A TRAVELING-WAVE ANALYSIS OF STANDING-WAVE

SUPERCONDUCTING STRUCTURES

P.B. Wilson[*]

*) On leave from the High Energy Physics Laboratory,
Stanford University, Stanford, California, U.S.A.

SUMMARY

The properties of resonant, periodic structures are studied using an equivalent circuit representation. A circuit consisting of a chain of coupled resonators is chosen. The complex propagation constant for an infinite chain of resonators is derived, and the field in a finite length of structure is then obtained by summing a traveling wave which undergoes multiple reflections between the planes forming the end boundaries of the structure.

The analysis is applied to study the effect of loss and detuning in high Q superconducting structures operating in the $\pi/2$ and π modes. It is shown that, even for the π mode, the effect of loss on the amplitude and phase of the field is in general negligible for the superconducting case. The effect of a badly detuned cavity is more serious. A principle conclusion is that, using a single cavity tuner located near the center of a structure, it is difficult to achieve a reasonable tuning range in a π - mode structure which is more than about 5 wavelengths long, or in a $\pi/2$- mode structure which is more than about 25 wavelengths long, without badly perturbing the field flatness.

The properties of the π-mode resonance are discussed in some detail. The somewhat complex behavior of this mode results from the fact that dispersion diagram and the attenuation as a function of phase shift per period change rapidly near the band edges. The group velocity and energy velocity are found to differ by a factor of two at the π -mode resonance. It is shown, however, that the group velocity as defined by $d\omega/d\beta$ becomes meaningless at this mode.

The principle expressions obtained in this report concerning the effect of loss and detuning on field flatness are, for convenience, collected together in Table I, Appendix D.

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I.   INTRODUCTION

Two quite distinct methods have been used to investigate the properties of resonant periodic structures. In the first method, the normal modes of the resonant structure are considered. The field profile (accelerating or deflecting field on the axis of the structure as a function of axial position) is obtained by an appropriate super-position of the normal mode fields. The normal mode approach has been quite thoroughly exploited in a number of papers and reports (see, for example, Refs. 1-4). An analysis of this type is well adapted to the use of perturbation theory to consider the case when tuning errors are present in the individual cavities along the structure.

A second quite different approach can also be useful in the analysis of resonant structures. Consider a wave which is coupled through a small aperture into a periodic structure, and which travels along the structure in both directions away from the excitation point. If two reflecting planes are now inserted at appropriate planes of symmetry within the structure, the initial traveling wave will bounce back and forth within the resulting cavity formed between the planes. The wave undergoes attenuation and phase shift as the total distance away from the excitation point increases, but if the length of the structure between the reflecting planes is chosen correctly, the traveling wave will return in phase after each reflection, and can build eventually to a large amplitude. If the properties of the structure for the propagation of a traveling wave are known, the resonant properties can be obtained by summing the traveling wave components at any point in the structure. This infinite sum must take into account both the amplitudes and phases of the individual components, and also the fact that some of the wave leaks back out of the coupling aperture each time it passes by.

From the nature of this approach, the resonant properties of a structure are seen to depend directly on the traveling-wave properties. Since the traveling-wave properties of a wide variety of structures have been extensively studied, this existing body of information can be applied rather directly to the study of resonant cavities formed from shorted sections of such structures.

The traveling-wave approach to the analysis of standing-wave structures has not previously been exploited to the same extent as the normal mode approach, and yet the traveling-wave view point has two rather considerable advantages. First, information concerning the transient behavior of the field during the filling of the structure is obtained as readily as the steady-state behavior. The process by which the field builds up to the resonant value as the sum of the traveling-wave components is simple and easy to visualize at each step of the way[(*)]. This leads to the second advantage. The method can lead to some quite subtle results with a very minimum of mathematical complexity. A disadvantage of the traveling-wave approach is that it is not as well adapted as normal mode theory to consider the case when there are many perturbations present in the structure, although, as we will see in Sec. VI, the case of a single detuned cavity can readily be taken into account.

One of the main purposes of this report will be to investigate the properties of the π-mode resonance in periodic structures, basing the development on the traveling-wave point of view. The term π-mode resonance will be used throughout, although, as is well known by now, the actual phase shift per period for this mode is slightly less than π if loss is present in the structure. The traveling- wave viewpoint provides a rather different physical picture as to how and why this phase shift arises.

It is of interest to investigate the π-mode because the shunt impedance is higher, and the peak electric and magnetic fields at the surface of the structure are lower for a given accelerating or deflecting gradient, than is the case for other modes well within the passband[(**)].

(*) This method for obtaining information concerning the transient behavior of the fields in a resonant structure has recently been exploited by Knapp[5]. By displaying the results in the form of a motion picture, the build-up of the field by the vector addition of successive traveling-wave components is clearly seen.

(**) For the case of a deflecting structure, this statement is valid only for small disk aperture diameters. As the aperture diameter increases, the peak magnetic field in such a structure increases rapidly for the case of the π-mode[6] .

The question of peak fields is of particular importance in the case of a superconducting structure. In such a structure the maximum gradient will be limited either by flux penetration in a region of peak magnetic field, or by electron emission in a region of peak electric field. If π-mode operation is possible, the advantages of a traveling-wave resonant ring (traveling-wave accelerator with feedback) are largely obtained, but the problems involved in the construction of such a complex device are avoided. The phase and amplitude of the fields along a structure operating in the π-mode are, of course, more strongly affected by perturbations in the frequencies of the individual cavities making up the structure than is the case for, say, the $\pi/2$ mode. However, for structures which are not too long (roughly if the length is less than about five wave lengths) we will show that the difference between π- and $\pi/2$- mode operation is not overwhelming. Operation at the π-mode resonance may then be the best choice when all factors are considered.

The philosophy of the present report may be summarized by saying that, if we want to understand a standing-wave structure, we must first know the behavior of a wave traveling on a infinite length of the same structure. Toward this end it is helpful to have an equivalent circuit which can accurately represent the behavior of physically real periodic structures. The choice of an appropriate equivalent circuit is considered in the next section.

II. EQUIVALENT CIRCUITS FOR PERIODIC STRUCTURES

The key to the analysis to be carried out in the following sections is provided by a knowledge of the complex dispersion relation for a traveling-wave structure. A detailed knowledge of the dispersion relation near the edges of the passband is particularly important, if we are to understand the behavior of the 0 and π modes in a resonant length of structure. There are a number of models which can give a theoretical account of the propagation characteristics for periodic structures. In this section we look briefly at several such models, examine their limitations, and chose the most suitable one for detailed analysis.

A uniform transmission line periodically loaded by lossless shunt susceptances, as shown in Fig. 1a, would seem to be a reasonable representation of a periodic structure. Several authors[7, 8] give the dispersion relation for such a network. The result is

$$\cos \gamma l = \cos \gamma_0 l - (B/2Y_0) \sin \gamma_0 l . \tag{1}$$

Here Y_0 is the characteristic admittance of the unloaded transmission line, l is the length of one period, $\gamma = \beta - j\alpha$ is the complex propagation constant for the loaded line, and $\gamma_0(\omega)$ is the known propagation constant for the unloaded line (complex for a lossy line). The traveling-wave fields then vary as $\exp[j(\omega t - \gamma z)]$. If the lossy case is considered, the real and imaginary parts of Eq. (1) provide two equations for αl and βl. By eliminating αl between these two equations, the dispersion relation is obtained. It is easily shown that Eq. (1) gives rise to the usual passbands and stopbands, and it is seen that as B approaches zero γ passes over smoothly into γ_0, as it must for a smooth guide.

The preceding approach can be made more general by using a more complex network to represent the discontinuities. For example, the effect of loss in the discontinuities can be accounted for by using complex shunt admittances. However, more fundamental limitations are encountered as the degree of loading becomes heavier. First, if the unloaded transmission line is capable of propagating in more

than one mode, the obstacles can couple power from the dominant mode
into higher-order propagating modes. Second, and more important,
the spacing between obstacles must be large enough to eliminate any
interaction between the local non-propagating modes excited in the
neighborhood of the discontinuities. This latter condition is most
certainly violated in the case of the heavy loading normally used in
typical disk-loaded structures. In order to get around these
difficulties, Feuchtwang [9] has given a generalized method for
analyzing periodically loaded waveguides based on a representation
by N periodically-coupled transmission lines. In this approach the
section of uniform waveguide is represented by N uncoupled transmission
lines which take into account the dominate mode and N-1 higher-order
modes, which may be either propagating or non-propagating. The
obstacles are represented by 2N-port networks. The case for two
modal lines loaded and coupled by shunt susceptances is shown in
Fig 1b. The diagonal elements of the matrix $\bar{B}$ represent loading
in the two lines considered separately. The off-diagonal elements
may be interpreted as providing the coupling between the two lines.
As loading becomes heavier, the number of model lines needed to provide
an accurate representation of the structure increases, and the mathematical
difficulty involved in obtaining a detailed solution also obviously
increases. For this reason the method is useful in practice only for
values of N equal to two or perhaps three.

In the case of very heavy loading an alternative description
is possible which again leads to a simple method of analysis. If the
loading obstacles close off the waveguide completely, the regions
between adjacent obstacles form resonant cavities. If the cavities have
a reasonably high Q and are excited in only a single mode, it is
well known that the behavior of each cavity is adequately represented
by a parallel resonant circuit. If small coupling holes are now opened
up between the cavities, a series of coupled resonant circuits, as
shown in Fig. 1c, provides an accurate representation of the resulting
loading transmission line. This equivalent circuit has been used
previously in the normal mode type of analysis[1-4], and it is also
the circuit which will be used in the following section to obtain the
complex dispersion relation for an infinite traveling-wave structure.
Some qualifying remarks need to be made, however, about this particular
representation. If the coupling coefficient k (where kL is the
mutual inductance between adjacent circuits) becomes too large, that is,
if the holes between adjacent cavities become large, the basic

assumption underlying the circuit shown in Fig. 1c is violated, and the model can no longer be trusted to give an accurate representation of the fields in a physical structure. In a way analogous to the case of the lightly loaded line discussed previously, the circuit can be patched up by introducing additional parameters. In this case, as k gets larger, additional coupling coefficients which take into account next nearest neighbor coupling can be introduced. Again, however, the mathematical complexity involved in analyzing the circuit is greatly increased. The complexity becomes overwhelming if still higher-order coupling parameters are required.

In this report we consider only the case for which a single coupling coefficient is sufficient. Without a much more extensive analysis, it is not clear in what detailed way the present treatment will break down as k increases due to the fact that higher-order coupling has been ignored. In any case, the results obtained can be considered valid in the limit as k approaches zero, and we will assume that applications of the results are meaningful up to k's on the order of 0.03.

III.   DISPERSION RELATION FOR A CHAIN OF COUPLED CAVITIES

The dispersion relation for the chain of coupled resonant circuits shown in Fig. 1c has been derived previously (see for example Refs. 1-4). In this section we will review the calculation and extend it to obtain, in particular, the complex dispersion relation near the edges of the passband.

