\text{coherent effect}} - \underbrace{\sum_{\text{others}} \lambda_i x_i}_{\text{incoherent effect}} \tag{4.2a}$$

$$\Delta x = -\lambda x - \sum_{\text{others}} \lambda_i x_i \,.$$

In our rectangular response model, $\lambda_i = \lambda$ is the same for all sample members. Hence, we can also write

$$\begin{aligned} x_c &= x - \lambda x - \lambda \sum_{\text{others}} x_i \,, \\ \Delta x &= -\lambda x - \lambda \sum_{\text{others}} x_i \,. \end{aligned} \tag{4.2b}$$

Equations (4.2) clearly exhibit the 'coherent' and the 'incoherent' effects mentioned above. The sum labelled 'others' includes all particles in the sample *except* the test particles.

You may want to rewrite this sum including the test particle (this sum will be labelled 'sample') and interpret it in terms of the average sample error (the sample centre of gravity if you like), which is by definition

$$\langle x \rangle_s = \frac{1}{N_s} \sum_{\text{sample}} x_i \,. \tag{4.3}$$

Equations (4.2b) then become

$$\begin{aligned} x_c &= x - (N_s \lambda)\langle x \rangle_s \,, \\ \Delta x &= -(\lambda N_s)\langle x \rangle_s \equiv -g\langle x \rangle_s \,. \end{aligned} \tag{4.4}$$

This introduces the second picture: What the cooling system does is to measure the average sample error and to apply a correcting kick, proportional to $\langle x \rangle_s$, to the test particle. (Up to now the sample is defined with respect to a specific test particle; however, to the extent that any beam slice of length $T_s$ has the same average error $\langle x \rangle_s$, our considerations apply to any test particle. This is true on a statistical basis, as will become clear later.)

A word about notation. It has become customary to write $\lambda N_s = g$, and to call g the 'gain'. Remember that this g is proportional to the amplification (the electronic gain) of the system *and* proportional to $N_s$. As from Eqs. (4.4), $-g = \Delta x/\langle x\rangle_s$, a more precise (but longer) name is 'fraction of observed sample error corrected per turn'.

## 5. A FIRST VERY CRUDE APPROXIMATION TO THE COOLING RATE

All we have to do to get the cooling rate from Eqs. (4.2) or (4.4) is to find acceptable (or optimal) values for the gain, and -- closely related to this -- to estimate the adverse effect of the incoherent term. A crude estimate can be made on the basis of the following observations.

Interpreting g as the fractional correction, we intuitively accept that it is unhealthy to correct more than the observed sample error, i.e. we assume $g \leq 1$. Let us put $g = 1$ to make an estimate of the upper limit. Secondly, we neglect the incoherent effect, assuming that the influence of the other particles averages to zero. This leads to the single passage correction [from Eq. (4.2) with $g = \lambda N_s = 1$:]

$$\Delta x = -\frac{1}{N_s} x \ . \tag{5.1}$$

This will immediately give us the cooling rate per turn which is in general:

$$1/\tau_n = -\frac{1}{x}\frac{dx}{dn} \simeq -\frac{\Delta x}{x} \tag{5.2}$$

and hence in the present approximation:

$$1/\tau_n = \frac{1}{N_s} \quad \text{per turn} \ .$$

To derive Eq. (5.2), differentiate the decay law $x = x_0 \exp(-n/\tau_n)$ describing exponential damping with the number of turns n. Assume that the correction per turn is small so that the differential dx/dn can be approximated by the difference $\Delta x/\Delta n$. But as $\Delta x$ is the change per turn, $\Delta x/\Delta n$ is simply equal to $\Delta x$: This is because n increases by 1 per turn, i.e. $\Delta n = 1$.

We now wish to obtain the cooling rate per second; we know that there are $f_0 = 1/T$ passages ($f_0$: revolution frequency), hence

$$1/\tau = f_0 \cdot 1/\tau_n \ .$$

$$\left[\text{Formally: } \frac{1}{x}\frac{dx}{dt} = \frac{dn}{dt}\left(\frac{1}{x}\frac{dx}{dn}\right) = f_0\left(\frac{1}{\tau_n}\right)\right] .$$

Finally, it is convenient to express $N_s = N(T/T_s) = N(f_0/2W)$ by the total number of particles N and by the system bandwidth W using Eqs. (3.1) and (3.2). We then obtain:

$$\boxed{\frac{1}{\tau} = \frac{2W}{N} \quad \text{per second} \ .} \tag{5.3}$$

Amazingly enough, this simple relation overestimates the optimum cooling rate by only a factor of 2. However, to gain confidence, we have to justify some of our assumptions, especially the restriction $g \leq 1$ and the neglect of the incoherent term. In fact, an evaluation of this term will clarify both assumptions and provide guidance on how to include other adverse effects such as amplifier noise.

## 6. TOWARDS A BETTER EVALUATION OF THE INCOHERENT TERM

To be able to deal with the incoherent term, we make a detour into statistics to recall (or to learn about) a few elementary 'sampling relations'[7]. Consider the following problem.

Given a beam of N particles characterized by an average $\langle x \rangle = 0$ and a variance $\langle x^2 \rangle \equiv x^2_{rms}$ of some error quantity x, say the deviation $\Delta p/p$ from the average momentum, to fix our ideas. Suppose we take a random sample of $N_s$ particles and do statistics on the sample population -- rather than on the whole beam -- to determine

i) the sample average $\langle x \rangle_s$;
ii) the sample variance $\langle x^2 \rangle_s$;
iii) the square of the sample average $(\langle x \rangle_s)^2$, i.e. the square of (i).

*Question:* What are the most probable values [the expectation values, denoted by $E(\langle x \rangle_s)$, etc.] of these sample characteristics?

*Answer:* For random samples the most probable values are:

i) sample average → beam average;
ii) sample variance → beam variance;
iii) square of sample average → beam variance/sample population.

Or, in more mathematical language,

$$E(\langle x \rangle_s) = \langle x \rangle = 0 \quad (6.1a)$$
$$E(\langle x^2 \rangle_s) = \langle x^2 \rangle = x^2_{rms} \quad (6.1b)$$
$$E[(\langle x \rangle_s)^2] = x^2_{rms}/N_s \quad (6.1c)$$

Results (6.1a) and (6.1b) are in agreement with common sense, which expects that the sample characteristics are true approximations of the corresponding population characteristics. This is the basis for sampling procedures. Equation (6.1c) is more subtle as it specifies the error to be expected when one replaces the population average by the sample average. In fact from Eqs. (6.1) the mean square of this error is

$$(\langle x \rangle_s)^2 - (\langle x \rangle)^2 \rightarrow x^2_{rms}/N_s$$

or symbolically

$$\langle x \rangle \approx \langle x \rangle_s \pm x_{rms}/\sqrt{N_s} \ .$$

In other words: The larger the beam variance and the smaller the sample size ($N_s$), the more imprecise is the sampling. In this form, Eqs. (6.1) are used in statistics to determine the required sample size for given accuracy and presupposed values for the beam variance $x^2_{rms}$.

A slightly different interpretation is useful in the present context: Suppose we repeat the process of taking beam samples and working out $\langle x \rangle_s$ many times. Although the beam has zero $\langle x \rangle$, the sample average will in general have a finite (positive or negative) $\langle x \rangle_s$. The sequence of sample averages will fluctuate around zero (around $\langle x \rangle$ in general) with a mean deviation $x^2_{rms}/N_s$. This is the fluctuation (or, if you prefer, the noise) of the sample average due to the finite particle number, sketched in Fig. 7b.

A simple example to illustrate the sampling relations and to familiarize us further with $\langle x^2 \rangle_s$ and $(\langle x \rangle_s)^2$ is given in Table 2. It is amusing to note that in this example 'the most probable values' [which agree with Eqs. (6.1)] never occur for any of the possible samples -- just another instance of statistics dealing with averages and being unjust to the individual.

Table 2

An example of the sampling relations

Assume a discrete distribution such that the values x = -1, 0, 1 occur with equal probability. Hence, beam average: $\langle x \rangle = 0$, and beam variance: $\langle x^2 \rangle = x^2_{rms} = 1/3\,[(-1)^2 + 0^2 + 1^2] = 2/3$; sample size: $N_s = 2$. To work out the most probable values of the sample characteristics, we write down all possible samples of size $N_s = 2$, determine $\langle x \rangle_s$, $(\langle x_s \rangle)^2$, and $\langle x^2 \rangle_s$, and take the average of these averages to find the expectations.

| Sequence | Sample averages $\langle x \rangle_s$ | $(\langle x \rangle_s)^2$ | $\langle x^2 \rangle_s$ |
|---|---|---|---|
| -1 -1 | -1 | 1 | 1 |
| -1 0 | $-\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| -1 1 | 0 | 0 | 1 |
| 0 -1 | $-\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 0 0 | 0 | 0 | 0 |
| 0 1 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 1 -1 | 0 | 0 | 1 |
| 1 0 | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ |
| 1 1 | 1 | 1 | 1 |
| Expectation = average of above values | 0 = $\langle x \rangle$ | 1/3 = $\langle x^2 \rangle/2$ | 2/3 = $\langle x^2 \rangle$ |

*Note for fun:* The expectation values agree with the prediction of the sampling theorems but the values 1/3 and 2/3 never occur (in this example) for any sample.

To conclude our detour, let us mention that the sampling relations (6.1) are a consequence of the more general 'central limit theorem'[7] of statistics. For the present purpose we can quote this theorem as follows.

When a large number of random samples of size $N_s$ are taken from a population with statistics $\langle x \rangle = 0$ and $\langle x^2 \rangle = x^2_{rms}$, then the distribution of the sample averages is approximately *Gaussian* with a mean equal to the population mean and a standard deviation $\sigma = x_{rms}/\sqrt{N_s}$.

7. A BETTER APPROXIMATION TO THE COOLING RATE (TEST PARTICLE PICTURE)

We can now attempt a less hazardous evaluation of Eqs. (4.2) and (4.4) to determine the cooling rate. To profit from the sampling relations, it is useful to regard the change $\Delta(x^2) = x_c^2 - x^2$ of the squared error rather than $\Delta x$. From Eq. (4.4) we obtain

$$\Delta(x^2) = -2gx \frac{1}{N_s} \sum_{\text{sample}} x_i + \left( g \frac{1}{N_s} \sum_{\text{sample}} x_i \right)^2 . \tag{7.1}$$

We anticipate that cooling is a slow process which requires many turns. Since we are (mainly) interested in this long-term behaviour, we regard the average of Eq. (7.1) over many turns. Doing so, we replace the sample averages by their expectation values for random samples to obtain the 'expected cooling rate' per turn.

The second term in Eq. (7.1) gives immediately

$$\left(g \frac{1}{N_s} \sum_{\text{sample}} x_i\right)^2 \rightarrow g^2(\langle x\rangle_s)^2 + \frac{g^2}{N_s} x_{rms}^2 , \tag{7.2}$$

where we have used the sampling relation (6.1c) to express the expected variance of the sample average in terms of the beam variance $x_{rms}^2$. To work out the first term we separate the test particle (once again) from the sum and write

$$x \cdot \frac{1}{N_s} \sum_{\text{sample}} x_i = \frac{x^2}{N_s} + \frac{x}{N_s} \sum_{\text{others}} x_i .$$

Next we apply the sampling relation (6.1a) to the remaining sum, i.e. we take

$$E\left(\frac{1}{N_s-1} \sum_{\text{others}} x_i\right) = E\left(\langle x\rangle_{\text{others}}\right) = 0$$

under the assumption that the sample (labelled 'others') without the test particle is a random sample such that (6.1a) applies. Then

$$E\left(x \cdot \frac{1}{N_s} \sum_{\text{sample}} x_i\right) = \frac{x^2}{N_s} . \tag{7.3}$$

Putting together the terms, the expected change is then

$$\Delta(x^2) \rightarrow -\frac{2g}{N_s} x^2 + \frac{g^2}{N_s} x_{rms}^2 . \tag{7.4}$$

Equation (7.4) applies to any test particle. Taking as typical a particle with an error equal to the beam r.m.s. we can write especially:

$$\frac{1}{x_{rms}^2} \Delta(x_{rms}^2) \rightarrow -\frac{1}{N_s} (2g - g^2) . \tag{7.5}$$

This gives the cooling rate (per second) for the beam variance:

$$\boxed{\frac{1}{\tau_{x^2}} = -f_0 \frac{\Delta(x_{rms}^2)}{x_{rms}^2} = \frac{f_0}{N_s} (2g - g^2) = \frac{2W}{N} (2g - g^2) .} \tag{7.6}$$

Clearly the term 2g presents the coherent effect already identified. The $g^2$ term represents the incoherent heating by the other particles. The inclusion of this term is the improvement obtained in the statistical evaluation of this section.

It emerges quite naturally from Eq. (7.6) that g should not be too large! In fact, optimum cooling (maximum of $2g - g^2$) is obtained with $g = 1$, and antidamping ($1/\tau$ negative) occurs if $g > 2$.

It should be remembered that Eq. (7.6) gives the cooling rate $1/\tau_{x^2}$ for $x^2$; the rate $1/\tau$ for x is 1/2 of this, as can easily be verified by comparing $x^2 \equiv x_0^2 \exp(-t/\tau_{x^2})$ and $x^2 = [x_0 \exp(-t/\tau)]^2$. Also, to recover the more familiar form of $1/\tau$ we used $f_0/N_s = 2W/N$, as in Eq. (5.3).

## 8. HEREWARD'S WAY

For those who were not pleased with the way in which we separated the test particle from its sample and regarded the remainder as a random sample of size $N_s$-1, we give yet another derivation of Eq. (7.6) which is due to Hereward (unpublished notes 1976, see also Ref. 8).

We restart from Eq. (7.1), which we write as

$$\Delta(x^2) = -2gx \cdot \langle x\rangle_s + g^2(\langle x\rangle_s)^2 \ . \qquad (8.1)$$

This is the change for one test particle and one turn. We now take the average of this *over the sample* of the test particle (before, we took the average for one particle *over many turns*).

A slight complication arises from the fact that strictly speaking each particle defines its own sample, as sketched in Fig. 8. We can assume, however, that the long-term behaviour of any sample (i.e. any beam slice of length $T_s$) is the same, so that expectation values are independent of the choice of the sample.

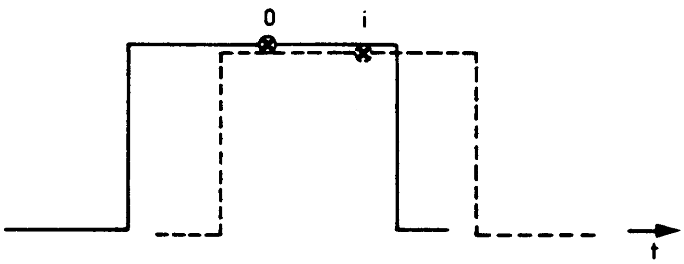


Fig. 8 Sample of the original test-particle (0) and of a particle passing earlier (i). Working out the average $\langle x_i\langle x\rangle_s\rangle_s$ of $x_i\langle x\rangle_s$ each particle has to be associated with its own sample. To the extent that all beam samples have the same statistical properties, all long-term averages are the same: $\langle x_i\langle x\rangle_s\rangle_s \to (\langle x\rangle_s)^2$.

Then the only variable on the r.h.s. involved in averaging over the original sample is the x in the first term, and we obtain

$$\langle\Delta(x^2)\rangle_s \to -2g(\langle x\rangle_s)^2 + g^2(\langle x\rangle_s)^2 \ . \qquad (8.2)$$

Next we use the sampling relations (6.1b) and (6.1c). We include the fact that the correction (8.2) is applied to all beam samples once per turn. Thus,

$$\langle\Delta(x^2)\rangle_s \to \Delta x^2_{rms} \ ,$$

$$(\langle x\rangle_s)^2 \to x^2_{rms}/N_s \ ,$$

and the expected correction of beam variance per turn is

$$\Delta(x^2_{rms}) = -\frac{1}{N_s}(2g - g^2)x^2_{rms} \ .$$

This leads to the same cooling rate as that given by the previous approach, but the derivation lends itself to the following formulation of the "sampling picture".

Take a random beam sample of $N_s$ particles. Measure and correct its average error $\langle x\rangle_s$ by giving a kick $-g\langle x\rangle_s$ to all its particles. Owing to the finite particle number, the beam variance appears as a fluctuation with "noise" $(\langle x\rangle_s)^2 \to x^2_{rms}/N_s$ of the centre of gravity $\langle x\rangle_s$. By correcting $\langle x_s\rangle$ to (1-g) of its value (i.e. to zero for full g = 1), one reduces the sample variance (on the average) by $1/N_s\ (2g-g^2)$. Repeat $N/N_s$ times per turn to reduce the beam variance by the same amount. Repeat for many turns.

Table 3

'Simulation' of a one-turn correction (with $g = 1$) using the example of Table 2

We note down all possible samples of size $N_s = 2$ and reduce the sample errors to zero by applying the same correction to both sample members. This reduces the beam variance from 2/3 to 1/3, i.e. $\Delta x^2_{rms}/x^2_{rms} = 1/N_s = 1/2$.

| Before correction | | | After correction | | |
|---|---|---|---|---|---|
| Sequence | Sample | | Sequence | Sample | |
| | Average $\langle x_s \rangle_s$ | Variance $\langle x^2 \rangle_s$ | | Average | Variance |
| -1 -1 | -1 | 1 | 0 0 | 0 | 0 |
| -1 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| -1 1 | 0 | 1 | -1 1 | 0 | 1 |
| 0 -1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $+\frac{1}{2}$ $-\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 0 0 | 0 | 0 | 0 0 | 0 | 0 |
| 0 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ $\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 1 -1 | 0 | 1 | 1 -1 | 0 | 1 |
| 1 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ $-\frac{1}{2}$ | 0 | $\frac{1}{4}$ |
| 1 1 | 1 | 1 | 0 0 | 0 | 0 |
| 'Beam variance' (average of all sample variances) | | 2/3 | | | 1/3 |

Thus, rather than treating single particles, one measures and corrects the centres of gravity of beam samples. It is amusing (but not too surprising) to note that the total number of measurements, namely the number of turns $n \approx N_s$ required for reasonable cooling multiplied by the number $\ell_s = N/N_s$ of samples per turn, is N, as if we treated the N particles individually.

It is easy to test this sampling prescription for simple distributions; in Table 3 we use the previous example (Table 2) to verify that the full correction ($g = 1$) reduces the variance by $1/N_s$ per turn. More generally, the sampling recipe can easily be simulated on a desk computer[4)] using a random number generator.

In the next two sections we will use the test particle and the sampling picture alternately to introduce two final ingredients, namely electronic noise of the amplifier and mixing of the samples due to the spread in revolution time.

## 9. SYSTEM NOISE

A large amplification of the error signals detected by the pick-up is necessary to give the required kicks to the beam. Electronic noise of the preamplifiers then becomes important. In Table 4 we anticipate some typical numbers pertaining to transverse cooling of $10^9$ $\bar{p}$ in LEAR. This example should convince us of the necessity to rewrite the basic

Table 4

Signal, noise, and amplification of a cooling system; orders of magnitude for $10^9$ particles and 50 s cooling time

| | |
|---|---|
| Pick-up signal | 50 nA |
| Preamplifier noise current | 150 nA |
| Kicker voltage per turn | 1 V |
| Corresponding current (into 50 Ω) | 20 mA |
| Power amplification | $\approx 2 \times 10^{10}$ |

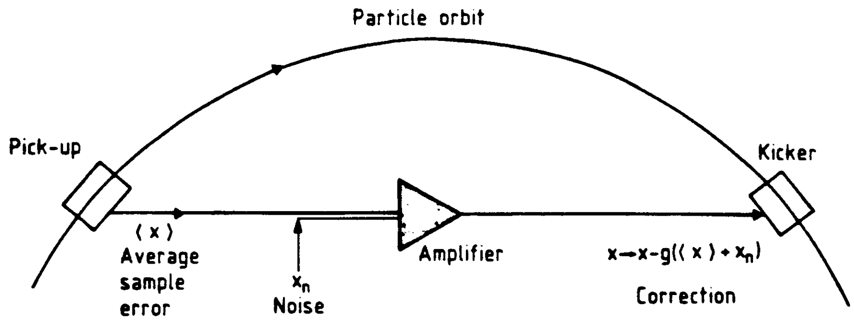


Fig. 9 Cooling loop including system noise. The noise is represented as an equivalent sample error $x_n(t)$ as observed at the pick-up.

equations to include noise. It is convenient[8]) to represent noise by an equivalent sample error (denoted by $x_n$) as observed at the pick-up. We then regard the system sketched in Fig. 9 and write

$$x_c = x - g\langle x\rangle_s - gx_n \ . \tag{9.1}$$

Going once again through our basic procedure, taking random noise uncorrelated with the particles we obtain the expected cooling rate

$$\boxed{\frac{1}{\tau} = \frac{2W}{N}\left[2g - g^2\ (1+U)\right]\ ,} \tag{9.2}$$

where $U = E(x_n^2)/E(\langle x\rangle_s^2)$ is the ratio of the expected noise to the expected signal power, called the 'noise-to-signal power ratio' or noise-to-signal ratio for brevity.

