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A COLLECTION OF ACCELERATOR ALGORITHMS FOR THE PERSONAL COMPUTER

PART I
(Fodo, linear lens, magnetic horn, etc)

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INTRODUCTION

A collection of Accelerator Algorithms for the Personal Computer

This file gives an introduction to the package of programs "Accelerator Algorithms for the Personal Computer". It gives also the contents of the chapters, and sums up the different ways to use the programs .

The Personal Computers (PC's) have made a very fast progress in the last years. The ability to perform large computations accurately is now present in small and cheap computers. In the same time a very great number of software applications have been developed in all fields, particularly for scientific and technical works.
Beside high-level languages used by programmers for all purposes, dedicated software applications are now available to perform efficiently scientific and technical calculations. By means of these most recent programs, it is now possible to replace the common scenario which was: programming, debugging, computation, presentation of results, reporting and editing, by only one task taken over by only one integrated software. These software applications now available on the market present a large choice of mathematical, graphical and statistical routines, and have replaced the arcane syntax of so-called high-level languages by real math notation. The possibility to help repetitive floating point computations by means of symbolic tools begins to be used.
It is impossible or inefficient to use the numerous and powerful codes developed on and for mainframe computers, on PC's and it is likely that these programs will continue to be used in the same way by the accustomed experts. The rare programs which have been ported from the mainframe to the PC don't show the common easiness of PC programs.

In the following a collection of algorithms (i.e. methods of computation) developed on and for PC's is presented to make various calculations, in several fields of particle accelerators physics, with a very good accuracy. Though it might be possible to describe the proposed algorithms in a way completely independant of any mathematical software or language, it has been decided to write them with Mathcad[R] in order to show that: they run as they are, provide accurate results, and can be exploited (curves, tables, files, reports, etc) in various ways. As they are written in real math notation, as transparently as possible, an eventual user who is not willing to change his or her software can easily rewrite them in any other language , or in any other dedicated math programs.

NOTA BENE: These programs are given as examples, to be modified if necessary by the user for his or her own problem...

BRIEF CONTENTS

PART 1:
Thick lens FODO
Beam distribution visualizations

Linear axisymmetric magnetic lens optics
Magnetic biconical horn shape
Magnetic horn skew trajectories
System of ordinary first order differential equtions

PART 2:
Production target optics (pbars, positrons)
Electron linac modes of acceleration
Solenoid fields
Converter skew trajectories
Multiple Coulomb Scattering

PART 3:
Magnet cost optimization
Superconducting magnet shapes
Design of kicker and septum magnets for injection and ejection
Synchrotron radiation fields
2D Laplace equation by finite differences

USAGE

For a reader of the written version:
Each part is made of independant chapters. Each chapter is made of algorithms treating close subjects in the same field, and their DOS names are similar, differing only by their last character (a number). Each chapter has its own introduction where some comments are given and advices specific to the chapter are reminded. Each algorithm is separated in two spaces: first one part for the purpose and input data of the program , and a second part (after a line of stars*********) where are the equations, the results, the curves, etc. The units have been omitted for the sake of brevity; they are those of the Systeme International (SI i.e. MKSA...), except when it is otherwise stated.

For a CERN user having an access to the PC server the previous comments stay the same. The access to the collection of algorithms is the following:
1) In the Windows program manager click twice onto the MathCad icone.
2) Open the File menu
3) Click onto the Open Document command.
4) Open the right drive and/or directory where is the collection of algorithms:
G:\home\s\schnurig\aapc
5) Open first the introduction file (FODO for the Fodo chapter...) and read it.
6) Choose a file, MathCad being in automatic mode by default, the program is executed by going down into the file (e.g. with the mouse clicking on the right-hand scroll bar). For the impatient browser or for anyone who is shocked by the density of equations it is possible to go directly to the results by CTRL END together. It is possible to change the input data in the first part of the program, before the line of stars. To change the assignements of the variables in the second part of the program could give erroneous results, but curves, tables of results can be added at will, very simply in that region.

The minimum hardware configuration is the same as required by Windows^R, but an arithmetic coprocessor is recommended, and four Mb RAM is needed. The programs will run at best on a 486 PC with 8 Mb RAM or more.

For a stand-alone user the previous remarks are the same. The majority of the programs can be run on smaller computers with the DOS version of MathCad^R, but as the Windows^R version of the programs cannot work on DOS, they have to be rewritten or can be provided upon request.

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FODO

Introduction

These files give the essential formulae to develop the algorithms for a periodic FODO cell in a accelerator ring or transfer line according to the known linear theory. It takes the exact lengthes of the various elements: bending magnets, quadrupoles and straight sections.

The calculation is made by means of products of 3x3 matrices obtained by passing the relevant parameters to a General-Purpose-Matrix. For instance $k>0$ gives the matrix of a focusing quadrupole whilst $k<0$ gives the matrix of a defocusing quadrupole. Another matrix built of 2 orthogonal trajectories plus the dispersion, multiplied by the relevant transfer matrix allows an automatic calculation of the amplitude function β, the phase advance μ, the dispersion D and their derivatives. The various integrals of the theory can then be calculated as for instance the chromaticity.

The accuracy of the results is only dependant of the computer used for the calculations and generally is of the order of 15 digits by computing with 8 bytes, but several optimization and solving algorithms may eventually give less accuracy. A special effort has been made to preserve the best obtainable precision by going back to the initial matrices whenever possible instead of using functions or integration algorithms where rounding-off introduces some inaccuracy.

The notations are the common ones except those which are otherwise stated.
Each file is split in two parts by a line made of *. The first part is where the input data are collected for a chosen example, but these data can be changed by the user to solve another problem : as indicated below some programs use data from another file which has to be run before. The second part is for the formulae and the results, eventually also for the curves of the example. There the formatting of results can be changed, curves can be added, but to change the assignments of the input data there would give erroneous results!

It is useful to add some remarks about the matrices: a name can be a scalar, a one-column vector or a 3x3 matrix: e.g. k is a scalar and kk is a vector needed for the same physical quantity. A name with a superscript is a column taken from a matrix, whilst a superscript on a transposed matrix (marked with a T) is a line of that matrix. A subscript applied to a vector gives the element. The function lengfh() gives the number of elements of a vector. The `vectorize` operator marked by an arrow above an operator or a function of a vector or a matrix apply the operation to all their elements.

Usage: these files give the algorithms i.e. the numerical methods to compute accurately a FODO periodic cell in an accelerator ring or in a transfer line. An example is completely treated to show that these algorithms are self-consistent and give very accurate results. An eventual user should first read them, run them on a Personal Computer (PC) if he has an access to the CERN server and modify them to solve his own problem. He can also run them on his PC if he has MathCad and the diskette with the programs or write the corresponding programs in the language of his choice.

To study a new lattice it is advised to run first a thin lens model FODO1, then put the lengthes of the elements in FODO3, and run the solving routine by introducing the thin lens model results as a first guess to find the main parameters needed in the further calculation.

N.B. ALL UNITS IN SI except otherwise stated.

FODO: this file.
FODO1: Thin lenses --> k, βmax, βmin, Dmax, Dmin, Φ per cell, chromaticity.
FODO2: Thick lenses, no bending magnets, k --> βh max, βv min.
FODO3: with bending magnets --> k, βh max, βv min, dispersion D max, phase advances per cell μh and μv.
FODO4: files from FODO3 --> Cos, Sin-like h trajectories, D outside elements.
FODO5: files from FODO3 --> Cos, Sin-like v trajectories outside elements.
FODO6: files from FODO4 and FODO5 --> β, μ, D (h and v), chromaticity .
FODO7: files from FODO3 and FODO5 --> exact h and v values everywhere.
FODO10: necessary and useful routines.

FODO1

Thin lens approximation

This program calculates beta max and beta min, and the dispersion in a FODO half cell with thin lenses.

$p := 400$ GeV/c

Bending magnets: $Nb := 744$ $L := 6.26$

Quadrupoles: $Nq := 216$ $lQ := 3.085$ $g := 19$ T*m

Half cell length: $l := 32$

****

$$B\rho := \frac{p}{0.299792458} \qquad B := \frac{2\cdot\pi}{Nb}\cdot\frac{B\rho}{L} \qquad \rho := \frac{B\rho}{B}$$

$B\rho = 1334.256$

$B = 1.8$

$\rho = 741.255$

$$k := \frac{g}{B\rho} \qquad N := \frac{Nq}{2}$$

$N = 108$

$k = 0.014$

$f = 45.526$

$$lq := \frac{lQ}{2} \qquad f := \frac{1}{k\cdot lq} \quad \text{(half quadrupole!)}$$

$$\sin\Phi := \frac{1}{f} \qquad \cos\Phi := \sqrt{1 - \sin\Phi^2} \qquad \Phi := \frac{180}{\pi}\cdot \operatorname{asin}\left(\frac{1}{f}\right)$$

$\Phi = 44.66$

$$\beta max := f\cdot\frac{1 + \sin\Phi}{\cos\Phi} \qquad \beta min := f\cdot\frac{1 - \sin\Phi}{\cos\Phi}$$

$\beta max = 108.993$

$\beta min = 19.016$

Estimated fraction of cell length filled with bending magnets ff:

$$ff := \frac{6.26\cdot 4}{32} \qquad ff = 0.783 \qquad R := \frac{\rho}{ff}$$

$R = 947.29$

$$Dmax := \left(\frac{f^2}{R}\right)\cdot\left(1 + \frac{\sin\Phi}{2}\right) \qquad Dmin := \left(\frac{f^2}{R}\right)\cdot\left(1 - \frac{\sin\Phi}{2}\right)$$

$Dmax = 2.957$

$Dmin = 1.419$

Momentum compaction α:

$$\alpha := \frac{f^2}{R\cdot\rho}$$

$\alpha = 0.003$

Chromaticity ksi = $dQ/d\delta$ (arcs, N cells):

$$N := \frac{Nq}{2}$$

$$ksi := \frac{N}{\pi}\cdot\left[\frac{-(\beta max - \beta min)}{2\cdot f}\right]$$

$N = 108$

$ksi = -33.971$

Sextupolar strength rls (with r=dg/ds) to cancel ksi (localized in the QF for x and in the QD for z):

$$rlsh := \frac{4 \cdot \pi \cdot ksi}{N \cdot \beta max \cdot Dmax}$$

$rlsh = -0.012$

$$rlsv := \frac{4 \cdot \pi \cdot ksi}{N \cdot \beta max \cdot Dmin}$$

$rlsv = -0.026$

Other relations:

$f^2 = 2.073 \cdot 10^3$ $\qquad$ $\beta max \cdot \beta min = 2.073 \cdot 10^3$

$$l := \frac{\sin(\Phi)}{g \cdot lq} \cdot B\rho$$

i.e. g being Bmax/a, with a the half aperture, the product l.lq/a.ρ = sin(Φ) approximately.

A FODO cell with quadrupoles

This program calculates the beta max (H) and beta min (V), k being given in a FODO period with quadrupoles only.

$p := 400$ $N := 744$ $Nq := 216$ $B := 1.8$

$\delta p := 0 \cdot p$

Quadrupole length:

$lQ := 3.085$

Straight sections lengthes:

$L := 6.26$

$l := 4 \cdot L + 0.36 + 0.4 + 0.39 + 0.38 + 2.3427$

$k := 0.015$

****

$B\rho := \dfrac{p}{0.299792458}$ $\rho := \dfrac{B\rho}{B}$

$N := \dfrac{Nq}{2}$ $lq := \dfrac{3.085}{2}$ $N = 108$ $\rho = 741.254$ $\dfrac{\delta p}{p} = 0$

Quadrupole matrices:

$\phi(K) := lq \cdot \sqrt{|K|}$

$cos\phi(K) := \cos(\phi(K))$ $sin\phi(K) := \sin(\phi(K))$

$ch\phi(K) := \cosh(\phi(K))$ $sh\phi(K) := \sinh(\phi(K))$

$$F(K) := \begin{bmatrix} cos\phi(K) & \dfrac{lq}{\phi(K)} \cdot sin\phi(K) & 0 \\ \dfrac{-\phi(K)}{lq} \cdot sin\phi(K) & cos\phi(K) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D(K) := \begin{bmatrix} ch\phi(K) & \dfrac{lq}{\phi(K)} \cdot sh\phi(K) & 0 \\ \dfrac{\phi(K)}{lq} \cdot sh\phi(K) & ch\phi(K) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Straight sections:

$$O := \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$l = 28.913$

Matrices for the cell :

$Mx(k) := F(k) \cdot O \cdot D(k) \cdot D(k) \cdot O \cdot F(k)$

$Mz(k) := D(k) \cdot O \cdot F(k) \cdot F(k) \cdot O \cdot D(k)$

Two orthogonal trajectories:

$$y1h(k,\beta h0) := Mx(k)\cdot\begin{pmatrix}\sqrt{\beta h0}\\0\\0\end{pmatrix} \qquad y2h(k,\beta h0) := Mx(k)\cdot\begin{bmatrix}0\\\sqrt{\frac{1}{\beta h0}}\\0\end{bmatrix}$$

$$\beta h(k,\beta h0) := \left[\left(y1h(k,\beta h0)_0\right)^2 + \left(y2h(k,\beta h0)_0\right)^2\right]$$

The condition of periodicity is now achieved (within a given tolerance):

$TOL := 10^{-10}$

guess: $\beta h0 := 100$

$\beta h0 := root((\beta h(k,\beta h0) - \beta h0),\beta h0)$

Result for the horizontal coordinate:

$\beta h(k,\beta h0) = 108.58$ $\quad k = 0.015 \quad \beta h0 = 108.58$

Similarly for the vertical coordinate:

$$y1v(k,\beta v0) := Mz(k)\cdot\begin{pmatrix}\sqrt{\beta v0}\\0\\0\end{pmatrix} \qquad y2v(k,\beta v0) := Mz(k)\cdot\begin{bmatrix}0\\\sqrt{\frac{1}{\beta v0}}\\0\end{bmatrix}$$

$$\beta v(k,\beta v0) := \left[\left(y1v(k,\beta v0)_0\right)^2 + \left(y2v(k,\beta v0)_0\right)^2\right]$$

guess: $\beta v0 := 20$

$\beta v0 := root((\beta v(k,\beta v0) - \beta v0),\beta v0)$

$\beta v(k,\beta v0) = 18.382$ $\quad \beta v0 = 18.382$

A FODO cell with dipoles and quadrupoles

This program calculates beta max and beta min, k, max dispersion and phase advance in a FODO periodic cell with bending magnets, by means of a general purpose matrix. It writes 4 data files: Lengthes, values of k, values of B, and βh0, βv0, Do...etc

$p := 400$ GeV/c

Bending magnets: $Nb := 744$

Bending magnet length: $L := 6.26$ m

Quadrupoles: $Nq := 216$

Quadrupole length: $lQ := 3.085$ m

Lengthes(the subscript is the number of the element or straight section):

$l_0 := \frac{lQ}{2}$ $\quad l_1 := 0.36$ $\quad l_2 := L$ $\quad l_3 := 0.4$ $\quad l_4 := L$ $\quad l_5 := 0.39$

$l_6 := L$ $\quad l_7 := 0.38$ $\quad l_8 := L$ $\quad l_9 := 2.3427$

$l_{10} := lQ$ $\quad l_{11} := 0.35$ $\quad l_{12} := L$ $\quad l_{13} := 0.38$ $\quad l_{14} := L$ $\quad l_{15} := 0.39$

$l_{16} := L$ $\quad l_{17} := 0.4$ $\quad l_{18} := L$ $\quad l_{19} := 2.3527$

$l_{20} := \frac{lQ}{2}$

$\text{length}(l) = 21$ $\qquad \sum l = 63.99539999999999$

****

$B\rho := \frac{p}{0.299792458}$ $\qquad B := \frac{2\cdot\pi}{Nb}\cdot\frac{B\rho}{L}$ $\qquad \rho := \frac{B\rho}{B}$

$B\rho = 1334.256$

$B = 1.799997442330278$

$\rho = 741.254598153917$

Quadrupole half length: $lq := \frac{lQ}{2}$

A General-Purpose-Matrix M(K,s,B) is built which can give the matrix of a focusing quadrupole, or a defocusing quadrupole, a bending magnet, a straight section (eventually a combined-function magnet):

$\phi(K,s) := s\cdot\sqrt{K}$ $\qquad \cos\phi(K,s) := \cos(\phi(K,s))$ $\qquad \sin\phi(K,s) := \sin(\phi(K,s))$

$$M(K,s,B) := \begin{bmatrix} \cos\phi(K,s) & \text{if}\left(K=0,s,s\cdot\frac{\sin\phi(K,s)}{\phi(K,s)}\right) & B\cdot\left(\frac{1-\cos\phi(K,s)}{B\rho\cdot K}\right) \\ \frac{-\phi(K,s)}{s}\cdot\sin\phi(K,s) & \cos\phi(K,s) & \left(\frac{B}{B\rho\cdot\sqrt{K}}\right)\cdot\sin\phi(K,s) \\ 0 & 0 & 1 \end{bmatrix}$$

$Kx(k) := -\left(k - \frac{1}{\rho^2}\right)$ $\qquad Kz(k) := k$

Quadrupole matrices:
K is kept as a parameter:

$QF(K) := M(K, lq, 0)$

$QD(K) := M(-K, lq, 0)$

Straight sections:

$$O(s) := \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Bending magnets:
The focusing strength is 0: $kb := 0$

$$Kx := -\left[kb - \left(\frac{B}{B\rho}\right)^2\right]$$

Matrices for the dipoles: $Kz := 0$ $Kx = 0.000001819974065$

$BFx(L) := M(Kx, L, B)$ $BFx(L) = \begin{pmatrix} 1 & 6.26 & 0.026 \\ 0 & 1 & 0.008 \\ 0 & 0 & 1 \end{pmatrix}$

$B1x(L) := BFx(L) \cdot O(l_7) \cdot BFx(L) \cdot O(l_5) \cdot BFx(L) \cdot O(l_3) \cdot BFx(L)$

$B2x(L) := BFx(L) \cdot O(l_{17}) \cdot BFx(L) \cdot O(l_{15}) \cdot BFx(L) \cdot O(l_{13}) \cdot BFx(L)$

$lB1 := 4 \cdot L + l_3 + l_5 + l_7$ $lB1 = 26.21$

$lB2 := 4 \cdot L + l_{13} + l_{15} + l_{17}$ $lB2 = 26.21$

$B1z := O(lB1)$

$B2z := O(lB2)$

Matrices for one period:

$Mx(K) := QF(K) \cdot O(l_{19}) \cdot B2x(L) \cdot O(l_{11}) \cdot QD(K) \cdot QD(K) \cdot O(l_9) \cdot B1x(L) \cdot O(l_1) \cdot QF(K)$

$Mz(K) := QD(K) \cdot O(l_{19}) \cdot B2z \cdot O(l_{11}) \cdot QF(K) \cdot QF(K) \cdot O(l_9) \cdot B1z \cdot O(l_1) \cdot QD(K)$

Verification:

$2 \cdot lQ + lB1 + lB2 + l_1 + l_9 + l_{11} + l_{19} = 63.9954$ $\sum l = 63.99539999999999$

Two orthogonal trajectories (cos-like and sin-like) in the horizontal plane are computed:

$y1h(K, \beta h0) := Mx(K) \cdot \begin{pmatrix} \sqrt{\beta h0} \\ 0 \\ 0 \end{pmatrix}$ $y2h(K, \beta h0) := Mx(K) \cdot \begin{bmatrix} 0 \\ \frac{1}{\sqrt{\beta h0}} \\ 0 \end{bmatrix}$