The circuit equation for the network of Fig. 1c can be written

$$[2j\omega L + 1/(j\omega C) + R] i_n + j\omega kL (i_{n-1} + i_{n+1}) = 0 . \tag{2}$$

We assume that the current in the nth circuit can be represented by a wave traveling in the direction of larger n and is given by

$$i_n = Ae^{-jn\gamma\ell} = i_{n-1} e^{-j\gamma\ell} = i_{n+1} e^{j\gamma\ell} . \tag{3}$$

Here ℓ is the periodic length and $\gamma = \beta - j\alpha$.

The factor $e^{j\omega t}$ is understood whenever it is not written explicitly. Since γ is complex , there will be both a phase shift by an amount $\beta\ell$ and an attenuation by an amount $\exp(-\alpha\ell)$ for a wave traveling through one period. It is assumed that the i_n's are proportional to the field at some arbitrary point in a given cavity, and that this point is identical from one cavity to the next. If Eq. (3) is used in Eq. (2) to eliminate i_{n-1} and i_{n+1} we obtain

$$1 - \omega_0^2 / \omega^2 - j Q^{-1}(\omega_0/\omega) + k \cos \gamma\ell = 0 . \tag{4}$$

We introduce $\omega_0^2 = 1/(2LC)$, and $Q = 2\omega_0 L/R$ is defined as the unloaded Q . By substituting $\gamma = \beta - j\alpha$ into Eq. (4), and setting the real and imaginary parts separately equal to zero, we obtain the following pair of equations,

$$\omega^2 = \omega_0^2/(1 + k \cos \beta\ell \cosh \alpha\ell) \tag{5}$$

$$\sinh \alpha \ell = \omega_0/(kQ\omega \sin \beta \ell) \ . \tag{6}$$

As α tends to zero, Eq. (5) reduces to the usual dispersion relation. For small k this can be written

$$\omega = \omega_0/(1 + k \cos \beta \ell)^{\frac{1}{2}} \approx \omega_0(1 - \tfrac{1}{2} k \cos \beta \ell) \ . \tag{7}$$

We note that the preceding relation indicates propagation in a forward-wave structure; propagation in a backward-wave structure can be taken into account by changing the sign of k . From Eq. (6) we see that away from the edges of the passband the attenuation will be small if $kQ \gg 1$. For small k we have also that $\omega \approx \omega_0$,and therefore

$$\alpha \ell \approx (kQ \sin \beta \ell)^{-1} \ . \tag{8}$$

Near the edges of the passband, where $\beta \ell$ approaches zero or π , the effect of attenuation becomes important even if the product kQ is large. Equations (5) and (6) are then coupled together and the dispersion relation becomes somewhat more complicated. By eliminating $\alpha \ell$ between Eqs. (5) and (6) we obtain for the general case

$$\omega^2 = \frac{\omega_0^2}{1 + \tfrac{1}{2}\left(\dfrac{1}{Q\tan\beta\ell}\right)^2 \pm \left\{k^2\cos^2\beta\ell + \left(\dfrac{1}{Q\tan\beta\ell}\right)^2 + \tfrac{1}{4}\left(\dfrac{1}{Q\tan\beta\ell}\right)^4\right\}^{\frac{1}{2}}} \ . \tag{9}$$

In this expression the sign in front of the brackets in the denominator is chosen to be plus for $0 < \beta\ell < \pi/2$ and minus for $\pi/2 < \beta\ell < \pi$ for forward-wave propagation. If Q becomes infinite Eq. (9) reverts to Eq. (7), as it should.

In order to examine Eq. (9) in the neighborhood of the zero or π modes, we let $\beta\ell = \Delta$ and $\beta\ell = \pi - \Delta$ for the two cases, where Δ is small. We then obtain in the limit as Δ approaches zero,

$$\omega \to \omega_0/(Q\Delta) \to \infty \qquad (\beta\ell \to \pi)$$
$$\omega \to \omega_0 Q\Delta \to 0 \qquad (\beta\ell \to 0) \ .$$

From Eq. (6) the attenuation parameter becomes in these two limits,

$$\sinh \alpha\ell \rightarrow 1/k \qquad\qquad (\beta\ell \rightarrow \pi)$$
$$\sinh \alpha\ell \rightarrow 1/(kQ^2\Delta^2) \rightarrow \infty \qquad (\beta\ell \rightarrow 0) \quad .$$

The variation of frequency and attenuation as a function of $\beta\ell$ is shown in Fig. 2.

IV.   FIELD PROFILE FOR A CHAIN OF COUPLED CAVITIES

IV.1.   Derivation of the Field Profile Function

Now that the phase shift and attenuation per period have been calculated for a wave traveling on an infinite periodic structure, we can proceed as discussed previously to derive the properties of a resonant section of such a structure. The method is illustrated by the diagram given in Fig. 3. A structure of length $L = N\ell$ is excited by a wave of amplitude A_0 launched at $z = 0$. For simplicity the points $z = n\ell$ are taken to be at the center of each cavity. Two shorting planes have been introduced at planes of symmetry passing through cavity centers, and the two end cavities are therefore half cavities. The first few reflections of an initial wave launched at $z = 0$ are also shown in the diagram. For the moment we assume in addition that the coupling to the structure is very weak and that power re-radiated back out the coupling apperature need not be taken into account. Both the forward and reverse waves take the form of geometric series which are easily summed to obtain

$$S^+ = A_0 [e^{j(\omega t - n\gamma\ell)}][1 - e^{-j(2NT\ell)}]^{-1}$$

$$S^- = A_0 [e^{j(\omega t + n\gamma\ell - 2N\gamma\ell)}][1 - e^{-j(2N\gamma\ell)}]^{-1} .$$

The total standing-wave field is obtained by adding S^+ and S^-, and with a little rearranging we obtain

$$S = \frac{2A_0 [e^{j(\omega t - N\gamma\ell)}] \cos[(N-n)\gamma\ell]}{1 - e^{-j(2N\gamma\ell)}} .$$

Recalling that the complex propagation constant is defined by $\gamma = \beta - j\alpha$, the preceding expression can be written in the form $S = A_N F(n) e^{j\omega t}$ where

$$A_N = \frac{2A_0 \, e^{-\tau} \, e^{-jN\beta\ell}}{1 - e^{-2\tau} e^{-j2N\beta\ell}}$$

(10)

$$F(n) = \cos[\beta\ell(N-n)]\cosh[\alpha\ell(N-n)]$$

$$+ j \sin[\beta\ell(N-n)]\sinh[\alpha\ell(N-n)] \quad . \tag{11}$$

The attenuation parameter, $\tau = N\alpha\ell$, has also been introduced. We note that the above summation has been written as the product of the amplitude of the field in the Nth cavity, A_N , and a field profile function , $F(n)$, which contains the dependence of the field on n . The field profile function at n = N reduces to $F(N) = 1$. The two expressions given by Eqs. (10) and (11) contain a wealth of information. The analysis to follow in this section, and in section VI for the case of a detuned cavity, will consist primarily in an examination of these two expressions under a number of different conditions.

An examination of the amplitude function for the case where τ is not a rapidly varying function of $\beta\ell$ (i.e., for modes away from the band edges) shows that resonances exist when $N\beta\ell$, the total phase shift through the structure, is equal to an integral multiple of π . In order to explore the amplitude function in the neighborhood of these resonances we introduce the phase shift parameter Δ , defined by

$$\beta\ell = (q\pi/N) - \Delta \quad , \tag{12}$$

where q is an integer such that $0 < q < N$. By substituting (12) into (10) we obtain for the amplitude function,

$$A_N = \frac{2 A_0 e^{-\tau} e^{-j(q\pi-N\Delta)}}{1- e^{-2\tau} e^{j2N\Delta}} \tag{13}$$

We now assume a high Q structure such that $\tau \gg 1$. By differentiating Eq. (7), using $\delta(\beta\ell) = -\Delta$, we obtain $(\delta\omega/\omega) = -\frac{1}{2} (k\Delta \sin\beta\ell)$. From Eq. (8) we have that

$\tau \approx N(kQ\sin\beta\ell)^{-1}$. By eliminating $\beta\ell$ between these two results we find (away from the edges of the passband),

$$N\Delta = -2Q\tau(\delta\omega/\omega) \; .$$

In the immediate neighborhood of a resonance $Q(\delta\omega/\omega) \approx 1$. From the preceding expression we see that the condition $\tau \ll 1$ then implies $N\Delta \ll 1$ near a resonance. Under these conditions Eq. (13) can be written

$$A_N \approx \frac{A_0 \; e^{-jq\pi}}{\tau - jN\Delta} \; . \tag{14}$$

Substituting the preceding relation for $N\Delta$ in Eq. (14), we have

$$A_N(\delta\omega) \approx \frac{A_N(0)}{1 + j2Q(\delta\omega/\omega)} \tag{15}$$

$$A_N(0) = (A_0/\tau) \; e^{-jq\pi} \; . \tag{16}$$

Equation (15) is the standard expression giving the variation in the amplitude and phase of the response of a resonant cavity as a function of frequency near resonance. Equation (16) shows that the field amplitude at resonance is enhanced over the amplitude of the initial traveling wave by a factor $1/\tau$.

The field profile function near resonance is obtained from Eqs. (11) and (12) as

$$F(n) = \cos[(\frac{q\pi}{N} - \Delta)(N-n)]\cosh[\alpha\ell(N-n)]$$

$$+ \; j \; \sin \; [(\frac{q\pi}{N} - \Delta)(N-n) \; \sinh \; \lceil\alpha\ell(N-n)], \tag{17}$$

where we have from Eq. (8) that

$$\alpha\ell \approx \lceil kQ \sin (\frac{q\pi}{N} -\Delta) \rceil^{-1} . \tag{18}$$

Equation (17) shows that in general there will be both an amplitude change and a phase shift of the field along the length of the structure as the result of loss.

The preceding development was carried through ignoring the effect of finite coupling to an external generator, and without specifically taking into account the effect of power transfer to the beam in the case of an accelerating structure. The effect of coupling can be considered by a straight-forward extension of the analysis, and this is done in Appendix A. The conclusion is that the field profile function is not changed by taking coupling into account, although the amplitude function now specifically involves the cavity coupling coefficient. The calculation of the effect of beam loading presents more of a problem. To a first approximation, however, the power transfer to the beam is β_c times as great as the power dissipation in the walls for a superconducting accelerating structure operating at maximum efficiency, where β_c is the coupling coefficient as measured with the beam off. Thus if we use $Q \approx Q_0/\beta_c$ in the expressions developed so far containing Q, the behavior of a beam loaded cavity should be reasonably well represented in the steady-state condition at resonance. An exact analysis would involve adding a current generator to each separate resonant circuit in the chain of resonant circuits given in Fig. 1c, in order to take into account the voltage induced by the beam in each cavity.

IV.2.   Field Profile for the $\pi/2$ Mode

Consider now the case of the $\pi/2$ mode at resonance. Letting $q = N/2$ and $\Delta = 0$ in Eq. (17), we have, using also the value of $\alpha\ell \approx 1/kQ$ obtained from Eq. (18),

$$F(n) \approx \cos[(\pi/2)(N-n)] \cosh[(N-n)/(kQ)]$$

$$+ j \sin[(\pi/2)(N-n)] \sinh[(N-n)/(kQ)] .$$

Under the assumption that $\tau = N(kQ)^{-1} \ll 1$, we have for the fields in the first and second cavities, assuming also that N is even,

$$F(0) \approx 1 + \frac{1}{2} \left(\frac{N}{kQ}\right)^2 \tag{19a}$$

$$F(1) \approx j \left(\frac{N-1}{kQ}\right) \quad . \tag{19b}$$

We have ignored the sign of the fields with respect to the field in the Nth cavity, given by $\cos(N\pi/2)$ and $\sin[(N-1)\pi/2]$ respectively. Since $F(N) = 1$, the first of these expressions shows that the field droop along the structure, defined as $\delta F = F(0)-1$, is

$$\delta F = \frac{1}{2} \left(\frac{N}{kQ}\right)^2 \quad . \tag{20}$$

By comparing Eqs. (19b) and (20) it is seen that the field droop in the presence of loss is a second-order effect compared to the field introduced into the nominally unexcited cavities.