This introduces the noise into our pictures: it increases the incoherent term by (1+U). System noise and the disturbance caused by the other particles enter in much the same way; the latter is therefore also called particle noise.

Several things can be observed from Eq. (9.2). Cooling remains possible despite very poor signal-to-noise ratios ($1/U \ll 1$). All we have to do is to choose g small enough ($g \lesssim g_0 = 1/(1 + U) \approx 1/U$), which unavoidably means slow cooling ($\tau \gtrsim NU/2W$). In other words, we have to be patient and give the system a chance to distil a signal out of the noise.

In the initial cooling experiment (ICE)[9] with 200 circulating $\bar{p}$'s the system worked with signal-to-noise ratios as low as $10^{-6}$.

Secondly, U has a tendency to increase as cooling proceeds: namely the noise tends to remain the same, whereas the signal decreases as the beam shrinks. This is the case unless the pick-up plates are mechanically moved to stay close to the beam edge -- this will be done in the new antiproton collector ACOL[10] to be built at CERN.

With changing U, cooling is no longer exponential. Equation (9.2) gives a sort of instantaneous rate, and cooling stops completely ($1/\tau \to 0$) when U has increased such that $(1+U) = 2/g$. In this situation, equilibrium is reached between heating by noise and the damping effect of the system. To avoid this 'saturation' it is sometimes advantageous to decrease g during cooling in order to work always close to the optimum gain [maximum of Eq. (9.2)] $g_0 = 1/(1 + U)$.

In all cases it is important to obtain a good signal-to-noise ratio. Frequently, this means having a large number of pick-ups as close as possible to the beam, as well as high quality, low-noise preamplifiers often working at cryogenic temperatures.

## 10. MIXING

So far, all our considerations have been based on the assumption of random samples. This is a good hypothesis for an undisturbed beam. However, the cooling system is designed to correct the statistical error of the samples. Just after correction, samples will no longer be random. For full correction the centre of gravity $\langle x \rangle_S$ will be zero rather than $\sqrt{x^2_{rms}/N_S}$ as expected for random conditions. Cooling will then stop as no error signal is observable.

As a simple example, consider a sample of two particles with position errors $x_1 = 2$ cm and $x_2 = 0$, respectively. The system will change the positions to $x_1 = 1$ cm and $x_2 = -1$ cm and thus reduce the variance $\frac{1}{2}(x_1^2 + x_2^2)$ from 2 cm$^2$ to 1 cm$^2$, but from thereon the sample will be invisible as the centre-of-gravity signal $\langle x \rangle_S = \frac{1}{2}(x_1 + x_2)$ is zero.

Fortunately, owing to momentum spread, particles in a storage ring go round at slightly different speeds, and the faster ones continuously overtake the slower ones. Because of this mixing, the sample population changes and the sample error reappears, until ideally all particles have zero error.

If mixing is fast so that complete re-randomization has occurred on the way from kicker to pick-up, then the assumption of random samples made in the previous sections is valid. If, however, mixing is incomplete, cooling is slower. In fact, if it takes M turns for a particle of typical momentum error to move by one sample length with respect to the nominal particle ($\Delta p/p = 0$), then intuitively one expects an M times slower cooling rate.

A slightly different way of looking at imperfect re-randomization suggests itself in the frame of the test particle picture: bad mixing means that a particle stays too long -- namely, M rather than 1 turns -- together with the same noisy neighbours. This increases the incoherent heating by the other particles by a factor M.

We thus generalize the basic equation (9.2) (a rigorous derivation will be given in Part II of this lecture),

$$\boxed{1/\tau = \frac{2W}{N}\left[2g - g^2\ (M+U)\right]\ ,} \tag{10.1}$$

and call $M \geq 1$ the mixing factor. Equation (10.1) has the optimum

$$g = g_0 = 1/(M + U) \ ,$$
$$\tau = \tau_0 = \frac{N}{2W} (M + U) \ . \tag{10.2}$$

This underlines the importance of having good mixing -- $M \to 1$ -- on the way from correction to the next observation, but ...

What about mixing between observation and correction? Surely if the sample as observed is very different from the sample as corrected, then adverse effects can happen. Let us again resort to the test particle description and try to imagine how the coherent and the incoherent effects change. As to the latter, we expect that it is to first order not affected. We can just assume that the perturbing kicks are due to a new sample which has the same statistical properties as the original beam 'slice'.

The coherent effect will, however, change because the system will be adjusted in such a way that the correction pulse will be synchronous with the nominal particle ($\Delta p/p = 0$). Particles that are too slow or too fast on the way from pick-up to kicker will therefore slip with respect to their self-induced correction (Fig. 10). In fact, in the rectangular response model used above, the coherent effect will be completely zero if the particle slips by more than half the sample length ($|\Delta T_{PK}| > T_s/2$). At this stage, it is more realistic to use a parabolic response model of the form $1 - (\Delta T/T_c)^2$, where $T_c$, the useful width of the correction pulse, is about equal to the sample length $T_s$ for a low-pass system. But $T_c$ is shorter than $T_s$ for a high-frequency band-pass system with $f_{min} \gtrsim W$, with a response as sketched in Fig. 4b; $\Delta T$ is the time-of-flight error of the particle between pick-up and kickers. Introducing the typical error $\Delta T_{PK}$ and calling $\Delta T_{PK}/T_c = 1/\tilde{M}$, we can modify the coherent term $g \to g[1 - \tilde{M}^{-2}]$ to account for unwanted mixing between observation and correction. In a regular lattice the flight time from pick-up to kicker is a fixed fraction of the time from kicker to pick-up, and the two mixing factors $M$ and $\tilde{M}$ are proportional to each other, $M = \alpha\tilde{M}$, with $\alpha$ being the ratio of the corresponding distances -- hence the interest in having a short beam path from pick-up to kicker. By a clever choice of the bending and focusing properties of the storage ring it is possible, in principle, to make $\Delta T_{PK} \to 0$ independent of momentum, and $\Delta T_{KP}$ large to approach the desired situation sketched

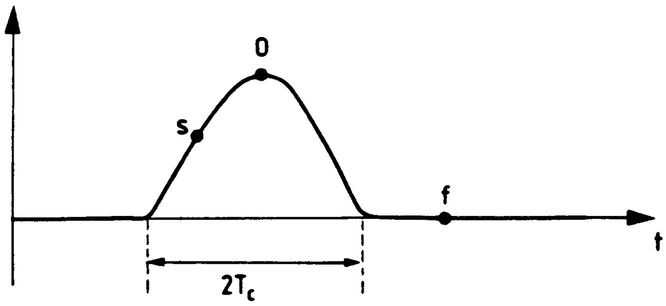


Fig. 10 Synchronism between particles and their correcting pulse on their way from pick-up to kicker. The response of the cooling system to a particle (the 'coherent effect') is approximated by a 'parabola' $s(t) = 1 - (\Delta t/T_c)^2$ of width $\pm T_c$ instead of the 'rectangle' used in Figs. 5 and 6. A nominal particle (0) arrives at the kicker simultaneously with the correction kick. The particle f is much too fast and advances its correction pulse. The particle s is slightly too slow. Thus, the three particles receive full correction, no correction, or partial correction, respectively.

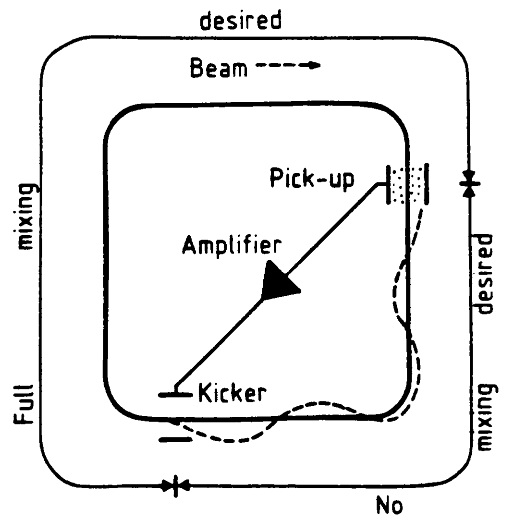


Fig. 11 The mixing dilemma: No mixing is desired between the pick-up and the kicker, but good mixing is needed on the way from the kicker to the pick-up. In a 'regular machine' the two mixing effects are proportional to the corresponding path length and it is desired to have the path from pick-up to kicker as short as possible. By clever choice of the focusing properties, the time of flight from pick-up to kicker can (in principle) be made independent of the particle energy, and the time kicker to pick-up strongly dependent on E. This has, however, not been adopted in existing designs as it leads to complications for the machine. The usual way out of the mixing dilemma is to accept some mixing between pick-up and kicker and imperfect mixing between kicker and pick-up.

in Fig. 11. This lattice choice was considered (but finally not adopted) in the design of the $\bar{p}$ collector ACOL.

Taking both mixing effects into account, we put our basic equation in the final form:

$$\boxed{\frac{1}{\tau_{x^2}} = \frac{2W}{N}\left[2g(1 - \tilde{M}^{-2}) - g^2(M+U)\right] .} \tag{10.3}$$

## 11. SOME LIMITS

Following convention, we now return to the cooling rate for x rather than $x^2$ (using $1/\tau_x = \frac{1}{2}\, 1/\tau_{x^2}$). Including both mixing effects as well as amplifier noise, we write

$$\frac{1}{\tau_x} = \frac{W}{N}\left[2g(1 - \tilde{M}^{-2}) - g^2(M+U)\right] , \tag{11.1}$$

Optimum

$$g_0 = \frac{1 - \tilde{M}^{-2}}{M + U} ,$$

$$\frac{1}{\tau_x} = \frac{W}{N}\left(\frac{(1 - \tilde{M}^{-2})^2}{M + U}\right) ,$$

with $M \geq 1$, $U > 0$, $\tilde{M}^{-2} \leq 1$. As an example of relatively straightforward technology, we take $W = 250$ MHz.

Then, in the best of all cases ($M \to 1$, $U \to 0$, $\tilde{M}^{-2} \to 0$) this gives

$$1/\tau = W/N , \tag{11.2}$$

or $\tau = 1$ s at $2.5 \times 10^8$ p or $\tau \approx 1$ day at $10^{13}$ p.

To include mixing, we assume that the time-of-flight dispersion between pick-up and kicker and between kicker and pick-up and the system response are such that the unwanted mixing $\tilde{M}$ is one half of the wanted mixing, i.e. we put (as an example) $\tilde{M}^{-1} = \frac{1}{2} M^{-1}$. We further assume that the sensitivity and the number of pick-ups are such that $U = 1$ (little is gained in going to more pick-ups, such that $U \ll 1$). Then the best cooling, obtained with $M \approx 1.5$, is

$$1/\tau_x \approx 0.32\ W/N\ , \tag{11.3}$$

This is about three times slower than the rate (11.2) with $\tilde{M}^{-2} \to 0$, $U \to 0$.

Further examples can be cooked up, especially for very low intensity beams where it is hard to have sufficient signal. Take, for instance, $U = 10$, $\tilde{M} = 2M$ to find an optimum $1/\tau_x \approx 0.07$ W/N for $M = 3.5$, etc.

In all cases, large bandwidth and a good signal-to-noise ratio are important. The relevant ring parameters have to be chosen such that, with the given $\Delta p/p$, reasonable mixing factors M and $\tilde{M}$ are obtained. Other limitations are imposed by the available broad-band power.

From Fig. 12 we conclude that existing cooling systems follow a 'working line' with $1/\tau \approx 0.1$ to 0.3 W/N. A bandwidth of 250 to 500 MHz is (more or less) standard; 2 to 4 GHz will be used in the CERN-ACOL and the Fermilab $\bar{p}$ source. Bands of 4 to 8 GHz or higher have been contemplated for sources accumulating $10^{13}$ $\bar{p}$ in a few hours, as desirable for multi-TeV colliders.

## 12. PRACTICAL DETAILS

So far we have, in a general way, discussed a system for correcting 'some error x'. We shall now briefly go into some details of the systems which cool momentum deviation or betatron oscillation, and which are listed in Table 5. The first two will be discussed below. The last two variants will be more logically treated in the frequency domain analysis (Part II).

Table 5

Stochastic cooling systems in use or proposed

| Type | Pick-up | Corrector | Optimum distance pick-up to kicker | Tested on | Used on |
|---|---|---|---|---|---|
| Betatron cooling, horizontal or vertical | Difference pick-up | Transverse kicker | $(2k+1)\lambda_\beta/4$ | ISR, ICE, Fermilab | AA, LEAR |
| Momentum cooling, Palmer-Hereward type | Horizontal difference pick-up | RF gap (acceleration/ deceleration) | $(2k+1)\lambda_\beta/2$ for simultaneous horizontal betatron cooling | ISR | AA: stack cooling |
| Momentum cooling, filter method | Longitudinal (sum) pick-up + comb filter | RF gap | | ICE Fermilab | AA: pre-cooling, LEAR |
| Momentum cooling, transit time method | Longitudinal pick-up + Differentiator | RF gap | | Fermilab | |

### 12.1 Palmer cooling

A horizontal position pick-up is used to detect the orbit displacement $y = D(\Delta p/p)_s$ concurrent with the momentum error of the sample; D (also denoted by $\alpha_p$ or $x_p$) is simply a constant, namely the value of the orbit 'dispersion function' at the pick-up. Its value (measured in metres per unit $\Delta p/p$ or in millimetres per permille) is determined by the focusing properties of the storage ring. You may think of it as the (local) change of orbit

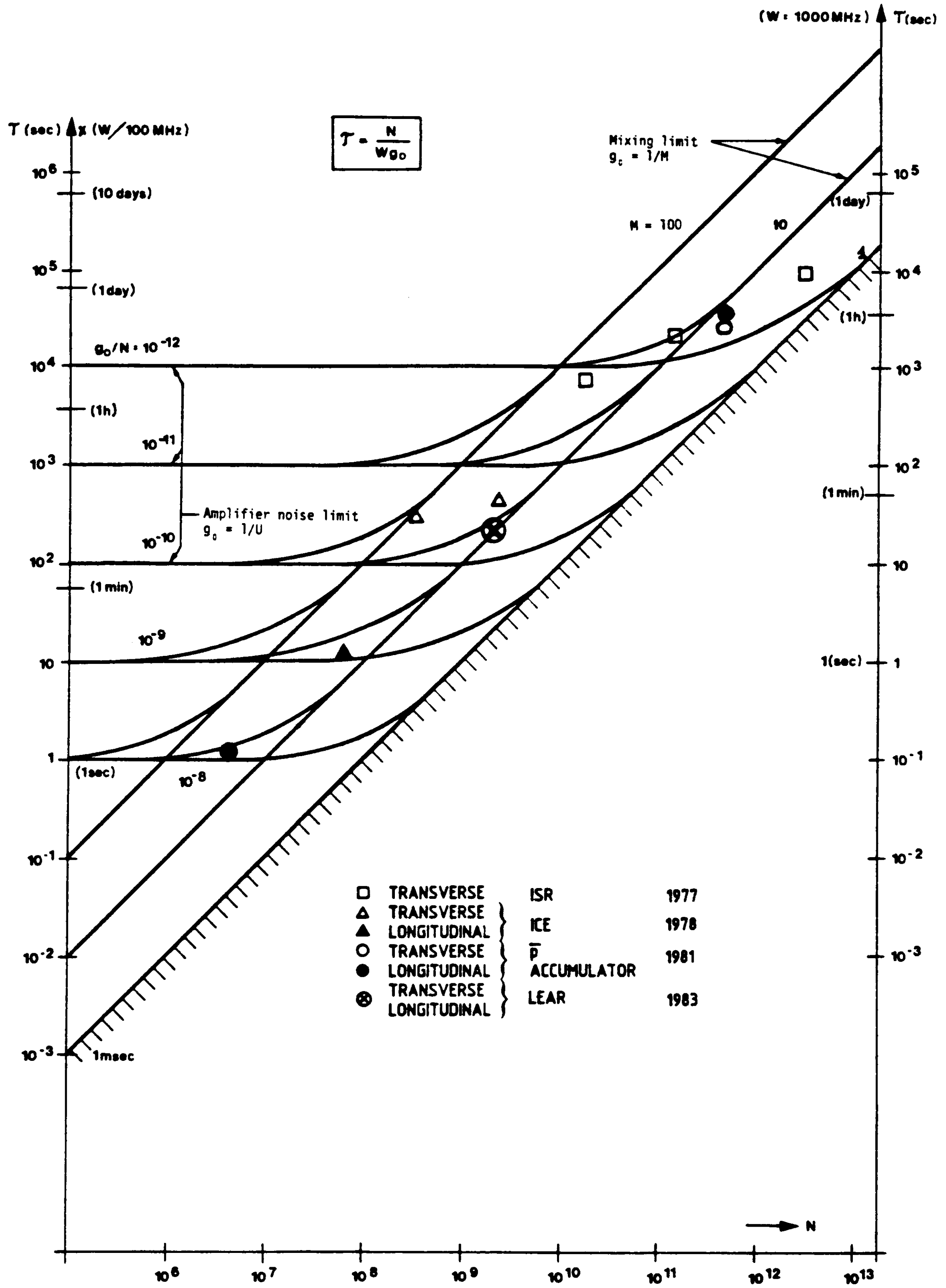


Fig. 12 Normalized cooling time versus intensity. The inclined lines represent the mixing limit. For low intensity the cooling time levels off because of noise. The points represent initial cooling in various machines. These points roughly follow a line with $\tau \approx 10$ N/W. During cooling, noise and/or mixing tend to become more important and the cooling time longer. Note that the vertical scale is normalized for 100 MHz bandwidth. Future systems (ACOL, Fermilab) aim at 2-5 GHz bandwidth.

radius which results from the balance of centrifugal force and restoring force as the momentum of a particle deviates from nominal.

In general there are further contributions to the position error, especially the betatron oscillation ($y_\beta$) of the particles. We shall neglect this contribution, assuming that by a clever choice of the focusing arrangement $\langle D(\Delta p/p)\rangle_s$ dominates over $\langle y_\beta\rangle_s$. We are then in a situation where momentum cooling as envisaged by R. Palmer of BNL (private communication to L. Thorndahl and H.G. Hereward in 1975) is possible. At the RF gap the particle receives a 'kick' of momentum and hence a change of y proportional to the detected error.

The basic one-passage equation (including noise) is written as

$$y_c = y - g[D\langle\Delta p/p\rangle_s + y_n] \, .$$

This is completely equivalent to Eq. (9.1), thus leading to the cooling rate (for $y^2$ and $\Delta p^2$, i.e. for the mean square of the momentum deviation):

$$\frac{1}{\tau_{\Delta p^2}} = \frac{2W}{N}\left[2g(1 - \tilde{M}^{-2}) - g^2(M+U)\right] \, , \tag{12.1}$$

where $U = E(y_n^2)/E[(D\langle\Delta p/p\rangle_s)^2]$ is the noise-to-signal ratio; $y_n$ is the system noise expressed as the equivalent pick-up signal $D(\Delta p/p)$, and $E(y_n^2)$ the expectation (i.e. the long-term average) of $y_n^2$.

Thus, the analysis has completely followed the line sketched in Sections 3 to 9.

12.2 Betatron oscillation cooling

The system to be discussed is the one considered by van der Meer in the first paper on stochastic cooling. To simplify the discussion, we use the so-called 'smooth approximation', i.e. we represent the betatron oscillation of particle 'i' by a sinusoidal motion with constant amplitude $a_i$ and constant wavelength $\beta = R/Q$:

$$\begin{aligned} y &= a_i \cos\left[Q(s/R) + \phi_i\right] \, , \\ \tilde{y}' &= \frac{R}{Q}\, y' = -a_i \sin\left[Q(s/R) + \phi_i\right] \, . \end{aligned} \tag{12.2}$$

A pick-up at, say, s = 0 measures the position $y_{PU} = a_i \cos(\phi_i)$ of the particles; a kicker $\Delta\psi = Q(\Delta s/R)$ downbeam corrects the 'angle' $\tilde{y}'_K = -a_i \sin(\phi_i + \Delta\psi)$. For simplicity, we now assume that the focusing properties are such that the orbit dispersion at the pick-up $[D(\Delta p/p)]_{PU}$ is negligible compared with the betatron displacement $y_\beta = y$.