$\beta h(K, \beta h0) := \left[\left(y1h(K, \beta h0)_0\right)^2 + \left(y2h(K, \beta h0)_0\right)^2\right]$

gamma h:

$gh(K, \beta h0) := \left[\left(y1h(K, \beta h0)_1\right)^2 + \left(y2h(K, \beta h0)_1\right)^2\right]$

phase advance:

$$\mu h(K) := \frac{180}{\pi}\cdot acos\left(\frac{tr(Mx(K)) - 1}{2}\right) \qquad \mu h(K) := if(\mu h(K)>0, \mu h(K), \mu h(K) + 180)$$

Two orthogonal trajectories in the vertical plane:

$$y1v(K,\beta v0) := Mz(K)\cdot\begin{pmatrix}\sqrt{\beta v0}\\ 0\\ 0\end{pmatrix} \qquad y2v(K,\beta v0) := Mz(K)\cdot\begin{bmatrix}0\\ \frac{1}{\sqrt{\beta v0}}\\ 0\end{bmatrix}$$

$$\beta v(k,\beta v0) := \left[\left(y1v(k,\beta v0)_0\right)^2 + \left(y2v(k,\beta v0)_0\right)^2\right]$$

$$gv(K,\beta v0) := \left[\left(y1v(K,\beta v0)_1\right)^2 + \left(y2v(K,\beta v0)_1\right)^2\right]$$

$$\mu v(K) := \frac{180}{\pi}\cdot acos\left(\frac{tr(Mz(K)) - 1}{2}\right) \qquad \mu v(K) := if(\mu v(K)>0, \mu v(K), \mu v(K) + 180)$$

The dispersion trajectory is given by the following periodicity condition:

$$Dh(K,D0,D'0) := Mx(K)\cdot\begin{pmatrix}D0\\ D'0\\ 1\end{pmatrix}$$

To start the solving routine which will fit the periodicity conditions the needed tolerance on the accuracy of the results and some guess values (cf FODO1) are given in the following:

$TOL := 10^{-9}$

guess: $K := 0.015$ $\qquad \beta h0 := 110$ $\qquad \beta v0 := 20$ $\qquad D0 := 3$ $\qquad D'0 := 0$

Given

$$\beta h(K,\beta h0) = \beta h0$$

$$gh(K,\beta h0) = \frac{1}{\beta h0}$$

$$\beta v(K,\beta v0) = \beta v0$$

$$gv(K,\beta v0) = \frac{1}{\beta v0}$$

$$Dh(K,D0,D'0)_0 = D0$$

$$Dh(K,D0,D'0)_1 = D'0$$

$$\mu h(K) < 92$$

$$\mu v(K) < 92$$

$R := Find(K, \beta h0, \beta v0, D0, D'0)$

The results are placed in the matrix R:

$$R = \begin{bmatrix}0.014999993897474\\ 108.498446789616\\ 18.38196702058286\\ 2.846009239778958\\ -0.001066804140302\end{bmatrix}$$

$K := R_0 \qquad \beta h0 := R_1 \qquad \beta v0 := R_2 \qquad D0 := R_3 \qquad D'0 := R_4$

Verifications:

$K = 0.014999994$

Phase advance for one period:

$\mu h(K) = 91.59987161571343$

$\mu v(K) = 91.46407925448619$

$\beta h(K, \beta h0) = 108.4984467896161$ $\beta h0 = 108.498446789616$

$gh(K, \beta h0) = 0.009216722739578$ $\frac{1}{\beta h0} = 0.009216721801917$

$\beta v(K, \beta v0) = 18.38196702058287$ $\beta v0 = 18.38196702058286$

$gv(K, \beta v0) = 0.054401142101945$ $\frac{1}{\beta v0} = 0.054401142101945$

$Dh(K, D0, D'0)_0 = 2.846009239778958$ $D0 = 2.846009239778958$

$Dh(K, D0, D'0)_1 = -0.001066804140302$ $D'0 = -0.001066804140302$

The phase advances per cell are added to the results:

$R_5 := \mu h(K)$ $R_6 := \mu v(K)$

$$R = \begin{bmatrix} 0.014999994 \\ 108.49844679 \\ 18.381967021 \\ 2.84600924 \\ -0.001066804 \\ 91.599871616 \\ 91.464079254 \end{bmatrix}$$

The results for the cell are stored in the file FODO3.PRN:

WRITEPRN(FODO3) := R

It is now possible to store the values of l, k and B for each element. According to the common practice, k is negative for a horizontally focusing quadrupole (but then K is positive):

$l_0 = 1.5425$	$l_1 = 0.36$	$l_2 = 6.26$	$l_3 = 0.4$	$l_4 = 6.26$	$l_5 = 0.39$
$kk_0 := -K$	$kk_1 := 0$	$kk_2 := kb$	$kk_3 := 0$	$kk_4 := kb$	$kk_5 := 0$
$BB_0 := 0$	$BB_1 := 0$	$BB_2 := B$	$BB_3 := 0$	$BB_4 := B$	$BB_5 := 0$
		$l_6 = 6.26$	$l_7 = 0.38$	$l_8 = 6.26$	$l_9 = 2.3427$
		$kk_6 := kb$	$kk_7 := 0$	$kk_8 := kb$	$kk_9 := 0$
		$BB_6 := B$	$BB_7 := 0$	$BB_8 := B$	$BB_9 := 0$
$l_{10} = 3.085$	$l_{11} = 0.35$	$l_{12} = 6.26$	$l_{13} = 0.38$	$l_{14} = 6.26$	l_{15}
$kk_{10} := K$	$kk_{11} := 0$	$kk_{12} := kb$	$kk_{13} := 0$	$kk_{14} := kb$	$kk_{15} := 0$
$BB_{10} := 0$	$BB_{11} := 0$	$BB_{12} := B$	$BB_{13} := 0$	$BB_{14} := B$	$BB_{15} := 0$
		$l_{16} = 6.26$	$l_{17} = 0.4$	$l_{18} = 6.26$	$l_{19} = 2.3527$
		$kk_{16} := kb$	$kk_{17} := 0$	$kk_{18} := kb$	$kk_{19} := 0$
		$BB_{16} := B$	$BB_{17} := 0$	$BB_{18} := B$	$BB_{19} := 0$

$l_{20} = 1.5425$ $kk_{20} := -K$ $BB_{20} := 0$

The various results are stored in the files LENGTHES.PRN, KK.PRN and BB.PRN:

PRNPRECISION := 15

PRNCOLWIDTH := 8

WRITEPRN(LENGTHES) := 1

WRITEPRN(kk) := kk   length(kk) = 21

WRITEPRN(BB) := BB

Horizontal trajectories, dispersion

This program calculates the horizontal cos-like y1h and sin-like y2h trajectories and their derivatives, the dispersion at beginning and exit of each element by means of a general purpose matrix, in a FODO cell with bending magnets. It uses the data of FODO3.

$p := 400$ GeV/c

$B\rho := \frac{p}{0.299792458}$ $B := 1.8$ $\rho := \frac{B\rho}{B}$ $\rho = 741.254$

The data from FODO3 are reintroduced. Lengthes(the subscript is the number of the element or straight section) for the lengthes of all elements, kk for the focusing coefficients and BB for the bending magnet field:

$l := \mathrm{READPRN(LENGTHES)}$

$kk := \mathrm{READPRN}(kk)$

$BB := \mathrm{READPRN}(BB)$

$\sum l = 63.995$ $\mathrm{length}(l) = 21$ $i := 0 .. \mathrm{length}(l) - 1$

The results of the solving routine of FODO3 are also called back:

$R := \mathrm{READPRN(Fodo3)}$

$$R = \begin{bmatrix} 0.015 \\ 108.5 \\ 18.38 \\ 2.846 \\ -0.001 \\ 91.6 \\ 91.46 \end{bmatrix}$$

$K := R_0$ $\beta h0 := R_1$ $\beta v0 := R_2$ $D0 := R_3$ $D'0 := R_4$ $\mu h0 := R_5$ $\mu v0 := R_6$

A General-purpose-matrix M(K,s,B) is necessary for the trajectory calculation:

$\phi(K,s) := s \cdot \sqrt{K}$

$\cos\phi(K,s) := \cos(\phi(K,s))$ $\sin\phi(K,s) := \sin(\phi(K,s))$

$$M(K,s,B) := \begin{bmatrix} \cos\phi(K,s) & \mathrm{if}\left(K=0, s, s\cdot\frac{\sin\phi(K,s)}{\phi(K,s)}\right) & B\cdot\left(\frac{1-\cos\phi(K,s)}{B\rho\cdot K}\right) \\ \frac{-\phi(K,s)}{s}\cdot\sin\phi(K,s) & \cos\phi(K,s) & \left(\frac{B}{B\rho\cdot\sqrt{K}}\right)\cdot\sin\phi(K,s) \\ 0 & 0 & 1 \end{bmatrix}$$

$$KKx_i := -\left[kk_i - \left(\frac{BB_i}{B\rho}\right)^2\right]$$

SS: distance from the middle of the 1st quad.

$\mathrm{length}(l) = 21$ $i := 0 .. \mathrm{length}(l) - 1$

$SS_0 := 0$ $SS_{i+1} := SS_i + l_i$ $\mathrm{length}(SS) = 22$

Initializations:

$$y1h^{<0>} := \begin{pmatrix} \sqrt{\beta h0} \\ 0 \\ 0 \end{pmatrix} \quad y2h^{<0>} := \begin{bmatrix} 0 \\ \frac{1}{\sqrt{\beta h0}} \\ 0 \end{bmatrix} \quad Dh^{<0>} := \begin{pmatrix} D0 \\ D'0 \\ 1 \end{pmatrix}$$

Iterative matrix multiplication:

$$y1h^{<i+1>} := M\left(KKx_i, l_i, BB_i\right) \cdot y1h^{<i>}$$

$$y2h^{<i+1>} := M\left(KKx_i, l_i, BB_i\right) \cdot y2h^{<i>}$$

$$Dh^{<i+1>} := M\left(KKx_i, l_i, BB_i\right) \cdot Dh^{<i>}$$

Verifications and results:

$$\left[\left(y1h^{<21>}\right)_0\right]^2 + \left[\left(y2h^{<21>}\right)_0\right]^2 = 108.497 \qquad \beta h0 = 108.5$$

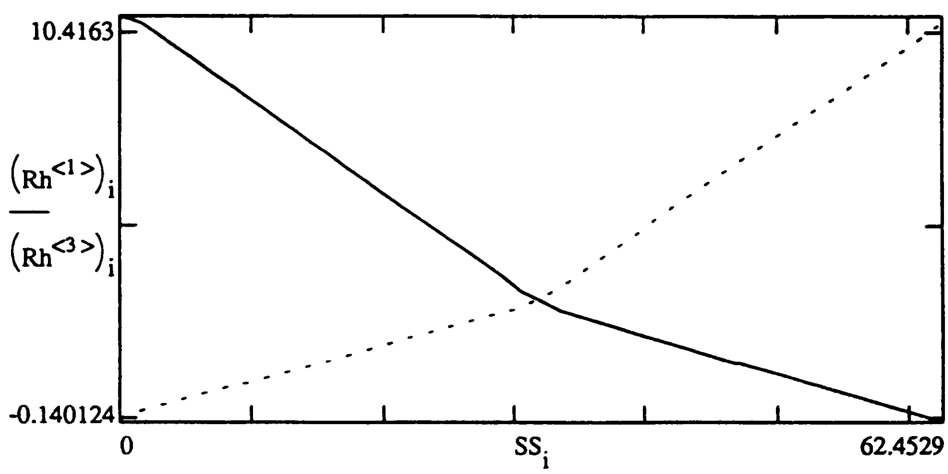
$$\left[\left(y1h^{<21>}\right)_1\right]^2 + \left[\left(y2h^{<21>}\right)_1\right]^2 = 0.009 \qquad \frac{1}{\beta h0} = 0.009$$

$$\left[\left(y1h^{<21>}\right)_2\right]^2 + \left[\left(y2h^{<21>}\right)_2\right]^2 = 0$$

$$\left(Dh^{<21>}\right)_0 = 2.846 \qquad D0 = 2.846$$

The results: abscissa, cos-like trajectory and its derivative, sin-like trajectory and its derivative, dispersion and its derivative are put in the table Rh:

$$Rh := \text{augment}\left[SS, \left(y1h^T\right)^{<0>}\right]$$
$$Rh := \text{augment}\left[Rh, \left(y1h^T\right)^{<1>}\right]$$
$$Rh := \text{augment}\left[Rh, \left(y2h^T\right)^{<0>}\right]$$
$$Rh := \text{augment}\left[Rh, \left(y2h^T\right)^{<1>}\right]$$
$$Rh := \text{augment}\left[Rh, \left(Dh^T\right)^{<0>}\right]$$
$$Rh := \text{augment}\left[Rh, \left(Dh^T\right)^{<1>}\right]$$



Cos-like and sin-like trajectories in the cell

Saving of the cos-like, sin-like, dispersion trajectories and their derivatives in the file FODO4.PRN:

PRNPRECISION := 15

WRITEPRN(FODO4) := Rh

Rh =

0	10.416	0	0	0.096	2.846	-0.001
1.543	10.231	-0.24	0.147	0.094	2.794	-0.067
1.903	10.145	-0.24	0.181	0.094	2.77	-0.067
8.163	8.645	-0.24	0.771	0.094	2.38	-0.058
8.563	8.549	-0.24	0.809	0.094	2.357	-0.058
14.823	7.048	-0.24	1.399	0.094	2.019	-0.05
15.213	6.955	-0.24	1.436	0.094	2	-0.05
21.473	5.453	-0.24	2.026	0.094	1.715	-0.041
21.853	5.362	-0.24	2.062	0.094	1.7	-0.041
28.113	3.861	-0.24	2.652	0.094	1.468	-0.033
30.455	3.299	-0.24	2.873	0.094	1.391	-0.033
33.54	2.779	-0.101	3.378	0.237	1.388	0.031
33.89	2.744	-0.101	3.461	0.237	1.398	0.031
40.15	2.112	-0.101	4.945	0.237	1.617	0.039
40.53	2.074	-0.101	5.036	0.237	1.632	0.039
46.79	1.442	-0.101	6.52	0.237	1.903	0.048
47.18	1.402	-0.101	6.612	0.237	1.922	0.048
53.44	0.77	-0.101	8.096	0.237	2.246	0.056
53.84	0.73	-0.101	8.191	0.237	2.269	0.056
60.1	0.097	-0.101	9.674	0.237	2.645	0.064
62.453	-0.14	-0.101	10.231	0.237	2.797	0.064
63.995	-0.292	-0.096	10.412	-0.003	2.846	-0.001

Vertical trajectories

This program calculates the vertical cos-like y1v and sin-like y2v trajectories and their derivatives, at beginning and exit of each element by means of a general purpose matrix, in a FODO cell with bending magnets. It uses the data of FODO3.

$p := 400$ GeV/c

$B\rho := \dfrac{p}{0.299792458}$ $\quad B := 1.8 \quad \rho := \dfrac{B\rho}{B} \quad \rho = 741.254$

The data from FODO3 are reintroduced. Lengthes(the subscript is the number of the element or straight section) for the lengthes of all elements, kk for the focusing coefficients and BB for the bending magnet field:

$l :=$ READPRN(LENGTHES)

$kk :=$ READPRN(kk)

$BB :=$ READPRN(BB)

$\sum l = 63.995 \qquad \text{length}(l) = 21$

$R :=$ READPRN(Fodo3)

$$R = \begin{bmatrix} 0.015 \\ 108.5 \\ 18.38 \\ 2.846 \\ -0.001 \\ 91.6 \\ 91.46 \end{bmatrix}$$

$K := R_0 \quad \beta h0 := R_1 \quad \beta v0 := R_2 \quad D0 := R_3 \quad D'0 := R_4 \quad \mu h0 := R_5 \quad \mu v0 := R_6$

General purpose matrix M(K,s,B):

$\phi(K,s) := s\cdot\sqrt{K} \qquad \cos\phi(K,s) := \cos(\phi(K,s)) \qquad \sin\phi(K,s) := \sin(\phi(K,s))$

$$M(K,s,B) := \begin{bmatrix} \cos\phi(K,s) & \text{if}\left(K=0, s, s\cdot\dfrac{\sin\phi(K,s)}{\phi(K,s)}\right) & B\cdot\left(\dfrac{1-\cos\phi(K,s)}{B\rho\cdot K}\right) \\ \dfrac{-\phi(K,s)}{s}\cdot\sin\phi(K,s) & \cos\phi(K,s) & \left(\dfrac{B}{B\rho\cdot\sqrt{K}}\right)\cdot\sin\phi(K,s) \\ 0 & 0 & 1 \end{bmatrix}$$

SS: distance from the middle of the 1st quad.

$\text{length}(l) = 21 \qquad i := 0 .. \text{length}(l) - 1$

$SS_0 := 0 \qquad SS_{i+1} := SS_i + l_i \qquad \text{length}(SS) = 22$

Initializations:

$$y1v^{<0>} := \begin{pmatrix} \sqrt{\beta v0} \\ 0 \\ 0 \end{pmatrix} \qquad y2v^{<0>} := \begin{bmatrix} 0 \\ \dfrac{1}{\sqrt{\beta v0}} \\ 0 \end{bmatrix}$$

Iterative matrix multiplication:

$$y1v^{<i+1>} := M(kk_i, l_i, 0)\cdot y1v^{<i>}$$

$$y2v^{<i+1>} := M(kk_i, l_i, 0)\cdot y2v^{<i>}$$

Verifications and results:

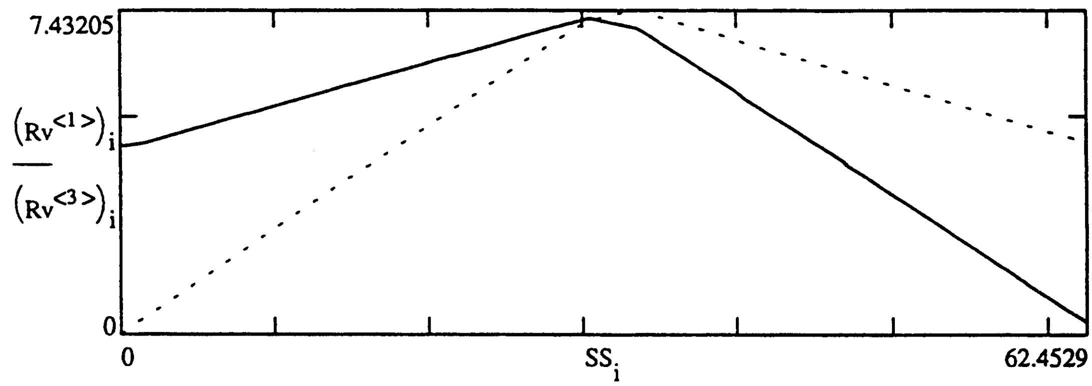
$$\left[\left(y1v^{<21>}\right)_0\right]^2 + \left[\left(y2v^{<21>}\right)_0\right]^2 = 18.384 \qquad \beta v0 = 18.38$$

$$\left[\left(y1v^{<21>}\right)_1\right]^2 + \left[\left(y2v^{<21>}\right)_1\right]^2 = 0.054 \qquad \frac{1}{\beta v0} = 0.054$$

$$\left[\left(y1v^{<21>}\right)_2\right]^2 + \left[\left(y2v^{<21>}\right)_2\right]^2 = 0$$

Saving of results in the file FODO4.PRN:

$$Rv := \text{augment}\left[SS, \left(y1v^T\right)^{<0>}\right]$$
$$Rv := \text{augment}\left[Rv, \left(y1v^T\right)^{<1>}\right]$$
$$Rv := \text{augment}\left[Rv, \left(y2v^T\right)^{<0>}\right]$$
$$Rv := \text{augment}\left[Rv, \left(y2v^T\right)^{<1>}\right]$$



Vertical cos-like and sin-like trajectoriesin the cell

PRNPRECISION := 15

WRITEPRN(FODO5) := Rv

Results for the vertical trajectories:

$$Rv = \begin{bmatrix} 0 & 4.287 & 0 & 0 & 0.233 \\ 1.543 & 4.364 & 0.1 & 0.362 & 0.237 \\ 1.903 & 4.4 & 0.1 & 0.447 & 0.237 \\ 8.163 & 5.025 & 0.1 & 1.934 & 0.237 \\ 8.563 & 5.064 & 0.1 & 2.029 & 0.237 \\ 14.823 & 5.689 & 0.1 & 3.515 & 0.237 \\ 15.213 & 5.728 & 0.1 & 3.608 & 0.237 \\ 21.473 & 6.353 & 0.1 & 5.094 & 0.237 \\ 21.853 & 6.391 & 0.1 & 5.184 & 0.237 \\ 28.113 & 7.015 & 0.1 & 6.67 & 0.237 \\ 30.455 & 7.249 & 0.1 & 7.227 & 0.237 \\ 33.54 & 7.038 & -0.235 & 7.432 & -0.106 \\ 33.89 & 6.956 & -0.235 & 7.395 & -0.106 \\ 40.15 & 5.486 & -0.235 & 6.733 & -0.106 \\ 40.53 & 5.397 & -0.235 & 6.692 & -0.106 \\ 46.79 & 3.927 & -0.235 & 6.03 & -0.106 \\ 47.18 & 3.836 & -0.235 & 5.989 & -0.106 \\ 53.44 & 2.366 & -0.235 & 5.326 & -0.106 \\ 53.84 & 2.272 & -0.235 & 5.284 & -0.106 \\ 60.1 & 0.803 & -0.235 & 4.621 & -0.106 \\ 62.453 & 0.25 & -0.235 & 4.372 & -0.106 \\ 63.995 & -0.11 & -0.233 & 4.286 & -0.006 \end{bmatrix}$$

Interpolated results, curves, integrals.