Consider the application of these relations to a super-conducting structure 3 m long operating at a wave length of 10.5 cm, giving $N = 114$. If $Q = 5 \times 10^7$ (assuming the case of an accelerating structure when the Q is dominated by power transfer to the beam), and if $k = 2 \times 10^{-2}$, then we calculate that $F(1) \approx 10^{-4}$ and $\delta F \approx 5 \times 10^{-9}$. Both effects are negligible, and for the case of a separator structure where there is no beam loading to contend with, the corresponding perturbations in the field profile are smaller still. For a room temperature structure, however, the quantity $N(kQ)^{-1}$ can easily be of the order of unity.

IV.3. Field Profile for the π-Mode

Many of the results in the preceding sections are not valid for the 0 or π modes because the attenuation changes very rapidly in the neighborhood of the passband edges (see Fig. 2b). In this section we will limit the discussion to the case of the π- mode, and will show that the maximum response as a function of phase shift Δ , where $\Delta = \pi-\beta\ell$, occurs not at $\Delta = 0$, but at a phase shift per cavity which is slightly less than π . The basic method of solution is to maximize the amplitude A_N as a function of Δ , after having also expressed the attenuation τ as a function of Δ . To simplify

the problem we assume again that both the attenuation and the phase shift are small that is, $\tau \ll 1$ and $N\Delta \ll 1$. If these conditions are not met, the concept of a resonance is not very meaningful. Under these conditions Eq. (18) is valid, and we have for $q = N$,

$$\tau \approx N/(kQ\Delta) \ . \tag{21}$$

Substituting this relation in Eq. (14) we obtain

$$A_N \approx \frac{(A_0 kQ\Delta)e^{-jN\pi}}{N(1-jkQ\Delta^2)} \ . \tag{22}$$

Taking the absolute value of this expression we find that

$$|A_N| \approx \frac{A_0 kQ\Delta}{N(1+k^2 Q^2 \Delta^4)^{\frac{1}{2}}} \ .$$

The value of Δ which maximizes this relation is readily found to be

$$\Delta_\pi = \frac{1}{(kQ)^{\frac{1}{2}}} \ . \tag{23}$$

From Eq. (21) we see also that

$$\tau_\pi = \frac{N}{(kQ)^{\frac{1}{2}}} \ . \tag{24}$$

This can be compared with the attenuation for the case of the $\pi/2$ mode obtained from Eq. (8) as $\tau = N(kQ)^{-1}$. The relative values for the attenuation in the two modes is shown qualitatively in Fig. 2b.

The preceding derivation was based on the assumption that both τ and $N\Delta$ are small compared to unity in the neighborhood of the π mode resonance. From Eqs. (23) and (24) these assumptions are seen to be valid if

$$\frac{N}{(kQ)^{\frac{1}{2}}} \quad << \quad 1 \quad .$$

Using the fact that τ as given by Eq. (21) must be small compared to unity, we have the following conditions on Δ in the neighborhood of the π mode resonance,

$$1 >> N\Delta >> \frac{N^2}{kQ} \quad . \tag{25}$$

The amplitude of the π mode resonance can now be calculated from Eq. (22) as

$$A_N(\pi) = \frac{A_0}{2\tau} (1+j) e^{-jN\pi} \quad .$$

By comparison with Eq. (16) we see that the phase of the π-mode resonance is shifted by 45° with respect to the response for modes within the passband. In addition, the absolute value of $A_N(\pi)$ is $|A_N(\pi)| = (1/\sqrt{2})(A_0/\tau)$. The amplitude of the π-mode response is therefore lower by a factor $1/\sqrt{2}$ for a given excitation level at the coupling aperature. This difference in response between the π-mode and other modes within the passband can be observed experimentally by a swept-frequency display of the modes of the structure.

The field profile for the π-mode resonance can be investigated by letting $q = N$ in Eq. (17) using also $\alpha \ell = (kQ)^{-\frac{1}{2}}$ and $\Delta = (kQ)^{-\frac{1}{2}}$. The result is

$$F(n) = (-1)^n \{\cos[(N-n)/(kQ)^{\frac{1}{2}}] \cosh[(N-n)/(kQ)^{\frac{1}{2}}]$$

$$-j \sin[(N-n)/(kQ)^{\frac{1}{2}}] \sinh[(N-n)/(kQ)^{\frac{1}{2}}]\} \quad .$$

This can be approximated by

$$F(n) \sim (-1)^n [1 - \frac{(N-n)^4}{6(kQ)^2} - j \frac{(N-n)^2}{kQ}] \quad . \tag{26}$$

The field is seen to change in both phase and amplitude as n varies from 0 to N . The total phase shift Θ is obtained from the

imaginary part of the above expression at $n = 0$,

$$\Theta_\pi \approx (N\Delta_\pi)^2 = N^2/(kQ) \quad . \tag{27}$$

This is identical to the result obtained using the normal mode analysis.
By taking the absolute value of Eq. (26) at $n = 0$ we calculate the
droop in the amplitude of the field as

$$\delta F = |F(0)| - 1 \approx \frac{1}{3} N^4/(kQ)^2 = \Theta_\pi^2 \quad . \tag{28}$$

Thus the field droop is seen to be a second-order effect compared to the
phase shift.

Consider now the implication of these relations for some
typical structures. A superconducting structure 3 m in length operating
in the π mode near 2850 MHz has about 57 cavities. If $k = 3 \times 10^{-2}$
and if the beam-loaded Q is 5×10^7 , then Eq. (27) predicts a total
phase shift of 2×10^{-3} radians, which is quite tolerable. For the case
of a separator structure, or at lower frequencies, the phase shift is even
smaller. According to Eq. (28), the field droop is completely negligible.
For a room temperature structure the situation is radically different.
For $Q = 10^4$ and $k = 10^{-2}$, the phase shift is on the order of unity for
a structure only 10 cavities long.

From these examples it might appear that π-mode operation is
feasible for a superconducting structure several meters in length, even
at S-band frequencies. We will see in Section VI , however, that the
effect on the field profile produced by a perturbation (detuning a cavity)
imposes a much stricter limitation on the length of a structure, expressed in
number of cavities, than does the effect of loss as calculated in this
section.

V. THE DISPERSION RELATION AND MODE SPACING NEAR THE π- MODE RESONANCE

V.1. Dispersion Relation

We have shown that if $(kQ)^{\frac{1}{2}} >> N$, then a π-mode resonance exists at a frequency such that the phase shift per period for a wave traveling on an infinite structure is $\Delta = (kQ)^{-\frac{1}{2}}$ less than π . Next we investigate the behavior of the dispersion relation in the immediate vicinity of this resonance. If we expand Eq. (9) in the neighborhood of the π-mode resonance using the fact that $kQ\Delta >> 1$, as given by Eq. (25), and that $kQ\Delta^2 \approx 1$, we obtain

$$\omega^2 \approx \frac{\omega_0^2}{1-k\left[1-(\Delta^2/2)+(2Q^2 k^2 \Delta^2)^{-1}\right]} \quad . \tag{29}$$

When $\Delta = \Delta_\pi = (kQ)^{-\frac{1}{2}}$ we see that $(\omega_\pi/\omega_0)^2 = 1/(1-k)$.

The denominator of this expression is accurate to within a term of order k/Q . The frequency of the π-mode resonance is, then, very closely the same as that expected in the case of a lossless structure.

V.2. Phase Shift and Frequency at the q = N-1 Resonance

The phase shift per period in the neighborhood of the mode closest to the π mode (q = N-1) is obtained from Eq. (12) as

$$\beta\ell = \pi - \pi/N - \Delta \quad .$$

If N is large, then $\pi/N << 1$ and $\sin\beta\ell \approx \Delta + \pi/N$. Assuming as usual that $\alpha\ell << 1$, Eq. (8) now gives

$$\tau \approx \frac{N}{kQ(\Delta + \pi/N)} \quad .$$

After substituting this expression in Eq. (14) and optimizing the amplitude with respect to Δ , we find that the phase shift at resonance is not quite zero. This small residual phase shift, denoted by Δ_r , is the solution to

$$N^3 - k^2 Q^2 \Delta_r (\pi + N \Delta_r)^3 = 0 \ .$$

The residual phase shift will be small compared to the mode spacing, that is , $\Delta_r \ll \pi/N$. If we use this approximation together with the preceding expression for τ , we obtain

$$\Delta_r \approx N^3/(\pi^3 k^2 Q^2) \approx \Delta_\pi \ (\tau/\pi)^{\frac{3}{2}} \ .$$

Thus resonance occurs essentially at $\Delta = 0$, as compared to the case of the π mode, if $\tau \ll \pi$. If a similar derivation is carried through for the $\pi/2$ mode, the result is that $\Delta_r = 0$. This is a direct consequence of the fact that τ as a function of $\beta\ell$ has a minimum at $\beta\ell = \pi/2$.

The frequency of the $q = N-1$ mode is obtained by substituting $\beta\ell = \pi(N-1)/N$ in Eq. (7), again assuming $\pi/N \ll 1$, to obtain

$$\omega(q = N-1) \approx \omega_0 \ [1 + \frac{k}{2} \ (1 - \frac{\pi^2}{2N^2})] \quad . \tag{30}$$

V.3.   Mode Spacing

The frequency difference between the π mode and the next nearest mode for the case of large N is obtained directly from Eq. (30) as

$$(\delta\omega/\omega)_{MS} \approx k\pi^2/4N^2 \ . \tag{31}$$

It is of interest to compare this value for the mode spacing with the bandwidth at resonance, $(\delta\omega/\omega)_{BW} = 1/Q$. The ratio of the two quantities is

$$\frac{(\delta\omega/\omega)_{BW}}{(\delta\omega/\omega)_{MS}} = \frac{4N^2}{\pi^2 kQ} \quad .$$

Using Eq. (24) for the attenuation at the π mode we have

$$\frac{(\delta\omega/\omega)_{BW}}{(\delta\omega/\omega)_{MS}} = \frac{4\tau^2}{\pi^2} \quad .$$

If the concept of a resonance is to be meaningful, then $\tau \ll \pi$ and the bandwidth will necessarily be small compared to the mode spacing. For the case of the $\pi/2$ mode it is readily shown that

$$(\delta\omega/\omega)_{MS} = k\pi/2N \quad . \tag{32}$$

The ratio of bandwidth to mode spacing is

$$\frac{(\delta\omega/\omega)_{BW}}{(\delta\omega/\omega)_{MS}} = \frac{2N}{\pi kQ} = \frac{2\tau}{\pi} \quad .$$

Again, the condition that the bandwidth should be small compared to the mode spacing is equivalent to the condition that the total one-way attenuation along the structure should be small. If the cavity is heavily overcoupled, then $Q \approx Q_0/\beta_c$ and the condition $\tau\beta_c \ll 1$ must be met in order to insure that the bandwidth remains small compared to the mode spacing.