The basic equation for a test particle is written:

$$\begin{aligned} \Delta y_K &= 0 \, , \\ \Delta\tilde{y}'_K &= -g[\langle y\rangle_s + y_n]_{PU} \, . \end{aligned} \tag{12.3}$$

Eq. (12.3) expresses the kick at the corrector as given by the error observed at the pick-up. We now assume that the distance pick-up to kicker is optimum so that $\Delta\psi = \pi/2$ $(+k\pi)$. Then the correction *at the kicker* is equivalent to a change of position *at the pick-up*, as sketched in Fig. 13, given by:

$$\begin{aligned} \Delta y_{PU} &= \Delta\tilde{y}'_K = -g[\langle y_s\rangle + y_n]_{PU} \, , \\ \Delta y'_{PU} &= 0 \, . \end{aligned} \tag{12.4}$$

[Use Eq. (12.2) or refer to Fig. 13 to calculate the equivalent jump (12.4) at the pick-up.]

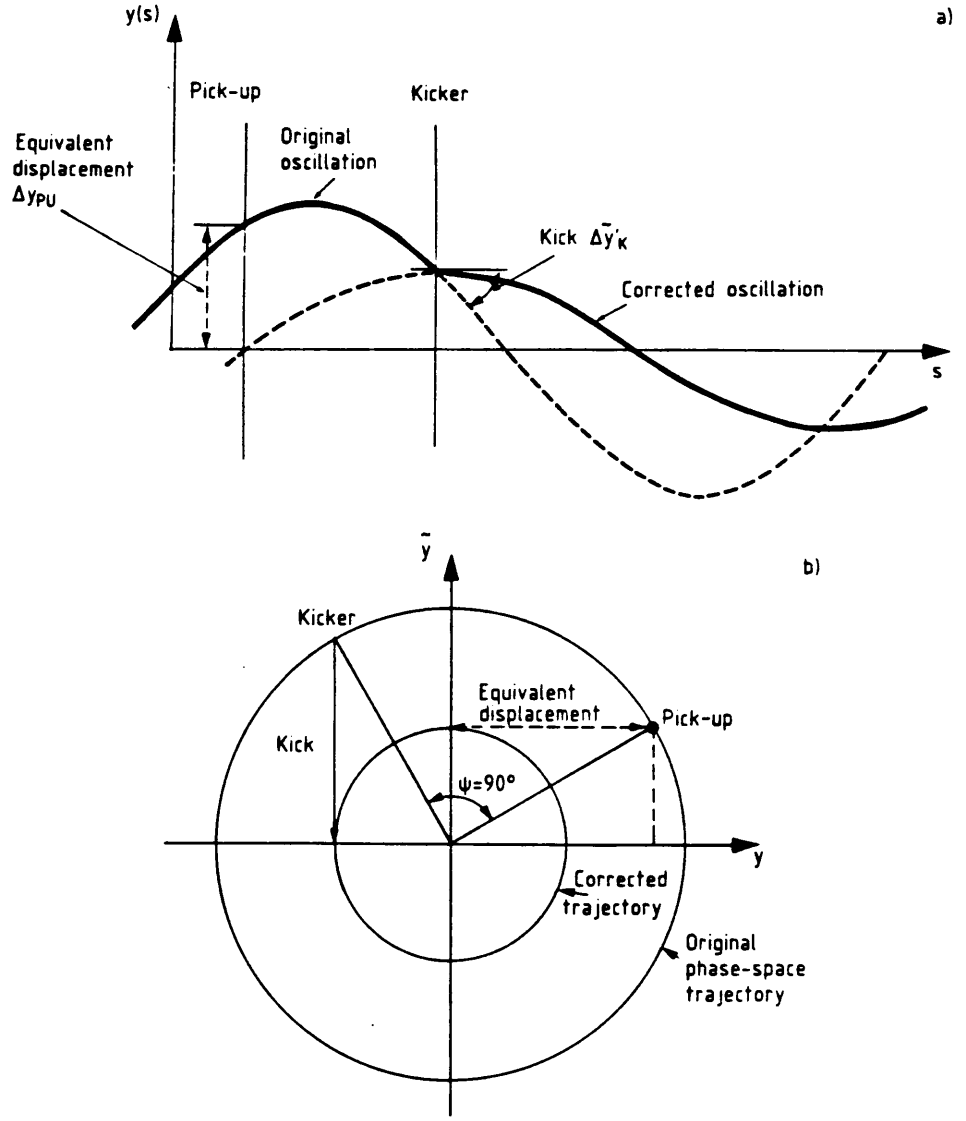


Fig. 13 Real-space diagram (a) and phase-space diagram (b) of betatron oscillation between pick-up and kicker. In smooth approximation the change $\Delta\tilde{y}'$ of normalized angle $[y'(R/Q)]$ is equivalent to a change of position $\Delta y_{PU} = \Delta\tilde{y}'$ at the pick-up at a phase $\Delta\psi = Q(\Delta s/R) = 90°$ $(+ k \cdot 180°)$ upstream.

Now that all quantities are used at the same location in the machine, we can apply the basic procedure to Eq. (12.4) to find the change of variance $y^2_{rms}$ per turn:

$$
\begin{aligned}
-\Delta(y^2_{rms}) &= y^2_{rms} \frac{1}{N_s} [2g - g^2(1+U)] , \\
\Delta(\tilde{y}'^2_{rms}) &= 0 ,
\end{aligned}
\qquad (12.5)
$$

where the noise-to-signal ratio U is defined by the expectation values $E(y^2_n)$ and $E[(\langle y \rangle_s)^2]$ of noise power and signal power, $U = E(y^2_n)/E[(\langle y \rangle_s)^2]$, referred to at the pick-up. We wish to find the reduction of r.m.s. betatron *amplitude*. Assuming that the initial phases $\phi_i$ [Eq. (12.2)] of the particles are random and uncorrelated with amplitude $a_i$ we have on the r.h.s. of Eq. (12.5):

$$y^2_{rms} = (a_i \cos \phi_i)^2_{rms} = a^2_{rms} \frac{1}{2} \qquad (12.6)$$

as the average of $\cos^2 \phi \rightarrow \frac{1}{2}$. In other words, owing to the oscillatory nature of the betatron motion the measurement of $y^2$ reveals, on the average, only half the information about $a^2$.

To find the change of amplitude from the l.h.s. of Eq. (12.5) we use [see Eq. (12.2)]

$$a^2_i = y^2_{iPU} + \tilde{y}'^2_{iPU} \; . \qquad (12.7)$$

Hence

$$\Delta(a^2) = \Delta(y^2) + \Delta(\tilde{y}'^2) \; .$$

Thus, collecting terms, we finally have (including mixing and using $N_s = N/2WT$) the cooling rate for $a^2$:

$$1/\tau_{a^2} = \frac{-\Delta(a^2)}{Ta^2} = \frac{1}{2} \frac{2W}{N} \left[2g(1 - \tilde{M}^{-2}) - g^2(M+U)\right]$$

$$\text{(per second)} \; . \qquad (12.8)$$

Compared with Eq. (10.3), the factor $\frac{1}{2}$ in the cooling rate (12.8) for the square of the betatron amplitude (i.e. of emittance $E \propto a^2_{rms}$) is new. Going to the rate $1/\tau_a$ for 'a' introduces another factor of $\frac{1}{2}$.

If the pick-up to kicker distance is different from optimum ($\pi/2$ modulo $\pi$) then the transformation (12.4) from kicker back to pick up becomes slightly more involved, and the final result is a factor $\sin(\Delta\psi)$ which multiplies the coherent term (2g) in the cooling rate (12.8). For those who want to verify this we sketch the calculation:

Correction referred from kicker to pick-up [generalization of Eq. (12.4)]:

$$\begin{aligned} \Delta y_{PU} &= \Delta\tilde{y}'_K \sin(\Delta\psi) \; , \\ \Delta\tilde{y}'_{PU} &= \Delta\tilde{y}'_K \cos(\Delta\psi) \; . \end{aligned} \qquad (12.4')$$

Use Eq. (12.3) for $\Delta\tilde{y}'_K$ as before. Apply the basic procedure to work out the expected change of variance for both equations (12.4'). In the second equation the coherent term is given by the expectation of

$$y_{PU} \cdot \langle \tilde{y}'_{PU} \rangle_s \; ,$$

which for random phases $\phi_i$ [Eq. (12.2)] is zero whereas the corresponding term in the first equation is

$$y_{PU} \cdot \langle y_{PU} \rangle_s \rightarrow \frac{1}{N_s} y^2_{rms} \; ,$$

as in Eq. (7.3). We then have

$$-\Delta(y^2_{rms}) = \frac{y^2_{rms}}{N_s} \left[2g \sin(\Delta\psi) - g^2 \sin^2(\Delta\psi) \; (1+U)\right]$$

$$-\Delta(y'^2_{rms}) = \frac{y^2_{rms}}{N_s} \left[-g^2 \cos^2(\Delta\psi) \; (1+U)\right]$$

which, using Eqs. (12.6) and (12.7), leads to the indicated modifications of the cooling rate.

### 12.3 Hereward cooling

In the analysis of momentum cooling, we assumed that the orbit dispersion $y_p = D(\Delta p/p)$ dominates at the pick-up, whereas for horizontal cooling we assumed that the betatron part $y_\beta = a_i \cos \phi_i$ is most important. The more realistic case, where both $y_p$ and $y_\beta$ are present, was analysed by Hereward. He showed[8] the mutual heating and at the same time the possibility of using the Palmer system for simultaneous longitudinal and horizontal cooling by a suitable choice of the pick-up to kicker distance.

Here we briefly sketch the derivation of the two cooling rates, hoping that the simple analysis will provide some insight into the important topic of mutual longitudinal horizontal influence. The first question that comes to mind is: How can the acceleration gap used in the Palmer system change the betatron oscillation? The answer is sketched in Fig. 14: a particle of nominal momentum oscillates around the central orbit, whereas an off-momentum particle bounces around the off-momentum orbit displaced by $D(\Delta p/p)$. If at a gap the momentum changes abruptly, the particle continues its oscillation around this displaced orbit with initial conditions $y = y_0 - D(\Delta p/p)$, $y' = y_0'$, as its oscillation centre has jumped by $\Delta y = D(\Delta p/p)$. [For simplicity, we assume that D(s) is constant at the kicker, $D'(s) = 0$.] Thus, the mechanism is similar to changing the oscillation of a pendulum by jumping its point of suspension.

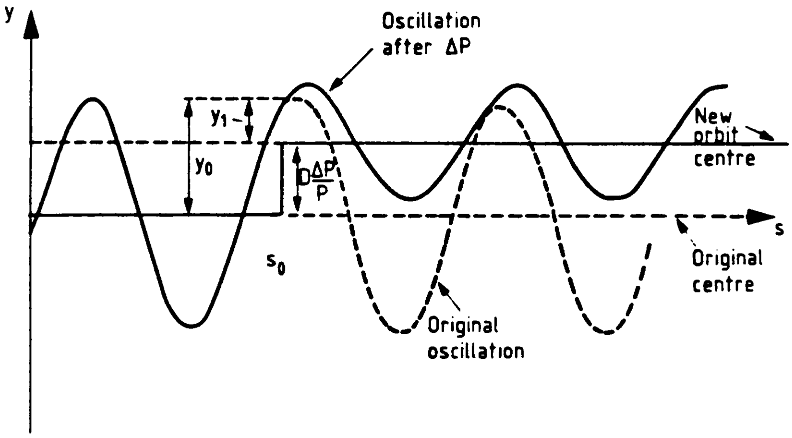


Fig. 14 Reduction of horizontal betatron oscillations by a momentum jump. At $s_0$ the momentum is abruptly changed by $\Delta p/p$. The particle continues its betatron oscillation around the new orbit centre, which is displaced by $D \cdot (\Delta p/p)$. Initial conditions for this new oscillation are $y_1 = y_0 - D \cdot (\Delta p/p)$, $y_1' = y_0'$ [change of lattice functions $D'(s)$, $\beta'(s)$ neglected!].

We can now write down the basic equations for Hereward cooling. The RF gap changes the momentum by an amount proportional to the observed position error of the sample. The basic equation for the change at the gap of the momentum displacement [$y_p = D(s)\Delta p/p$] and the betatron displacement from the instantaneous orbit is written

$$\left(\frac{1}{D}\,\Delta y_p\right)_K = -g\left(\frac{1}{D}\,\langle y\rangle_s\right)_{PU} ,$$

$$\left(\frac{1}{D}\,\Delta y_\beta\right)_K = g\left(\frac{1}{D}\,\langle y\rangle_s\right)_{PU} , \qquad (12.9)$$

$$(\Delta\tilde{y}_\beta')_K = 0 ,$$

where $\langle y\rangle_s = \langle y_p + y_\beta\rangle_s$ is the total pick-up signal of a sample containing both the betatron and the momentum contribution. In Eq. (12.9) we simplify, again assuming zero derivative of D(s) at the kicker, but we allow for different D at pick-up and kicker. Again the l.h.s. of Eqs. (12.9) is the correction at the kicker; the r.h.s. is given by the observation at the pick-up. To apply our basic procedure, we once more refer the corrections back to the PU to obtain for the first equation [using $(y_p/D)_{PU} = (y_p/D)_K$],

$$\Delta y_p = -g \langle y\rangle_s ,$$

which by our standard procedure leads to the cooling rate for the mean square momentum deviation:

$$\frac{1}{\tau_{\Delta p^2}} = \frac{2W}{N}\left[2g - g^2\left(1 + \frac{E(\langle y_\beta\rangle_s^2 + y_n^2)_{PU}}{E(D_{PU}^2 \langle \Delta p/p\rangle_s^2)}\right)\right] \qquad (12.10)$$

Here noise $(y_n)$ is included but perfect mixing is assumed. This equation shows heating of momentum cooling by the betatron oscillation as observed at the pick-up.

The betatron correction referred back to the pick-up is

$$(\Delta y_\beta)_{PU} = (\Delta y_\beta)_K \cos \Delta\psi ,$$

$$(\Delta \tilde{y}_\beta')_{PU} = (\Delta y_\beta)_K \sin \Delta\psi .$$

Using this to rewrite Eqs. (12.9) and carrying through the analysis for betatron cooling, as in the preceding section, we find the cooling rate for the mean square amplitude

$$\frac{1}{\tau_{a^2}} = \frac{1}{2}\,\frac{2W}{N}\left[-2g_\beta \cos \Delta\psi - g_\beta^2\left(1 + \frac{E(D_{PU}^2 \langle \Delta p/p\rangle_s^2 + y_n^2)}{E(\langle y_\beta\rangle_s^2)_{PU}}\right)\right] , \qquad (12.11)$$

where $g_\beta = g(D_K/D_{PU})$[11]. Hence for a distance pick-up to RF gap such that $\cos \Delta\psi < 0$, simultaneous cooling is possible. The desired distance is $\Delta\psi = \pi$ (modulo $2\pi$) in order to have the best efficiency ($\cos \Delta\psi = -1$). Also large dispersion $D_K$ at the kicker is useful in this case for good betatron cooling.

If the distance cannot be chosen so as to have $\cos \Delta\psi < 0$, then $D_K = 0$ avoids heating of horizontal oscillations by momentum cooling (the more general condition is $D_K = 0$ and $D_K' = 0$). This applies to all momentum-cooling methods which use an RF gap as a kicker; hence $D_K = 0$, $D_K' = 0$ is frequently aimed for at the location of the momentum kicker.

The condition $y_{\beta PU} \gg D_{PU}(\Delta p/p)$ ensures good betatron cooling. This is also generally true for all methods which use a horizontal position pick-up, such as the van der Meer method discussed in the previous section; $D_{PU} = 0$, $D_{PU}' = 0$ is therefore frequently aimed for at the position of the horizontal pick-up for transverse cooling. For Palmer momentum cooling, the inverse [$D_{PU}(\Delta p/p) \gg y_\beta$] is favoured for minimizing the 'betatron noise'. For combined horizontal and longitudinal cooling by the Hereward method a compromise ($y_p \approx y_\beta$ at the pick-up) has to be found.

These considerations have (hopefully) provided us with some insight into the problems of 'cross-heating'. We mention two other tricks which have been considered for avoiding horizontal heating by momentum cooling: i) splitting the momentum kicker into two units spaced by $\Delta\psi = \pi$, or ii) a horizontal kick proportional to the momentum kick. In the second case the compensator has to be displaced by $\Delta\psi = \pi/2$ from the momentum kicker if D is large, or $\Delta\psi = 0$ if D' is large at the $\Delta p$ kicker.

When the mixing is imperfect, it is (sometimes) also possible to decouple the horizontal-longitudinal heating, as the two signals have different frequency bands. This becomes clearer in the frequency domain analysis.

## 14. HISTORY

For a long time the principle of stochastic cooling was regarded as too far-fetched to be practical. A first experimental demonstration was tried only seven years after its invention and three years after the first publication of the idea (Table 6). The inventor, S. van der Meer, and the early workers had (mainly?) emittance cooling of high-intensity beams in mind with a view to improving the luminosity in the CERN ISR.

A new era began in 1975 when Strolin, coming back from a visit to Novosibirsk, and Thorndahl realized the interest of stochastic cooling -- for both emittance and momentum -- of low-intensity $\bar{p}$ beams for the purpose of stacking. Stochastic cooling at low intensity is different from the original van der Meer cooling. The extension of the theory [to $g < 1$ in Eq. (9.2)] first done by Hereward and Thorndahl, and the design of the momentum cooling hardware (Thorndahl, Carron), are perhaps as fundamental as the original invention and the earlier feasibility studies by van der Meer and Schnell.

Following upon this broadening of the field of application, in 1975 Strolin et al. worked out $\bar{p}$ collection schemes of the ISR using stacking in momentum space, and Rubbia et al. made their first proposals for the $p\bar{p}$ scheme for the SPS using similar techniques of stochastic cooling and accumulation. This work gave new life to the idea at a time when the ISR was routinely stacking such high proton currents that proton beam-cooling became unnecessary -- or even impossible. Further milestones between 1975 and 1980 were the invention of the filter method of momentum cooling, the refinement of the stochastic stacking schemes, the results of the initial cooling experiment (ICE), and last, but not least, the success of the AA. The ICE ring was used to make a careful comparison of cooling theory with reality, including bunched beam cooling. By the middle of 1978 all systems worked so well that beam lifetimes at 2 GeV/c of the order of a week were reached, compared with lifetimes of a few hours without cooling. This permitted a measurement of the stability of the antiproton, and this experiment improved the lower limit in one go from 120 μs to 80 h.

Running-in of the AA started in the summer of 1980, and since 1981/82, stacks of $10^{11}$ $\bar{p}$ are routinely accumulated from batches of a few $10^{6}$ $\bar{p}$ per second. The AA uses a total of seven cooling systems for longitudinal cooling of different 'regions' of the beam and for horizontal and vertical emittance cooling. Time constants are of the order of a second at up to $5 \times 10^{6}$ $\bar{p}$ or 30 minutes for $10^{11}$ $\bar{p}$, thus nearing design specifications. The AA is at the heart of CERN's antiproton programme, which culminated in the observation of the Intermediate Vector Bosons predicted by the unifying electroweak theory.

From about 1978 onwards, other laboratories, especially the Novosibirsk group, an ANL-LBL-Fermilab collaboration and, more recently, a group at the INS Tokyo, have done work (both experimental and theoretical) on stochastic cooling. This work has placed emphasis on various important aspects such as low-noise cryogenic amplifiers, very high frequency systems, cooling of heavy ions, or cooling of bunches.

In 1983 the Tevatron-1 project at Fermilab and the $\bar{p}$ collector ACOL to be added to the AA at CERN were approved. Both systems aim at stochastic cooling and stacking of antiprotons at a rate of several $10^{7}$ $\bar{p}$ per second.