This file uses the data from FODO4, and FODO5. It calculates, by means of the vectorize operator, the amplitude functions βh and βv, the phase advances μh and μv. It then interpolates these values, as well as the horizontal dispersion Dh, all along the periodic cell and draws their curves. As an example of an integral of these betatron functions, the natural chromaticity is calculated. The accuracy of the results can be raised at will by adding exact results in a chosen region of the cell by means of FODO7.

$l := \mathrm{READPRN}(\mathrm{LENGTHES})$

$R := \mathrm{READPRN}(\mathrm{FODO3})$

$Rh := \mathrm{READPRN}(\mathrm{FODO4})$

$Rv := \mathrm{READPRN}(\mathrm{FODO5})$

$Smax := \max(Rh^{\langle 0 \rangle})$

$\mathrm{length}(Rh^{\langle 0 \rangle}) = 22$ $\quad Smax = 63.995$ $\quad \mathrm{length}(Rv^{\langle 0 \rangle}) = 22$ $\quad S := 0, 0.5 .. Smax$

$$\beta h := \left(\overrightarrow{\left[(Rh^{\langle 1 \rangle})^2\right]}\right) + \left(\overrightarrow{\left[(Rh^{\langle 3 \rangle})^2\right]}\right)$$

$$\mu h := \left(\frac{180}{\pi}\right) \cdot \overrightarrow{\left[\mathrm{acos}\left[\frac{Rh^{\langle 1 \rangle}}{\sqrt{(Rh^{\langle 1 \rangle})^2 + (Rh^{\langle 3 \rangle})^2}}\right]\right]}$$

$$\beta v := \left(\overrightarrow{\left[(Rv^{\langle 1 \rangle})^2\right]}\right) + \left(\overrightarrow{\left[(Rv^{\langle 3 \rangle})^2\right]}\right)$$

$$\mu v := \left(\frac{180}{\pi}\right) \cdot \overrightarrow{\left[\mathrm{acos}\left[\frac{Rv^{\langle 1 \rangle}}{\sqrt{(Rv^{\langle 1 \rangle})^2 + (Rv^{\langle 3 \rangle})^2}}\right]\right]}$$

$$gammah := \left(\overrightarrow{\left[(Rh^{\langle 2 \rangle})^2\right]}\right) + \left(\overrightarrow{\left[(Rh^{\langle 4 \rangle})^2\right]}\right) \qquad gammav := \left(\overrightarrow{\left[(Rv^{\langle 2 \rangle})^2\right]}\right) + \left(\overrightarrow{\left[(Rv^{\langle 4 \rangle})^2\right]}\right)$$

Interpolations (the linear interpolation is used in that example but the parabolic pspline or the cubic cspline might be used as well):

$\beta h(S) := \mathrm{interp}(\mathrm{lspline}(Rh^{\langle 0 \rangle}, \beta h), Rh^{\langle 0 \rangle}, \beta h, S)$

$gammah(S) := \mathrm{interp}(\mathrm{lspline}(Rh^{\langle 0 \rangle}, gammah), Rh^{\langle 0 \rangle}, gammah, S)$

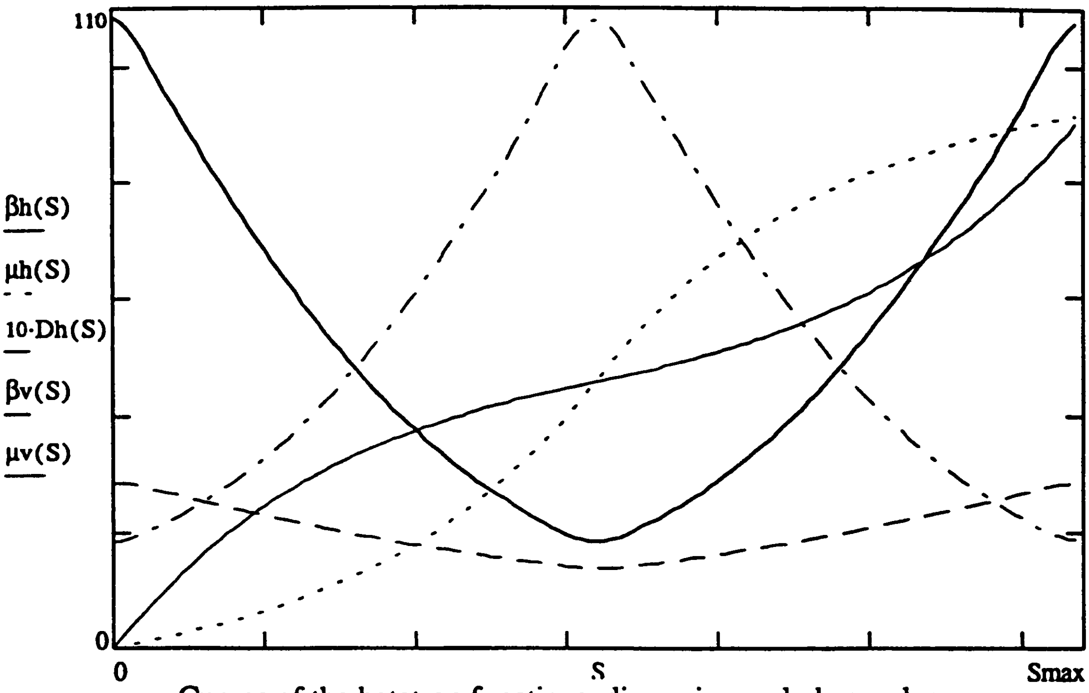
$\mu h(S) := \mathrm{interp}(\mathrm{lspline}(Rh^{\langle 0 \rangle}, \mu h), Rh^{\langle 0 \rangle}, \mu h, S)$

$Dh(S) := \mathrm{interp}(\mathrm{lspline}(Rh^{\langle 0 \rangle}, Rh^{\langle 5 \rangle}), Rh^{\langle 0 \rangle}, Rh^{\langle 5 \rangle}, S)$

$\beta v(S) := \mathrm{interp}(\mathrm{lspline}(Rv^{\langle 0 \rangle}, \beta v), Rv^{\langle 0 \rangle}, \beta v, S)$

$gammav(S) := \mathrm{interp}(\mathrm{lspline}(Rv^{\langle 0 \rangle}, gammav), Rv^{\langle 0 \rangle}, gammav, S)$

$\mu v(S) := \mathrm{interp}(\mathrm{lspline}(Rv^{\langle 0 \rangle}, \mu v), Rv^{\langle 0 \rangle}, \mu v, S)$



Curves of the betatron functions, dispersion and phase advances.

Various results:

$$R = \begin{bmatrix} 0.015 \\ 108.5 \\ 18.38 \\ 2.846 \\ -0.001067 \\ 91.6 \\ 91.46 \end{bmatrix}$$

$\beta h(32) = 18.441016081$ $\quad \beta v(32) = 107.864597752$

$\mu h(32) = 45.811413863$ $\quad \mu v(32) = 45.736382685$

$\mathrm{gammah}(32) = 0.066429756$ $\quad \mathrm{gammav}(32) = 0.066323673$

$Dh(32) = 1.369139407$

$\beta h(Smax) = 108.496896051$ $\quad \beta v(Smax) = 18.383931683$

$\mu h(Smax) = 91.609032116$ $\quad \mu v(Smax) = 91.463922654$

$Dh(Smax) = 2.845988257$

Quadrupole chromaticity:

$SS := Rh^{<0>}$ $\qquad K := R_0$ $\qquad K = 0.015$

$$ksih := \left(\frac{-1}{4\cdot\pi}\right)\cdot\left[\int_{SS_0}^{SS_1} K\cdot\beta h(S)\,dS + \int_{SS_{10}}^{SS_{11}} (-K)\cdot\beta h(S)\,dS \ldots + \int_{SS_{20}}^{SS_{21}} K\cdot\beta h(S)\,dS\right]$$

$$ksiv := \left(\frac{-1}{4\cdot\pi}\right)\cdot\left[\int_{SS_0}^{SS_1} (-K)\cdot\beta v(S)\,dS + \int_{SS_{10}}^{SS_{11}} (K)\cdot\beta v(S)\,dS \ldots + \int_{SS_{20}}^{SS_{21}} (-K)\cdot\beta v(S)\,dS\right]$$

Cell: $ksih = -0.324808$

$ksiv = -0.324504$

$N := 108$ $\quad Ksih := ksih \cdot N$ $\quad Ksiv := ksiv \cdot N$

Ring: $Ksih = -35.079$

$Ksiv = -35.046$

By assuming a sextupole of length ls is located at the beginning of the QF for x and at the beginning of the QD for z, the sextupolar strengthes rls (with r=dg/ds) to cancel ksi become:

$$rlsh := \frac{ksih \cdot (4 \cdot \pi)}{\beta h\left(SS_{20}\right) \cdot Dh\left(SS_{20}\right)}$$

$SS_{20} = 62.453$

$rlsh = -0.014$

$\beta h\left(SS_{20}\right) = 104.691$

$Dh\left(SS_{20}\right) = 2.797$

$$rlsv := \frac{ksiv \cdot (4 \cdot \pi)}{\beta v\left(SS_{10}\right) \cdot Dh\left(SS_{10}\right)}$$

$SS_{10} = 30.455$

$rlsv = -0.028$

$\beta v\left(SS_{10}\right) = 104.772$

$Dh\left(SS_{10}\right) = 1.391$

Miscellaneous:

$\mu h0 := R_5$ $\quad \mu v0 := R_6$

$Qh := \dfrac{N \cdot \mu h0}{360}$ $\quad Qv := \dfrac{N \cdot \mu v0}{360}$

$Qh = 27.48$

$Qv = 27.438$

$frac(x) := x - floor(x)$

$$\frac{1}{frac(Qh)} = 2.083333333$$

$$\left(\frac{180}{\pi}\right) \cdot \int_0^{Smax} \frac{1}{\beta h(s)}\, ds = 91.578$$

$\mu h0 = 91.6$

FODO7

Exact results inside elements.

This program calculates the horizontal and vertical y1 and y2 trajectories and their derivatives, βh, βv, μh ,μv and the dispersion at distance S inside any element by means of a general purpose matrix, in a FODO cell with bending magnets. It uses the data of FODO3.PRN, kk.PRN, BB.PRN and LENGTHES.PRN. The value of S can be changed in random order and the calculation repeated to get accurate results in the region of interest. These results can be added after reordering in the files FODO4.PRN obtained by FODO4, and in the file FODO5.PRN obtained by FODO5.

Enter a new abscissa where the betatron functions have to be calculated:

$S := 63.995$ m

****

$p := 400$ GeV/c

$B\rho := \frac{p}{0.299792458}$ $\quad B := 1.8$ $\quad \rho := \frac{B\rho}{B}$ $\quad \rho = 741.254$

$l := \mathrm{READPRN}(\mathrm{LENGTHES})$ $\quad \Sigma l = 63.995$

SS: distance of the elements from the middle of the 1st quad.

$\mathrm{length}(l) = 21$ $\quad i := 0..\mathrm{length}(l) - 1$ $\quad j := 0..\mathrm{length}(l)$

$SS_0 := 0$ $\quad SS_{i+1} := SS_i + l_i$ $\quad \mathrm{length}(SS) = 22$

The data from previous calculations are called back:

$R := \mathrm{READPRN}(\mathrm{Fodo3})$

$$R = \begin{bmatrix} 0.015 \\ 108.5 \\ 18.38 \\ 2.846 \\ -0.001 \\ 91.6 \\ 91.46 \end{bmatrix}$$

$K := R_0$ $\quad \beta h0 := R_1$ $\quad \beta v0 := R_2$ $\quad D0 := R_3$ $\quad D'0 := R_4$ $\quad \mu h0 := R_5$ $\quad \mu v0 := R_6$

$kk := \mathrm{READPRN}(KK)$

$BB := \mathrm{READPRN}(BB)$

$$KKx_i := -\left[kk_i - \left(\frac{BB_i}{B\rho}\right)^2\right]$$

Table of abscissa, lengthes, focusing coefficients and dipole field

SS_j	l_i	kk_i	KKx_i	BB_i
0	1.543	-0.014999994	0.014999994	0
1.543	0.36	0	0	0
1.903	6.26	0	0.00000182	1.8
8.163	0.4	0	0	0
8.563	6.26	0	0.00000182	1.8
14.823	0.39	0	0	0
15.213	6.26	0	0.00000182	1.8
21.473	0.38	0	0	0
21.853	6.26	0	0.00000182	1.8
28.113	2.343	0	0	0
30.455	3.085	0.014999994	-0.014999994	0
33.54	0.35	0	0	0
33.89	6.26	0	0.00000182	1.8
40.15	0.38	0	0	0
40.53	6.26	0	0.00000182	1.8
46.79	0.39	0	0	0
47.18	6.26	0	0.00000182	1.8
53.44	0.4	0	0	0
53.84	6.26	0	0.00000182	1.8
60.1	2.353	0	0	0
62.453	1.543	-0.014999994	0.014999994	0
63.995				

Number of the element at the chosen distance S from the 1st quad:

$Smax := max(SS)$

$Sc := if(S<Smax, S, mod(S, Smax))$

$II := SS - Sc$ $\qquad III_i := if(II_i<0, i, 0)$ $\qquad I := max(III)$ $\qquad I = 20$

A general-purpose-matrix is necessary:

$$\phi(K,s) := s\cdot\sqrt{K}$$

$$cos\phi(K,s) := cos(\phi(K,s)) \qquad sin\phi(K,s) := sin(\phi(K,s))$$

$$M(K,s,B) := \begin{bmatrix} cos\phi(K,s) & if\left(K=0, s, s\cdot\dfrac{sin\phi(K,s)}{\phi(K,s)}\right) & B\cdot\left(\dfrac{1 - cos\phi(K,s)}{B\rho\cdot K}\right) \\ \dfrac{-\phi(K,s)}{s}\cdot sin\phi(K,s) & cos\phi(K,s) & \left(\dfrac{B}{B\rho\cdot\sqrt{K}}\right)\cdot sin\phi(K,s) \\ 0 & 0 & 1 \end{bmatrix}$$

A matrix Yh is built for the matrix calculation. Its first two columns are the two orthogonal trajectories, the third column is the dispersion Dh.
Initialization of Yh, Mx, y1h, y2h, Dh:

$$Yh0 := \begin{bmatrix} \sqrt{\beta h0} & 0 & D0 \\ 0 & \dfrac{1}{\sqrt{\beta h0}} & D'0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Yh := Yh0 \qquad Mx := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$y1h^{<0>} := Yh^{<0>} \qquad y2h^{<0>} := Yh^{<1>} \qquad Dh^{<0>} := Yh^{<2>}$$

Similarly a matrix Yv is created for the matrix calculation. Its first two columns are the two orthogonal trajectories, the third column is the dispersion Dv (here=0). Initialization of Yv, Mz, y1v, y2v, Dv:

$$Yv0 := \begin{bmatrix} \sqrt{\beta v0} & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{\beta v0}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Yv := Yv0 \qquad Mz := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$y1v^{<0>} := Yv^{<0>} \qquad y2v^{<0>} := Yv^{<1>}$$

####################

The matrix to the beginning of the Ith element is the product of the matrices of all elements from 0 to I-1 in the reverse order:

$$ii := I - 1 .. 0 \qquad I = 20$$

$$Mx := \prod_{ii} M\left(KKx_{ii}, l_{ii}, BB_{ii}\right)$$

The matrix down to S is:

$$Mx := M\left(KKx_I, Sc - SS_I, BB_I\right)\cdot Mx$$

$$Yh := Mx \cdot Yh0 \qquad Yh = \begin{pmatrix} -0.292 & 10.412 & 2.846 \\ -0.096 & -0.003 & -0.001 \\ 0 & 0 & 1 \end{pmatrix}$$

A additional row Rha made of the results of Yh is added to Rh:

$$Rha := \left[S \;\; \left(Yh^{<0>}\right)_0 \;\; \left(Yh^{<0>}\right)_1 \;\; \left(Yh^{<1>}\right)_0 \;\; \left(Yh^{<1>}\right)_1 \;\; \left(Yh^{<2>}\right)_0 \;\; \left(Yh^{<2>}\right)_1 \right]$$

$$Rha = (\, 63.995 \;\; -0.292 \;\; -0.096 \;\; 10.412 \;\; -0.003 \;\; 2.846 \;\; -0.001 \,)$$

$$Rh := READPRN(FODO4)$$

$$Rh := csort\left(\left(augment\left(Rh^T, Rha^T\right)\right)^T, 0\right)$$

Similarly for the vertical coordinate:

$$Mz := \prod_{ii} M\left(kk_{ii}, l_{ii}, 0\right)$$

The matrix down to S is:

$$Mz := M\left(kk_I, Sc - SS_I, 0\right) \cdot Mz$$

$$Yv := Mz \cdot Yv0 \qquad Yv = \begin{pmatrix} -0.109 & 4.286 & 0 \\ -0.233 & -0.006 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A additional row Rva made of the results of Yv is added to Rv:

$$Rva := \left[S \quad (Yv^{<0>})_0 \quad (Yv^{<0>})_1 \quad (Yv^{<1>})_0 \quad (Yv^{<1>})_1 \right]$$

$$Rva = (\,63.995 \;\; -0.109 \;\; -0.233 \;\; 4.286 \;\; -0.006\,)$$

$$Rv := READPRN(FODO5)$$

$$Rv := csort\left(\left(augment\left(Rv^T, Rva^T\right)\right)^T, 0\right)$$

Results:

$$\beta h := \left[\left[(Yh^{<0>})_0\right]^2 + \left[(Yh^{<1>})_0\right]^2\right]$$

$$\beta v := \left[\left[(Yv^{<0>})_0\right]^2 + \left[(Yv^{<1>})_0\right]^2\right]$$

$$\mu h := \left(\frac{180}{\pi}\right) \cdot \left[atan\left[\frac{(Yh^{<1>})_0}{(Yh^{<0>})_0}\right]\right] \qquad \mu h := if(\mu h > 0, \mu h, \mu h + 180)$$

$$\mu v := \left(\frac{180}{\pi}\right) \cdot \left[atan\left[\frac{(Yv^{<1>})_0}{(Yv^{<0>})_0}\right]\right] \qquad \mu v := if(\mu v > 0, \mu v, \mu v + 180)$$

The various results for the chosen abscissa are in the second column:

		$S = 63.995$
	$\beta h0 = 108.5$	$\beta h = 108.497$
	$\beta v0 = 18.38$	$\beta v = 18.384$
$Dh := (Yh^{<2>})_0$	$\mu h0 = 91.6$	$\mu h = 91.609$
$Dv := (Yv^{<2>})_0$	$\mu v0 = 91.46$	$\mu v = 91.463$
	$D0 = 2.846$	$Dh = 2.846$
		$Dv = 0$

If the present results must be added to the results of FODO4 and FODO5, to get a better accuracy in the region of interest, the following commands have to be activated. By executing again the present program several times, it is possible to add many accurate results in the region where the accuracy must be better.