VI. EFFECT OF A DETUNED CAVITY ON THE FIELD PROFILE

VI.1. Derivation of the Amplitude and Field Profile Functions

If there is more than one resonant structure in a super-conducting linac or separator system, it will be necessary to introduce tuning devices so that the resonant frequencies of the separate structures can be accurately matched to within a small fraction of the loaded Q bandwidth. One method for accomplishing this is to adjust the frequency of a single cavity in the structure by a tuner, which will also shift the resonant frequency of the structure as a whole. The tuning range for the complete structure will then be roughly $1/N_e$ times the tuning range of the individual tuned cavity, where N_e is the number of excited cells in the structure. It is important to investigate the effect of this tuning process on the field profile; in particular we will be interested in comparing the effect for the $\pi/2$ and π modes. It will also be of interest to compare the results with those obtained using the normal mode approach.

The effect of a tuner on the field profile will depend in general on the location of both the feed point and the tuner. For the purpose of the calculation to be made here, the simplest situation is to assume that both the feed point and tuner are located in an end cavity of the structure, although from the method of analysis it is easy to see that if the feed point and tuner are located in a cavity close to the center of the structure, the resulting perturbation in the field profile will be smaller. To begin with, however, let us consider the simpler situation as shown by the diagram in Fig. 4. Both the feed point and tuner are assumed to be located in the initial half-cavity (n=0). Assume also that the effect of the tuner is to lengthen the one-way phase shift through the cavity by an amount $\varphi/2$.

If the forward and reverse traveling waves are separately summed and then added, as in the earlier analysis, the total field obtained for the nth cavity is

$$ S = \frac{2A_0 [e^{j(\omega t - N\gamma\ell)}] \cos [(N-n)\,\gamma\ell]}{1 - e^{-j(2N\gamma\ell + \varphi)}} \quad . $$

This is the same result as obtained earlier in Section IV.1 except for the phase factor φ. By introducing the complex propagation constant $\gamma = \beta - j\alpha$ we again write the above expression as the product of an amplitude function at $n = N$ and a field profile function in the form $S = A_N F(n) e^{j\omega t}$. The field profile function turns out to be the same as that obtained before in Eq. (11), and the amplitude function is now given by

$$A_N = \frac{2A_0 \; e^{-\tau} \; e^{-jN\beta\ell}}{1 - e^{-2\tau} \; e^{-j(2N\beta\ell + \varphi)}} \; . \tag{33}$$

From the amplitude function it is seen that resonances occur (away from the passband edges) when $2N\beta\ell + \varphi = 2q\pi$, or

$$\beta\ell = q\pi/N - \Delta_0 \tag{34}$$

where $\Delta_0 = \varphi/2N$.

The basic assumption implicit in this model is that all of the cavities except the one with the tuner are tuned to precisely the same frequency. Thus at resonance a phase shift $-\varphi/2$ is distributed equally among all the other cavities, giving a phase shift $-\varphi/2N$ per cavity.

VI.2. Effect of Detuning for the Case of the $\pi/2$ Mode

The field profile for the $\pi/2$ mode at resonance is now calculated by letting $q = N/2$ in Eq. (34) and substituting the result in Eq. (11). Let us compute the effect first for the lossless case. The result is

$$F(n) = \cos \left[(\pi/2 - \Delta_0)(N-n) \right] \; .$$

Assuming that Δ_0 is small, we have for the fields at $n = 0$ and $n = 1$, using also the fact that N is even,

$$F(0) = \cos (N\Delta_0) \approx 1 - \frac{(N\Delta_0)^2}{2}$$

$$F(1) = \sin\,[(N-1)\Delta_0] \approx (N-1)\Delta_0 \ .$$

In these expressions we have ignored the sign of the field, which is given by $\cos(N\pi/2)$ and $\sin[(N-1)(\pi/2)]$ for the two cases. By differentiating Eq. (7) we can relate the phase shift Δ_0 to a frequency shift $\delta\omega$ using the fact that $\delta(\beta\ell) = -\Delta_0$,

$$\frac{\delta\omega}{\omega_0} = -\frac{k\Delta_0\ \sin\beta\ell}{2} \ . \tag{35}$$

For the $\pi/2$ mode we have $\Delta_0 = -2\,(\delta\omega/\omega)\,k^{-1}$ and the preceding expressions for the fields in the first and second cavities give, again ignoring signs and introducing $\delta F(0) = 1-F(0)$,

$$\delta F(0) \approx \frac{2N^2\,(\delta\omega/\omega)^2}{k^2} \tag{36}$$

$$F(1) \approx \frac{2(N-1)\,(\delta\omega/\omega)}{k} \tag{37}$$

It is interesting to consider these relations in terms of the mode spacing in the neighborhood of the $\pi/2$ mode, giving by Eq. (32) as $(\delta\omega/\omega)_{MS} = k\pi/2N$. Thus if the frequency shift is to be kept small compared to the mode spacing, then necessarily $\delta F(0) \ll F(1)$. But the same condition also imposes the limit $F(1) \ll 1$. The condition that the field in the unexcited cavities must remain small compared to the field in the excited cavities is equivalent to the condition that the change in frequency produced by detuning a cavity must be kept small compared to the mode spacing.

The principal result of detuning a cavity in a $\pi/2$ mode structure is therefore to introduce field in the nominally unexcited cavities. The biggest effect is in the cavity next to the detuned cavity, and the effect decreases linearly in going toward the end of the structure. From the method of analysis it is readily seen that the effect of a detuned cavity will be less if the cavity is located near the center of the structure. From symmetry considerations the effective length of the structure is then only $N/2$, so that Eq. (37) becomes, assuming large N ,

$$F(1) \approx \frac{N(\delta\omega/\omega)}{k} \tag{38}$$

Equation (38) differs by a factor $4/\pi$ from the result obtained by Smith[4] , who has applied first-order perturbation theory to the normal modes of the network shown in Fig. 1c.

It is easy to extend the calculation to the case of a lossy structure. The result is

$$F(0) \approx 1 - \frac{N^2 \Delta_0^2}{2} + \frac{N^2}{2k^2 Q^2} + j \frac{N^2 \Delta_0}{kQ}$$

$$F(1) \approx (N-1)\Delta_0 [1 + \frac{(N-1)^2}{2k^2 Q^2}] + j \frac{(N-1)}{kQ} [1 - \frac{(N-1)^2 \Delta_0^2}{2}]$$

If we again express Δ_0 in terms of $\delta\omega$, and assume $N \gg 1$, $N\Delta_0 \ll 1$,and $N/(kQ) \ll 1$, we obtain

$$F(0) \approx 1 - \frac{2N^2}{k^2} \left[(\frac{\delta\omega}{\omega})^2 - \frac{1}{4Q^2} + \frac{j}{Q} (\frac{\delta\omega}{\omega}) \right] \tag{39}$$

$$F(1) \approx \frac{2N}{k} \left[(\frac{\delta\omega}{\omega}) + \frac{j}{2Q} \right] \tag{40}$$

For $\delta\omega = 0$, these relations reduce to case of no detuning as given by Eqs. (19a) and (19b), while if Q approaches infinity we obtain the result of Eqs. (36) and (37) . The relative sizes of the terms in Eqs. (39) and (40) depend on whether the frequency shift involved is large or small compared to the bandwidth. Normally we will be concerned with the case in which $(\delta\omega/\omega) \gg 1/Q$. Under this condition the first term in the brackets of each expression is seen to be dominant, and the effect of loss can be neglected.

These results can be applied to calculate the perturbation produced when we attempt to tune a superconducting structure by means

of a tuner located in a single cavity. Assume a structure 3 m in length
operating at λ = 10.5 cm (N = 114). The structure can probably be
tuned at room temperature to an accuracy of perhaps one part in 10^5.
There may also be variations in relative contraction when different
structures are cooled to liquid helium temperatures. If these variations
are on the order of 0.3 % of the total frequency shift on cool-down,
a frequency uncertainty on the order of one part in 10^5 is again
indicated. To allow a margine of safety, chose a tuning range
$\delta\omega/\omega$ = 3 x 10^{-5}. Then from Eq. (38), assuming mid-structure tuning,
we calculate that F(1) $\approx$ 0.11 for k = 3 x 10^{-2}. Thus for this
example the loss in the nominally unexcited cavity is about 1 % of the
loss in an excited cavity. In addition there may be a multipactor
problem at this field level, especially for the case of a biperiodic
structure where the unexcited cavities are reduced in length. In
summary, the problem of tuning a long superconducting structure is a
serious one, and it may be difficult to accomplish with a single-cavity
tuner.

VI.3. Effect of Detuning for the Case of the π -Mode

We must investigate the dependence of the field profile
function on phase shift in the neighborhood of the π -mode resonance.
Substituting $\beta\ell$ = $\pi-\Delta$ in Eq. (11) for the case n = 0 , and
using $\alpha\ell \approx 1/kQ\Delta$, as obtained from Eq. (18), we have

$$F(0) = \cos(N\Delta)\cosh(N/kQ\Delta) -j \sin(N\Delta)\sinh(N/kQ\Delta) .$$

For convenience we introduce a new phase shift parameter x defined as

$$x = kQ\Delta^2 . \tag{41}$$

Thus if x = 1 , $\Delta = \Delta_\pi$ where Δ_π is the phase shift per cavity
calculated previously for the unperturbed π-mode resonance. Now
introduce the parameter   x in the preceding expression for F(0)
and then expand carefully including all terms to the order of
N^4/k^2Q^2 (assuming x is of the order of unity). We obtain

$$F(0) \approx \left[1 + \frac{N^2}{2kQ} \left(\frac{1}{x} - x \right) + \frac{N^4}{24k^2Q^2} \left(x^2 + \frac{1}{x^2} \right) - \frac{N^4}{4k^2Q^2} \right]$$

$$- j \left[\frac{N^2}{kQ} + \frac{N^4}{6k^2Q^2} \left(\frac{1}{x} - x \right) \right] .$$

For the case $x = 1$ this reduces, as it should, to Eq. (26) with $n = 0$. The field droop and phase shift are obtained from the absolute value and imaginary part of the preceding expression,

$$\delta F = \frac{N^2}{2kQ} \left(\frac{1}{x} - x \right) + \frac{N^4}{6k^2Q^2} \left(\frac{1}{x^2} + x^2 \right) \tag{42}$$

$$\Theta = \frac{N^2}{kQ} + \frac{N^4}{6k^2Q^2} \left(\frac{1}{x} - x \right) . \tag{43}$$

We must next relate the phase shift parameter x to a frequency shift $\delta\omega$ from the unperturbed π-mode resonance. If we introduce the phase shift parameter, Eq. (41), into the dispersion relation in the neighborhood of the π mode, as given by Eq. (29), we obtain

$$\frac{\delta\omega}{\omega} = \frac{\omega - \omega_\pi}{\omega_0} \approx \frac{1}{4Q} \left(\frac{1}{x} - x \right) .$$