Table 6

HISTORY

| | | |
|---|---|---|
| | Prehistory | |
| Liouville | 1838 | Invariance of phase space area |
| Schottky | 1918 | Noise in d.c. electron beams |
| | History | |
| van der Meer | 1968 | Idea of stochastic cooling |
| ISR staff (Borer, Bramham, Hereward, Hübner, Schnell, Thorndahl) | 1972 | Observation of proton beam Schottky noise |
| van der Meer | 1972 | Theory of emittance cooling |
| Schnell | 1972 | Engineering studies |
| Hereward | 1972-74 | Refined theory, low-intensity cooling |
| Bramham, Carron, Hereward, Hübner, Schnell, Thorndahl | 1975 | First experimental demonstration of emittance cooling |
| Palmer (BNL), Thorndahl | 1975 | Idea of low-intensity momentum cooling |
| Strolin, Thorndahl | 1975 | $\bar{p}$ accumulation, schemes for the ISR using stochastic cooling |
| Rubbia | 1975 | $\bar{p}$ accumulation, schemes for the SPS |
| Thorndahl, Carron | 1976 | Experimental demonstration of p cooling |
| Thorndahl, Carron | 1977 | Filter method of p cooling |
| Sacherer, Thorndahl, van der Meer, | 1977-78 | Refinement of theory; imperfect mixing, Fokker-Planck equations |
| ICE team | 1978 | Detailed experimental verification |
| Herr | 1978 | Demonstration of bunched beam cooling |
| AA team | 1981-82 | Accumulation of several $10^{11}$ $\bar{p}$ from batches of several $10^{6}$, cooling times close to design specifications |
| LEAR team | 1982 | Stochastic cooling to permit loss-free deceleration of $\bar{p}$. Interest in combining stochastic and electron cooling. |
| Kilian | 1982 | Stochastic cooling of heavy ions (proposals) |
| Novosibirsk group | 1980/82 | Stochastic cooling experiments, work on cooling theory |
| ANL-LBL-Fermilab group | 1979/83 | Theoretical and hardware studies, stochastic cooling experiments at FNAL |
| TARN group at INS Tokoyo | 1983/84 | Stochastic cooling experiments in the TARN ring |
| Fermilab group | 1983 | Design report of the Tevatron-1 project using a debuncher ring and an accumulated ring for fast stochastic cooling and stacking of $4 \times 10^{7}$ $\bar{p}$ per second |
| AA team | 1983 | Design report of the $\bar{p}$ collector ACOL using fast momentum and emittance cooling of $4 \times 10^{7}$ $\bar{p}$ per second |
| Berkeley group | 1983 | $\bar{p}$ collection scheme for 20 TeV colliders. Collection of $5 \times 10^{8}$ $\bar{p}$ per second. |
| SPS $\bar{p}$ team | 1983 | Feasibility study of bunched beam cooling in the SPS collider |

Table 7

Development of cooling rates:
number of particles that can be cooled
with time constants of 1s.

| Machine | Maximum number of particles for cooling time of 1 s | Cooling system bandwidth (GHz) |
|---|---|---|
| ISR 1975-77 | $10^5$ | 0.1 |
| ICE 1978 | $10^5$-$10^6$ | 0.25 |
| AA 1982 | $10^6$-$10^7$ | 1 |
| ACOL, TEVATRON 1985/86 | $10^7$-$10^8$ | 2 |
| $\bar{p}$-source for 20 TeV collider (1990?) | $10^9$ ($10^{10}$) | 4 (10) |

The rapid development of stochastic cooling over the last decade is illustrated by Table 7, which gives the cooling rate obtained or aimed at. Roughly one order of magnitude was gained every four years in the cooling power, i.e. in the number of particles which can be cooled with a time constant of 1 s. Probably an 'absolute' limit in the range $10^9$ to $10^{10}$ per second will soon be reached (in the projects!) unless frequencies much above 10 GHz can be used where (most) vacuum chambers transmit waveguide modes and the beam size becomes comparable to the RF-wavelength.

Since about 1980, interest in cooling of heavy ion beams is rapidly increasing, and the combination of stochastic 'precooling' with post-cooling by electrons looks attractive for some applications at the low-energy end. At the very highest energies, ideas on bunched beam cooling have been followed up, and a thorough study on stochastic cooling of bunches in the SPS collider is being carried out.

* * *

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2) See, for example, H. Bruck, Accélérateurs circulaires de particules, Bibliothèque des Sciences nucléaires, Paris, 1966.
M. Sands, SLAC report 121 (1970); or other textbooks on particle accelerators.

3) J. Borer, Proceedings of this School.

4) C. Taylor, ibid.

*) A Bibliography on stochastic cooling will be given at the end of Part II.

5) The bandwidth/pulse length relation was introduced by Nyquist and independently by Küpfmüller in 1928. For a discussion see, for example, W. Meyer Eppler, Grundlagen und Anwendungen der Informationstheorie, Springer Verlag Berlin, 1969. This theorem is closely-related to the more general sampling theorem of communication theory: If a function S(t) contains no frequencies higher than W cycles per second, it is completely described by its values $S(mT_s)$ at sampling points spaced by $\Delta t = T_s = 1/2W$ (i.e. taken at the 'Nyquist rate' 2W); see, for example, J.A. Betts, Signal processing and noise, (English Universities Press, London, 1970), or other textbooks on signal and communication theory.

6) LEAR Design Study Team: Design of a facility for experiments with low-energy antiprotons (LEAR), internal report CERN-PS-DL 80-1 (1980).
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9) G. Carron et al., Phys. Lett. 77B, 353 (1978).
M. Bregman et al., Phys. Lett. 78B, 174 (1978).

10) E.J.N. Wilson (ed.), Design study of an antiproton collector for the antiproton accumulator, CERN-83-10 (1983) and B. Autin, Proc. of this School.

11) If the values of the focusing function ($\beta_H(s)$) are also different at pick up and kicker then

$$g_\beta = g \frac{(D/\sqrt{\beta_H})_K}{(D/\sqrt{\beta_H})_{PU}}$$

has to be used in Eq. (12.10). If, in addition, the derivatives of D(s) and $\beta_H(s)$ are non-zero at the gap, the change of betatron oscillation by a momentum kick is

$$\Delta y_\beta = -D_K \frac{\Delta p}{p}$$

$$\Delta y'_\beta = -D' \frac{\Delta p}{p}$$

The corresponding change in emittance ($E_H$) can be calculated from the betatron equations

$$y_\beta = \sqrt{\beta E/\pi} \sin \psi$$

$$y'_\beta = \sqrt{E/(\beta\pi)} \left(-\tfrac{1}{2} \beta' \sin \psi + \cos \psi\right) .$$

The result is a more involved expression for the betatron heating and a change of the optiumum spacing for simultaneous cooling.

# PART II: DESCRIPTION IN FREQUENCY DOMAIN

## 1. INTRODUCTION

In Part II we restart from the frequency spectrum of the pick-up signals, the so-called Schottky noise spectrum of a coasting beam which was discovered and pioneered at the ISR. This approach to stochasic cooling, first used by Hereward and Sacherer in 1978, permits a clearer picture of mixing and of heating by the ampflifier and by the other particles. We then briefly try to appreciate signal suppression due to 'the feedback via the beam', also first observed at the ISR and included in cooling theory by F. Sacherer. Although probably not too important for the cooling itself, the signal reduction has become an important diagnostic tool for optimizing the system, harmonic by harmonic. Finally, we try to get a first feeling for the more general cooling equations which describe the evolution of the particle distribution. The filter method of Thorndahl and Carron, which is a vital ingredient in all low-intensity cooling rings, is introduced in this context.

Part II (like Part I) is devoted to the beginner. Those who want to design a cooling system will have to consult the more technical literature.

## 2. NOISE SIGNALS FROM A COASTING BEAM

Imagine the pick-up signal from a single particle going around at constant revolution frequency in a storage ring. On each traversal the particle induces a short current pulse $i(t)$ with a length given by the transit time $\Delta t$ in the pick-up (Fig. 1). The area under this pulse is determined by the charge which passed the current pick-up, i.e. $\int i(t)\,dt = e$.

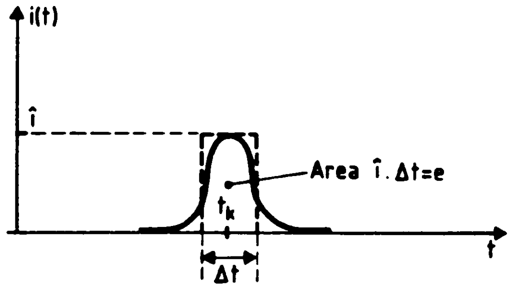



Fig. 1 The current pulse due to a single particle passing a longitudinal pick up and its equivalent square pulse. The area under the pulse $\int i(t)dt = \hat{i} \cdot \Delta t = e$ is given by the charge which has passed the pick up. The transit time $\Delta t$ depends on the length and the shape of the electrodes as well as on the particle velocity. Except for very high frequency, $i(t)$ can be approximated by a δ-function $e\delta(t-t_k)$.

This current pulse can be approximated by a delta-function (Fig. 1), unless we are interested in its very high-frequency content ($f \gtrsim 1/\Delta t$). Viewed over many revolutions, the pick-up signal (Fig. 2a) is then a train of short pulses spaced by the revolution time $T = 1/f_0$. It can be approximated using the periodic delta-function $\delta(t - mT)$:

$$i(t) = e\,\delta(t - t_k - mT) \quad . \tag{2.1}$$

Here the 'passage time' $t_k$ is given by the initial conditions, namely the time at which the particle was injected.

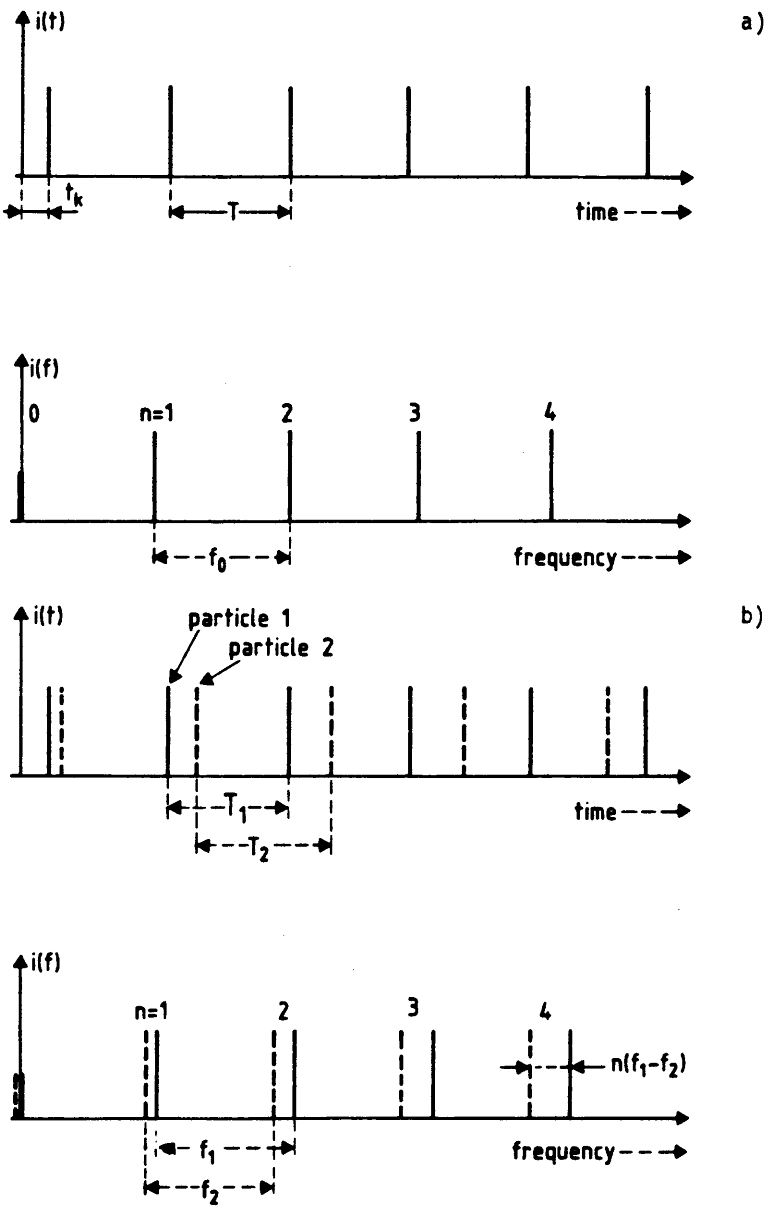



Fig. 2 a) Time and frequency spectrum of the pick up signal from a single particle coasting at revolution $f_0 = 1/T$. b) Time and frequency spectrum for a particle of frequency $f_1 = 1/T_1$ and a second particle with $f_2 = f_1 - \Delta f_0 < f_1$.

Using the Fourier expansion of the periodic delta-function, Eq. (2.1) can also be written as

$$i(t) = i_0 + \sum_{n=1}^{\infty} i_n \cos n\omega_0(t - t_k) \qquad (2.2)$$

where

$$i_0 = e/T = ef_0 \ , \quad i_n = 2i_0 = 2ef_0 \ , \quad \omega_0 = 2\pi/T = 2\pi f_0 \ .$$

Thus in the frequency domain the single-particle current is described by a line spectrum (Fig. 2a) with a ray at each harmonic $nf_0$ of the revolution frequency. In the delta approximation all lines have the same height, $2ef_0$, except for the d.c. component $i_0 = ef_0$.

It is sometimes useful to write Eq. (2.2) in terms of its sine and cosine components:

$$i(t) = i_0 + \sum a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \qquad (2.3)$$

where

$$a_n = i_n \cos n\psi_k = 2i_0 \cos n\psi_k \ ,$$
$$b_n = i_n \sin n\psi_k = 2i_0 \sin n\psi_k \ ,$$

and

$$\psi_k = \omega_0 t_k \ .$$

Next, assume a particle of a slightly different frequency $f_1$. It will have a similar time and frequency spectrum with lines at $kT_1$ and $nf_1$, respectively (Fig. 2b). Hence, for a beam with a distribution of revolution frequencies between $f_1 = f_c - \Delta f_0/2$ and $f_2 = f_0 + \Delta f_0/2$, one expects a spectrum with bands around each harmonic $nf_0$ of the revolution frequency (Fig. 3). The $m^{th}$ band extends from $mf_1 = mf_0 - m\Delta f_0/2$ to $mf_2 = mf_0 + m\Delta f_0/2$. Its width is therefore $m\Delta f_0$.

This simple picture introduces the possible frequency content of a beam in a storage ring. To calculate the strength of these bands, a slightly more involved analysis is needed, taking the initial phases ($\omega t_k$ in Eq. 2.1) of the particles into account[1,2].

Let us assume a distribution of initial times $t_k$ [Eq. (2.2)] between 0 and the revolution time $T = 2\pi/\omega$. In the time domain the pick-up signal is then a dense series of pulses which looks like a d.c. current with, superimposed on it, a high-frequency current with a wavelength spectrum given by the spacing of the particles. For *uniformly* spaced particles in typical beams with $10^5$ to $10^{10}$ particles per metre the a.c. component is in the $10^{13}$-$10^{18}$ Hz

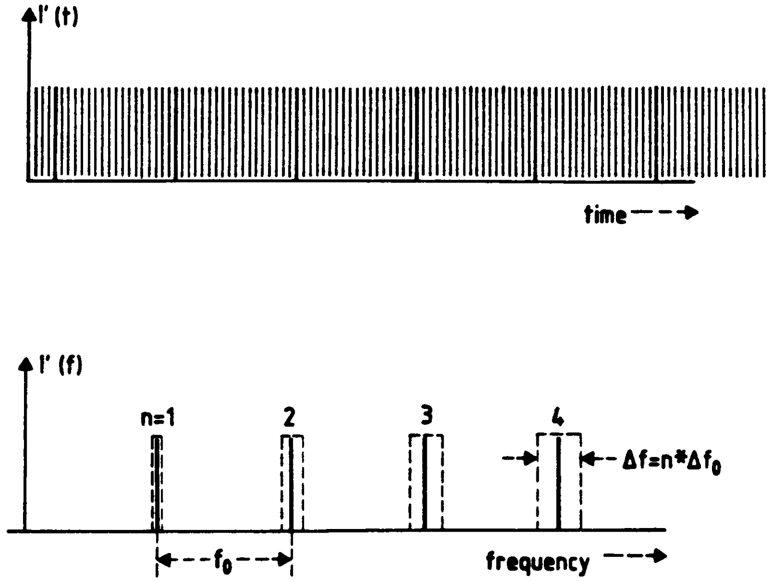



Fig. 3 Time and frequency spectrum of a group of particles with a distribution of revolution frequencies $f_0 \pm \Delta f_0/2$. The nth band of the frequency spectrum has a width $n\Delta f_0$. The height of the bands is arbitrary at this stage.

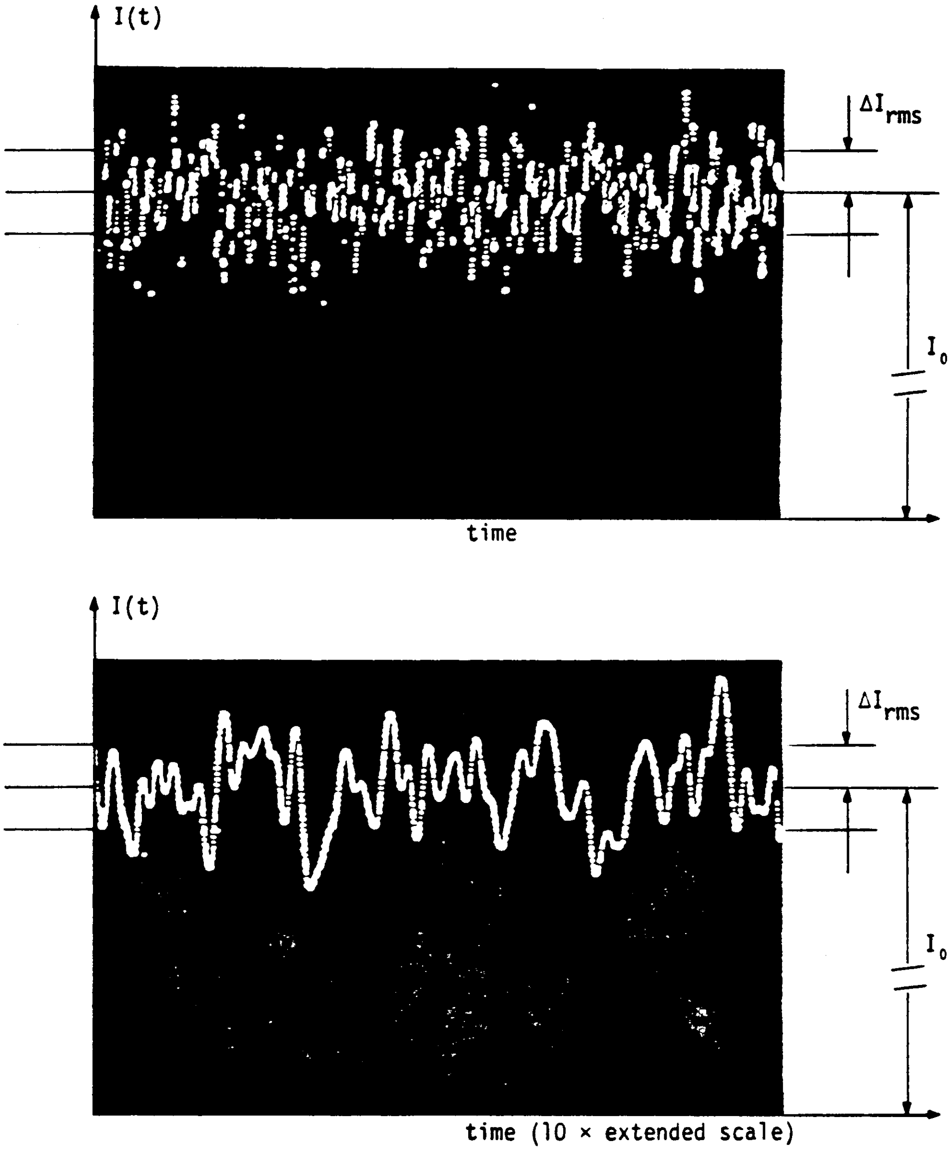



Fig. 4 The pick up current I(t) of a coasting beam consists of a DC current $I_0 \propto N$ and a fluctuating (noise) component with an r.m.s. $\sqrt{\overline{\Delta(I)^2}} = \Delta I_{rms} \propto \sqrt{N}$. The highest frequencies resolved are given by the bandwidth of the acquisition system.

region ($\lambda = 10^{-5}$ to $10^{-10}$ m) and is not resolved by the pick-up. If, on the contrary, initial phases are distributed in a *random* manner, as they are for instance in a beam emerging from a cathode, then the statistical fluctuation of the number of particles per time interval $\Delta t_k$ leads to a noise-like current (Fig. 4) with frequency components in the bands discerned above.