PRNPRECISION := 15▫

WRITEPRN(FODO4) := Rh▫

WRITEPRN(FODO5) := Rv▫

Useful routines

The calculation of a FODO cell cannot be achieved without a set of routines at hand (if not they have to be written). First for the arrays (one-column vector, matrices) it is required to have the functions: min, max, length, last element and sorting of a vector, number of rows or columns, min, max, trace of a matrix. It is needed also to join matrices. All operations on matrices: multiplication, cross products, inverse, integer power, transposition, sum of vector elements, sorting, vectorization (to apply a function or operator to all elements of an array), eigenvalues should be available.It is requested to calculate complex numbers.
Rectangular arrays of numbers have to be stored in ASCII files for further use, without loss of accuracy (15 digits is a must).
In the following some simple routines relevant to the analysis of complicated cell are given.

Function for the focusing strength k(S):

$l := \mathrm{READPRN}(lengthes)$

$kk := \mathrm{READPRN}(kk)$

$Rh := \mathrm{READPRN}(FODO4)$ $\qquad SS := Rh^{<0>}$

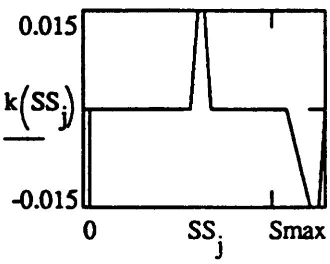
$\mathrm{length}(l) = 21$ $\qquad i := 0..\mathrm{length}(l) - 1$

$\mathrm{length}(SS) = 23$ $\qquad j := 0..\mathrm{length}(SS) - 1$

$Smax := \max(SS)$ $\qquad S := 132$

$S := \mathrm{if}(S > Smax, \mathrm{mod}(S, Smax), S)$

$$k(S) := \sum_i \left[\mathrm{if}\left[(S - SS_i)\cdot(S - SS_{i+1}) \le 0, kk_i, 0\right]\right]$$



$SS_{20} = 60.1$

$SS_{21} = 62.453$

$k(30.454) = 0$

$k(30.456) = 0.015$

$k(33.55) = 0$

$k(SS_{20}) = -0.015$

$k(SS_{21}) = -0.015$

$k(50) = 0$

Number I of the element corresponding to an abscissa S:

$S := 132$ $\qquad Sc := \mathrm{if}(S > Smax, \mathrm{mod}(S, Smax), S)$ $\qquad Sc = 4.009$

$II := SS - Sc$ $\qquad III_i := \mathrm{if}(II_i < 0, i, 0)$ $\qquad I := \max(III)$ $\qquad I = 2$

Numbers In of the elements for a regular range of abscissa Sn:

$N := 10$ $\qquad n := 0..N$ $\qquad S_n := \left(\frac{n}{N}\right)\cdot 100$

$Sc_n := \mathrm{if}(S_n > Smax, \mathrm{mod}(S_n, Smax), S_n)$

$II^{<n>} := SS - Sc_n$ $\qquad III_{i,n} := \mathrm{if}\left[(II^{<n>})_i < 0, i, 0\right]$ $\qquad I_n := \max(III^{<n>})$

i	SS_i	S_n	Sc_n	I_n
0	0	0	0	0
1	1.543	10	10	4
2	1.903	20	20	6
3	8.163	30	30	9
4	8.563	40	40	13
5	14.823	50	50	17
6	15.213	60	60	19
7	21.473	70	6.005	2
8	21.853	80	16.005	6
9	28.113	90	26.005	8
10	30.455	100	36.005	13
11	32			
12	33.54			
13	33.89			
14	40.15			
15	40.53			
16	46.79			
17	47.18			
18	53.44			
19	53.84			
20	60.1			

Matrix made of elements equal to i up to i:

$I := 5 \qquad i := 0..I \qquad j := 0..I$

$iii_{i,j} := \text{if}(i \le j, i, 0)$

$$iii = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad i = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Matrix made of elements equal to i up to i in the reverse order:

$I := 5 \qquad i := 0..I \qquad j := 0..I$

$iii_{i,j} := \text{if}(i \le j, j - i, 0)$

$$iii = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad i = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

General-Purpose-Matrix:

The linear theory of the betatron oscillations in an accelerator is commonly made by means of specific matrix types: focusing, defocusing, bending magnet, combined function, straight section matrices. It is possible to work with a single type of matrix (General-Purpose-Matrix) able to replace all the previous types, where the type is obtained by passing an argument to it. For this it is only necessary to point out that:

$$\sin(i\phi) = i\sinh(\phi)$$
$$\cos(i\phi) = \cosh(\phi)$$

Commonly the K coefficient is kept positive and the focusing matrix is made of trigonometric functions whilst the defocusing matrix is made of hyperbolic functions. From now onwards K is considered positive for the focusing matrix and NEGATIVE for the defocusing one. Then ϕ is real for the focusing case and imaginary for the defocusing case, e.g. for the most general case:

$s := 3$ $\quad k := -0.1$ $\quad \rho := 100$

$Kx := -\left(k - \frac{1}{\rho^2}\right)$ $\quad \phi := s\cdot\sqrt{Kx}$ $\quad \phi = 0.949$

$s := 3$ $\quad Kz := k$ $\quad \phi := s\cdot\sqrt{Kz}$ $\quad \phi = 0.949i$

The focusing matrix is:

$$F(K,s) := \begin{bmatrix} \cos\left(\sqrt{K}\cdot s\right) & \left(\frac{1}{\sqrt{K}}\right)\cdot\sin\left(\sqrt{K}\cdot s\right) & \frac{1-\cos\left(\sqrt{K}\cdot s\right)}{\rho\cdot K} \\ \left(-\sqrt{K}\right)\cdot\sin\left(\sqrt{K}\cdot s\right) & \cos\left(\sqrt{K}\cdot s\right) & \frac{\sin\left(\sqrt{K}\cdot s\right)}{\rho\cdot\sqrt{K}} \\ 0 & 0 & 1 \end{bmatrix}$$

According to the sign of K, one gets for instance:

$K := Kx$ $\quad \phi := s\cdot\sqrt{K}$ $\quad \phi = 0.949$

$$F(K,s) = \begin{pmatrix} 0.582 & 2.569 & 0.042 \\ -0.257 & 0.582 & 0.026 \\ 0 & 0 & 1 \end{pmatrix}$$

$K := Kz$ $\quad \phi := s\cdot\sqrt{K}$ $\quad \phi = 0.949i$

$$F(K,s) = \begin{pmatrix} 1.485 & 3.471 & 0.048 \\ 0.347 & 1.485 & 0.035 \\ 0 & 0 & 1 \end{pmatrix}$$

This last matrix was commonly obtained with:

$K := k$ $\quad k = -0.1$ $\quad \phi := s\cdot\sqrt{|Kz|}$

$$D(\phi) := \begin{bmatrix} \cosh(\phi) & \left(\frac{s}{\phi}\right)\cdot\sinh(\phi) & -\frac{1-\cosh(\phi)}{\rho\cdot|K|} \\ \left(\frac{\phi}{s}\right)\cdot\sinh(\phi) & \cosh(\phi) & \frac{\sinh(\phi)}{\rho\cdot\sqrt{|K|}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$D(\phi) = \begin{pmatrix} 1.485 & 3.471 & 0.048 \\ 0.347 & 1.485 & 0.035 \\ 0 & 0 & 1 \end{pmatrix}$$

Then taking into account that ρ is infinite for a straight section and that there is a small focusing term in a bending magnet for x, the General-Purpose-Matrix is:

$B\rho := 200 \qquad B := 2$

$$Kx := -\left[k - \left(\frac{B}{B\rho}\right)^2\right] \qquad k = -0.1 \qquad Kx = 0.1001$$

$Kz := k \qquad Kz = -0.1$

##

$\phi(K,s) := s \cdot \sqrt{K} \qquad \cos\phi(K,s) := \cos(\phi(K,s)) \qquad \sin\phi(K,s) := \sin(\phi(K,s))$

$$M(K,s,B) := \begin{bmatrix} \cos\phi(K,s) & \text{if}\left(K = 0, s, s \cdot \frac{\sin\phi(K,s)}{\phi(K,s)}\right) & B \cdot \left(\frac{1 - \cos\phi(K,s)}{B\rho \cdot K}\right) \\ \frac{-\phi(K,s)}{s} \cdot \sin\phi(K,s) & \cos\phi(K,s) & \left(\frac{B}{B\rho \cdot \sqrt{K}}\right) \cdot \sin\phi(K,s) \\ 0 & 0 & 1 \end{bmatrix}$$

GENERAL-PURPOSE-MATRIX

##

Examples:

Combined function magnet:

$$M(0.1,3,2) = \begin{pmatrix} 0.583 & 2.57 & 0.042 \\ -0.257 & 0.583 & 0.026 \\ 0 & 0 & 1 \end{pmatrix} \qquad F(Kx,3) = \begin{pmatrix} 0.582 & 2.569 & 0.042 \\ -0.257 & 0.582 & 0.026 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(Kx,3,2) = \begin{pmatrix} 0.582 & 2.569 & 0.042 \\ -0.257 & 0.582 & 0.026 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(-0.1,3,2) = \begin{pmatrix} 1.485 & 3.471 & 0.048 \\ 0.347 & 1.485 & 0.035 \\ 0 & 0 & 1 \end{pmatrix} \qquad F(Kz,3) = \begin{pmatrix} 1.485 & 3.471 & 0.048 \\ 0.347 & 1.485 & 0.035 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(Kz,3,2) = \begin{pmatrix} 1.485 & 3.471 & 0.048 \\ 0.347 & 1.485 & 0.035 \\ 0 & 0 & 1 \end{pmatrix}$$

Bending magnet:

$$k := 0 \qquad Kx := -\left[k - \left(\frac{B}{B\rho}\right)^2\right] \qquad Kx = 1 \cdot 10^{-4}$$

$$Bx := M(Kx,3,2) \qquad Bx = \begin{pmatrix} 1 & 3 & 0.045 \\ 0 & 1 & 0.03 \\ 0 & 0 & 1 \end{pmatrix}$$

$Bz := M(0,3,0)$

$$B := 0 \qquad Bz = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Pure quadrupole:

$$QF := M(0.1,3,0) \qquad QF = \begin{pmatrix} 0.583 & 2.57 & 0 \\ -0.257 & 0.583 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$QD := M(-0.1,3,0) \qquad QD = \begin{pmatrix} 1.485 & 3.471 & 0 \\ 0.347 & 1.485 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Straight section:

$$O(s) := M(0,s,0) \qquad O(3) = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The General-Purpose-Matrix can be used efficiently in various routines: see for instance FODO3 to FODO7.

Gaussian truncated distribution of a beam in an elliptic or rectangular vacuum chamber

This program simulates a gaussian beam distribution in an elliptic or a rectangular vacuum chamber.

Enter a: horizontal half aperture of the vacuum chamber
- b: vertical half aperture of the vacuum chamber
- N: number of points in x and y
- σx: horizontal rms
- σy: vertical rms
- δx:horizontal error
- δy:vertical errror
- L: horizontal width
- H: vertical height

N.B. All units in SI except otherwise stated.

$a := 2 \qquad b := 2 \qquad N := 30$

$\sigma x := 1 \qquad \sigma y := 1 \qquad \delta x := 1 \qquad \delta y := 0$

$L := 8 \cdot \sigma x \qquad H := 8 \cdot \sigma y$

****

General 2D distribution:

$$F(x,y) := \frac{\exp\left[-\left(\frac{1}{2}\right)\cdot\left[\left(\frac{x-\delta x}{\sigma x}\right)^2 + \left(\frac{y-\delta y}{\sigma y}\right)^2\right]\right]}{2\cdot\pi\cdot\sigma x\cdot\sigma y}$$

Truncated distribution in an elliptic vacuum chamber:

$$f(x,y) := \mathrm{if}\left[\left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right] > 1, 0, F(x,y)\right]$$

Truncated distribution in a rectangular vacuum chamber:

$$ff(x,y) := \mathrm{if}(|x| > a, 0, \mathrm{if}(|y| > b, 0, F(x,y)))$$

Generation of the 3D diagrams:

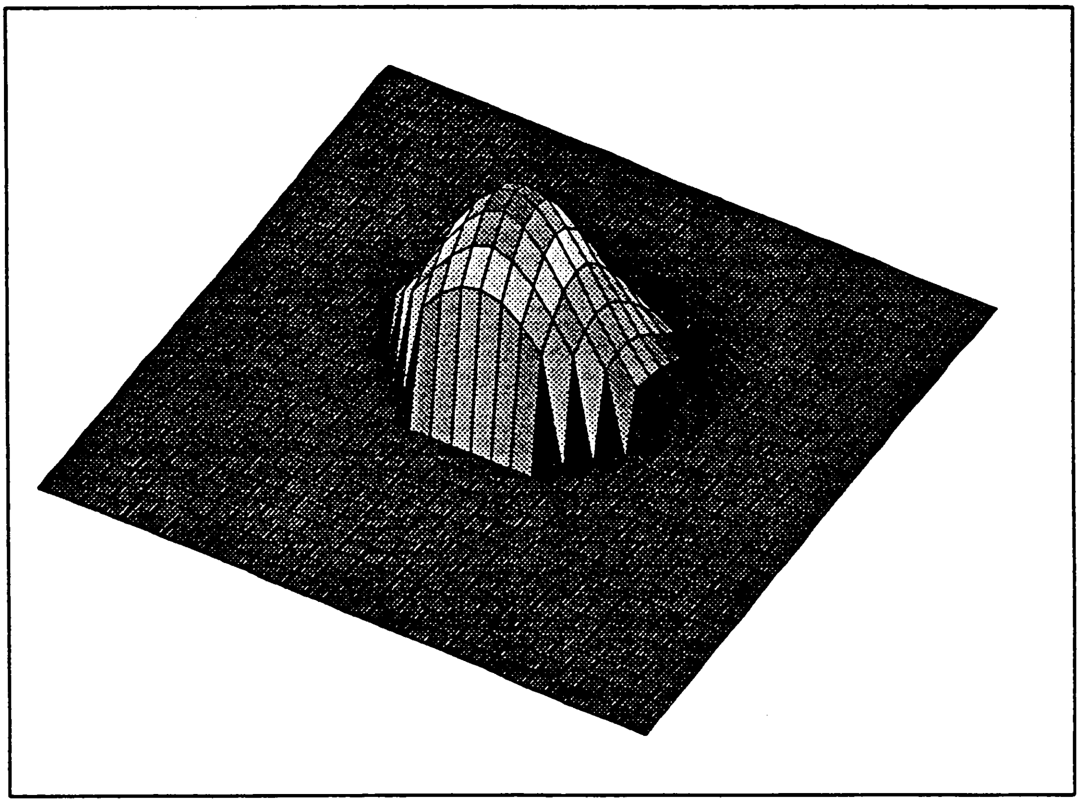
$i := 0..N \qquad j := 0..N$

$x_i := \left(\frac{i}{N} - \frac{1}{2}\right)\cdot L + \delta x \qquad y_j := \left(\frac{j}{N} - \frac{1}{2}\right)\cdot H + \delta y$

$m_{i,j} := f(x_i, y_j)$ (elliptic)

$mm_{i,j} := ff(x_i, y_j)$ (rectangular)

$M_{i,j} := F(x_i, y_j)$ (not truncated)



m

3D diagram of the truncated distribution

(To obtain the density distribution in an elliptic vacuum chamber enter m in the lower left corner of the diagram, or mm for a rectangular vacuum chamber):

Transmission::

Elliptic vacuum chamber: $\left[\sum_i \left(\sum_j m_{i,j}\right)\right] \cdot \left(\frac{L \cdot H}{N^2}\right) = 0.726$

Rectangular vacuum chamber:: $\left[\sum_i \left(\sum_j mm_{i,j}\right)\right] \cdot \left(\frac{L \cdot H}{N^2}\right) = 0.787$

$\sum_i \sum_j M_{i,j} \cdot \left(\frac{L \cdot H}{N^2}\right) = 1$ (verification: the result must be 1)

SHOWBEAM

Simulation of a beam distribution

This program simulates the distribution of a particle beam in a vacuum chamber.