Using this expression in Eqs. (42) and (43) we have,

$$\delta F = \frac{2N^2}{kQ} \left[Q \left(\frac{\delta\omega}{\omega} \right) + \frac{4N^2Q}{3k} \left(\frac{\delta\omega}{\omega} \right)^2 + \frac{N^2}{6kQ} \right] \tag{44}$$

$$\Theta = \frac{N^2}{kQ} \left[1 + \frac{2N^2}{3k} \left(\frac{\delta\omega}{\omega} \right) \right] . \tag{45}$$

For $\delta\omega = 0$, these expressions reduce to Eqs. (27) and (28) obtained previously for the unperturbed case. It is of interest to compare the terms in these relations under various limiting conditions. Let us assume that $\delta\omega$ is large compared to the bandwidth of the resonance, but small compared to the mode spacing. Recall also that we are assuming $\tau \ll 1$. Then from Eqs. (24) and (31) we have the following inequalities,

$$\delta\omega \gg (\delta\omega)_{BW} \quad : \quad Q(\delta\omega/\omega) \gg 1$$

$$\delta\omega \ll (\delta\omega)_{MS} \quad : \quad (N^2/k)(\delta\omega/\omega) \ll 1$$

$$\tau \ll 1 \quad : \quad N^2/kQ \ll 1$$

Using the second inequality, we see from Eq. (45) that the effect of detuning does not increase the phase shift along the structure appreciably over that already present because of loss. From Eq. (44) we see that the quadratic term in $\delta\omega$ is small compared to the linear term because of the first inequality above. The third term is small compared to the first term because of the first and third inequalities. Finally, the first term of Eq. (44) is large compared to the dominant term in Eq. (45). Thus when a detuned cavity is present, such that the detuning shifts the frequency more than a bandwidth, the perturbation of the field profile is a more significant effect than the phase shift due to finite attenuation. We have then for this case

$$\delta F = \frac{2N^2}{k} \left(\frac{\delta\omega}{\omega}\right)$$

as the principle result of the calculation. From the second inequality above we see also that $\delta F \ll 1$ is equivalent to the requirement that the detuning be kept small compared to the mode spacing. Recall that for the $\pi/2$ mode this same condition on allowable detuning resulted in $F(1) \ll 1$. From the normal mode view point, distortion of the field profile arises because of the mixing of normal modes coupled together by the perturbation. The effect is expected to be proportional to the ratio $(\delta\omega)/(\delta\omega)_{MS}$, with only a numerical constant in doubt.

For the case when the detuned cavity is close to the mid-point of the structure, N can be replaced in the preceding relation by N/2 to give

$$\delta F = \frac{N^2}{2k} \left(\frac{\delta\omega}{\omega}\right) \ .$$

(46)

This differs from the result obtained by Smith[4] by a factor $8/\pi^2$.

The important result given by Eq. (46) is that the perturbation in the field profile varies as the square of the number of cavities for the case of the π -mode resonance. This is in contrast to the case of the $\pi/2$ mode, where the dependence of the perturbation on N was linear. It is interesting to compare the two cases for a structure of given length. If N_e is the number of excited cavities (or the number of half-wavelengths) in the structure, then from Eqs. (38) and (46) we have

$$\frac{\delta F(\pi)}{\delta F(\pi/2)} = \frac{N_e}{4} = \frac{L}{2\lambda}$$

Thus for a structure with $N_e = 10$ (L = 5λ), the π mode is worse than the $\pi/2$ mode by only a factor of 2.5 . For relatively short structures then, the advantages of the $\pi/2$ mode are not overwhelming with regard to sensitivity to tuning errors.

As an application of Eq. (46), consider a structure 10 cavities long having $k = 3 \times 10^{-2}$. These are roughly the parameters chosen by Carne, et al.[10], for a π-mode superconducting deflecting structure under consideration at the Rutherford High Energy Laboratory. If the frequency is to be pulled by $\delta\omega/\omega = 3 \times 10^{-5}$ by a tuner located in a central cavity, the field droop will be about 5 %, which seems quite acceptable.

As another example, choose a structure with L = 3m, $k = 3 \times 10^{-2}$ and $\lambda = 10$ cm. The droop will be of the order of unity for $\delta\omega/\omega = 4 \times 10^{-6}$. Thus π-mode operation appears to be ruled out for structures having the length and frequency contemplated for a superconducting separator at CERN.

As a final example we compute the field droop expected for $N = 20$, $k = 10^{-2}$, and $\delta\omega/\omega = 10^{-6}$. The result is D = 2 % . This agrees with the result of a computer calculation carried out by

Smith[4], using perturbation theory, for the same parameters. We also note from Smith's calculation that the droop is indeed reduced by a factor of four if the tuner is moved from the end of the structure to the center.

VI.4.   Concluding Remarks on the Effect of Loss and Detuning

In the preceding analysis several loose ends have been left hanging. These are gathered up in the Appendices. In Appendix A the effect of finite coupling to an external generator is considered. In Appendix B the calculation of the effect of loss and detuning is repeated for the case of a $\pi/2$ mode structure terminated in full end cells. In this situation each cell in the structure is equally excited in the absence of loss or detuning. There is, however, no reduction in the sensitivity to the effect of a tuning error. The perturbation in the field of the cavity next to the detuned cavity is identical to that given by Eq. (38) for half end cell terminations. In Appendix C the effect of arbitrary tuner and feedpoint locations is considered. In Appendix D results are given for the effect of loss and detuning in the case of an arbitrary mode.

The simple traveling-wave analysis used in this report probably cannot be persued much further in the investigation of the effect of perturbations on the field profile. In particular, the case of a structure with random tuning errors in the individual cavities is undoubtedly better studied by a normal mode analysis such as that carried out by Smith[4]. In such an analysis the mathematics becomes quite complex however and it is necessary to use a computer to obtain detailed solutions.

VII. GROUP VELOCITY AND ENERGY VELOCITY AT THE π-MODE RESONANCE

VII.1. Calculation of v_g and v_E

The group velocity is given by definition as

$$v_g = d\omega/d\beta = -\ell(d\omega/d\Delta) \ .$$

In the neighborhood of the π-mode resonance Eq. (29) can be differentiated to obtain v_g . The result is

$$v_g \approx (\omega_0 k\ell\Delta/2)(1 + 1/k^2 Q^2 \Delta^4) \ .$$

At the π mode resonance $kQ\Delta^2 = 1$ and the preceding relation gives

$$v_g(\pi) = \omega_0 \ell (k/Q)^{\frac{1}{2}} \ . \tag{47}$$

Well away from the π-mode, Eq. (7) is valid and we obtain by differentiation,

$$v_g = (\omega_0 k\ell/2) \sin\beta\ell \ . \tag{48}$$

The group velocity at the $\pi/2$ mode is therefore

$$v_g(\pi/2) = \omega_0 k\ell/2 \ .$$

Another useful relation valid at the $\pi/2$ mode is

$$v_g/v_p = k\pi/4 \ ,$$

where v_p is the phase velocity. The ratio between v_g at the π and $\pi/2$ modes is

$$\frac{v_g(\pi)}{v_g(\pi/2)} = \frac{2}{(kQ)^{\frac{1}{2}}} \ .$$

The velocity for energy propagation is obtained from power flow considerations. The energy velocity is defined by $v_E = P/w$, where w is the stored energy per unit length and P is the power flow. The Q is defined as

$$Q = \frac{\omega w}{-(dP/dz)} = \frac{\omega w}{2\alpha P} \; .$$

The energy velocity then becomes $v_E = \omega/(2\alpha Q)$. Combining this expression with Eq. (8) for the case $\alpha \ell << 1$ we have

$$v_E = (\omega k \ell/2) \sin\beta\ell \; . \tag{49}$$

This is identical, in the limit of small k when $\omega \sim \omega_0$, to the group velocity as given by Eq. (48) for modes away from the ends of the passband. However, at the π-mode resonance $\sin\beta\ell \approx \Delta_\pi = (kQ)^{-\frac{1}{2}}$ and Eq. (49) becomes

$$v_E(\pi) = (\omega\ell/2)(k/Q)^{\frac{1}{2}} \; . \tag{50}$$

By comparing Eq. (50) and Eq. (47) we see that there is a factor of two difference between v_g and v_E at the π-mode resonance. In the next section we will show that we should not necessarily expect agreement between v_g and v_E for this mode.

VII.2. Breakdown in the Meaning of v_g near the π-Mode Resonance

For the case of a low-loss, dispersionless structure, $d\omega/d\beta$ defines the velocity of propagation of a wave packet in an unambiguous manner. It is then reasonable to identify the energy transport velocity, v_E , with the group velocity. However, if the structure has an attenuation which is varying rapidly as a function of the propagation constant, that is if $d\alpha/d\beta$ is sufficiently large, then the waves composing the packet will be attenuated at significantly different rates and the packet will distort rapidly. Likewise, if the group velocity is varying rapidly as a function of propagation constant, that is, if $d^2\omega/d\beta^2$ is sufficiently large, then waves within the packet will be traveling at significantly different velocities and again the packet will undergo rapid distortion. In either case, it is no longer clear how to

associate energy propagation with the motion of the packet.

Let us apply the first criterion for distortion, namely large $d\alpha/d\beta$, to the case of propagation at the π-mode resonance. Specifically, we expect a breakdown in the identity of v_g and v_E when the component of the packet with propagation constant $\beta + \delta\beta$ is attenuated by a factor e with respect to the component with propagation constant β in moving through a distance δL, where $\delta\beta$ is the range of wave numbers in the packet and δL is the spatial width of the packet. If α is the attenuation parameter at propagation constant β , and if $\alpha + \delta\alpha$ is the attenuation parameter at $\beta + \delta\beta$, then the ratio of attenuation for the two components in moving through a length δL is

$$e^{(\alpha+\delta\alpha)(\delta L)} / e^{\alpha(\delta L)} = e^{(\delta\alpha)(\delta L)} .$$

The condition for which the packet distorts badly in moving through its own width is then

$$(\delta\alpha)(\delta L) \approx 1 . \tag{51}$$

For a wave packet with spatial extent δL , transform theory states that the packet must be composed of waves with a range of wave numbers $\delta\beta$ such that

$$(\delta\beta)(\delta L) \approx 1 . \tag{52}$$

By combining Eqs. (51) and (52) we obtain as the condition for breakdown that

$$\delta\alpha/\delta\beta \approx 1 . \tag{53}$$

We can approximate this condition by $d\alpha/d\beta$ and apply it to the case of the π-mode. From Eq. (21) we have $\alpha\ell = (kQ\Delta)^{-1}$ and

$$d\alpha/d\beta = -d(\alpha\ell)/d\Delta = (kQ\Delta^2)^{-1} .$$

However, the condition for the π-mode resonance is just that $kQ\Delta^2 = 1$, and therefore $d\alpha/d\beta = 1$.

Thus the breakdown condition as given by Eq. (53) is fulfilled at the π-mode resonance. In the neighborhood of this resonance a wave packet diffuses as rapidly as it propagates, and the concept of group velocity as given by $d\omega/d\beta$ is meaningless. The factor of two difference between v_g and v_E for this case should not therefore be cause for concern.

ACKNOWLEDGEMENTS

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       Report No. BNL 50155 (C-55), p. 165.