We shall now try to estimate the density of the noise in these 'Schottky bands'[1] of a coasting beam.

As a first step let us single out a group of N′ particles all having the same revolution frequency $f = \omega/2\pi$. The contributions (2.3) of these particles can be added up to give the current I′(t) of the subgroup:

$$I'(t) = I_0' + \sum_n A_n \cos n\omega t + B_n \sin n\omega t \ , \tag{2.4}$$

where each of the Fourier coefficients is the sum of the corresponding N′ single-particle contributions:

$$\begin{aligned} A_n &= \sum_{k=1}^{N'} a_{nk} = \sum_k 2i_0 \cos n\psi_k \\ B_n &= \sum_k 2i_0 \sin n\psi_k \\ I_0' &= N'i_0 \end{aligned} \tag{2.5a}$$

and

$$\begin{aligned} i_0 &= ef_0 \\ \psi_k &= \omega t_k \end{aligned}$$

Equivalently:

$$\begin{aligned} A_n &= 2i_0 \ \mathrm{Re} \sum e^{in\psi_k} \ , \\ B_n &= 2i_0 \ \mathrm{Im} \sum e^{in\psi_k} \ . \end{aligned} \tag{2.5b}$$

In general, we only know the statistical properties of the distribution of initial phases. Our aim is then to derive the corresponding 'probable' values of the Fourier coefficients in order to determine the statistical properties of the current (2.4). We are especially interested in the average current $I_{av}$ and the mean squared fluctuation $(\Delta I')^2 = (\Delta I'_{rms})^2$. This flutuating noise current -- or rather those of its components that fall into the bandwidth of the system -- leads to the heating of a test particle during stochastic cooling (incoherent effect, as discussed in Part I) as we shall see later. Also, a spectrum analyser will record the frequency components $(I')^2_n$ of $(\Delta I')^2$. We can define

$$\begin{aligned} I'_{av} &\equiv \langle I'(t) \rangle_T \\ (\Delta I')^2 &\equiv \langle [I'(t) - I'_{av}]^2 \rangle_T \ , \end{aligned} \tag{2.6}$$

where $\langle \ldots \rangle_T$ is the average over a sufficiently long time, say the revolution time T (or multiples of it). Obviously then from Eq. (2.4):

$$I'_{av} \to I'_0 = N'ef_0 \tag{2.7}$$

$$(\Delta I')^2 \to \sum_n (A_n^2/2) + (B_n^2/2) \tag{2.8}$$

Equation (2.8) gives the mean square current fluctuation in terms of the contributions of its spectral lines. The quantity $I_0'^2 + (\Delta I')^2$ is the (average) power of the current [Eq. (2.4)] dissipated in a 1Ω resistor, and $(\Delta I')^2$ [Eq. (2.8)] defines the power spectrum of the noise of $I'(t)$. The spectral line at $n\omega$ contributes

$$(I')^2_n = (A_n^2/2) + (B_n^2/2) \ . \tag{2.9}$$

So far we have not specified the distribution of the initial phases $\psi_k$. We now assume a *random* distribution such that in working out the probably values of $A_n^2$ and $B_n^2$ from Eq. (2.5a) the sums over cross-terms cancel and

$$A_n^2 = (2i_0)^2 \left(\sum_{k=1}^{N'} \cos n\psi_k\right)^2 \to (2i_0)^2 \sum_{k=1}^{N'} (\cos n\psi_k)^2 , \tag{2.10a}$$

and in a similar fashion

$$B_n^2 \to (2i_0)^2 \sum_{k=1}^{N'} (\sin n\psi_k)^2 . \tag{2.10b}$$

Then from Eq. (2.9):

$$(I')_n^2 = (2i_0)^2 \sum_{k=1}^{N'} (\sin n\psi_k)^2/2 + (\cos n\psi_k)^2/2 \to (2i_0)^2 N'/2 ,$$

that is

$$\boxed{(I')_n^2 = 2e^2f^2N' .} \tag{2.11}$$

In other words, for random $\psi_k$ the Fourier coefficients [Eq. (2.5a)] are sums of random numbers, and the probable value of the result is obtained by quadratic addition of the single-particle contributions; or equivalently [see Eq. (2.5b)], the random walk problem for the vector $e^{in\psi k}$, i.e. the addition of N′ vectors with unit length and random angle, leads to a sum vector with probable length $\sqrt{N'}$ and arbitrary direction.

Equation (2.11) specifies the intensity of the $n^{th}$ spectral line of the mean-squared noise current for a group of N′ monoenergetic (equal $f_{rev}$) particles. The Fourier expansion of the current [Eq. (2.4)] consistent with this is

$$\boxed{\Delta I'(t) = 2ef\sqrt{N'} \sum_{n=1} \cos n\omega t + \phi_n .} \tag{2.12}$$

This follows immediately, writing Eq. (2.4) in 'phase amplitude' form:

$$I'(t) - I_0' = \sum_n \sqrt{A_n^2 + B_n^2} \cos n\omega t + \phi_n$$

and taking the probable values of $A_n^2$, $B_n^2$ from Eq. (2.10). The unknown phases $\phi_n$ are of no importance for the present considerations. Our main result is expressed by Eqs. (2.11) and (2.12), which indicate a noise current with spectral lines at all harmonics of the revolution frequency and a (probable) 'power contribution' $2e^2f^2N'$ and Fourier amplitude $2ef\sqrt{N'}$ for each harmonic (up to a cut-off due to trasit time!).

This is for a beam with equal f and random $\psi_k$. In reality we are faced with a distribution of revolution frequencies ($f_r$) between, say, $f_0 \pm \Delta f_0/2$. Clearly, then, the different components of $f_r$ contribute current in the range $n(f_0 \pm \Delta f_0/2)$ around each harmonic n, as discussed above. We thus have a band spectrum, the so-called 'Schottky noise spectrum[1])' (or 'Schottky band spectrum') of a coasting beam. For a subgroup with a very narrow range $df_r$ of frequencies we can still apply Eq. (2.11). Let $N'(f_r)df_r = (dN/df_r)df_r$ be the fraction of particles with $f_r \pm df_r/2$, and call their contribution to the $n^{th}$ band $d(I)_n^2$. Then from Eq. (2.11)

$$d(I)_n^2 = 2e^2 f_r^2 \frac{dN}{df_r} df_r \ ,$$

or

$$\boxed{\frac{d(I)_n^2}{df_r} = 2e^2 f_r^2 \frac{dN}{df_r} \ .} \qquad (2.13)$$

This gives the spectral density of the noise (of the mean squared current fluctuation) in the $n^{th}$ Schottky band. By virtue of Eq. (2.13) the observation of a Schottky band on a spectrum analyser provides the frequency distribution $dN/df_r$ which is proportional to the momentum distribution of the beam (as $df_0/f_0 \propto dp/p$). An example is given in Fig. 5, which shows the Schottky noise spectrum before and after momentum cooling of a 600 MeV/c LEAR beam.

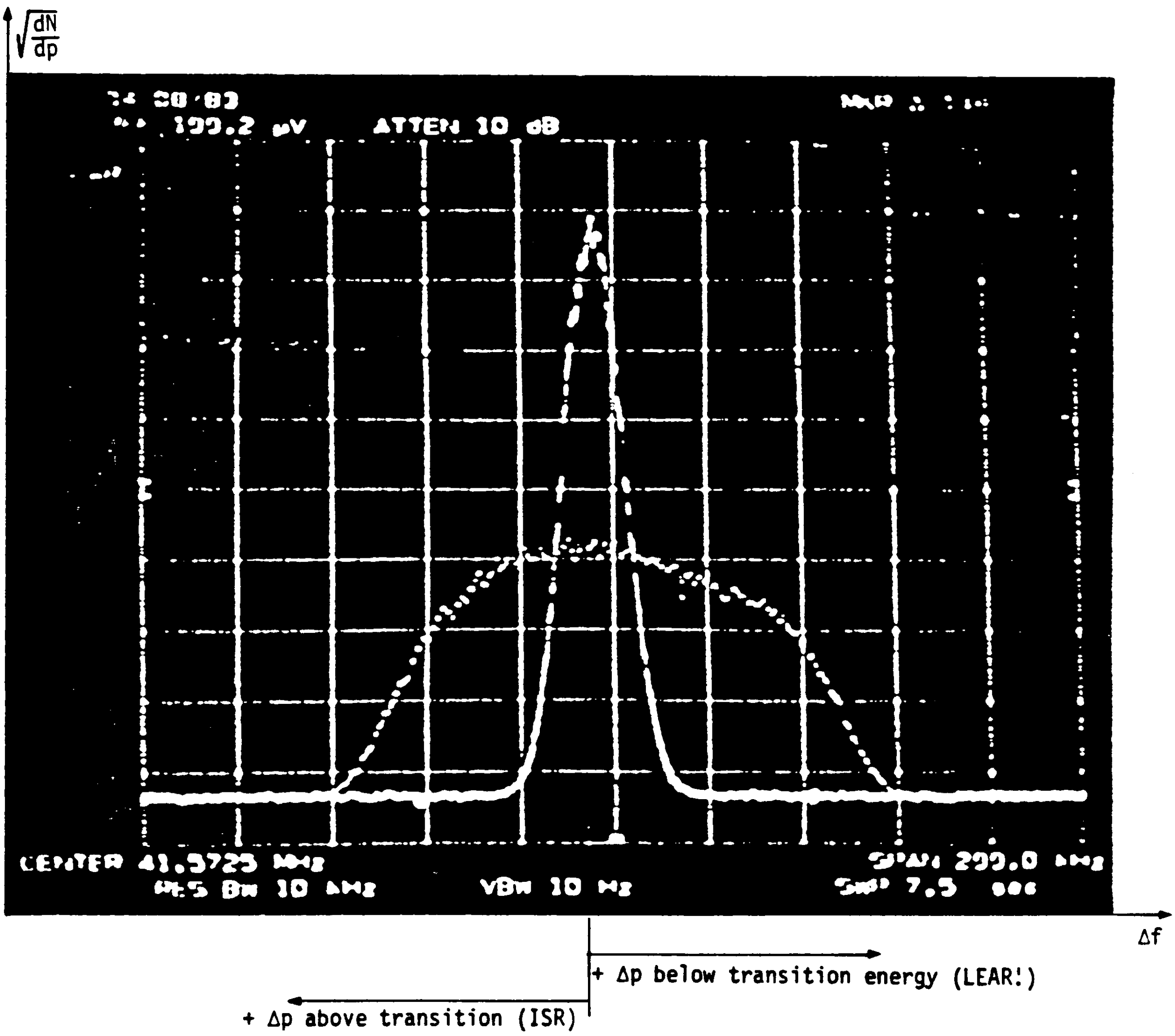



Fig. 5 'Schottky scan' of a coasting beam in LEAR at 600 MeV/c before and after momentum cooling. The signal from a longitudinal pick up is analysed using a spectrum analyser which records the amplitude of the noise current v.s. frequency. The scan is centred at the 20th harmonic of the revolution frequency ($f_0 \approx 2$ MHz) and covers 1/10 of the spacing between two adjacent bands. The horizontal coordinate is momentum ($\Delta p/p \approx 7 \times 10^{-4}$/div). The vertical coordinate is proportional to the square root of the momentum distribution i.e. to $\sqrt{dN/dp}$. During cooling the momentum spread decreases and the density increases by a factor $\sim 4$.

The band near 40 MHz ($n = 20$) is scanned. The horizontal coordinate is frequency or, equivalently, momentum ($\Delta f/f = 0.5 \times 10^{-3}$ per division, corresponding to $\Delta p/p \approx 0.7 \times 10^{-3}$ for the given machine conditions). The vertical coordinate is $\sqrt{dN/df} \propto \sqrt{dN/dp}$ because the analyser records 'current', i.e. $\sqrt{dI^2/df}$ versus frequency. Let us now have a closer look at the behaviour of these bands for different harmonic number n.

Integrating Eq. (2.13) over a band ($f_r$ going from $f_0 - \Delta f_0/2$ to $f_0 + \; f_0/2$ and simplifying by using $\Delta f_0 \ll f_0$), we have for the total noise per band,

$$(I)_n^2 = 2e^2 f_0^2 N = 2eI_0 f_0 , \qquad I_0 = Nef_0 . \tag{2.14}$$

In other words: the area of each Schottky noise band (in a $dI^2/df$ versus f-diagram) is constant. Since the $n^{th}$ band has a width $\Delta f = n\Delta f_0$, the spectral density decreases with $1/n$. This is also obvious from Eq. (2.13) as we now regard the band on its proper frequency scale $f = nf_r$, so that $dI^2/df = (1/n)\, dI^2/df_r$. We thus have the situation sketched in Fig. 6 where the width of the bands increases and the height decreases linearly with n.

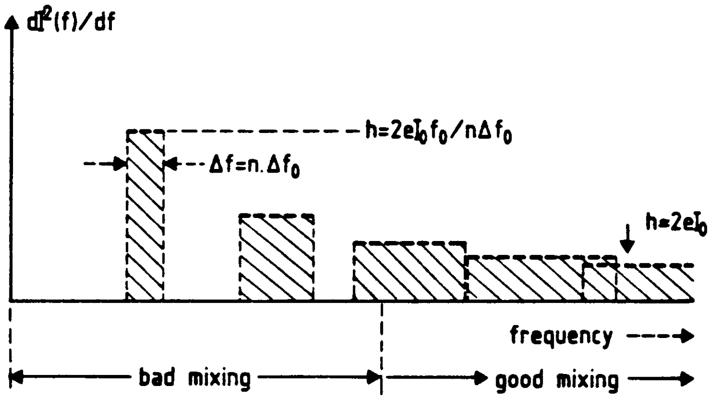



Fig. 6 Schottky noise bands (rectangular approximation) of a coasting beam. The nth band, centred at $nf_0$, has a width $n\Delta f_0$ and an effective height $2Ne^2 f_0 \cdot f_0/n\Delta f_0$. The bands describe the power spectrum of the current fig. 4; $f_0 \pm \Delta f_0/2$ is the range of revolution frequencies. At high harmonics ($n > f_0/\Delta f_0$) bands overlap and the height (spectral density) of the noise power is $2Ne^2 f_0 = 2eI_0$ as for classical shot noise. The width of the bands is exagerated.

It can be useful to approximate the noise spectrum by rectangular bands with width $n\Delta f_0$ and height equal to the average noise density (Fig. 6), where

$$\boxed{\left\langle \frac{dI^2}{df} \right\rangle_{band} = \frac{(I)_n^2}{n\Delta f_0} = 2eI_0 \frac{f_0}{n\Delta f_0}} \tag{2.15}$$

by virtue of Eq. (2.14). Observe that at low frequencies the bands are well separated and Eq. (2.15) approximates the density for a frequency inside a band.

At high frequencies ($n\Delta f_0 > f_0$), bands overlap and several bands contribute to the noise at a given point f. If adjacent bands just touch the density, Eq. (2.15) becomes

$$dI^2/df = 2eI_0 \; . \tag{2.16}$$

Equation (2.16) remains true for the density at higher frequencies where many bands overlap, because (qualitatively) near $f = mf_0$, with $m\Delta f_0 = \ell f_0$, $\ell$ bands contribute each with a density $\approx 2eI_0/\ell$.

Equation (2.16) agrees with the spectral density of the 'shot noise' (or 'Schottky noise') of a 'continuous' electron beam emerging from a cathode[3]. It was calculated in a classical paper by Schottky in 1918. In contrast to a coasting beam the classical shot noise has a continuous spectrum.

To summarize, for a coasting beam we have (cf. fig. 6)

at low frequency: separated bands with a noise density in the $n^{th}$ band higher by $f_0/n\Delta f_0$ than for the classical shot noise;

at high frequency: overlapping bands with a noise density as for the classical Schottky noise.

Remember that [from Eq. (2.10) onwards] results are for a beam with a random distribution of particle positions (initial phases). If the positions are ordered, the Schottky signals change. A 'signal suppression' is frequently observed during the action of the cooling system (see Section 7). Also, in a bunched beam, strong signals (with a current proportional to N instead of $\sqrt{N}$) occur at harmonics of the bunch frequency up to a cut off region determined by the bunch length. Our main results [Eqs. (2.13) to (2.16)] specify the *power spectrum* of the current of a coasting beam. The current (Fig. 4) itself can be imagined by interpreting Eq. (2.4) as the contribution $dI/df_r$ of a group with a narrow range $f_r \pm df_r/2$ of revolution frequencies. Hence, for the beam [integrating Eq. (2.4) over the frequency content]

$$I(t) = I + \sum_n \int_{f_0-\Delta f}^{f_0+\Delta f} I_n'(f_r) \cos (n\omega t + \phi_n) df_r .$$

This gives a Fourier expansion type of representation of the current sketched in Fig. 4.

To conclude our consideration of the coasting beam noise, let us sketch yet another (qualitative) derivation which reveals the close relation to the fluctation of the number of particles per unit length.

Suppose we centre a band-pass filter with a bandwidth $W = f_0$ equal to the revolution frequency on the first Schottky band $f = f_0$, $n = 1$. From the bandwidth/pulse-length relation discussed in Part I of this lecture, we know that such a system resolves the beam into samples of a length $T_s = 1/(2f_0) = T/2$, i.e. into two samples per turn. For a random distribution, the number of particles in each sample is $N/2$ with a probable deviation of $\pm\sqrt{N/2}$, as given by Poisson's law.

As a simple model, assume that the density varies sinusoidally:

$$\frac{dN'}{dt} = \frac{N'}{T} + \sqrt{\frac{N'}{2}}\,\frac{\omega}{2} \sin \omega t ,$$

so that in the first sample ($0 \le t < T/2$) the number $\int (dN'/dt)dt = N'/2 + \sqrt{N'/2}$, and in the second, $N'/2 - \sqrt{N'/2}$. The corresponding current is

$$I'(t) = \dot{N}'e = (N'e/T) + \pi e\sqrt{N'/2}/T \sin \omega t$$

with the mean square variation for the first band:

$$(I')_1^2 = \left\langle [I'(t) - I_0']^2 \right\rangle_T = \frac{\pi^2}{8}\, 2N'(ef_0)^2 .$$

This is the same as Eq. (2.11) except for the factor $\pi^2/8$. The transition from the mono-energetic group (N′) to a beam can proceed as before. Consideration of the noise near higher harmonics n can start from the possible number of particles in the corresponding 2n samples as given by the Poisson distribution.

## 3. COOLING BY HARMONICS

Suppose now that the cooling loop acts proportionally to the error x of a single particle times the longitudinal current, i.e. it is convenient to work in terms of

$$y = S_1 \cdot x \cdot i(t) \;, \tag{3.1}$$

where $S_1$ is a proportionality factor and i(t) is the single-particle current [Eq. (2.1)]. To fix our ideas, we may think of Palmer cooling, where the pick-up signal is proportional to the particle current *and* the radial displacement $x = D \cdot (\Delta p/p)$.

Now transform to the frequency domain, introduce the Schottky noise ($y_{Sch}$) due to the other particles and assume for the moment a bandwidth $W = f_0$ including only one band. The single passage correction is

$$y_c(f) - y(f) = -\lambda_n y(f) - a\lambda_n y_{Sch}(f) \;, \tag{3.2}$$

where $\lambda_n$ is the 'gain' in the band under consideration and 'a' is a constant which describes the response of the particle to the Schottky noise. Rather than to derive this response, we adjust 'a' to recover results of the time domain analysis. We invoke that a particle will mainly respond to components of the noise with a frequency very close to its revolution harmonic[2,4]. Other frequency components will rapidly fall out of synchronism with the particle. The perturbation is then proportional to the noise density at the corresponding frequency $nf_r$.