All units are in SI, except otherwise stated, δp is the momentum error, δx, δy are the closed orbit errors and αp is the dispersion):
Enter:

$\beta x := 100$ m $\quad \varepsilon x := 5\cdot 10^{-6}$ m*rad $\quad \delta x := 0$ m $\quad \alpha p := 5$ m

$\beta y := 50$ m $\quad \varepsilon y := 2\cdot 10^{-6}$ m*rad $\quad \delta y := 0$ m $\quad \delta p := 10^{-3}$

Dimensions of vacuum chamber: $L := 0.1$ $\quad H := 0.05$

Radial position of scraper: $s := 0.01$

Number of calculated points: $N := 2500$

****

Calculation of the rms horizontal and vertical dimensions of the beam:

$$\sigma x := \frac{\sqrt{\varepsilon x\cdot\beta x}}{2} \qquad \sigma y := \frac{\sqrt{\varepsilon y\cdot\beta y}}{2} \qquad \sigma x = 0.01118 \qquad \sigma y = 0.005$$

Number of particles between 2 σ and 3 σ:

$$Nb3 := \exp\left(\frac{-2^2}{2}\right) - \exp\left(\frac{-3^2}{2}\right) \qquad N3 := \mathrm{floor}(Nb3\cdot N) \qquad N3 = 310$$

$$i := 0..N3 \qquad \theta3_i := \mathrm{rnd}(2\cdot\pi) \qquad r3_i := 2 + \mathrm{rnd}(1)$$

$$x3_i := r3_i\cdot\cos\left(\theta3_i\right)\cdot\sigma x + \alpha p\cdot\delta p + \delta x \qquad y3_i := r3_i\cdot\sin\left(\theta3_i\right)\cdot\sigma y + \delta y$$

Number of particles between σ and 2 σ:

$$Nb2 := \exp\left(\frac{-1}{2}\right) - \exp\left(\frac{-2^2}{2}\right) \qquad N2 := \mathrm{floor}(Nb2\cdot N) \qquad N2 = 1.177\cdot 10^3$$

$$j := 0..N2 \qquad \theta2_j := \mathrm{rnd}(2\cdot\pi) \qquad r2_j := 1 + \mathrm{rnd}(1)$$

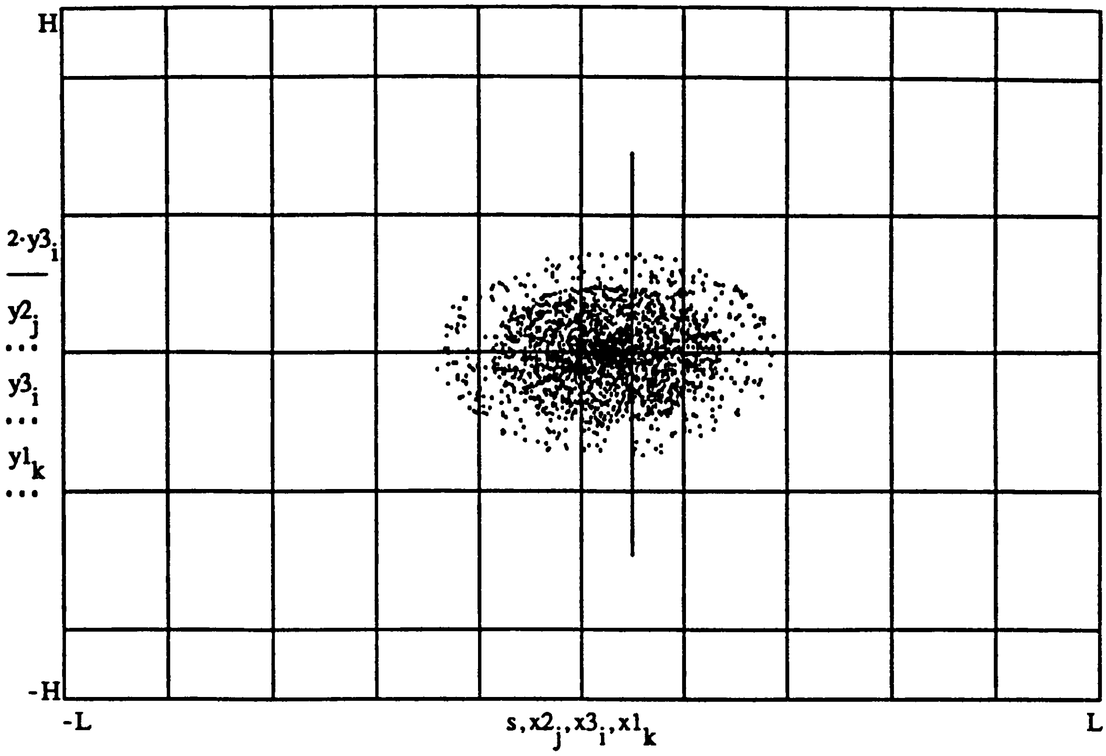
$$x2_j := r2_j\cdot\cos\left(\theta2_j\right)\cdot\sigma x + \alpha p\cdot\delta p + \delta x \qquad y2_j := r2_j\cdot\sin\left(\theta2_j\right)\cdot\sigma y + \delta y$$

Number of particles between 0 and σ:

$$Nb1 := 1 - \exp\left(\frac{-1}{2}\right) \qquad N1 := \mathrm{floor}(Nb1\cdot N) \qquad N1 = 983$$

$$k := 0..N1 \qquad \theta1_k := \mathrm{rnd}(2\cdot\pi) \qquad r1_k := \mathrm{rnd}(1)$$

$$x1_k := r1_k\cdot\cos\left(\theta1_k\right)\cdot\sigma x + \alpha p\cdot\delta p + \delta x \qquad y1_k := r1_k\cdot\sin\left(\theta1_k\right)\cdot\sigma y + \delta y$$



Simulation of the distribution of the beam
(the vertical line shows the scraper location)

Losses due to the scraper in %:
1)in a transfer line:

$$\frac{100}{2}\cdot\left(1-\operatorname{erf}\left(\frac{\frac{s}{100}-\alpha p\cdot\delta p-\delta x}{\sqrt{2}\cdot\sigma x}\right)\right)=66.94 \quad \%$$

2)in a ring:

$$\operatorname{if}\left[\frac{s}{100}>(\alpha p\cdot\delta p+\delta x),100\cdot\exp\left[\frac{-\left(\frac{\frac{s}{100}-\alpha p\cdot\delta p-\delta x}{\sigma x}\right)^{2}}{2}\right],100\right]=100 \quad \%$$

SHOWSTAC

Simulation of a stacked beam distribution

This program simulates the distribution of a stacked beam in a vacuum chamber.
All units are in the SI, except otherwise stated, δp is the momentum error δx, δy are the closed orbit errors and αp is the dispersion:

$\beta x := 100$ m $\quad \varepsilon x := 4 \cdot 10^{-6}$ m*rad $\quad \delta x := 0.002$ m $\quad \delta p := 10^{-2}$

$\beta y := 20$ m $\quad \varepsilon y := 2 \cdot 10^{-6}$ m*rad $\quad \delta y := 0$ m $\quad \alpha p := 5$ m

Dimensions of vacuum chamber: $L := 0.15$ $\quad H := 0.05$

Number of points $N := 2000$

****

Calculation of the rms beam dimensions:

$$\sigma x := \frac{\sqrt{\varepsilon x \cdot \beta x}}{2} \qquad \sigma y := \frac{\sqrt{\varepsilon y \cdot \beta y}}{2} \qquad \sigma x = 0.01 \qquad \sigma y = 0.003$$

Number of particles between 2 σ and 3 σ:

$$Nb3 := \exp\left(\frac{-2^2}{2}\right) - \exp\left(\frac{-3^2}{2}\right) \qquad N3 := \mathrm{floor}(Nb3 \cdot N) \qquad N3 = 248$$

$$i := 0..N3 \qquad \theta 3_i := \mathrm{rnd}(2 \cdot \pi) \qquad r3_i := 2 + \mathrm{rnd}(1)$$

$$x3_i := r3_i \cdot \cos(\theta 3_i) \cdot \sigma x + \alpha p \cdot (\mathrm{rnd}(\delta p) + \mathrm{rnd}(-\delta p)) + \delta x$$

$$y3_i := r3_i \cdot \sin(\theta 3_i) \cdot \sigma y + \delta y$$

Number of particles between 1 σ and 2 σ:

$$Nb2 := \exp\left(\frac{-1}{2}\right) - \exp\left(\frac{-2^2}{2}\right) \qquad N2 := \mathrm{floor}(Nb2 \cdot N) \qquad N2 = 942$$

$$j := 0..N2 \qquad \theta 2_j := \mathrm{rnd}(2 \cdot \pi) \qquad r2_j := 1 + \mathrm{rnd}(1)$$

$$x2_j := r2_j \cdot \cos(\theta 2_j) \cdot \sigma x + \alpha p \cdot (\mathrm{rnd}(\delta p) + \mathrm{rnd}(-\delta p)) + \delta x$$

$$y2_j := r2_j \cdot \sin(\theta 2_j) \cdot \sigma y + \delta y$$

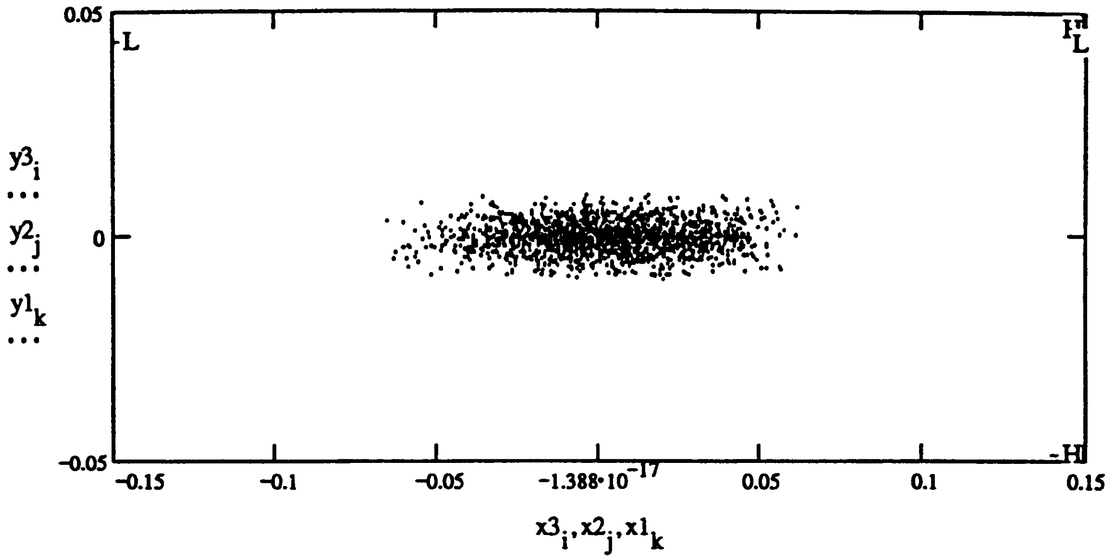
Number of particles between 0 and σ:

$$Nb1 := 1 - \exp\left(\frac{-1}{2}\right) \qquad N1 := \mathrm{floor}(Nb1 \cdot N) \qquad N1 = 786$$

$$k := 0..N1 \qquad \theta 1_k := \mathrm{rnd}(2 \cdot \pi) \qquad r1_k := \mathrm{rnd}(1)$$

$$x1_k := r1_k \cdot \cos(\theta 1_k) \cdot \sigma x + \alpha p \cdot (\mathrm{rnd}(\delta p) + \mathrm{rnd}(-\delta p)) + \delta x$$

$$y1_k := r1_k \cdot \sin(\theta 1_k) \cdot \sigma y + \delta y$$



Simulation of the stacked beam

LILENS

Linear axisymmetric lens optics

These programs develop the simple algorithms for the optics of a linear axisymmetric lens as the lenses used to collect the pbars in the target zone of the CERN-ACOL: Lithium lens, magnetic horn , plasma lens (as far as they can be considered as linear). The magnetic field is assumed here to rise (linearly) with the distance to the axis, and to be constant on a circle drawn around this axis. The common laws of the light optics and of the particle optics in the small angles approximation can then be used. Because these lenses are always placed in a transfer line, it is very useful to link the traditional concepts of light optics e.g. the focal length, the depth of focus, with the common variables used in the particles optics: betatron amplitude functions, phase advances, etc...

The role of any collecting lens being to focuse the highly divergent beam of secondary particles issued from the production target into a quasily parallel beam, its focal length will naturally be defined as the ratio of this exiting beam radius upon the divergence angle of the incoming beam. In the following it is therefore assumed that the beam is parallel to the axis in the exit plane of the lens, but there is a non-zero divergence of the beam in that plane: the product of max radius times the divergence angle is according to the Liouville theorem (or better to the Lagrange-Helmoltz law of old light optics) constant, and this applied to the focus corresponds to a non-zero size of the source, which even within the present assumptions of linearity and small angles, is definitely not point-like in the radial direction as well as in the longitudinal direction .

The complete treatement of a production target and lens optimization needs to introduce the angular distribution of secondary particles as well as the effects of the reabsorption and scattering,etc and for this, numerical calculations have been necessary: the main result of these calculations is the rule of the three diameters which is sufficient for the scope of this programs. The primary proton beam diameter, the target diameter and the secondary beam diameter reflected backwards to the focus of the lens have to be equal; the focus has to be located slightly upstream of the middle of the (long) target. The first equality is obvious, the second corresponds to the fact that secondary particles produced beyond the limits of the betatron amplitude limits of the transfer line are lost; the last rule about the location of the focus may be explained by the natural decay of the primary beam in the highly dense material of the target. It is very unfortunate that these rules cannot always be fully matched in real conditions, for practical reasons: for instance the target may be so long that it is impossible to place it sufficiently close to the Lithium lens. Another drastic limitation is the maximum current which can be achieved in any type of lens, which definitely state the maximum collected angle for this type of lens, for a given secondary particle energy.
In the programs of this chapter the following matters are covered:

LILENS: this introduction.
LILENS1: Relations between betatron functions at focus and exit plane.
LILENS2: Relations of optical parameters and lens currrent.
LILENS3: Limitations due to the distance from the focus to the lens
LILENS4: Optimisation of the length of the lens.
LILENS5: Maximum collected angles.

Transfer relations between the source and the lens exit

This file collects the formulae linking the source optical parameters and the similar quantities at exit of the lens, particularly the betatron functions which are requested to match the source to the transfer line.
Enter A: acceptance of the transfer line.
Rmax: beam radius at exit of the lens
Rmin: beam radius at source

$A := 200\cdot 10^{-6}$ Π*rad*m

$Rmax := 0.01$ $Rmin := 0.0015$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

The exit plane divergence is:

$$\alpha min := \frac{A}{Rmax} \qquad \alpha min = 0.02$$

The source divergence is:

$$\alpha max := \frac{A}{Rmin} \qquad \alpha max = 0.133$$

The focal length defined as the ratio of Rmax upon αmax becomes:

$$f := \frac{Rmax}{\alpha max} \qquad f = 0.075$$

The focal length is also Rmin divided by αmin:

$$f := \frac{Rmin}{\alpha min} \qquad f = 0.075$$

The betatron function at exit is (because the amplitude of a matched beam is equal to Rmax):

$$\beta max := \frac{Rmax^2}{A} \qquad \beta max = 0.5$$

The betatron function slope is zero at this point because the beam is parallel to the axis in that plane.

The betatron function at source is then:

$$\beta min := \frac{Rmin^2}{A} \qquad \beta min = 0.011$$

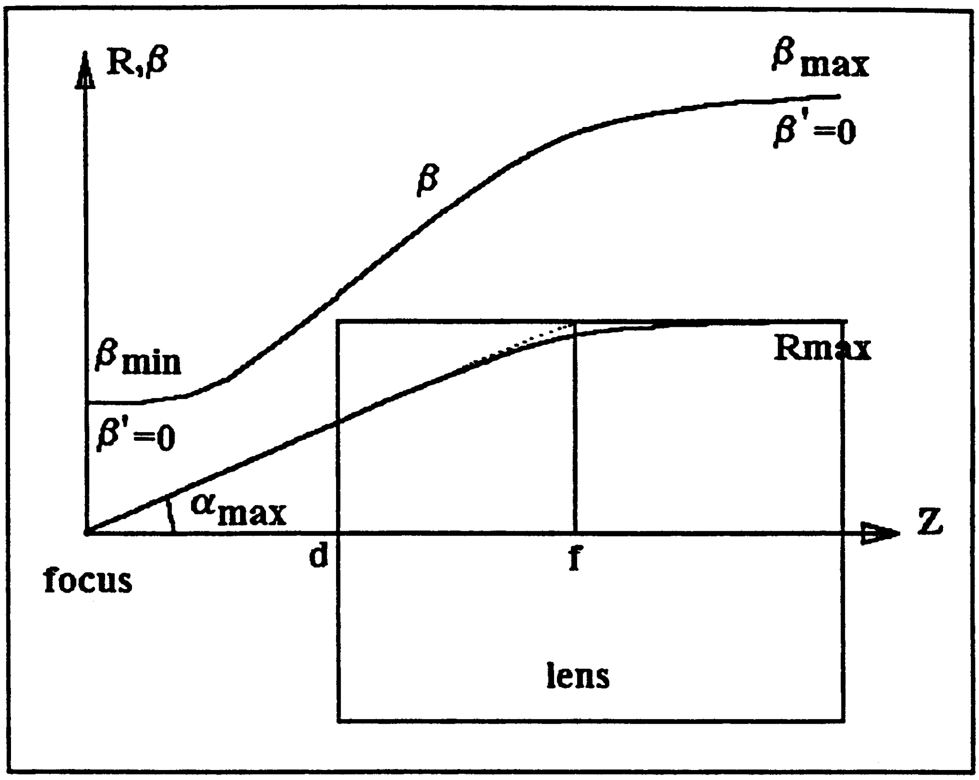
or otherwise:

$$\beta min := \frac{A}{\alpha max^2} \qquad \beta min = 0.011$$

One can point out a very simple relation:

$$f := \sqrt{\beta max\cdot\beta min} \qquad \beta max\cdot\beta min = 0.006$$

$$f^2 = 0.006$$



Definition of LILENS betatron parameters

Relations between the focal length and the lens current

This file shows the relation between the main optical parameter: the focal length and the intensity of current feeding the lens.
Enter A: acceptance of the transfer line.
Rmax: beam radius at exit of the lens
Rmin: beam radius at source
L: lens length
d: distance from focus to lens
Bρ: magnetic rigidity of the particles
I: lens current

$A := 200\cdot 10^{-6}$ Π*rad*m $\qquad L := 0.2$

$Rmax := 0.01 \qquad Rmin := 0.0015 \qquad B\rho := 11.926 \qquad I := 400000$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

In a transfer line the optics is calculated by means of products of matrices. Here to find the focal length one considers a drift space of length d followed by a lens where the gradient is g or where the common focusing coefficient is k with the definitions:

$$Bmax := \frac{\mu 0\cdot I}{2\cdot\pi\cdot Rmax} \qquad g := \frac{Bmax}{Rmax} \qquad k := \frac{g}{B\rho} \qquad k = 67.08$$

The matrix for a focusing lens is the same as the matrix of a quadrupole (for the coordinate where this quad is focusing):

$$Mx(k,L) := \begin{bmatrix} \cos\left(\sqrt{k}\cdot L\right) & \sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right) \\ \dfrac{-\sin\left(\sqrt{k}\cdot L\right)}{\sqrt{k}} & \cos\left(\sqrt{k}\cdot L\right) \end{bmatrix}$$

The matrix for a drift space is:

$$O(d) := \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

The product of matrices in the reverse order is:

$$M(k,L,d) := \begin{bmatrix} \cos\left(\sqrt{k}\cdot L\right) & \dfrac{\sin\left(\sqrt{k}\cdot L\right)}{\sqrt{k}} \\ -\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right) & \cos\left(\sqrt{k}\cdot L\right) \end{bmatrix}\cdot\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$M(k,L,d) := \begin{bmatrix} \cos\left(\sqrt{k}\cdot L\right) & \dfrac{\left(\cos\left(\sqrt{k}\cdot L\right)\cdot d\cdot\sqrt{k} + \sin\left(\sqrt{k}\cdot L\right)\right)}{\sqrt{k}} \\ -\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right) & -\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right)\cdot d + \cos\left(\sqrt{k}\cdot L\right) \end{bmatrix}$$