APPENDIX A

The Effect of Coupling to an External Generator

In Fig. 5 the traveling wave analysis is extended to the case of a structure coupled to an external generator. We assume that the coupling network is lossless, and that it reflects a fraction s of a unit incident wave. Planes A and C represent reference planes in the input transmission line and in the structure itself. Reference plane A is chosen such that the reflected wave is exactly 180° out of phase with the incident wave (i.e., it is at a voltage minimum). Reference plane C is chosen such that there is exactly a 180° phase shift between the incident wave at A and the transmitted wave at C. Plane C is also identical to the plane $z = 0$ in the previous analysis (see Fig. 3). As far as the present analysis is concerned, the coupling network can be a "black box" as long as these conditions defining the reference planes are met. The situation can, however, be described in more physical terms by considering the specific case of a small lossless inductive aperture located at plane B . A vector diagram showing the phase relations between the incident wave, E_i , the reflected wave, E_r , and the transmitted wave, E_t is given in Fig. 5 for this case.

In calculating phase changes using this diagram the convention is used that motion, at a fixed time in the same direction as a wave is traveling, is represented by a clockwise rotation of the phase vector through the appropriate phase angle. Thus the relative phase shift between the incident and reflected wave in going from A to B and back to A is $2\pi-\Theta$, which is seen from the vector diagram to be just the amount required to make E_r lie at an angle of π with respect to E_i . The asymmetry in the location of plane B is due to the fact that we are considering the case of a wave with maximum magnetic field at A and maximum electric field at C . Stated another way, there is a π phase shift in the phase of the electric field for a wave reflected at plane A , but no phase shift upon reflection at plane C .

Consider first the reflection coefficient, Γ , in the input transmission line. The waves at plane A traveling back toward the generator are readily summed to obtain

$$\Gamma = \frac{(1-s^2)\ e^{-j2N\gamma\ell}}{1-s\ e^{-j2N\gamma\ell}} - s \quad .$$

Let us examine this expression in the neighborhood of the qth resonance by letting $\beta\ell = q\pi-\Delta$. Under the assumptions also that $\tau \ll 1$ and $s \approx 1$, the preceding expression becomes

$$\Gamma = \frac{1-s-2\tau+j2N\Delta}{1-s+2\tau-j2N\Delta} = \frac{Z-1}{Z+1} \qquad (A.1)$$

where we have also introduced the definition of the normalized impedance, Z , terminating the transmission line. The load impedance at plane A is given therefore by

$$Z = \frac{1-s}{2\tau-j2N\Delta} \quad . \qquad (A.2)$$

In Sec. IV. it was shown that the phase shift Δ can be related (away from the band edges) to a frequency difference $\delta\omega$ by

$$N\Delta = -2\tau Q_0(\delta\omega/\omega) \quad . \qquad (A.3)$$

Let us also define a coupling coefficient by

$$\beta_c = (1-s)/2\tau \quad . \qquad (A.4)$$

Introducing Eqs. (A.3) and (A.4) in (A.2) we obtain

$$Z = \frac{\beta_c}{1+j2Q_0(\delta\omega/\omega)} \quad .$$

This is seen to be the standard expression for the impedance of a resonant cavity terminating a transmission line. From Eqs. (A.1) and (A.4) we see that if the cavity is critically coupled, that is if $\beta_c = 1$, then the reflected wave is zero at resonance.

The field at any point within the structure is also obtained by adding together the sums of the forward and reverse waves. The result can again be written in the form $S = A_N F(n) \ e^{j\omega t}$. The field profile function is the same as that previous obtained in Eq. (11), while the amplitude function for the present case becomes

$$A_N = \frac{2(1-s^2)^{\frac{1}{2}} \ e^{-\tau} \ e^{-N\beta \ell}}{1-s \ e^{-2\tau} \ e^{-j2N\beta \ell}} \ .$$

Using again the approximations $\tau \ll 1$ and $s \approx 1$, this reduces to

$$A_N = \frac{2\sqrt{2} \ (1-s)^{\frac{1}{2}}}{1-s+2\tau-j2N\Delta} \ .$$

Introducing Eqs. (A.3) and (A.4) in this expression we have

$$A_N = \frac{1}{\tau^{\frac{1}{2}}} \cdot \frac{2\beta_c^{\frac{1}{2}}}{1+\beta_c} \cdot \frac{1}{1+j2Q_L(\delta\omega/\omega)} \qquad (A.5)$$

where the loaded Q is defined as $Q_L = Q_0/(1+\beta_c)$. This relation shows that the amplitude of the field has the expected variation as a function of frequency near resonance, and the usual dependance on cavity coupling coefficient. We have also shown explicitly in Eq. (A.4) how β_c depends on the reflection coefficient of the input iris and on the attenuation along the structure. We see also that the field is enhanced by a factor $1/\tau^{\frac{1}{2}}$ by resonance. The amplitude of synchronous wave, S^+ is only one-half of the amplitude of A_N .

$$F(n) = -\sin(N-n)\pi \equiv 0 \,.$$

The field is identically zero in each cavity, and there is therefore no π mode for a structure with full end cells. There are not $N+2$ resonances, as we would expect from Eq. (B.1), but only $N+1$ resonances. This is also just equal to the total number of cells in the structure.

Consider next Eq. (B.2) for the $\pi/2$ mode in the limit of zero loss. We have

$$F(n) = (1/\sqrt{2}) \{\cos[(N-n)\pi/2]-\sin[(N-n)\pi/2]\} \,.$$

The excitation of the cavities is seen to be uniform along the structure, varying as $+1$, $+1$, -1, -1, etc. and there are no unexcited cells. The field profile appears, in fact, to be identical with that of the π-mode.

Next let us consider the case of the $\pi/2$ mode when loss is present. From Eq. (B.2) we obtain, again using $\alpha \approx 1/kQ$ and the fact that N must be odd to have a $\pi/2$ mode,

$$F(0)= \pm \; (1/\sqrt{2})[1+ \frac{1}{2} \; (\frac{N+1/2}{kQ})^2 -j(\frac{N+1/2}{kQ})]$$

$$F(1)= \pm \; (1/\sqrt{2})[1+ \frac{1}{2} \; (\frac{N-1/2}{kQ})^2 +j(\frac{N-1/2}{kQ})]$$

We see that the field droop is a second order effect compared to a phase shift along the structure. This is similar to the case of the π-mode in a structure terminated in half end cells, except in the present example the phase shift is linear in n. It is, in fact, of interest to compare power transfer for the two modes in more detail. It can be shown that the power transfer between the nth and $(n+1)$th cavities is proportional to $\text{Im}\{F(n)F^*(n+1)\}$, where the asterisk denotes the complex conjugate. In the present case we have that $F(0) \sim 1-j\delta_0$ and $F(1) \sim 1+j\delta_1$, and the power transfer is therefore $P_{01} \sim (\delta_0+\delta_1) \sim 2N/kQ$. A similar calculation for the case of the π-mode shows that $P_{01} \sim (\delta_0-\delta_1)$. Thus power transfer is only proprotional to the difference between the imaginary components while for the $\pi/2$ mode it is seen to be proportional to the sum. The fact that for the π-mode the power transfer is only proportional to the difference

APPENDIX   B

Analysis of a Structure Terminated in Full End Cells

In Fig. 6 a diagram is given showing a traveling wave
analysis for the case of a structure terminated in full end cells. By
comparing this diagram with Fig. 3, we see that the expressions obtained
previously for the case of termination in $h_a$lf end cells will be valid
if the following substitutions are made :

$$n \rightarrow n + 1/2$$
$$N \rightarrow N + 1$$
$$N-n \rightarrow N-n + 1/2$$

The amplitude function becomes, upon making these substitutions in
Eq. (10) ,

$$A_N = \frac{2A_0\ e^{-\tau}e^{-j(N+1)\beta\ell}}{1-\ e^{-2\tau}\ e^{-j2(N+1)\beta\ell}}\ .$$

Resonances are seen to occur when

$$\beta\ell = q\pi/(N+1)\ . \tag{B.1}$$

The field profile function is given as $F(n) = \cos\gamma\ell(N-n+1/2)$,
which can be expanded as

$$F(n) = \{\cos[\beta\ell(N-n)]\cos[\beta\ell/2)-\sin[\beta\ell(N-n)]\sin(\beta\ell/2)\}\cosh[\alpha\ell(N-n+\tfrac{1}{2})]$$

$$+j\{\sin[\beta\ell(N-n)]\cos(\beta\ell/2)+\cos[\beta\ell(N-n)]\sin(\beta\ell/2)\}\sinh[\alpha\ell(N-n+\tfrac{1}{2})\}\ . \tag{B.2}$$

Let us first examine this expression in the limit of zero loss
for the π and $\pi/2$ modes. Setting $\beta\ell = \pi$ we find

between the phase shifts in neighboring cells leads directly to the n^2 variation in phase shift. Although the field profile for the $\pi/2$ mode with full end cells is apparently identical to the π-mode field profile, the way in which power transfer occurs is seen to be quite different.

The effect of detuning in the case of a structure with full end cavities can be calculated by substituting $\beta\ell = (\pi/2)-\Delta$ in Eq. (B.2). The field in the first cavity for $N\alpha\ell \ll 1$ and $N\Delta \ll 1$ is

$$F(0) \simeq (1/\sqrt{2})(1-N\Delta-jN/kQ) \ . \tag{B.3}$$

We again express the phase shift in terms of a frequency difference at the $\pi/2$ mode using Eq. (35) to obtain,

$$\delta\omega/\omega = -k\Delta/2 \ .$$

We see also that if $Q(\delta\omega/\omega) = kQ\Delta/2$ is large, that is, if the detuning is large compared to the bandwidth, then $\Delta \gg 1/kQ$ and the imaginary part of Eq. (B.3) can be neglected. Thus we have

$$\delta F = N\Delta = \frac{2N}{k}\left(\frac{\delta\omega}{\omega}\right) \tag{B.4}$$

for the unflattening of the field resulting from a detuned end cavity. Equation (B.4) can be compared to Eq. (37) for the case of a structure terminated in half cells. The magnitude of the field error is the same, but in the present case it exists as a linear slope on the excitation of all the cells along the structure. In the former case (half end cells) it will be recalled that the result was a linearly rising excitation of the nominally unexcited cells of the structure.

APPENDIX   C

The Effect of Detuning and Loss for an Arbitrary Location of Feed
Point and Tuner

In Sec. VI.1 the calculation concerning the effect of
detuning was carried out for the case when both the tuner and the
feedpoint are located in an end cavity. It was then argued, from
a consideration of symmetry, that if both the tuner and feedpoint
are moved to the center cavity, it is only necessary to replace N
by N/2 in each expression.