As before (Part I), we work out the change of the squared error of a test particle

$$\Delta[y^2(f)] = y_c^2 - y^2 = -2\lambda_n y^2 + \lambda_n^2 y^2 + a^2\lambda_n^2 y_{Sch}^2 + \dots yy_{Sch} \;.$$

Average over many turns, take the noise and test-particle signal uncorrelated so that $\langle yy_{Sch}\rangle \to 0$, and anticipate $\lambda_n \ll 1$ (see below), i.e. neglect $\lambda_n^2\, y^2 \ll 2\lambda_n y^2$ to obtain the cooling rate (per second):

$$\frac{1}{\tau} = f_0 \, \frac{-\langle\Delta(y^2)\rangle}{\langle y^2\rangle} = f_0\left(2\lambda_n - a^2\lambda_n^2 \, \frac{\langle y_{Sch}^2\rangle}{\langle y^2\rangle}\right). \tag{3.3}$$

Now use Eqs. (2.14), (2.15), and (3.1) to write

$$\langle y_{Sch}^2\rangle = S_1^2 \, x_{rms}^2 \; 2Ne^2f_0^2 \cdot \begin{cases} \dfrac{f_0}{n\Delta f_0} & n\Delta f_0 < f_0 \quad \text{(separated bands)} \\ 1 & n\Delta f_0 \geq f_0 \quad \text{(overlapping)} \end{cases}$$

Take again as a 'typical' particle one with $y = y_{rms}$ to write [see Eq. (3.1) and (2.2)]:

$$y^2 = S_1^2 \, x_{rms}^2 (2ef_0)^2$$

Rewrite Eq. (3.3):

$$\frac{1}{\tau} = f_0\left[2\lambda_n - \frac{a^2}{2} N\lambda_n^2 M_n(\Delta f_0)\right]$$

$$M_n(\Delta f_0) = \begin{cases} \dfrac{f_0}{n\Delta f_0} & n\Delta f_0 < f_0 \\ 1 & n\Delta f_0 \geq f_0 \end{cases} \tag{3.4}$$

Identify $\lambda_n = 2g/N$ and $a = 1$, to recover the familiar result [Eq. (7.6) of Part I] in the limit of large $\Delta f_0$, i.e. for good mixing:

$$\frac{1}{\tau} = \frac{2f_0}{N} (2g_n - g_n^2) .$$

(one band, $M_n = 1$)

For $\ell = W/f_0$ bands we can add up the contributions (3.4) as the 'cross terms' between different bands average out in Eq. (3.3). Including finite frequency spread:

$$\boxed{\frac{1}{\tau} = \frac{2f_0}{N} \sum_{n=n_1}^{n_1+\ell} (2g_n - g_n^2 M_n)} \tag{3.5a}$$

For constant $g_n = g$ (independent of n) this simplifies to give

$$\frac{1}{\tau} = \frac{2W}{N} [2g - g (\frac{1}{\ell} \sum M_n)] . \tag{3.5b}$$

We thus identify the mixing factor

$$M = \frac{1}{\ell} \sum_n M_n , \tag{3.6a}$$

with $M_n$ given under Eq. (3.4). If all bands are non-overlapping we have $\ell = W/f_0$ bands contributing and we may write:

$$M = \frac{f_0^2}{W\Delta f_0} \sum \frac{1}{n} \approx \frac{f_0^2}{W\Delta f_0} \ln (f_{max}/f_{min}) . \tag{3.6b}$$

We may use the sample length $T_s = 1/(2W)$ (see Part I) and $|\Delta f_0/f_0| = |\Delta T/T|$ to express

$$\frac{f_0^2}{W\Delta f_0} = \frac{T_s}{\Delta T/2}$$

Equation (3.6) then agrees with the common sense expectation that a spread in revolution times equal to about one sample length assures good mixing ($M \to 1$).

The frequency domain analysis has thus provided us with more insight as we obtain:

i) A more general cooling equation which includes gain variation with n and mixing. It can be optimized, harmonic by harmonic.

ii) An interpretation of bad mixing as enhancement of the incoherent heating, because the noise density is increased by $M_n > 1$.

This is the interpretation introduced by Sacherer in 1978 [2] shortly after an expression similar to Eq. (3.6a) had been derived by van der Meer[5] using time domain analysis.

The approximation of 'rectangular' frequency distribution -- implicitly used in this subsection -- is slightly pessimistic if we are mainly interested in the cooling speed of particles with large error. This is because in the tails the noise density is lower than average.

## 4. NOISE-TO-SIGNAL RATIO BY HARMONICS

An (idealized) amplifier characterized by a noise figure $\nu$ (dB) and an input resistance R produces a noise current which -- referred to the input -- has a spectral density which is constant in the passband and is given by[3]:

$$\frac{I_\nu^2}{\Delta f} = 10^{\nu/10}\,\frac{kT_e}{R} \tag{4.1}$$

where k is the Boltzmann constant and $T_e$ the temperature (K). As an example,

$$\nu = 1.5 \text{ dB}, \quad T_e = 300 \text{ K}, \quad R = 50\ \Omega$$

$$\Delta f = 250 \text{ MHz} \curvearrowright I_\nu^2 = (170 \text{ nA})^2 \ .$$

Equation (4.1) is similar to the Nyquist-Johnson relation[3] for the thermal noise of a resistor:

$$\frac{I_R^2}{\Delta f} = 4\,\frac{kT_e}{R}\ , \tag{4.2}$$

and sometimes the equivalent noise temperature $4T_{eq} = 10^{\nu/10}\,T_e$ is used to specify amplifier noise by Eq. (4.2). The total (Schottky and amplifier) noise entering the cooling loop is shown in Fig. 7.

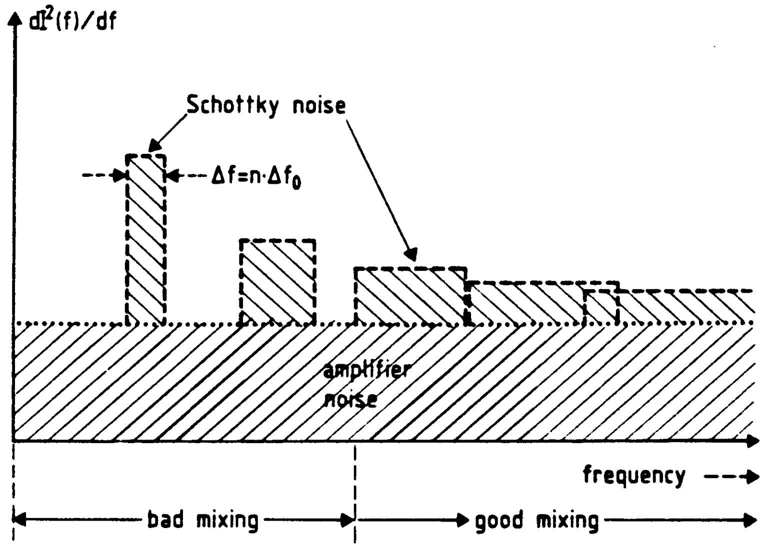



Fig. 7 Amplifier and Schottky noise which determine the incoherent heating. The amplifier noise has a continuous spectrum with a density $dI^2/df = 10^{\nu/10}\,k\,T_e/R$ given by the noise figure $\nu$ of the low level system. The Schottky noise has the band spectrum, fig. 6. The width of the bands is exagerated.

To work out the noise-to-signal ratio U we compare Eq. (4.1) with the average spectral density of the Schottky noise. In the definition of U, good mixing has been assumed, so we have to take Eq. (2.16),

$$\frac{I_{Sch}^2}{\Delta f} = 2Ne^2 f_0\ ,$$

as the noise density with respect to which U is defined. The pick-up current is only some fraction $S_1 < 1$ of the beam current $I_{Sch}$; $S_1$ contains construction details and has to be measured or calculated. For a difference pick-up as used for transverse and Palmer cooling, the ratio beam size/plate spacing enters. If the signal power from $n_{PU}$ gaps or plate pairs is added (using a combiner), U is decreased accordingly. It can be expressed as

$$\boxed{U = \frac{10^{\nu/10}\ kT_e/R}{S_1^2\ 2Ne^2 f_0 n_{PU}}} \tag{4.3}$$

For example:

$$N = 10^9\ ,\quad f_0 = 1\ \text{MHz},\quad \Delta f = 250\ \text{MHz},$$

$$\curvearrowright\ I^2_{Sch} = (120\ \text{nA})^2\ ,$$

$$n_{PU} = 5,\quad S_1 = 0.5,\quad I^2_\nu = (170\ \text{nA})^2$$

$$\curvearrowright\ U = 1.2\ .$$

The possible increase of U as the beam shrinks, and many construction details, are hidden in $S_1$. In addition, Eq. (4.3) is not valid for 'travelling-wave pick-ups'[6] which add the current rather than the signal power from several gaps. With current addition, $U \propto n_{PU}^{-2}$.

Equation (4.3) reveals that in the noise limit the cooling rate, Eq. (9.2) of Part I,

$$\frac{1}{\tau} \approx \frac{2W}{NU}$$

is independent of the intensity N.

## 5. MIXING PICK-UP TO KICKER

Let the 'typical' test particle pass the pick-up at t = 0 and the kicker at $t = T_{PK} + \Delta T_{PK}$. Let the correction take the nominal time $T_{PK}$ to travel from pick-up to kicker. Assume again a bandwidth $W = f_0$ covering one harmonic for the moment, and regard only the effect of the particle upon itself. From Eq. (3.1) and (3.2) the single passage correction is written as

$$\begin{aligned} y_c &= y - \lambda_n \cdot S_1 \cdot x \cdot i_n(\Delta T_{PK})\ , \\ &= y - \lambda_n \cdot y \cdot \cos\ (n \cdot \omega \cdot \Delta T_{PK})\ . \end{aligned} \tag{5.1}$$

Going through the standard procedure again to work out $\langle y_c^2 - y^2 \rangle$, summing over harmonics we find the 'coherent effect' as expressed by the first term in Eq. (3.5a) modified:

$$\frac{2}{\ell} \sum_{n_1}^{n_1+\ell} g_n \rightarrow \frac{2}{\ell} \sum_{n_1}^{n_1+\ell} g_n \cos\ (n\omega\Delta T_{PK}) \tag{5.2a}$$

or for constant $g_n = g$,

$$2g \rightarrow 2g\ \frac{1}{\ell} \sum_{n_1}^{n_1+\ell} \cos\ (n\omega\Delta T_{PK})\ . \tag{5.2b}$$

Hence, the harmonic method gives an improved expression also for the unwanted mixing, and we can identify the factor $(1-\tilde{M}^{-2})$ used in Part I,

$$g \cdot (1-\tilde{M}^{-2}) = \frac{1}{\ell} \sum_{n_1}^{n_1+\ell} g_n \cos\ (n\omega\Delta T_{PK})$$

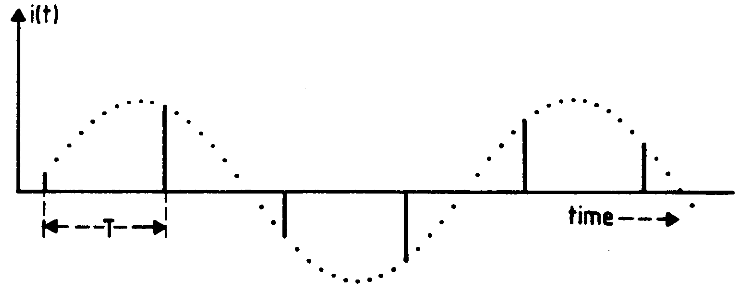
as the average cosine of the phase error in the passband.

The incoherent effect remains unchanged in our approximation as the Schottky noise is stationary and thus independent of the time of arrival.

## 6. BETATRON OSCILLATION AND TRANSVERSE SCHOTTKY SIGNALS

To look at transverse cooling in the frequency domain it is useful to understand the noise signals induced by a coasting beam on a *position*-sensitive pick-up[1]. Imagine a single particle of constant revolution $f_r$ which performs a betatron oscillation $x_k = a_k \cos Q\omega t + \phi_k$ around the ideal orbit $x = 0$. The time-domain signal is the series of short current pulses [Eq. (2.1)] modulated in strength by the oscillation (Fig. 8):

$$i(t) = e\delta(t-t_k-nT)\, a_k \cos Q\omega t + \phi_k \ . \tag{6.1}$$



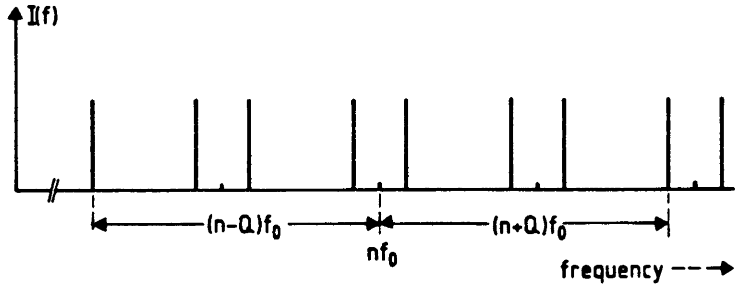




Fig. 8 Time and frequency domain signal of a particle performing a betatron oscillation. A position sensitive pick up records a short pulse at each traversal modulated in amplitude by the oscillation. The frequency spectrum contains lines at the two sideband frequencies $(n\pm Q)f_0$ of each revolution harmonic $nf_0$.

Using again the expansion (2.2) (as well as the trigonometric identity $\cos\alpha \cos\beta = 1/2[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$):

$$i(t) = a_k \cdot e \cdot f \cdot \left[\cos(Q\omega t + \phi_k) + \sum_{n=1} \cos(n+Q)\omega t + \psi_{n+} + \cos(n-Q)\omega t + \psi_{n-}\right]$$

$$\psi_{n\pm} = \pm\phi_k - n\omega t_k \ . \tag{6.2}$$

This defines the 'transverse frequency content' of the beam. It is characterized by two sidebands $(n\pm Q)\omega$ for each harmonic n of the revolution frequency -- similar to the two sidebands of an amplitude-modulated oscillator.

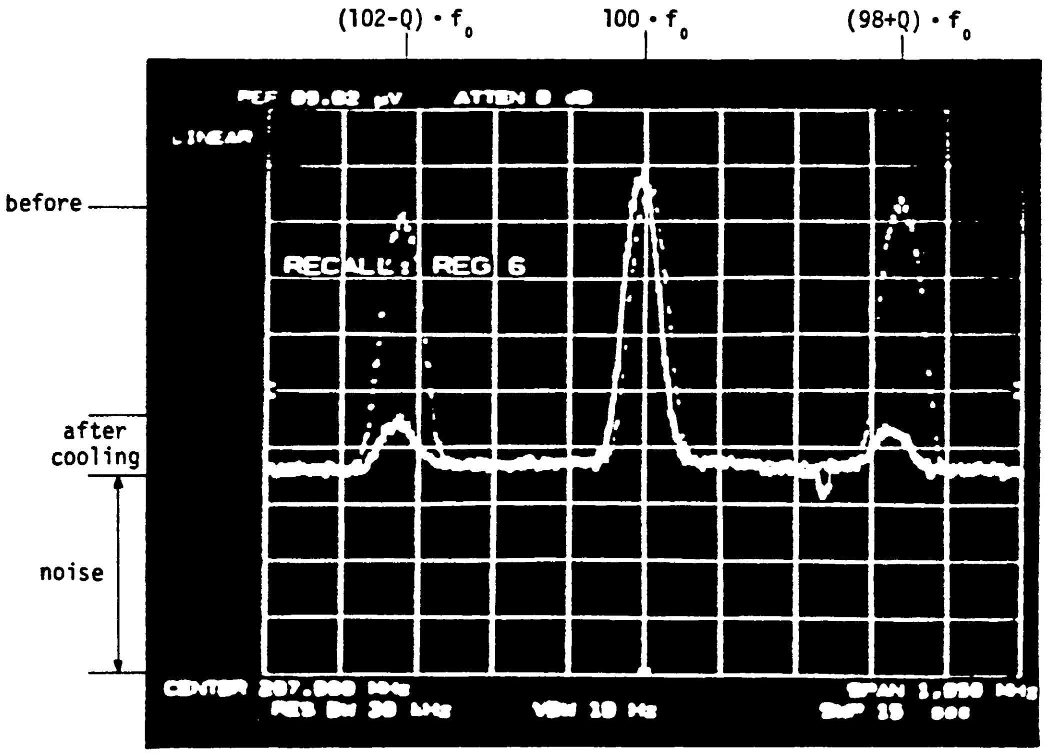



Fig. 9 Example of a horizontal Schottky scan in LEAR at 600 MeV/c. The central band, the harmonic n = 100 of the revolution frequency, is visible as the beam is not completely centred at the position pick up. The right and left bands are the sidebands $(98+Q)f_0$ and $(102-Q)f_0$ where $Q \approx 2.3$. During emittance cooling the sidebands decrease. The difference between the base line of the trace and the bottom line (zero signal) is given by the noise of the pick-up system. The span covers (approximately) half a revolution interval $f_0$.

A spectrum analyser will record two bands per frequency interval $f_0 = \omega/2\pi$ (see Fig. 9). Some confusion could arise from the following: In general, $Q > 1$ and the two bands next to a given $nf_0$ are *not* sidebands to this n but to a higher and a lower harmonic as sketched in Fig. 8. Some of this confusion can be avoided by interpreting (6.2) in terms of the non-integer part q of Q (e.g. q = 0.7 for Q = 2.7). For $q < 0.5$, the two bands nearest to $nf_0$ are the $n \pm q$ sidebands to the same n (Fig. 10a). For $q > 0.5$, the situation is shown in Fig. 10b.

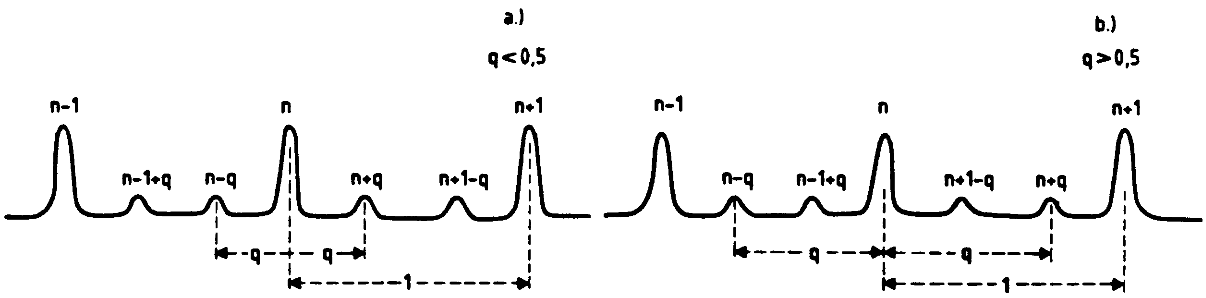



Fig. 10 Position of sidebands $(n \pm Q)f_0$ where q is the non-integral part of the betatron tune Q. For $q < 0.5$ the two bands right and left of a revolution harmonic n are the bands $(n \pm q)f_0$ (fig. 10a). For $q > 0.5$ the two bands closest to n are the $n + 1 - q$ and $n - 1 + q$ bands (fig. 10b).

Betatron sidebands are frequently used[1] to determine the Q-values. In fact, their observation provides only the distance q or 1-q to the nearest integer. Further information is needed in order to know whether Q is just above or below an integer and to determine the integral part.

Having identified the potential frequency content, we now wish to find the noise current. Assume a beam with again random longitudinal positions $0 < t_k < T$, random 'betatron phases' $0 < \phi_k < 2\pi$, a distribution $n'(a)$ of betatron amplitudes, and $N'(f)$ of the particles versus the sideband frequencies $f = (n \pm Q)f_r$. Take the distributions uncorrelated (thus neglecting the dependence of Q and/or $f_r$ on betatron amplitude) and proceed, as in Section 2, to evaluate the spectral density of the mean squared current in the bands. One finds:

$$\frac{dI^2_{n\pm Q}}{df} = \frac{dN}{df}\,\frac{e^2 f_0^2}{2}\,a^2_{rms} \,. \tag{6.3}$$

Here the r.m.s. amplitude $a^2_{rms} = \int n'(a)a^2\,da$ comes from the averaging over the betatron amplitudes.

The width of the $n \pm q$ band is given by the frequency spread $\Delta f_0 = f_0 \cdot \eta \cdot (\Delta p/p)$ and the chromatic q spread $\Delta q = \Delta Q = Q \cdot \xi \cdot (\Delta p/p)$:

$$\begin{aligned} \Delta f &= (n\pm q)\Delta f_0 \pm f_0 \Delta q \\ \Delta f &\approx n\Delta f_0 \quad \text{for large } n \,. \end{aligned} \tag{6.4}$$

The chromaticity $\xi$ and the off-energy parameter $\eta = 1/\gamma^2 - 1/\gamma^2_{tr}$ are the usual 'machine constants'.