The focal length is therefore obtained by applying this matrix to a trajectory issued from the origin with the coordinate 0 and slope αmax:

$$\begin{bmatrix} \cos\left(\sqrt{k}\cdot L\right) & \dfrac{\left(\cos\left(\sqrt{k}\cdot L\right)\cdot d\cdot\sqrt{k} + \sin\left(\sqrt{k}\cdot L\right)\right)}{\sqrt{k}} \\ -\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right) & -\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right)\cdot d + \cos\left(\sqrt{k}\cdot L\right) \end{bmatrix}\cdot\begin{pmatrix} 0 \\ \alpha max \end{pmatrix}$$

$$\begin{bmatrix} \dfrac{\left(\cos\left(\sqrt{k}\cdot L\right)\cdot d\cdot\sqrt{k} + \sin\left(\sqrt{k}\cdot L\right)\right)}{\sqrt{k}}\cdot\alpha max \\ -\left(\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right)\cdot d - \cos\left(\sqrt{k}\cdot L\right)\right)\cdot\alpha max \end{bmatrix}$$

The slope being zero at exit the second term of this matrix is zero and then:

$$d := \frac{1}{\sqrt{k}\cdot\tan\left(\sqrt{k}\cdot L\right)}$$

Replacing in the first term of the last matrix which is equal to Rmax, one gets f=Rmax/αmax:

$$f := \frac{1}{\sqrt{k}\cdot\sin\left(\sqrt{k}\cdot L\right)}$$

This holds for the other radial coordinate, and in fact as the lens is axisymmetric, for any radial direction. One may deduce from that, by substracting the two previous squared equations:

$$k := \frac{1}{f^2 - d^2}$$

or even by introducing the current density j:

$$k := \frac{\mu 0\cdot j}{2\cdot B\rho}$$

One may for instance draw the curve of f(I):

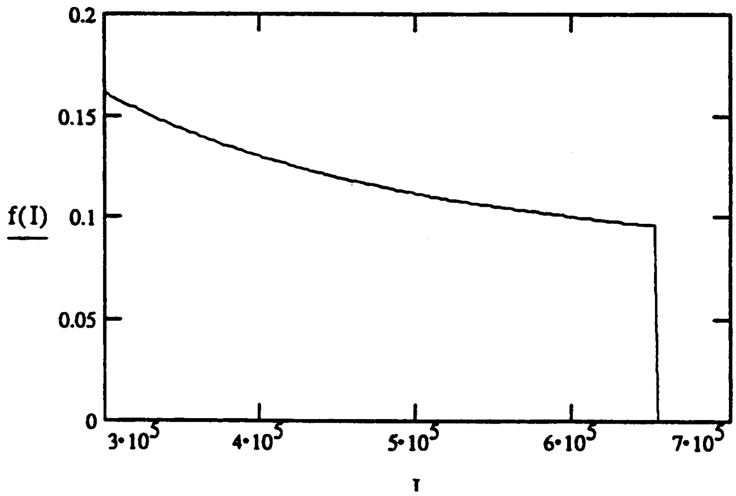
$$k(I) := \frac{\mu 0\cdot I}{2\cdot\pi\cdot B\rho\cdot Rmax^2}$$

$$d(I) := \frac{1}{\sqrt{k(I)}\cdot\tan\left(\sqrt{k(I)}\cdot L\right)}$$

$$f(I) := if\left(d(I)>0, \frac{1}{\sqrt{k(I)}\cdot\sin\left(\sqrt{k(I)}\cdot L\right)}, 0\right)$$

for a range: $I := 300000, 302000 .. 700000$

and constant length and radius of the lens: $L = 0.15$ $Rmax = 0.01$



Limitations due to the distance from the focus to the lens

This file states the limitation due to the distance d already introduced in LILENS2. As the target length is approximately equal to the absorption length of the metal of the target (e.g. 6 to 12 cm depending of the chosen material), it is difficult to place the target, for instance, in front of a Lithium lens, according to the requirement to have the focus slightly upstream of the middle of the target.

Enter A: acceptance of the transfer line.
Rmax: beam radius at exit of the lens
Rmin: beam radius at source
L: lens length
Bρ: magnetic rigidity of the particles
I: lens current

$A := 200\cdot 10^{-6}$ Π*rad*m $\quad Rmax := 0.01 \quad Rmin := 0.0015$

$B\rho := 11.926 \quad L := 0.15 \quad I := 400000$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

It is equivalent to use the focusing coefficient k or the characteristic angle of the lens θ as we have:

$$k := \frac{g}{B\rho} \qquad \text{or} \qquad k := \frac{\mu 0\cdot I}{2\cdot\pi\cdot B\rho\cdot Rmax^2}$$

and

$$\theta := \sqrt{\frac{\mu 0\cdot I}{2\cdot\pi\cdot B\rho}}$$

therefore:

$$\theta := \sqrt{k}\cdot Rmax$$

It is better to keep the characteristic angle as it is closely related to the maximum collected angle.

$$d := \frac{1}{\sqrt{k}\cdot\tan\left(\sqrt{k}\cdot L\right)}$$

Replacing k, one gets :

$$d := \frac{Rmax}{\theta}\cdot\frac{1}{\tan\left(\theta\cdot\frac{L}{Rmax}\right)} \qquad d = 0.044$$

For the given values of Rmax, L and θ (or current I) the front space d is too short :

One should change I to get d=0.06 for instance:

$L := 0.15 \quad Rmax := 0.01$

$$\theta := \mathrm{root}\left(\frac{Rmax}{\theta}\cdot\frac{1}{\tan\left(\theta\cdot\frac{L}{Rmax}\right)} - 0.06, \theta\right) \qquad \theta = 0.076$$

$$I := \frac{2\cdot\pi\cdot B\rho\cdot\theta^2}{\mu 0} \qquad I = 3.464\cdot 10^5$$

For a Lithium lens this yields a max current: $Imax := \frac{I}{0.8}$ $\qquad Imax = 4.33\cdot 10^5$

Also:

$$\alpha max := \theta\cdot\sin\left(\theta\cdot\frac{L}{Rmax}\right) \qquad \alpha max = 0.069$$

Optimization of the length of the lens

This file considers a possible way to get rid of the limitation due to the distance d already considered in LILENS3. It is difficult to place the target, for instance, in front of a Lithium lens, according to the requirement to have the focus slightly upstream of the middle of the target. The length of the lens is assumed here to be free at the design stage of the lens.
Enter A: acceptance of the transfer line.

Rmax: beam radius at exit of the lens
Rmin: beam radius at source
d: distance from focus to lens
$B\rho$: magnetic rigidity of the particles
I: lens current

$A := 200 \cdot 10^{-6}$ Π*rad*m $\qquad d := 0.06$

$Rmax := 0.01 \qquad Rmin := 0.0015 \qquad B\rho := 11.926 \qquad I := 400000$

****

$\mu 0 := 4 \cdot \pi \cdot 10^{-7}$

It is equivalent to use the focusing coefficient k or the characteristic angle of the lens θ as we have:

$$k := \frac{g}{B\rho} \qquad \text{or} \qquad k := \frac{\mu 0 \cdot I}{2 \cdot \pi \cdot B\rho \cdot Rmax^2}$$

and

$$\theta := \sqrt{\frac{\mu 0 \cdot I}{2 \cdot \pi \cdot B\rho}}$$

therefore:

$$\theta := \sqrt{k} \cdot Rmax$$

It is better to keep the characteristic angle as it is closely related to the maximum collected angle.

$$d := \frac{1}{\sqrt{k} \cdot \tan\left(\sqrt{k} \cdot L\right)}$$

Replacing k, one gets :

$$d := \frac{Rmax}{\theta} \cdot \frac{1}{\tan\left(\theta \cdot \frac{L}{Rmax}\right)}$$

As: $\quad \alpha max := \frac{Rmax}{f} \quad$ it is possible to express αmax:

$$\alpha max := \theta \cdot \sin\left(\theta \cdot \frac{L}{Rmax}\right)$$

Rmax, d, I being assumed to be fixed:

$Rmax := 0.01 \qquad d := 0.06 \qquad \theta = 0.082$

$$L := \mathrm{root}\left(\frac{Rmax}{\theta} \cdot \frac{1}{\tan\left(\theta \cdot \frac{L}{Rmax}\right)} - 0.06, L\right) \qquad L = 0.903$$

The length of the lens has to be shorter than in LILENS3 to get the focus in the right location and then:

$$\alpha max = 0.077$$

For a Lithium lens, this is for a max current Imax :

$$Imax := \frac{I}{0.8} \qquad Imax = 5 \cdot 10^5$$

Maximum collected angle

This file gives the maximum collected angle in a linear lens with the constraint due to the distance d already introduced in LILENS2. As already said the target length is approximately equal to the absorption length of the metal of the target (e.g. 6 to 12 cm depending of the chosen material), it is difficult to place the target, for instance, in front of a Lithium lens, according to the requirement to have the focus slightly upstream of the middle of the target. Here we investigate the problem to find the max collected angle for given d and Rmax(which corresponds to a fixed transfer line), and again with the assumption of a parallel beam at exit of the lens:

Enter A: acceptance of the transfer line.

Rmax: beam radius at exit of the lens

Rmin: beam radius at source

d: front space between focus and lens

Bρ: magnetic rigidity of the particles

$A := 200\cdot10^{-6}$ Π*rad*m $\quad Rmax := 0.01 \quad Rmin := 0.0015$

$B\rho := 11.926 \quad d := 0.06$

****

$\mu0 := 4\cdot\pi\cdot10^{-7}$

It is equivalent to use the focusing coefficient k or the characteristic angle of the lens θ as we have:

$$k := \frac{g}{B\rho} \qquad \text{or} \qquad k := \frac{\mu0\cdot I}{2\cdot\pi\cdot B\rho\cdot Rmax^2}$$

and

$$\theta := \sqrt{\frac{\mu0\cdot I}{2\cdot\pi\cdot B\rho}}$$

therefore:

$$\theta := \sqrt{k}\cdot Rmax$$

It is better to keep the characteristic angle as it is closely related to the maximum collected angle.

$$d := \frac{1}{\sqrt{k}\cdot\tan\left(\sqrt{k}\cdot L\right)}$$

Replacing k, one gets :

$$d := \frac{Rmax}{\theta}\cdot\frac{1}{\tan\left(\theta\cdot\frac{L}{Rmax}\right)}$$

Similarly we have:

$$f := \frac{Rmax}{\theta}\cdot\frac{1}{\sin\left(\theta\cdot\frac{L}{Rmax}\right)}$$

But f is Rmax/αmax, therefore:

$$\alpha max := \theta\cdot\sin\left(\theta\cdot\frac{L}{Rmax}\right)$$

Then:

$$Rmax := d\cdot\frac{\alpha max}{\sqrt{1-\frac{\alpha max^2}{\theta^2}}} \qquad \text{or inversely:} \qquad \alpha max := \frac{1}{\sqrt{\frac{d^2}{Rmax^2}+\frac{1}{\theta^2}}}$$

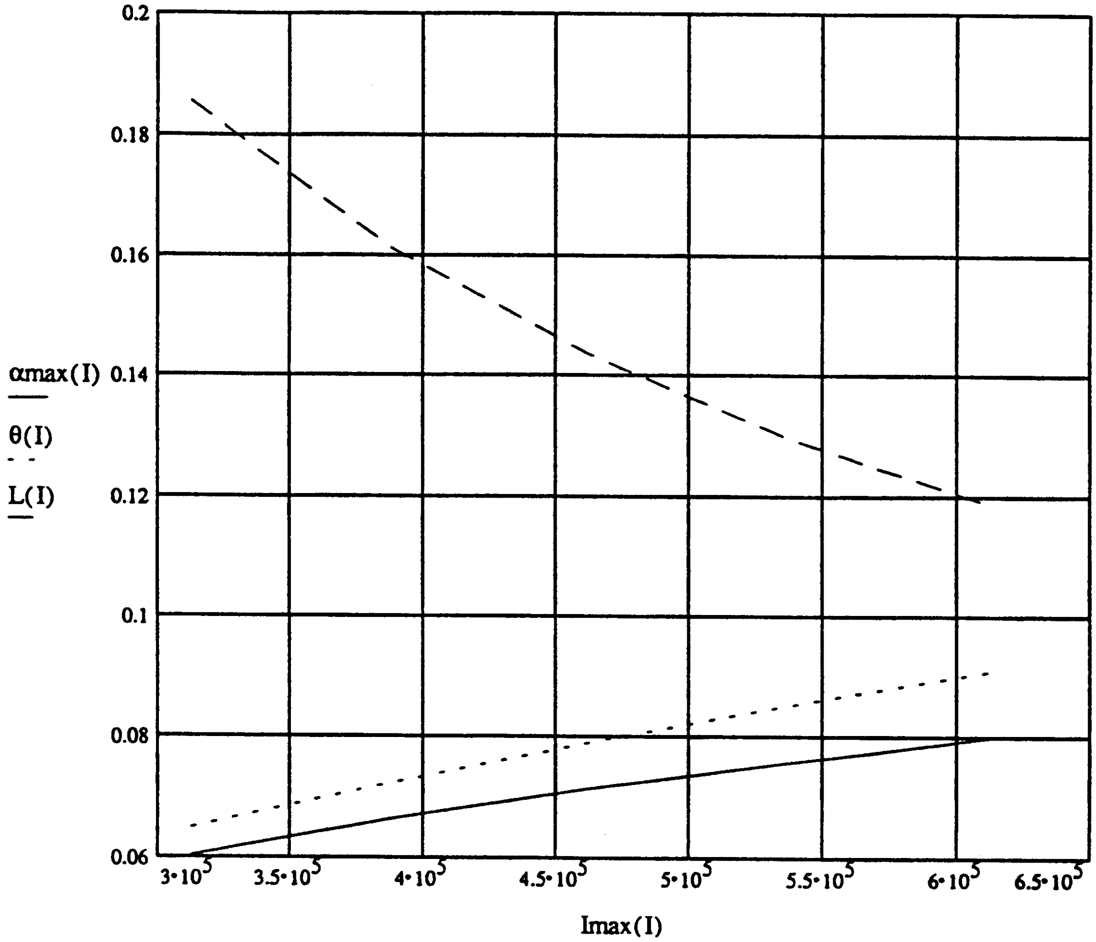
If d=0 αmax is equal to θ: for an immersed focus the maximum collected angle is the characteristic angle of the lens and is only dependant of the current for a given particle momentum. It is now possible to draw the curve of the max collected angle according to the current for a given front space d, with the requirement to get a parallel beam at exit. The length has to be adapted accordingly:

For a range: $I := 250000, 310000 .. 500000$

$d := 0.06 \qquad Rmax := 0.01$

$$\theta(I) := \sqrt{\frac{\mu 0 \cdot I}{2 \cdot \pi \cdot B\rho}} \qquad \alpha max(I) := \frac{1}{\sqrt{\frac{d^2}{Rmax^2} + \frac{2 \cdot \pi \cdot B\rho}{\mu 0 \cdot I}}}$$

$$L(I) := \frac{Rmax}{\theta(I)} \cdot atan\left(\frac{Rmax}{d \cdot \theta(I)}\right) \qquad Imax(I) := \frac{I}{0.8}$$



Max collected angle, characteristic angle and length
as functions of the max current in a Lithium lens

HORN

Magnetic horn for pbar collection

These files present the essential algorithms to prepare the design of a magnetic biconical horn similar to the one which has been used in the target zone of CERN-ACOL for pbar production.
The pbars are produced by a primary beam of protons hitting a metallic target, with large angles: it is the role of the magnetic horn to focuse these pbars and create a parallel beam. The shape is defined according to two constraints: to be not reentrant (i.e. its diameter is maximum at ends), and to have a constant focal length (not varying with the angle of the trajectory with the axis):

Horn1: Theoretical shape to keep the focal length constant.
Horn2:Shape including a neck of minimal radius.
Horn3:Shape and thickness in parametric form.
Horn4:Interpolated function for the shape.
Horn5:Estimated mechanical stress in a magnetic horn.

The trajectories in and around a magnetic horn present a real challenge to the designer: the magnetic field is decreasing as the inverse of the distance to the axis and the equations of the movement are therefore non-linear, and secondly the angles are so large that the common paraxial approximation is no longer valid. The treatement presented here use the basic differential equations of the movement of a particle in a general field and solves these equations by the classical Runge-Kutta (or similar) routines as the ones the reader may find in the enclosed chapter on this subject (cf. SRK).

Horntra1: Horn meridian trajectories (approximative solution).
Horntra2: Horn skew trajectories (rigorous solution making use of data from Horn4).
Horntra3: Traversal lengthes of trajectories issued from the focus.

Shape of a biconical magnetic horn

This program delivers the parametric equations of the theoretical shape of a biconical magnetic horn, with a focal length which does not vary with the trajectory angle.
All units in SI except otherwise stated,
Enter:

$I := 400000$ $\qquad$ $Rmax := 0.03$

$B\rho := 11.675$ $\qquad$ $n := 30$

****

$\mu0 := 4\cdot\pi\cdot10^{-7}$

Characteristic angle: $\theta := \sqrt{\left[\frac{\mu0\cdot I}{(2\cdot\pi\cdot B\rho)}\right]}$ $\qquad$ $\theta = 0.082778$

Focal length: $f := \frac{Rmax}{\theta}$ $\qquad$ $f = 0.362414$

Range: $i := 0..n$ $\qquad$ $\alpha_i := \frac{i}{n}\cdot\theta$

Parametric equations:

1st cone:

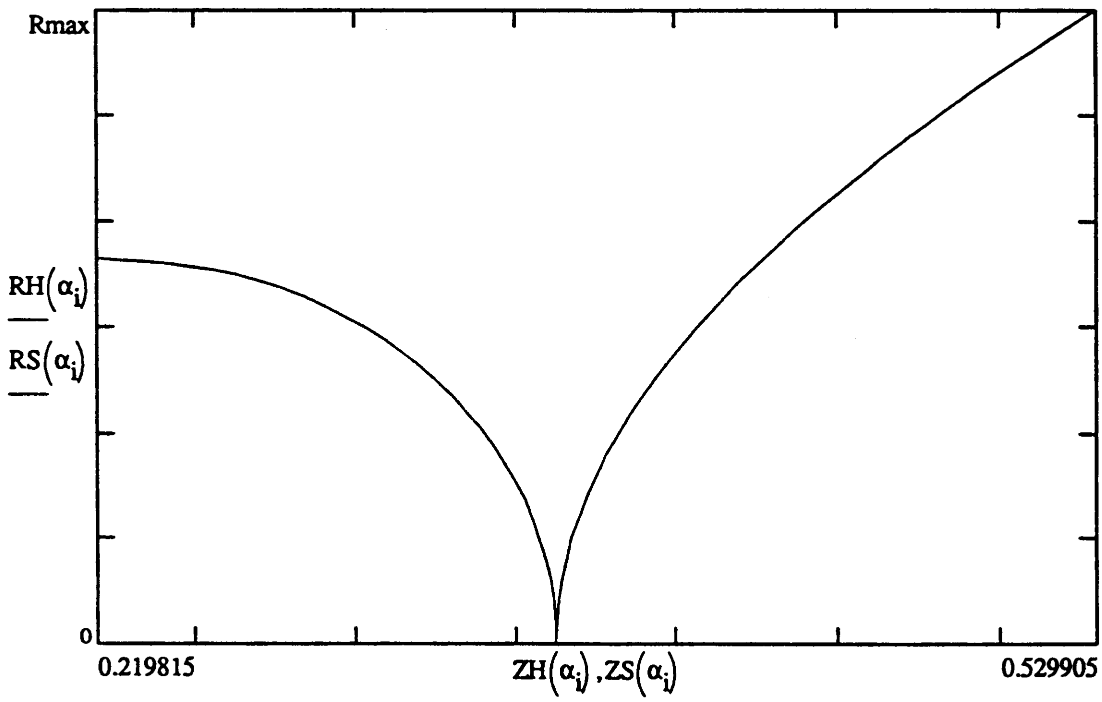
$$RH(\alpha) := f\cdot\alpha\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

$$ZH(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

2nd cone:

$$RS(\alpha) := f\cdot\alpha$$

$$ZS(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right) + f\cdot\left(\frac{\alpha}{\theta}\right)\cdot\sqrt{\frac{\pi}{2}}\cdot\mathrm{erf}\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)$$