The case in which tuner and feed point are located at
opposite ends of the structure is also easy to calculate. If we
use the method outlined in Fig. 4 but with the phase error $\varphi/2$
located in the Nth cavity, the result for the field profile function
is

$$F(n) = \cos[\beta\ell(N-n)+2N\Delta]\cosh[\alpha\ell(N-n)]$$

$$+j \sin[\beta\ell(N-n)+2N\Delta]\sinh[\alpha\ell(N-n)] \qquad (C.1)$$

where $\Delta = \varphi/2N$. For the $\pi/2$ mode in the limit of zero loss
this becomes

$$F(n) = \cos[(\pi/2)(N-n)]+n\Delta \sin[(\pi/2)(N-n)] . \qquad (C.2)$$

This can be compared with the result obtained with a tuner located
at n = 0 (see Sec. VI.2),

$$F(n) = \cos[(\pi/2)(N-n)]+(N-n)\Delta \sin[(\pi/2)(N-n)] . \qquad (C.3)$$

By comparing Eqs. (C.2) and (C.3) we see that n is replaced by
N-n in the error term of the second expression. The effect in both
cases is to introduce a linearly rising field in the unexcited cavities,
but with opposite slope. That is, the field error is maximum in the
unexcited cavity nearest the tuner, and falls linearly to zero in

moving toward the opposite end of the structure. From Eq. (C.¹) we see also that if the detuning error is zero, the field profile function reverts to that given by Eq. (11), and the field profile is then determined only by the location of the feedpoint. As discussed previously, the effects of loss and detuning on the field profile are comparable when $(\delta\omega/\omega) \approx 1/2Q$. For a detuning which is small compared to the bandwidth, the location of the feedpoint plays the dominant role. If the detuning is large compared to the bandwidth, only the location of the tuner is important.

We have calculated the results for tuner and feedpoint locations which are either at the center, at the same end, or at opposite ends of the structure. The results for an intermediate case can be estimated from the limiting cases discussed here.

APPENDIX   D

Effect of Loss and Detuning for other Modes, and Summary of Results

   The various relations which express the effect of loss
and detuning on the field profile are scattered throughout the
text. It is useful to collect there relations together so that
comparisons may be more readily made. In addition, the analysis
has been carried out only for the $\pi/2$ and π -modes. Other modes
may also be of interest. The results for the general case are
easily obtained by using

$$\alpha\ell \;=\; \frac{1}{kQ\,\sin\beta\ell}$$

$$\Lambda \;=\; -\,\frac{2(\delta\omega/\omega)}{k\,\sin\beta\ell}$$

in Eq. (17) for the field profile. A summary of results for the
general case, for the π mode, and for the $\pi/2$ mode with both
boundary conditions is given in Table 1. Since we are normally
interested in a given length of structure, rather than in the number
of cavities, the results have been written explicitly in terms of L.
All of the expressions given are valid for the case when both the tuner
and the feedpoint are located in a central cavity. If an end cavity
is used for either feeding or tuning, then L must be replaced by
2L. In addition, the relations are valid in the limit when N is
large. In the expressions listed for the general case, a parameter
N_1 has been introduced, where $N_1 = 2\pi/\beta\ell$ is the number of cavities
per wavelength. Recall also that $\delta F(0)$ gives the relative change in
the field in the cavity containing the tuner or feedpoint, while
$\delta F(1)$ refers to the field in the adjacent nominally unexcited cavity
for the case of the $\pi/2$ mode structure terminated in half end cells.

TABLE I

Summary of Results on the Effect of Loss and Detuning

Losses dominant $2Q\ (\delta\omega/\omega) \ll 1$	Detuning Dominant $2Q\ (\delta\omega/\omega) \gg 1$
$\pi/2$ mode with half end cells $\delta F(0) = 2(L/\lambda)^2(1/kQ)^2$ $\delta F(1) = 2(L/\lambda)(1/kQ)$	$\delta F(0) = 8(L/\lambda)^2(1/k)^2(\delta\omega/\omega)^2$ $\delta F(1) = 4(L/\lambda\ )(1/k)(\delta\omega/\omega)$
$\pi/2$ mode with full end cells $\delta F(0) = 2(L/\lambda)^2(1/kQ)^2$ $\Theta\ (0) = 2(L/\lambda)(1/kQ)$	$\delta F(0)\ = 4(L/\lambda)(1/k)(\delta\omega/\omega)$ $\Theta\ (0)\ = 0$
π mode $\delta F(0) = 1/3\ (L/\lambda)^4(1/kQ)^2$ $\Theta\ (0) = (L/\lambda)^2(1/kQ)$	$\delta F(0)\ = 2(L/\lambda)^2(1/k)(\delta\omega/\omega)$ $\Theta\ (0)\ = 0$
General case $\left.\begin{array}{c}\delta F(\text{max})\\ \Theta(\text{max})\end{array}\right\} \approx \dfrac{(N_1/2)(L/\lambda)}{kQ\ \sin\beta\ell}$ $N_1 \quad = 2\pi/\beta\ell$	$\delta F(\text{max}) \approx \dfrac{N_1\ (L/\lambda)(\delta\omega/\omega)}{k\ \sin\ \beta\ell}$

Fig. 1a :   Transmission line periodically loaded with shunt susceptances.

Modal line
No. 1
(γ_1, Y_{01})

Modal line No.2
(γ_2, Y_{02})

$$\bar{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$
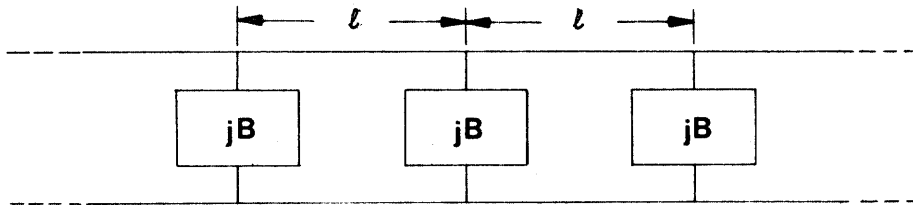
Fig. 1b :   Two modal lines periodically coupled by shunt four-terminal networks.

Fig. 1c :   Periodically loaded line represented by a chain of coupled resonant circuits.

$\underline{\text{Fig. 2a}}$: Dispersion diagram for a chain of coupled resonant circuits with loss.

$\underline{\text{Fig. 2b}}$: Attenuation per period for a chain of coupled resonant circuits.

$n =$ | 0 | 1 | 2 | ... | N-2 | N-1 | N | $\longrightarrow z$

$|\leftarrow \ell \rightarrow|$

$A_0 \; e^{j\omega t}$

$\longrightarrow \quad A_0 \; e^{j(\omega t - n\gamma\ell)}$

$\longleftarrow \quad A_0 \; e^{j(\omega t + n\gamma\ell - 2N\gamma\ell)}$

$\longrightarrow \quad A_0 \; e^{j(\omega t - n\gamma\ell - 2N\gamma\ell)}$

$\longleftarrow \quad A_0 \; e^{j(\omega t + n\gamma\ell - 4N\gamma\ell)}$

Fig. 3 : Diagram showing history of a traveling wave
launched at $z = 0$ in a shorted section of
periodic structure.

$$\mathrm{n} = \boxed{0 \quad 1 \quad 2 \quad \cdots \quad N-2 \quad N-1 \quad N}$$

$\varphi/2 \rightarrow$

$|\leftarrow \ell \rightarrow|$

$A_0 \ e^{j\omega t}$

$\longrightarrow \quad A_0 \ e^{j(\omega t - n\gamma\ell)}$

$\longleftarrow \quad A_0 \ e^{j(\omega t + n\gamma\ell - 2N\gamma\ell)}$

$\longrightarrow \quad A_0 \ e^{j(\omega t - n\gamma\ell - 2N\gamma\ell - \varphi)}$

$\longleftarrow \quad A_0 \ e^{j(\omega t + n\gamma\ell - 4N\gamma\ell - \varphi)}$

Fig. 4 : Diagram showing history of a traveling wave launched at $z = 0$ in a shorted section of periodic structure in which the first cavity is detuned.

Coupling network

A   **B**   **C**   **D**

Input trans-
mission line   Periodic structure

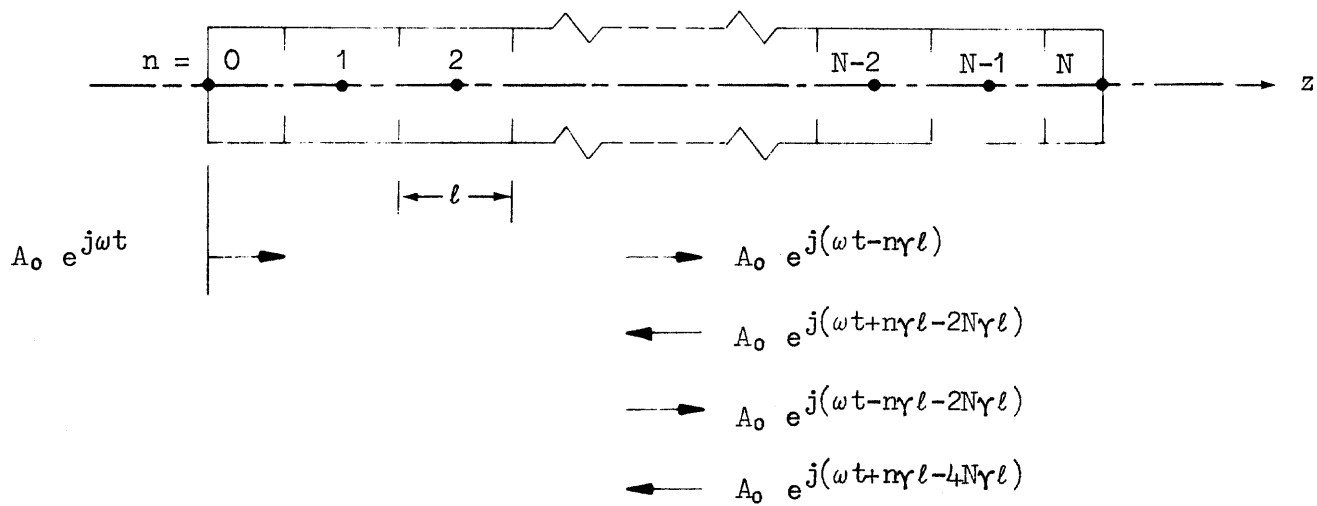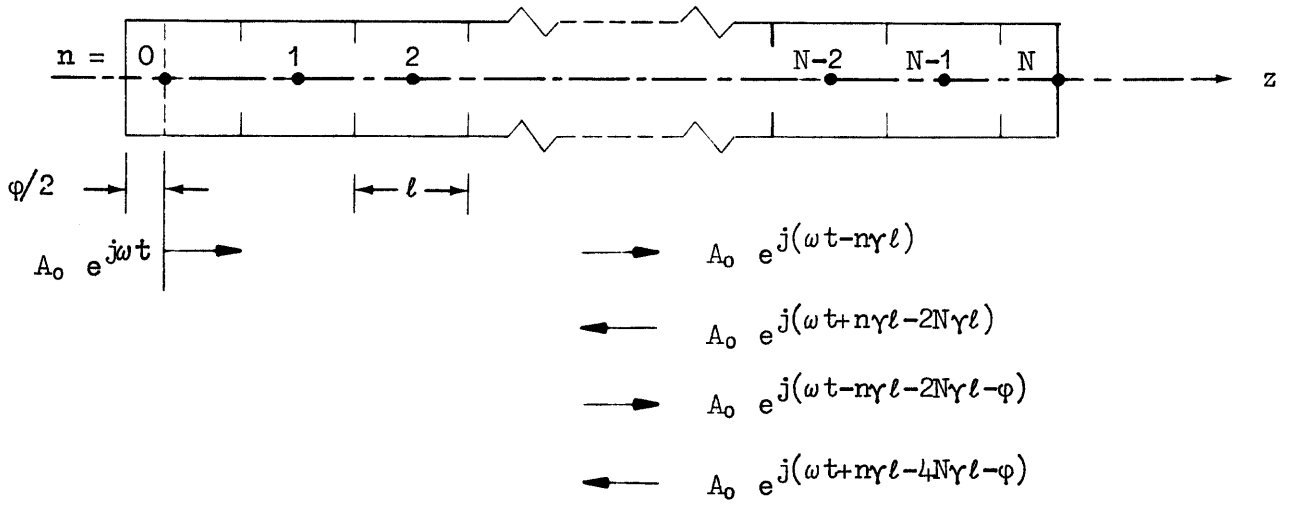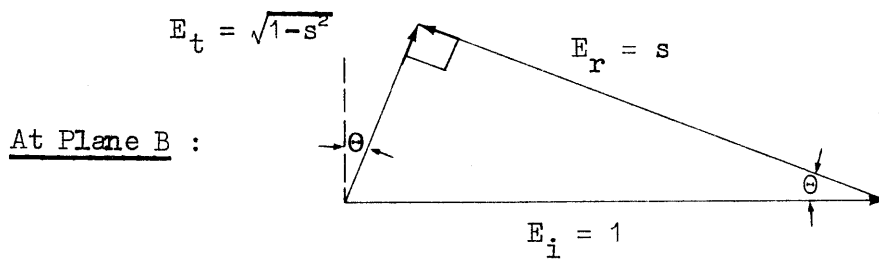$\left(\pi - \dfrac{\Theta}{2}\right)$ $\left(\dfrac{\pi}{2} - \dfrac{\Theta}{2}\right)$ Shorting plane