Integrating Eq. (6.3), the total noise per band is found:

$$I^2_{n\pm Q} = N\,\frac{e^2 f_0^2}{2}\,a^2_{rms} \,. \tag{6.5}$$

Approximating the shape of a band again by a rectangle of width (6.4), the effective height (noise density in the band) is

$$\frac{dI^2}{df} = N\,\frac{e^2 f_0}{2}\,a^2_{rms}\,\frac{f_0}{\Delta f} \,, \tag{6.6a}$$

with $\Delta f$ from Eq. (6.4). Equation (6.6a) gives (in the approximation of rectangular bands) the noise acting on a beam particle as long as bands are separated. At high frequencies, where many bands overlap, the density is

$$\frac{dI^2}{df} = N\,e^2 f_0\,a^2_{rms} \,. \tag{6.6b}$$

A factor of 1/2 drops out as the n-q and n+q bands contribute at a given f.

We can now define a mixing factor similar to the longitudinal case. If $g_n$ is constant and all bands are separated (the case of varying $g_n$ is only slightly more difficult and can be treated in analogy to the longitudinal case),

$$M_\beta = \frac{1}{\ell} \sum \frac{f_0}{2\Delta f} \approx \frac{1}{\ell} \sum \frac{f_0}{2n\Delta f_0} \,. \tag{6.7}$$

For harmonics where adjacent n±q bands overlap (as is frequently the case if q is close to 0.5 or 1), the two bands contribute at a given beam frequency and essentially the factor of 1/2 has to be left out from the corresponding part of the sum. For high harmonics

($n\Delta f_0 > f_0$), the contribution to the sum is again unity instead of $f_0/2n\Delta f_0$ or $f_0/n\Delta f_0$. Note that the summation goes over $\ell = 2W/f_0$ terms instead of $W/f_0$ in the longitudinal case. This is because we now have two sidebands per harmonic.

Having defined the transverse mixing factor, let us have a fresh look at horizontal/transverse 'cross-heating' via the cooling systems. If the beam is not well centred at the position pick-up, or if the position dispersion due to momentum spread is large, the longitudinal bands at $nf_0$ are also visible. They can lead to betatron oscillation heating at frequencies where transverse and longitudinal bands overlap. At lower frequencies where bands are separated, this 'cross-heating' is avoided as the longitudinal 'cross-noise' has components different from the transverse particle response frequencies. The cooling system does, however, process the unwanted noise unless special filters are included to cut the transmission at frequencies between the required bands. This is often advantageous, or even indispensable, for avoiding saturation of the amplifier by unwanted Schottky noise and thermal noise!

Similar considerations apply to the Palmer-Hereward cooling (Part I). As long as longitudinal and transverse bands are separated (which unavoidably means bad mixing!) the cross-heating included in Eqs. (12.10) and (12.11) of Part I can be avoided.

Finally, let us have a brief look at the question of beam diagnostics with transverse Schottky scans: By virtue of Eq. (6.3), the observation of a betatron sideband on a spectrum analyser (recording $\sqrt{dI^2/df}$ versus f) gives the r.m.s. betatron amplitude $a_{rms}$ and the height of the band *shrinks* with transverse cooling, in constrast to the longitudinal bands $\sqrt{dN/df} \propto \sqrt{dN/dp}$ which increase with momentum cooling. Normalizing to the longitudinal distribution (observable on a sum pick-up), the emittance $E \propto a_{rms}^2$ can be deduced and surveyed from the transverse Schottky scan. The position of the bands provides information on q-values and, comparing the width of adjacent $n \pm q$ bands (for small n), one can disentangle the Q-spread $\Delta q = \Delta Q$ due to chromaticity $\xi$.

In summary, the beam noise is a very rich source of information on the beam properties. One just has to 'listen'.

## 7. ENHANCEMENT OF SCHOTTKY SIGNALS OR FEEDBACK VIA THE BEAM

A complete theory for the case of bad mixing has to include the enhancement of beam signals by the cooling system. We shall find that this introduces a modification of the cooling equation, which is not very important, but a diagnostics tool which can be of great practical value.

Up to now we have assumed random distribution such that the pick-up signals are the Schottky signals of a free coasting beam. In reality the kicker imposes a coherent modulation and, in the case of imperfect mixing, part of the resulting order is still present at the pick-up. The approach -- first made by Sacherer[2] to cope with this situation -- is to assume a coherent beam modulation $x_b$ *in addition* to Schottky and amplifier noise ($x_s$ and $x_n$). The correction applied at the kicker is proportional to $x_b + x_s + x_n$.

Amplifier noise and Schottky noise are regarded as stationary random noise, whereas $x_b$ is given by the coherent beam response to $x_b + x_s + x_n$ (Fig. 11). If only one harmonic is present, $x_s = \tilde{x}_s e^{j\omega t}$, etc., a single particle responds with $\tilde{x}_i e^{j\omega t}$. In this simple case $\tilde{x}_b = (1/N) \Sigma \tilde{x}_i$. This 'average beam response' has to be determined from the cooling system equation in a self-consistent manner.

In the case of betatron cooling, one has to solve the betatron equation for a single particle[2,7].

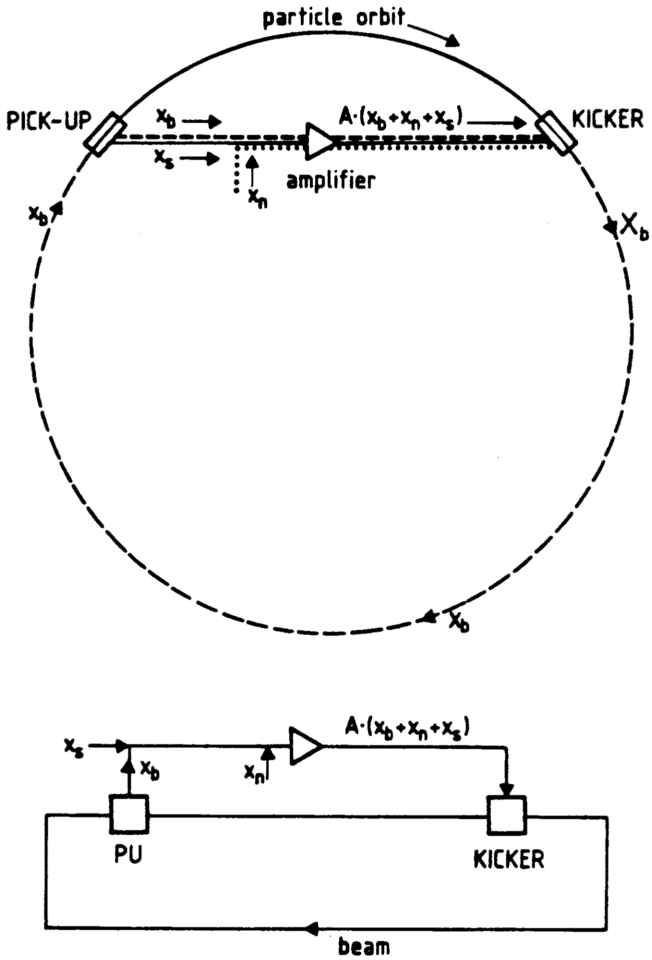



Fig. 11 Cooling system including the coherent beam modulation $x_b$ imposed at the kicker and Sacherers equivalent feedback loop. Amplifier noise ($x_n$) and Schottky noise ($x_s$) are stationary (time invariant) random noises whereas the coherent modulation is fed back via the beam from kicker to the pick-up. This feedback changes the open loop response to $x_n + x_s$ by a 1/1+s where the complex loop gain s depends on the amplification (cooling strength) and the degree of mixing between kicker and pick-up.

$$\ddot{x}_i + \omega_\beta^2 x_i = -2A(\dot{x}_b + \dot{x}_s + \dot{x}_n) \ , \qquad (7.1)$$

where A is proportional to the cooling strength. To recover our previous result for cooling of the betatron amplitude in the case where one band is used, $W = f_0/2$, we have to take $A \approx f_0 g_n/2$ so that e.g. in the limiting case of only one particle, $x_b = x_i$, $x_s = 0$, $x_n = 0$; we obtain $1/\tau_x = f_0 g_n/2$ from Eq. (7.1).

Inserting complex 'trial solutions' $x = \tilde{x}e^{j\omega t}$ for all $x_i$, $x_b$, ..., involved, the coherent response of Eq. (7.1) for one harmonic may be written:

$$\tilde{x}_b = -S(\omega)(\tilde{x}_b + \tilde{x}_s + \tilde{x}_n) \ ,$$

that is,

$$\tilde{x}_b = \frac{-S}{1 + S}(\tilde{x}_s + \tilde{x}_n) \ , \qquad (7.2)$$

where

$$S(\omega) = \frac{1}{N} \sum \frac{2j\omega A}{\omega_i^2 - \omega^2} \approx jA \int \frac{n(\omega_i)d\omega_i}{\omega_i - \omega} . \tag{7.3}$$

Here $n(\omega_i)$ is the distribution of the particles with respect to the sideband frequencies $\omega_i = (n \pm Q_i)\omega$. The dispersion integral involved[4], and hence S, is in general, complex owing to the resonant nature of the response. This 'Landau damping' effect[4] is important in the theory of coherent beam stability. The distribution $n(\omega_i)$ in (7.3) has unit area.

Using Eq. (7.2), the result is that both the pick-up and the kicker signal are modified. Instead of the beam and amplifier noise $\tilde{x}_s + \tilde{x}_n$, we now have:

$$\tilde{x}_b + \tilde{x}_s + \tilde{x}_n \to [1 - S/(1+S)](\tilde{x}_s + \tilde{x}_n) = [1/(1+S)](\tilde{x}_s + \tilde{x}_n) . \tag{7.4}$$

This expresses the signal suppression when the cooling loop is closed. The conclusion for cooling is that both the coherent effect and the incoherent effect are modified. In the simple case where S is real, the corresponding factors are $1/(1+S)$ and $[1/(1+S)]^2$. This can be concluded repeating, for example, the analysis of Section 3, taking a signal reduced by $(1+S)^{-1}$, i.e. taking $\lambda_n \to \lambda_n(1+S)^{-1}$. In the general case of complex S a slightly more complicated analysis gives Re $[(1+S)^{-1}]$ and $|1 + S|^{-2}$, respectively, as the two reduction factors.

The factor $(1+S)^{-1}$ is very similar to the closed-loop response factor of a feedback system. In our cooling problem $S(\omega)$ is purely real near the centre of the distribution, and approximately[2,8] [note that we use a sign convention in (7.3) different from Ref. 8, such that S is usually positive here]:

$$S = S_0 \approx A \frac{\pi}{\Delta\omega} \approx g_n \frac{f_0}{4\Delta f_n} , \tag{7.3a}$$

where $\Delta\omega = 2\pi\Delta f$ is the frequency spread in the band under consideration. The result (7.3a) is obtained because the dispersion integral in Eq. (7.3) has a typical residue $-\pi j/\Delta\omega$.

We can now modify the amplitude cooling equation. Summing over the Schottky bands we write for the centre of the distribution,

$$\frac{1}{\tau} = \frac{1}{4} \frac{f_0}{N} \sum_n \left[ 2 \frac{g_n}{1 + S_n} - M_n \frac{g_n^2}{(1+S_n)^2} - \frac{U_n g_n^2}{(1+S_n)^2} \right] . \tag{7.5}$$

Here $g_n$ is the gain in the n±Q bands, $S_n \approx g_n f_0/4\Delta f_n \approx g_n M_n/2$; $M_n = f_0/2\Delta f_n$ is the mixing factor (6.7) for the band; and the sum goes over all $2W/f_0$ bands within the passband. Two things of practical importance emerge. With $U_n \ll M_n$, i.e. with negligible amplifier noise and given mixing factor $M_n$, the gain for each band is optimized if:

$$(1 + S_n)^{-1} g_n M_n = 1, \quad \text{i.e. } S_n = 1 \text{ and } g_n M_n = 2 . \tag{7.6}$$

With the optimum gain (7.6) the signals are suppressed by a factor of 2 in the centre of the distribution. This is used as a diagnostics tool.

With strong amplifier noise the signal suppression is less important. Finally, the cooling equation (7.5) has still the familiar form -- we can just reinterpret

$$g_n \to \frac{g_n}{1 + S_n} .$$

However, $g_n$ and $S_n \approx g_n M_n/2$ are linked via the dispersion integral (7.3). Results of this section are approximate and the finer details depend on the distribution function entering into Eq. (7.3).

## 8. DISTRIBUTION FUNCTIONS AND PARTICLE FLUX

To follow the details of the cooling process, we (may) want to know more than the evaluation of the mean-square beam size and the r.m.s. momentum spread -- the only quantities used up to now to characterize cooling. In fact, a beam profile monitor records the particle distribution with respect to transverse position (see Fig. 12 as an example), and a longitudinal Schottky scan such as Fig. 5 gives the (square root of the) momentum distribution. These pictures are rich in fine information on peak densities, densities in the tails, asymmetries, and other practical details which are overlooked if only the r.m.s. is regarded.

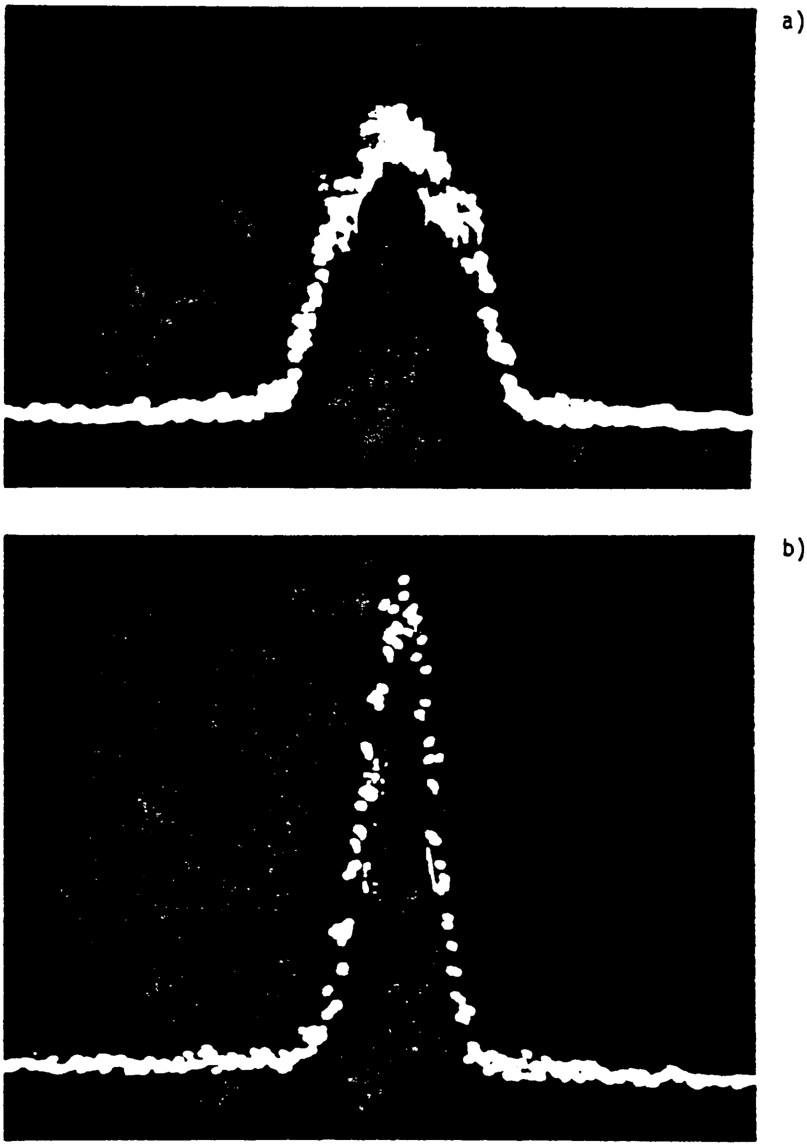



Fig. 12 Evolution of beam profile (number of particles vs. vertical position) during stochastic cooling test in "ICE". The scans were obtained with a profile monitor which records the position of electrons liberated by beam particles through collisions with the residual gas. a) Before cooling; b) after 4 min of cooling.

It is therefore challenging to find an equation which describes all that can be seen and that is of practical importance. Such an equation does in fact exist!

For stochastic cooling the problem was (to my knowledge) first tackled by Thorndahl[9], who already in 1976 worked with a Fokker-Planck type of equation for the particle density. This line was followed by virtually all subsequent workers[10], and computer codes for solving the distribution function equations are extensively used in the design of stochastic cooling and stacking systems.

The basic ideas behind this 'distribution function analysis' are simple, so that also the beginner can get -- hopefully without too much pain -- some first degree of familiarity with this powerful tool of cooling theory. I will first give the recipe and then try to justify it.

Let $\psi(x)$ (Fig. 13) be the particle distribution with respect to the error x (e.g. $x = \Delta p/p$). Define $\psi(x) = dN/dx$ so that $\psi(x)\,dx$ gives the number of particles with an error in the range x to x + dx. During cooling we find different distributions $\psi(x)$, taking snapshots at different times (see Fig. 5 as an example). We characterize this by letting $\psi = \psi(x,t)$ be a function of time also. The partial differential equation which describes the dynamics of $\psi(x,t)$ can be written in the following form:

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial x}\left(-F\psi + D\,\frac{\partial\psi}{\partial x}\right) \tag{8.1}$$

The cooling is completely characterized by the two coefficients F and D (which describe the cooling system) and the initial conditions $\psi(x, t = 0)$ (which describe the distribution at the start). Particle loss due to walls or influx during stacking can be included via appropriate boundary conditions $\psi(x_1) = 0$, $(\partial\psi/\partial x)(x_1) = \text{const.}$, etc. Two representative examples of results obtainable with Eq. (8.1) are given in Fig. 14, taken from Ref. 11, and Fig. 15 from Ref. 12.

To analyse a given system we have to find its coefficients F and D. These quantities are closely related to the coherent and incoherent effect, respectively, which we have identified before. In fact

$$F/f_0 = \langle \Delta x \rangle_t \tag{8.2}$$

is the expectation value (long-term average) of the coherent change $\Delta x$ per turn of the error, and

$$2D/f_0 = \langle (\Delta x)^2 \rangle_t \tag{8.3}$$

is the expectation of the square of this change. The quantities F and 2D alone are corresponding average changes per second. Note the difference between $(\Delta x)^2 = (x_c - x)^2$ used here, and $\Delta(x^2) = x_c^2 - x^2$ as frequently used before!

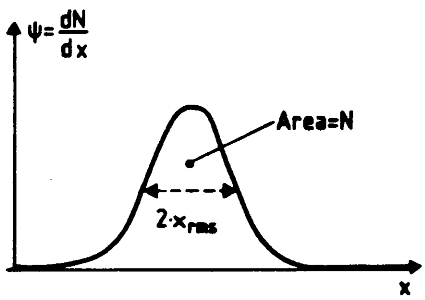



Fig. 13 A particle distribution function $\psi(x)$ defining the number of particles $dN = \psi(x)dx$ with an error in the interval from x to x + dx.