Theoretical horn shape

Shape of a biconical magnetic horn

This program delivers the parametric equations of a biconical magnetic horn, with a neck of radius Rmin.
All units in SI except otherwise stated,
Enter:

$I := 400000$ $\quad$ $Rmax := 0.03$ $\quad$ $Rmin := 0.006$

$B\rho := 11.675$ $\quad$ $n := 30$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

Characteristic angle: $\theta := \sqrt{\left[\dfrac{\mu 0\cdot I}{(2\cdot\pi\cdot B\rho)}\right]}$ $\quad$ $\theta = 0.082778$

Focal length: $f := \dfrac{Rmax}{\theta}$ $\quad$ $f = 0.362414$

Range: $i := 0..n$ $\quad$ $\alpha_i := \dfrac{i}{n}\cdot\theta$

Parametric equations:

1st cone:

$$RRH(\alpha) := f\cdot\alpha\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

$$RH(\alpha) := \text{if}(RRH(\alpha)>Rmin, RRH(\alpha), Rmin)$$

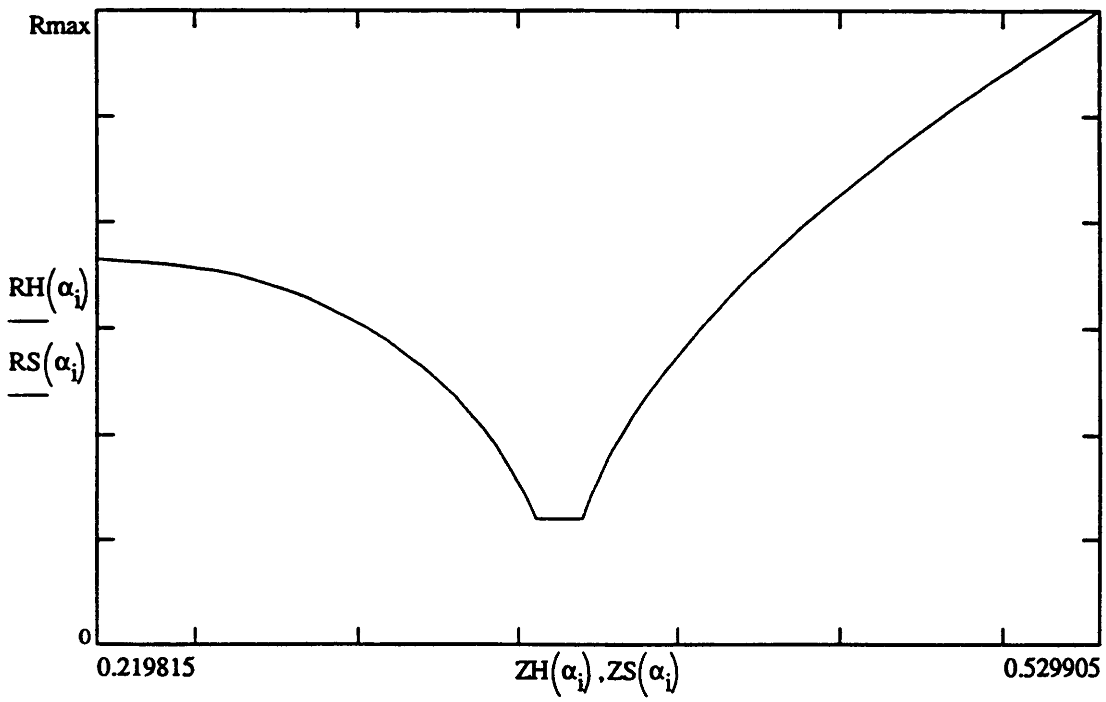
$$ZH(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

2nd cone:

$$RRS(\alpha) := f\cdot\alpha$$

$$RS(\alpha) := \text{if}(RRS(\alpha)>Rmin, RRS(\alpha), Rmin)$$

$$ZS(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right) + f\cdot\left(\frac{\alpha}{\theta}\right)\cdot\sqrt{\frac{\pi}{2}}\cdot\text{erf}\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)$$



Horn shape

HORN3

Shape and thickness of a biconical magnetic horn

This program delivers the parametric equations for the inner and the outer sides of a biconical magnetic horn.
All units in SI except otherwise stated,
Enter:

$I := 400000$ $\quad Rmax := 0.03$ $\quad Rmin := 0.006$

$B\rho := 11.675$ $\quad emax := 0.003$ $\quad emin := 0.001$ $\quad n := 30$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

Characteristic angle:

$$\theta := \sqrt{\left[\frac{\mu 0\cdot I}{(2\cdot\pi\cdot B\rho)}\right]} \qquad \theta = 0.082778$$

Focal length:

$$f := \frac{Rmax}{\theta} \qquad f = 0.362414$$

Range: $\quad i := 0..n \qquad \alpha_i := \frac{i}{n}\cdot\theta$

Parametric equations:

1st cone:

$$RRH(\alpha) := f\cdot\alpha\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

$$RH(\alpha) := if(RRH(\alpha)>Rmin, RRH(\alpha), Rmin)$$

$$ZH(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

2nd cone:

$$RRS(\alpha) := f\cdot\alpha$$

$$RS(\alpha) := if(RRS(\alpha)>Rmin, RRS(\alpha), Rmin)$$

$$ZS(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right) + f\cdot\left(\frac{\alpha}{\theta}\right)\cdot\sqrt{\frac{\pi}{2}}\cdot erf\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)$$

Thickness definition:

$$eeH(\alpha) := \frac{emin\cdot Rmax}{RH(\alpha)} \qquad eeS(\alpha) := \frac{emin\cdot Rmax}{RS(\alpha)}$$

$$eH(\alpha) := if(eeH(\alpha)<emax, eeH(\alpha), emax)$$

$$eS(\alpha) := if(eeS(\alpha)<emax, eeS(\alpha), emax)$$

Parametric equations of the inner side of the horn:
1st cone:

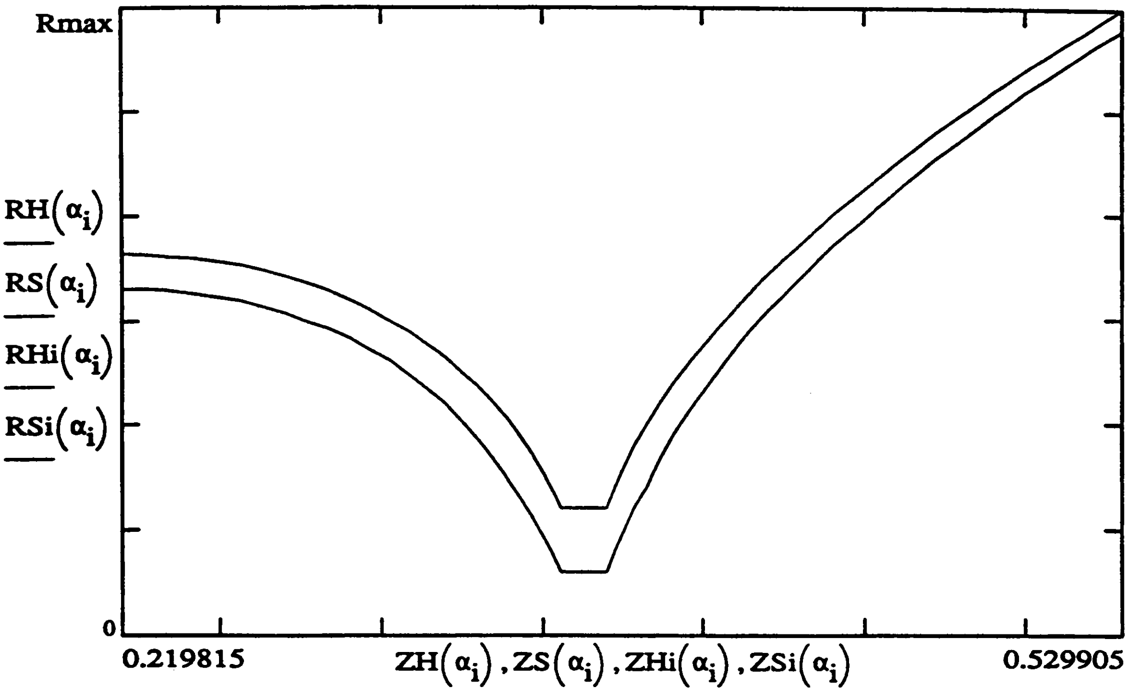
$$RHi(\alpha) := RH(\alpha) - eH(\alpha)$$

$$ZHi(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

2nd cone:

$$RSi(\alpha) := RS(\alpha) - eS(\alpha)$$

$$ZSi(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right) + f\cdot\left(\frac{\alpha}{\theta}\right)\cdot\sqrt{\frac{\pi}{2}}\cdot erf\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)$$



Horn shape and thickness

HORN4

Interpolated shape of a biconical magnetic horn

This program delivers a interpolation function R(Z) for a biconical magnetic horn.
All units in SI, except otherwise stated,
Enter:

$I := 400000$ $\quad$ $Rmax := 0.03$ $\quad$ $Rmin := 0.006$

$B\rho := 11.675$ T*m $\quad$ $emax := 0.003$ $\quad$ $emin := 0.001$ $\quad$ $n := 30$

****

$\mu 0 := 4 \cdot \pi \cdot 10^{-7}$

Characteristic angle:

$$\theta := \sqrt{\left[\frac{\mu 0 \cdot I}{(2 \cdot \pi \cdot B\rho)}\right]} \qquad \theta = 0.082778$$

Focal length:

$$f := \frac{Rmax}{\theta} \qquad f = 0.362414$$

Range: $\quad i := 0..n \qquad \alpha_i := \frac{i}{n} \cdot \theta$

Parametric equations:

1st cone:

$$RRH(\alpha) := f \cdot \alpha \cdot \exp\left(-\frac{\alpha^2}{2 \cdot \theta^2}\right)$$

$$RH(\alpha) := \mathrm{if}(RRH(\alpha) > Rmin, RRH(\alpha), Rmin)$$

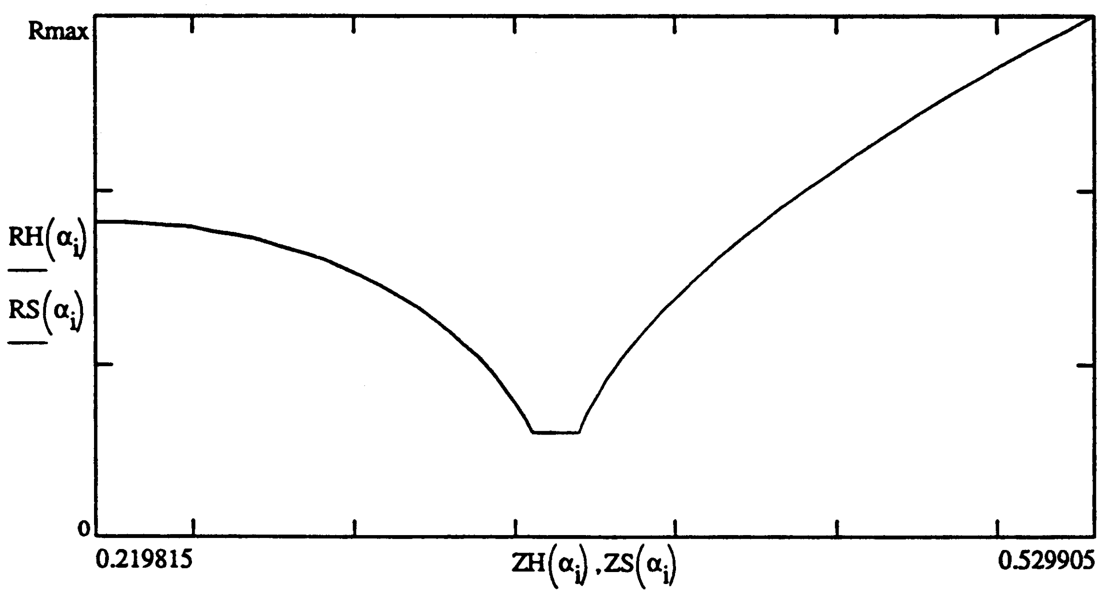
$$ZH(\alpha) := f \cdot \exp\left(-\frac{\alpha^2}{2 \cdot \theta^2}\right)$$

2nd cone:

$$RRS(\alpha) := f \cdot \alpha$$

$$RS(\alpha) := \mathrm{if}(RRS(\alpha) > Rmin, RRS(\alpha), Rmin)$$

$$ZS(\alpha) := f \cdot \exp\left(-\frac{\alpha^2}{2 \cdot \theta^2}\right) + f \cdot \left(\frac{\alpha}{\theta}\right) \cdot \sqrt{\frac{\pi}{2}} \cdot \mathrm{erf}\left(\frac{\alpha}{\sqrt{2} \cdot \theta}\right)$$



Horn shape according to the parametric equations

Auxiliary vectors for the interpolation:

$i := 0..n-1$

$Rh_i := RH(\theta - \alpha_i)$ $\quad Rs_i := RS(\alpha_i)$ $\quad Rs_n := RS(\alpha_n)$

$Zh_i := ZH(\theta - \alpha_i)$ $\quad Zs_i := ZS(\alpha_i)$ $\quad Zs_n := ZS(\alpha_n)$

Merging of the auxiliary vectors:

$RR := \text{augment}(Rh^T, Rs^T)^T$ $\quad ZZ := \text{augment}(Zh^T, Zs^T)^T$

New range: $\quad j := 0..\text{last}(RR)$ $\quad \text{last}(RR) = 60$

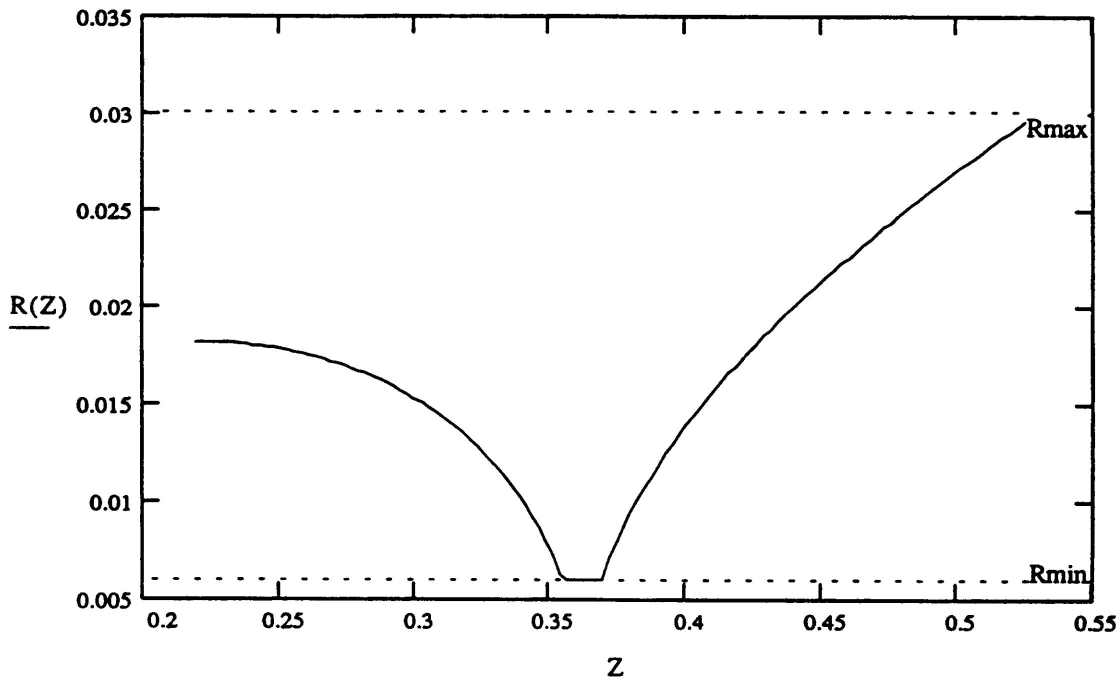
Interpolation:

$R(Z) := \text{interp}(\text{pspline}(ZZ, RR), ZZ, RR, Z)$

$R(0.3) = 0.015268$

Range of Z: $\quad ZZ_0 = 0.219815$ $\quad ZZ_{\text{last}(ZZ)} = 0.529905$

$$Z := ZZ_0, ZZ_0 + \frac{ZZ_{\text{last}(ZZ)} - ZZ_0}{4 \cdot n} .. ZZ_{\text{last}(ZZ)}$$



Horn shape according to the interpolation function

Force, pressure, stresses in a magnetic horn

This file gives a set of practical formulae relevant to the design of a magnetic horn, as well as remarks about the mechanical problems raised by a magnetic horn.

One considers first a cylindrical hollow conductor fed by a DC current of intensity I. The current density j being then constant between the inner and outer radii R0 and R1, the current in a cylinder of radius R between R0 and R1 is:

$$I(R) := j\cdot\pi\cdot(R^2 - R0^2)$$

By applying the Ampère theorem:

$$B(R) := \frac{\mu_0 \cdot I(R)}{2\cdot\pi\cdot R}$$

On the external surface of the conductor:

$$I(R1) := j\cdot\pi\cdot(R1^2 - R0^2)$$

which yields:

$$j := \frac{I(R1)}{\pi\cdot(R1^2 - R0^2)}$$

The magnetic field is then inside the conductor:

$$B(R) := \frac{\mu_0 \cdot I(R1)\cdot(R^2 - R0^2)}{2\cdot\pi\cdot R\cdot(R1^2 - R0^2)}$$

Outside the conductor and inside another coaxial return conductor the field is:

$$B_{ext}(R) := \mu_0 \cdot \frac{I(R1)}{2\cdot\pi\cdot R}$$

One has also:

$$I := I(R1)$$

The electromagnetic force is a volume force given by the cross product j*B. For an elementary volume between two meridian planes, by summing up between R0 and R1 and for one meter length:

$$F := \frac{\mu_0 \cdot I^2}{\left[2\cdot\pi^2\cdot(R1^2 - R0^2)^2\right]}\cdot\left[\left(\frac{R1^3 - R0^3}{3}\right) - R0^2\cdot(R1 - R0)\right]$$

$$F := \frac{(R1 + 2\cdot R0)\cdot\mu_0\cdot I^2}{6\cdot(R1 + R0)^2\cdot\pi^2}$$

The equivalent pressure on the external surface is then (by considering the material of the conductor as being uncompressible):

$$p := \frac{(R1 + 2\cdot R0)\cdot\mu_0\cdot I^2}{6\cdot R1\cdot(R1 + R0)^2\cdot\pi^2}$$

By introducing the magnetic field on the outer surface:

$$p := \frac{2\cdot(R1 + 2\cdot R0)\cdot R1}{3\cdot(R1 + R0)^2}\cdot\frac{B(R1)^2}{\mu_0}$$

which yields for a thin tube (R0~R1):

$$p := \frac{B(R1)^2}{2\cdot\mu_0}$$

For a thick tube (R0=0) the pressure becomes:

$$p := \frac{4}{3}\cdot\frac{B(R1)^2}{2\cdot\mu_0}$$

It is clearly seen that replacing the electromagnetic force by an external pressure can lead to an underestimate of the effect of this force (this is equivalent to considering only one term in the Maxwell tensor).
A rigorous analysis of the mechanical stress differential equation would need to express the elementary volume force, and to solve the resultant differential equation. One considers here only the simple solutions as given in textbooks about elasticity and resistance of materials for the model of a thick cylinder, under an external pressure, and keeping a constant length:

$$\sigma_t(R) := -p\cdot\left[\frac{\left(\frac{R1^2}{R0^2}+\frac{R1^2}{R^2}\right)}{\left[\frac{R1^{(2)}}{R0^2}-1\right]}\right]$$

$$\sigma_r(R) := -p\cdot\left[\frac{\left(\frac{R1^2}{R0^2}-\frac{R1^2}{R^2}\right)}{\left[\frac{R1^{(2)}}{R0^2}-1\right]}\right]$$

$$\sigma_z(R) := -p\cdot\left[\frac{\left(\frac{R1^2}{R0^2}\right)}{\left[\frac{R1^{(2)}}{R0^2}-1\right]}\right]$$

These are compression stresses, the circumferential stress being always higher than the others and achieving a max value on the inner surface of the tube:

$$\sigma_{max} := p\cdot\frac{(2\cdot R1^2)}{(R1^2 - R0^2)}$$

For a tube without a hole the max stress rises up to:

$$\sigma_{max} := 2\cdot p$$

For a thin tube (of thickness e):

$$\sigma_{max} := p\cdot\frac{R1}{e}$$

The different following cases can be considered:
1)Conductor without a hole, DC current:

$$\sigma_{max} := \frac{8}{3}\cdot\frac{B(R1)^2}{2\cdot\mu_0}$$

2)Conductor without a hole, short pulse current:

$$\sigma_{max} := 2\cdot\frac{B(R1)^2}{2\cdot\mu_0}$$

3) Hollow conductor:

$$\sigma_{max} := \frac{R1}{e}\cdot\frac{B(R1)^2}{2\cdot\mu_0}$$

but there may be buckling.