$1 \rightarrow$ $\rightarrow$ $-\sqrt{1-s^2}$ $\rightarrow$ $-\sqrt{1-s^2}\ e^{-j(n\gamma\ell)}$

$-s \leftarrow$

$(1-s^2)\ e^{-j(2N\gamma\ell)} \leftarrow$ $\leftarrow -\sqrt{1-s^2}\ e^{-j(2N\gamma\ell)} \leftarrow -\sqrt{1-s^2}\ e^{-j(2N\gamma\ell - n\gamma\ell)}$

$\rightarrow -s\sqrt{1-s^2}\ e^{-j(2N\gamma\ell)} \rightarrow -s\sqrt{1-s^2}\ e^{-j(2N\gamma\ell + n\gamma\ell)}$

$s(1-s^2)\ e^{-j(4N\gamma\ell)} \leftarrow$ $\leftarrow -s\sqrt{1-s^2}\ e^{-j(4N\gamma\ell)} \leftarrow -s\sqrt{1-s^2}\ e^{-j(4N\gamma\ell - n\gamma\ell)}$

$E_t = \sqrt{1-s^2}$ $E_r = s$

At Plane B :

Θ

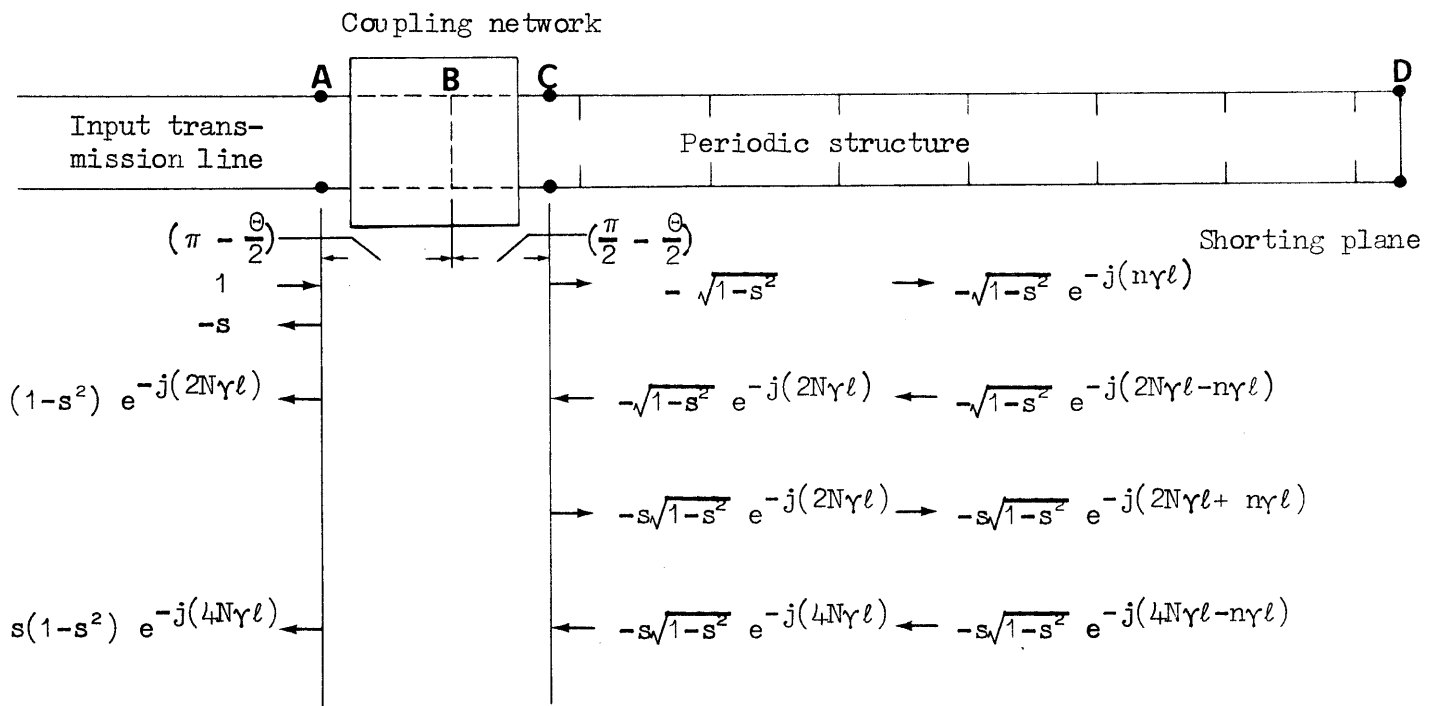$E_i = 1$ Θ

Fig. 5 :   Diagram showing the vector addition of traveling wave
components for a structure with coupling to an external
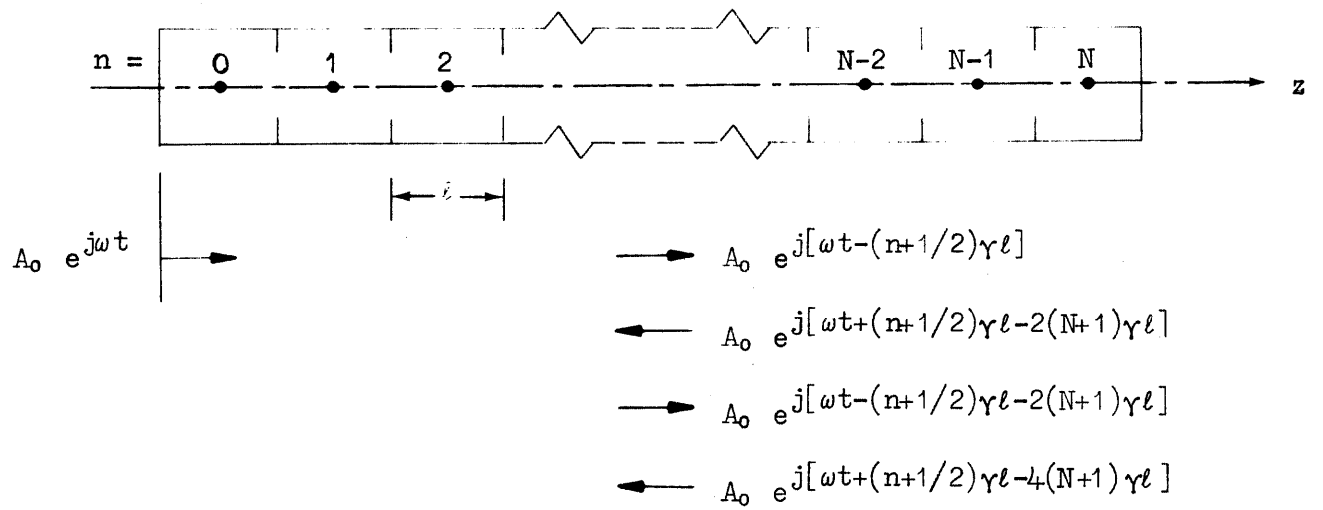generator. The factor $e^{j\omega t}$ has been suppressed.

$$n = \quad 0 \quad 1 \quad 2 \quad \cdots \quad N-2 \quad N-1 \quad N \qquad z$$

$$A_0 \, e^{j\omega t}$$

$$\longrightarrow \quad A_0 \, e^{j[\omega t - (n+1/2)\gamma \ell]}$$

$$\longleftarrow \quad A_0 \, e^{j[\omega t + (n+1/2)\gamma \ell - 2(N+1)\gamma \ell]}$$

$$\longrightarrow \quad A_0 \, e^{j[\omega t - (n+1/2)\gamma \ell - 2(N+1)\gamma \ell]}$$

$$\longleftarrow \quad A_0 \, e^{j[\omega t + (n+1/2)\gamma \ell - 4(N+1)\gamma \ell]}$$

Fig. 6 : Diagram showing history of a traveling wave
launched at $z = -\ell/2$ in a shorted section of
periodic structure terminated in full end cells.

ABSTRACT OF REPORT

A TRAVELING-WAVE ANALYSIS OF STANDING-WAVE

SUPERCONDUCTING STRUCTURES

by

P. Wilson [*])

(Ref.: CERN/MPS/MU-EP 69-3, 21.7.69)

Summary: The properties of resonant, periodic structures are studied using an equivalent circuit representation. A circuit consisting of a chain of coupled resonators is chosen. The complex propagation constant for an infinite chain of resonators is derived, and the field in a finite length of structure is then obtained by summing a traveling wave which undergoes multiple reflections between the planes forming the end boundaries of the structure.

The analysis is applied to study the effect of loss and detuning in high Q superconducting structures operating in the $\pi/2$ and π modes. It is shown that, even for the π mode, the effect of loss on the amplitude and phase of the field is in general negligible for the superconducting case. The effect of a badly detuned cavity is more serious. A principle conclusion is that, using a single cavity tuner located near the center of a structure, it is difficult to achieve a reasonable tuning range in a π-mode structure which is more than about 5 wavelengths long, or in a $\pi/2$-mode structure which is more than about 25 wavelengths long, without badly perturbing the field flatness.

The properties of the π-mode resonance are discussed in some detail. The somewhat complex behavior of this mode results from the fact that dispersion diagram and the attenuation as a function of phase shift per period change rapidly near the band edges. The group velocity and energy velocity are found to differ by a factor of two at the π-mode resonance. It is shown, however, that the group velocity as defined by $d\omega/d\beta$ becomes meaningless at this mode.

The principle expressions obtained in this report concerning the effect of loss and detuning on field flatness are, for convenience, collected together in Table I, Appendix D.

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[*]) On leave from the High Energy Physics Laboratory, Stanford University, Stanford, California, U.S.A.

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