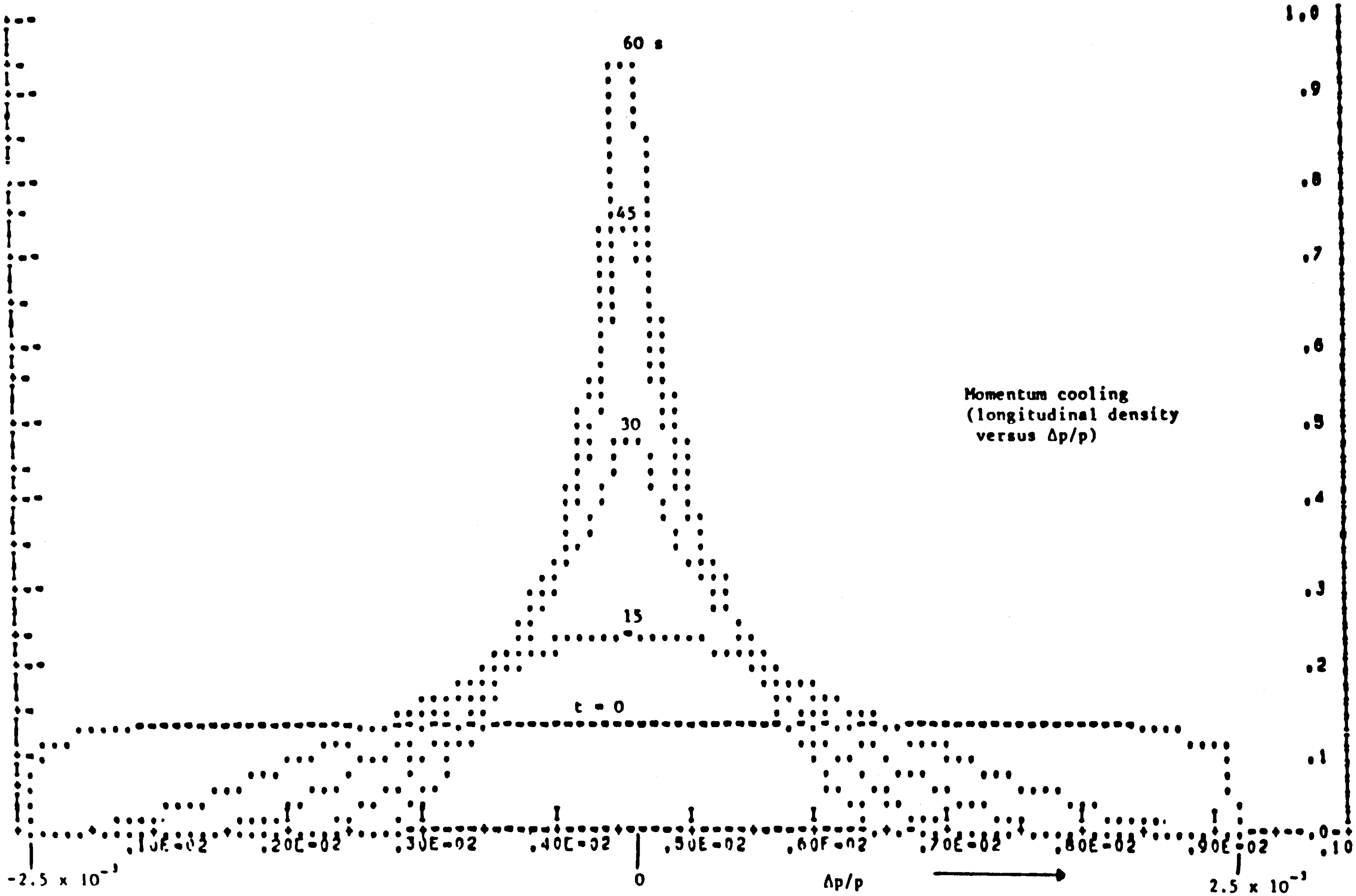



Fig. 14 Momentum cooling at 600 MeV/c in LEAR computed using equation (8.1). (Curves taken from Ref. 11.)

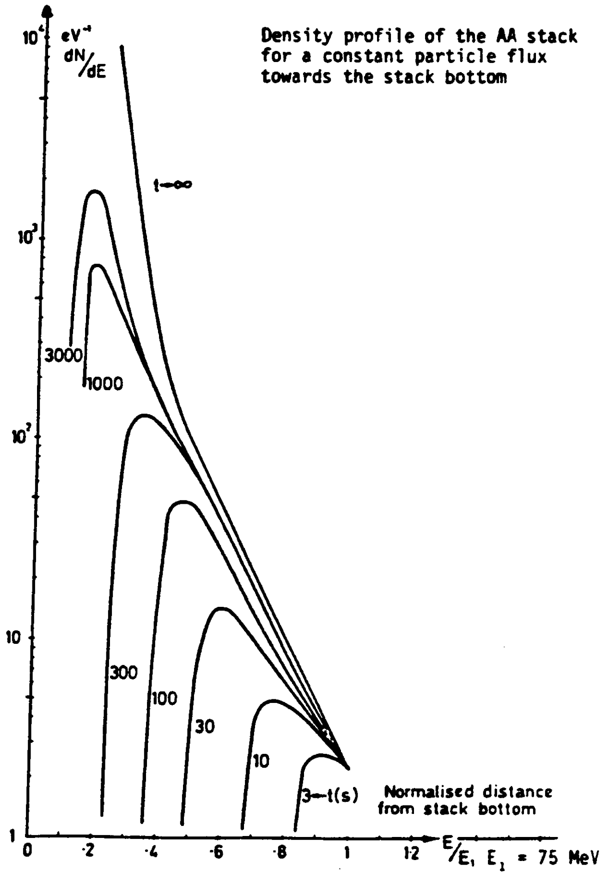



Fig. 15 Evolution of the stack in the AA during stochastic accumulation. Curves computed by S. van der Meer using the distribution function equation with the boundary condition of constant particle influx simulating the new $\bar{p}$ added every 2.6 sec. (Curves taken from Ref. 12.)

The important thing is that a distribution function equation (8.1) -- similar to the Fokker-Planck equation used in a variety of fields -- exists *and* that relatively simple prescriptions (8.2) and (8.3) permit us to establish the two coefficients F and D for any given stochastic cooling system. Incidentally, an equation similar to Eq. (8.1) had long been used (before 1976!) by the Novosibirsk Group to study the dynamics of electron cooling. Also the kinetic equations in plasma physics closely resemble our distribution equation.

Let us now try to follow a simple derivation of Eqs. (8.1)-(8.3). This derivation is due to Thorndahl[9]. It proceeds along the lines used in textbooks to derive the diffusion -- or heat transfer -- equations which resemble Eq. (8.1). Imagine a distribution function $\psi(x)$ and calculate, for a particular value $x_1$ of x, the number of particles per turn which are transferred from x-values below $x_1$ to values above $x_1$ (Fig. 16). If the correction per turn at the kicker is $\Delta x$, then particles with an error between $x_1$ and $x_0 = x_1 - \Delta x$ (cross-hatched area in Fig. 16) pass through $x_1$. Their number is

$$\Delta N = \int_{x_0}^{x_1} \psi(x)\, dx \ . \qquad (8.4)$$

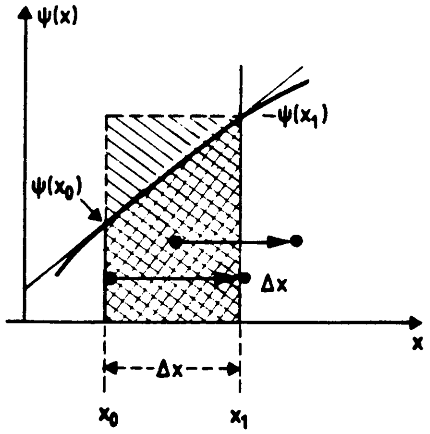



Fig. 16 A look at the distribution function fig. 13 through a magnifying glass. When the error for particles with a value near $x_1$ is changed by $\Delta x$ particles in the dark shaded area have the error value changed from values below to values above $x_1$, Eq. (8.6) expresses this area as the difference between the rectangle and the triangle sketched in the figure.

Expanding $\psi$ at $x_1$,

$$\psi \approx \psi(x_1) + \frac{\partial\psi(x_1)}{\partial x}(x-x_1) , \qquad (8.5)$$

the integration yields

$$\Delta N = \psi(x_1)\cdot\Delta x - \frac{1}{2}\frac{\partial\psi(x_1)}{\partial x}(\Delta x)^2 . \qquad (8.6)$$

The first and second terms can be interpreted as the area of the rectangle and the triangle, respectively, sketched in Fig. 16.

We now define the (average) particle flux

$$\phi = f_0\langle\Delta N\rangle_t$$

as the expected number of particles per second passing a given error value. Clearly, then, from Eq. (8.6), the instantaneous flux is:

$$\phi(x) = \underbrace{f_0\langle\Delta x\rangle_t}_{F}\ \psi(x) - \underbrace{\frac{f_0}{2}\langle(\Delta x)^2\rangle_t}_{D}\ \frac{\partial\psi}{\partial x} .$$

This gives the flux in terms of F and D as defined by Eqs. (8.2) and (8.3). The assumption has tacitly been made that the change $\Delta x$ per turn at the kicker is small and $\psi(x)$ smooth, so that higher expansion terms in Eq. (8.5) can be neglected.

Having found the flux we can immediately obtain Eq. (8.1) from the continuity equation

$$\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial t} = 0 . \qquad (8.7)$$

It states that the change per second of the density is given by the 'gradient'-$\partial\phi/\partial x$ of the flux. This is similar to continuity considerations in other fields like, for instance, the charge conservation law of electrodynamics:

$$\frac{\partial j}{\partial x} + \frac{\partial\rho}{\partial t} = 0 ,$$

relating current density j and charge density $\rho$.

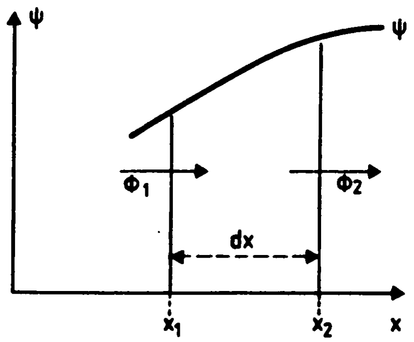



Fig. 17 The flux into and out of a narrow element of width dx in ψ-x space. An excess of incoming over outgoing flux leads to an increase with time of the density $\psi = \Delta N/dx$ of particles in the element.

Like other continuity equations, Eq. (8.7) can be obtained by looking at the flux going into and coming out of an element of width dx in ψ, x-space (Fig. 17):

Incoming flux per second : $\phi_1$

Outgoing flux per second : $\phi_2 \approx \phi_1 + \frac{\partial\phi}{\partial x} dx$

Surplus per second : $\Delta\phi = \phi_1 - \phi_2 = -\frac{\partial\phi}{\partial x} dx$ .

The resulting density increase (per second) in the element is thus

$$\frac{\Delta\phi}{dx} = -\frac{\partial\phi}{\partial x} ,$$

and conservation of the particle number requires a $\partial\psi/\partial t$ equal to this.

This completes the derivation. The resulting equation (8.1) agrees with observations made in the ISR and all subsequent machines using stochastic cooling. The reader who might have had some difficulty in appreciating the derivation may now be pleased to learn that the exact form of Eq. (8.1) has been a subject of discussion for quite some time. Looking at the derivation of the Fokker-Planck equation in textbooks[13]), one is tempted to put the coefficient D under the second derivative as it is correct for a variety of other stochastic processes. In 1977 a machine experiment[14]) was performed at the ISR to clear up this question for cooling and diffusion problems in storage rings. The experiment clearly indicated that in the present case the diffusion term should be $\partial/\partial x\,[D(\partial\psi/\partial x)]$ as in Eq. (8.1) and not $(\partial^2/\partial x^2)(D\psi)$.

## 9. EXAMPLES AND ASYMPTOTIC DISTRIBUTIONS

We may conclude from the preceding sections that it is relatively simple to determine the distribution equation pertaining to a given cooling problem. It is usually much more difficult to solve the equation. This is because in general the coefficients F and D are functions of x, t, and ψ itself. Analytical solutions have therefore only been obtained in a few simple cases.

As an example, let us briefly look at Palmer cooling with the following simplifying assumption: No unwanted mixing, and Schottky noise negligible compared with amplifier noise. Using $x \propto (\Delta p/p)$, the correction per turn is

$$\Delta x = -g[\langle x\rangle_s + x_n]$$

as given by Eq. (9.1) in Part I. In analogy to Eq. (5.1) in Part I, we assume that the long-term average of $\langle x \rangle_s = (1/N_s)\ \Sigma\ x_i$ is zero except for the contribution $x/N_s$ of the test particle upon itself. The noise $x_n$ has zero average. Hence

$$\langle \Delta x \rangle_t = -g\,\frac{x}{N_s} = -g\,\frac{2W}{Nf_0}\cdot x\ .$$

In a similar way (using the assumption that $\langle x_n^2 \rangle_t \gg \left\langle \langle x \rangle_s{}^2 \right\rangle_t$), i.e. amplifier noise dominating over Schottky noise)

$$\langle (\Delta x)^2 \rangle_t = g^2 \langle x_n^2 \rangle_t = g^2 x_{n,rms}^2 = \text{const.}$$

Hence in this simple case $F = -F_0'x$ and $D = D_0$, where $F_0' = (2W/N)g$ and $2D_0 = f_0 g^2 x_{n,rms}^2$ are constants. In this case, Eq. (8.1) is amenable to an analytic solution. Try

$$\psi = \frac{N}{\sqrt{2\pi}\ \sigma(t)}\ \exp\left[-x^2/2\sigma(t)^2\right]$$

i.e. a Gaussian with σ changing in time. Upon substitution, one obtains an ordinary differential equation for σ:

$$\dot{\sigma}/\sigma = -F_0' + D_0/\sigma^2\ .$$

Special cases:

$$D_0 = 0: \qquad \sigma^2 = \sigma_0^2\ e^{-2F_0't} \qquad \text{(continuous cooling)}\ ,$$

$$F_0' = 0: \qquad \sigma^2 = \sigma_0^2 + 2D_0 t \qquad \text{(diffusion)}\ .$$

General solution:

$$\sigma^2 = \sigma_1^2\ e^{-2F_0't} + D_0/F_0'\ .$$

This describes cooling towards an asymtotic (Gaussian) distribution with $\sigma_\infty = \sqrt{D_0/F_0'}$. In this situation an equilibrium between heating and cooling is reached. A similar result is arrived at from the simple cooling equations [e.g. Eq. (9.2), Part I] which suggest $1/\tau \to 0$ when the signal $(\langle x \rangle_s)^2$ has decreased so much that $gU = g[x_n^2/(\langle x \rangle_s)^2] \to 2$. The new information obtained from Eq. (8.1) is that the asymptotic ψ is Gaussian in the simple case considered.

The existence of asymptotic equilibrium distributions is a common feature also in more complicated cases of Eq. (8.1). The final distribution $\psi_\infty$ can be obtained putting $\partial\psi/\partial t = 0$, which converts Eq. (8.1) into a simpler ordinary differential equation:

$$-F\psi_\infty + D\,\frac{d\psi_\infty}{dx} = \text{const.} \tag{9.1}$$

The constant is frequently zero (e.g. when $F(x) = 0$ and $\partial\psi/\partial x = 0$ for $x = 0$). Equation (9.1) is important as it indicates the limiting density which can be reached.

## 10. MOMENTUM COOLING BY FILTER AND TRANSIT TIME METHODS

These methods measure the revolution frequency of particles or the time of flight between pick-up and kicker in order to detect their momentum error. The filter method of Carron and Thorndahl[15] (Fig. 18) uses a notch filter between the preamplifier and the power amplifier, with notches at all revolution harmonics in the passband (Fig. 19). In the simplest case the filter is a transmission line shorted at the far end (Fig. 19), with a length corresponding

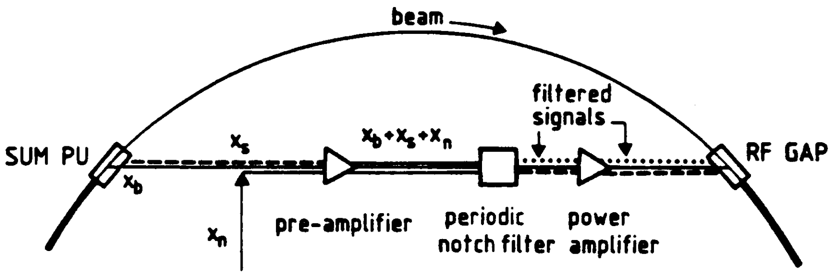



Fig. 18 The basic set up for momentum cooling by the filter method. An advantage of this method is that a sum pick up is used which is sensitive even to small beam signals. Secondly, Schottky and preamplifier noise are reduced by the filter.

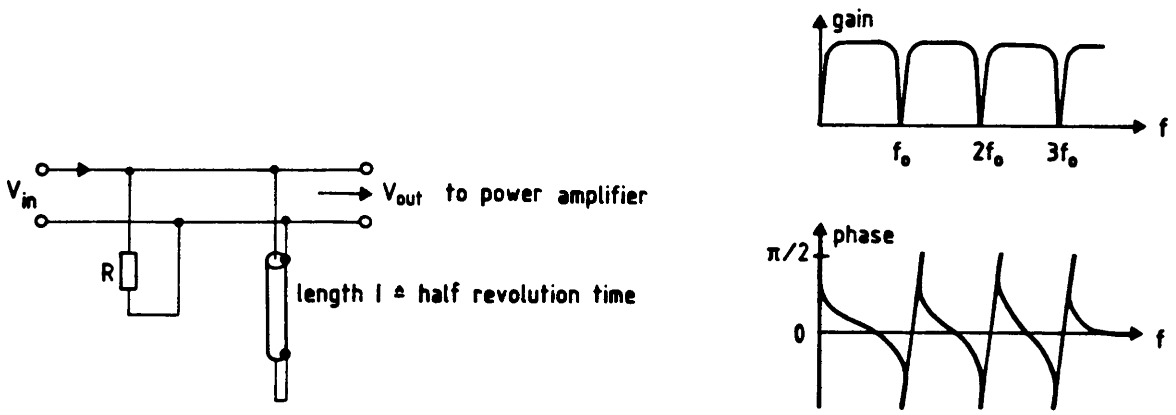



Fig. 19 A simple periodic notch filter namely a half wave low loss transmission line (used as a stub resonator). The (idealized) gain and phase characteristic are given by the half wave resonances at the multiples of the revolution frequency. Additional elements are usually added to reduce the gain between the harmonics.

to half the revolution time of the particles in the storage ring. The notches are produced by $\lambda/2$ resonances, where ideally the input impedance is zero and the phase changes sign. Because of these phase and amplitude characteristics, particles with a wrong revolution frequency are accelerated or decelerated until ideally all have 'fallen into the notches'. The price to pay for this is that all the Schottky bands used have to be well separated, so that particles 'know' the notches into which they have to fall. This means unavoidably imperfect mixing. However, this slight disadvantage could probably be circumvented by using the signal from a second pick-up -- rather than the reflection of the previous turn pulse via a cable -- to cancel signals of a particle with the correct time of flight between the two pick-ups and to accelerate/decelerate others.

Yet another time-of-flight method has been discussed at Fermilab[16]. Essentially, the idea is to differentiate the pick-up pulse and apply this signal on the kicker with a delay so that particles with the correct time of flight between pick-up and kicker are not affected, whereas slow or fast ones get a correction.

Both variants of the filter method are less efficient in noise suppression and have, therefore, not found applications so far.

We shall return to the time domain for a short moment to suggest slightly different explanations of the filter method: the pulse sent into the cooling system by a particle of nominal frequency will be cancelled by its pulse from the previous revolution reflected at the end of the line. For particles that are too slow or too fast, the cancellation is imperfect and acceleration or deceleration will result.

The filter method is important for the cooling of low-intensity beams, and in fact the whole antiproton complex at CERN would probably not have worked with stochastic cooling had this technique not been invented in due time. Sum pick-ups are used, and these produce a much larger signal than the difference devices that are necessary with other methods. The filter reduces not only the particle signals but also the preamplifier noise at the critical frequencies. This feature is important for fast cooling at low intensity. The filter method is usually analysed using the distribution equation (8.1). The coefficients F and D can be worked out theoretically and/or by measurements on the system. Usually, measurements and calculations are done harmonic by harmonic, including various ingredients such as imperfect mixing and signal suppression. However, this is not the subject of an introductory course. All we want to do is to write down the general form of the relevant coefficients F and D which, expanding up to second order in the error quantity $E \propto \Delta p/p$ take the following form:

$$F = -G_0 E \ ,$$
$$D = G_1^2 \psi(E^2 + \kappa_0) + G_2^2 \mu(E^2 + \kappa_1) \ ,$$

where E is the energy error; $G_0$ (proportional to the gain), $G_1^2(\propto g^2)$, and $G_2^2(\propto g^2)$ are given by the ideal filter, $\kappa_0$ and $\kappa_1$ by the losses; and $\mu$ relates to the amplifier noise. The first term of D (which is proportional to the density $\psi$) gives the Schottky noise filtered by the notches, and the second term the filtered preamplifier noise. For more details, the reader should consult the specialized literature[2,8,10].

## 11. CONCLUDING REMARKS

This concludes my survey of stochastic cooling. I hope that the reader for whom this presentation was prepared has found the length and the amount of repetition involved helpful rather than bothersome. For those interested in a more concise description or in technical details, a bibliography is included. It contains all references on stochastic cooling up to October 1983 known to me.

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*Special problems of*
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