The preceding formulae allow to calculate the order of magnitude of the max stress in a magnetic horn and to compare them with the max allowed stress in case of fatigue. In the case of vibrations or pulse loading, and for aluminium alloys these values are very inaccurate and long and careful tests are necessary. It is nevertheless interesting to consider the case of a hollow conductor, and by introducing the field, this yields:

$$\sigma_{max} := \frac{\mu_0 \cdot I^2}{8 \cdot \pi^2 \cdot e \cdot R1}$$

In order to have a constant stress along the axis for a varying tube radius, the product e*R1 must be constant. This rule has been chosen in the case of the biconical magnetic horn (cf HORN3), but this is no longer valid in the case of buckling for which the critical pressure is, for a hollow and thin tube:

$$p_k := \frac{1}{4} \cdot \left[\frac{E}{\left(1 - \nu^2\right)} \cdot \left(\frac{e}{R1}\right)^3 \right]$$

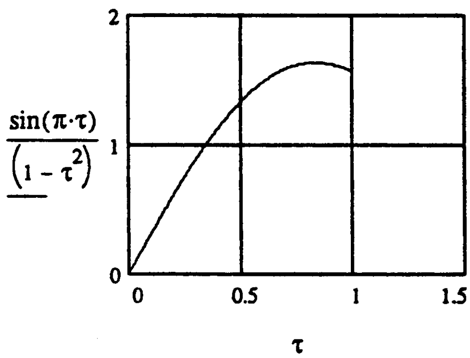
(E is the elasticity modulus, ν is the Poisson modulus)

from which, by introducing the current, it is the product $e^3/R1$ which must be constant. This is anyway approximative because in the case of the magnetic horn the magnetic pressure is not uniform and the buckling mode is not known; the curvature of the surface along the axis and the proximity of the flanges improve the withstanding to buckling. Here again tests are needed.

The following remarks about pulse loading are relevant: for a half-sinus pulse and for the axisymmetric mode:

$$\sigma_{max}(\tau) := \frac{10^{-7} \cdot I^2}{2 \cdot \pi \cdot R1 \cdot e} \cdot \frac{\sin(\pi \cdot \tau)}{\left(1 - \tau^2\right)}$$

where τ is the ratio of the pulse duration and the mechanical period (supposedly <1). The variation of pulse stress is given in the following curve:



This factor becomes proportional to τ for the small values of τ from which one gets:

$$\sigma_{max}(\tau) := \frac{10^{-7} \cdot I^2}{2 \cdot R1 \cdot e} \cdot \tau$$

It is then better to minimize the pulse duration (this holds also to reduce the Joule heating). For the lower mechanical axisymmetric mode the period is proportional to the radius, and therefore the product e^*R1^2 which should be kept constant. All these remarks apply to the case of the horn within the limits of the assumptions made and call for experimental verifications in real conditions.

Finally the longitudinal force applied on the radial connections between the inner conductor (radius R1) and the external coaxial conductor (radius R2) has to be considered. The current density is in one of these connections:

$$j_r(R) := \frac{I}{2 \cdot \pi \cdot R \cdot s}$$

where s is the current depth (along the axis).
The magnetic field is $B_{ext}(R)$ on the surface as given previously; its mean value in s is half of that amount, from which the force density is:

$$\frac{\mu_0 \cdot I^2}{8\cdot\pi^2\cdot R^2\cdot s}$$

and the force between R and R+dR is:

$$\frac{\mu_0 \cdot I^2}{8\cdot\pi^2\cdot R^2\cdot s}\cdot 2\cdot\pi\cdot R\cdot dR\cdot s$$

or:

$$\frac{1}{4}\cdot\mu_0\cdot\frac{I^2}{\pi\cdot R}\cdot dR$$

whose integral is:

$$\frac{1}{4}\cdot\mu_0\cdot\frac{I^2}{\pi}\cdot\ln(R)$$

and the total longitudinal force is then:

$$F_{long} := \frac{1}{4\cdot\pi}\cdot\mu_0\cdot I^2\cdot\ln\left(\frac{R2}{R1}\right)$$

The same force but reversed applies onto the other connection of the central conductor. These forces are supported by the external conductor (whose thickness may be as large as needed) and by the central conductor which is therefore subject to axial stresses which add to the previously given axial stress . To these electromagnetic forces the stress due to the assembly tolerances, the thermal stress due to Joule heating and to the residual proton beam coming from the production target have to be added. A complete and careful analysis of all these effects needs a very good design office, but only long and methodological tests on prototypes can give reliable results and are anyway necessary.

Meridian trajectories in a biconical magnetic horn

This program delivers the parametric equations for the meridian trajectories of a biconical magnetic horn(meridian means in a plane containing the axis).
All units in SI except otherwise stated,
Enter:

$I := 400000$ $\quad$ $Rmax := 0.03$ $\quad$ $Rmin := 0.006$

$B\rho := 11.675$ $\quad$ $n := 20$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

Characteristic angle: $\theta := \sqrt{\left[\dfrac{\mu 0\cdot I}{(2\cdot\pi\cdot B\rho)}\right]}$ $\quad$ $\theta = 0.082778$

Focal length: $f := \dfrac{Rmax}{\theta}$ $\quad$ $f = 0.362414$

Range: $i := 0..n$ $\quad$ $\alpha_i := \dfrac{i}{n}\cdot\theta$

Parametric equations:
1st cone:

$$RRH(\alpha) := f\cdot\alpha\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

$$RH(\alpha) := if(RRH(\alpha)>Rmin, RRH(\alpha), Rmin)$$

$$ZH(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

2nd cone:

$$RRS(\alpha) := f\cdot\alpha$$

$$RS(\alpha) := if(RRS(\alpha)>Rmin, RRS(\alpha), Rmin)$$

$$ZS(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right) + f\cdot\left(\frac{\alpha}{\theta}\right)\cdot\sqrt{\frac{\pi}{2}}\cdot erf\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)$$

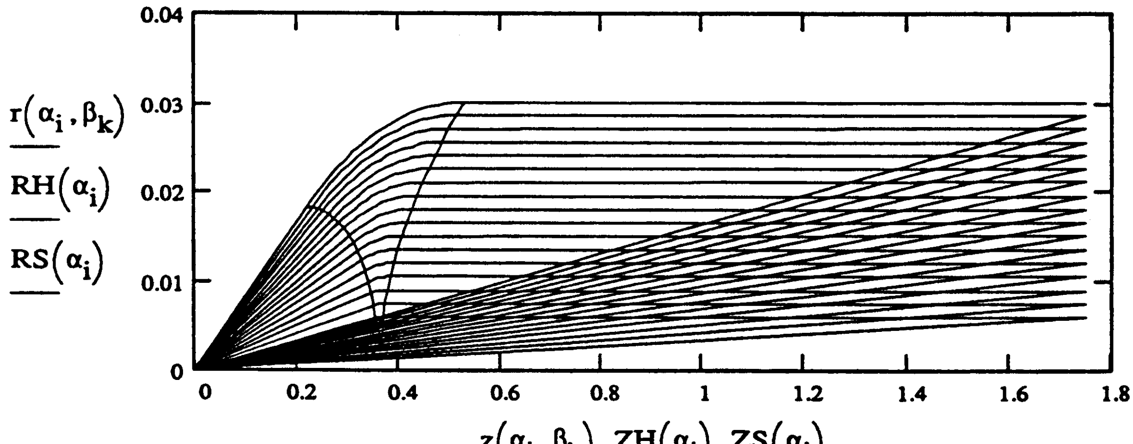
Meridian trajectory ending at radius RS(α):

$k := n..0$ $\quad$ $\beta_k := \dfrac{k}{n}\cdot\theta$

$$rr(\alpha,\beta) := RS(\alpha)\cdot\exp\left(-\frac{\beta^2}{2\cdot\theta^2}\right) \quad zz(\alpha,\beta) := ZS(\alpha) - \frac{RS(\alpha)}{\theta}\cdot\sqrt{\frac{\pi}{2}}\cdot erf\left(\frac{\beta}{\sqrt{2}\cdot\theta}\right)$$

$$r(\alpha,\beta) := if(\beta<\alpha, if(\beta=0, RS(\alpha), rr(\alpha,\beta)), 0)$$

$$z(\alpha,\beta) := if(\beta<\alpha, if(\beta=0, ZS(\theta)\cdot 3.3, zz(\alpha,\beta)), 0)$$



Meridian trajectories in a biconical horn

HORNTRA2

Skew trajectories in a biconical magnetic horn

This program provides an exact solution to the calculation of trajectories inside and outside of a magnetic horn without the common assumptions that this trajectories have small angles with the axis and are in meridian planes. The shape of the biconical horn is caculated by means of HORN4 whose data were stored in the files RR.PRN and ZZ.PRN.
All units in SI except otherwise stated,
Enter:

$I := 400000$ $\quad Rmax := 0.03$ $\quad Rmin := 0.006$

$B\rho := 11.675$ $\quad n := 20$

Initial values of the trajectory:

$s0 := 0$ $\quad sf := 0.5299$ $\quad x0 := 0$ $\quad y0 := 0$ $\quad z0 := 0$

$Vx0 := 0.01$ $\quad Vy0 := 0.08$ $\quad Vz0 := \sqrt{1 - Vx0^2 - Vy0^2}$

Number of intervals: $N := 200$

Number of points: $n := 20$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

Characteristic angle: $\theta := \sqrt{\left[\dfrac{\mu 0\cdot I}{(2\cdot\pi\cdot B\rho)}\right]}$ $\quad \theta = 0.083$

Focal length: $f := \dfrac{Rmax}{\theta}$ $\quad f = 0.362$

$RR := READPRN(RR)$
$ZZ := READPRN(ZZ)$

Range: $j := 0 .. last(RR)$ $\quad last(RR) = 60$

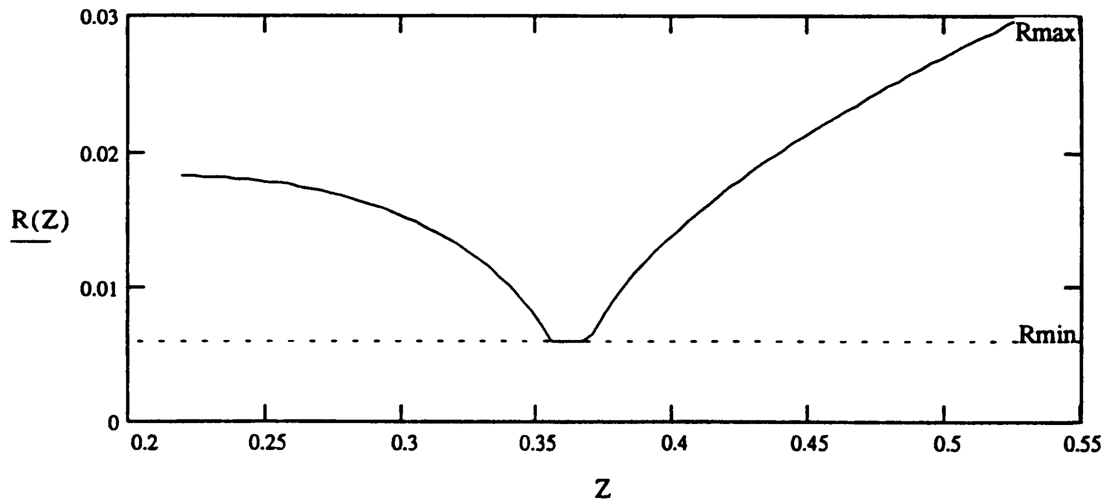
Interpolation:

$R(Z) := interp(pspline(ZZ, RR), ZZ, RR, Z)$

$R(0.3) = 0.015$

Range of Z: $ZZ_0 = 0.219815071$ $\quad ZZ_{last(ZZ)} = 0.529905143$

$$Z := ZZ_0, ZZ_0 + \frac{ZZ_{last(ZZ)} - ZZ_0}{4\cdot n} .. ZZ_{last(ZZ)}$$



Horn shape according to the interpolation function

The six right-hand sides of the differential equations for the trajectory of a particle in the field of a horn are written in the following vector below 1:

$$Fe(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ -\theta^2 \cdot Vz \cdot \left(\dfrac{x}{x^2+y^2}\right) \\ -\theta^2 \cdot Vz \cdot \left(\dfrac{y}{x^2+y^2}\right) \\ \theta^2 \cdot \dfrac{Vx \cdot x + Vy \cdot y}{x^2+y^2} \end{bmatrix}$$

These equations become the following where the field is zero:

$$Fi(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore the system of equations for the two domains in the range of Z previously defined is:

$$F(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ \text{if}\left[z<ZZ_0, 0, \text{if}\left[x^2+y^2<R(z)^2, 0, -\theta^2 \cdot Vz \cdot \left(\dfrac{x}{x^2+y^2}\right)\right]\right] \\ \text{if}\left[z<ZZ_0, 0, \text{if}\left[x^2+y^2<R(z)^2, 0, -\theta^2 \cdot Vz \cdot \left(\dfrac{y}{x^2+y^2}\right)\right]\right] \\ \text{if}\left(z<ZZ_0, 0, \text{if}\left(x^2+y^2<R(z)^2, 0, \theta^2 \cdot \dfrac{Vx \cdot x + Vy \cdot y}{x^2+y^2}\right)\right) \end{bmatrix}$$

The solution to this set of differential equations is the one developed in SRI2:

Interval size: $h := \dfrac{sf - s0}{N}$ $\quad h2 := \dfrac{h}{2}$ $\quad h4 := \dfrac{h}{4}$ $\quad h = 0.003$

Range: $i := 0 .. 2 \cdot N$

Initializations:

$s_0 := s0$ $\quad sf := ZZ_{last(ZZ)}$ $\quad x_0 := x0$ $\quad y_0 := y0$ $\quad z_0 := z0$

$Vx_0 := Vx0$ $\quad Vy_0 := \theta$ $\quad Vz_0 := Vz0$

First auxiliary point:

$$\begin{bmatrix} sa \\ xa \\ ya \\ za \\ Vxa \\ Vya \\ Vza \end{bmatrix} := \begin{bmatrix} s0 \\ x0 \\ y0 \\ z0 \\ Vx0 \\ Vy0 \\ Vz0 \end{bmatrix} + h4 \cdot F(s0, x0, y0, z0, Vx0, Vy0, Vz0)$$

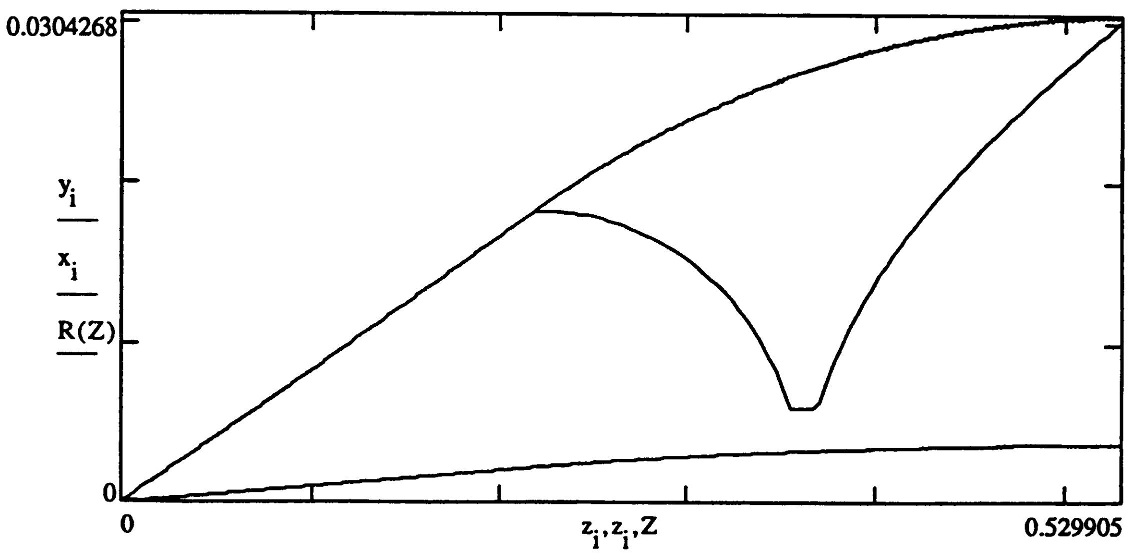
First middle point:

$$\begin{bmatrix} s_1 \\ x_1 \\ y_1 \\ z_1 \\ Vx_1 \\ Vy_1 \\ Vz_1 \end{bmatrix} := \begin{bmatrix} s_0 \\ x_0 \\ y_0 \\ z_0 \\ Vx_0 \\ Vy_0 \\ Vz_0 \end{bmatrix} + h2 \cdot F(sa, xa, ya, za, Vxa, Vya, Vza)$$

Interleaved iteration:

$$\begin{bmatrix} s_{i+2} \\ x_{i+2} \\ y_{i+2} \\ z_{i+2} \\ Vx_{i+2} \\ Vy_{i+2} \\ Vz_{i+2} \end{bmatrix} := \begin{bmatrix} s_i \\ x_i \\ y_i \\ z_i \\ Vx_i \\ Vy_i \\ Vz_i \end{bmatrix} + h \cdot F\left(s_{i+1}, x_{i+1}, y_{i+1}, z_{i+1}, Vx_{i+1}, Vy_{i+1}, Vz_{i+1}\right)$$

Results:



Skew trajectory: y(z) and x(z)

Table of results

Reduced number of points: $j := 0, \text{floor}\left(2 \cdot \frac{N}{n}\right) .. 2 \cdot N$

s_j	x_j	y_j	z_j	Vx_j	Vy_j	$\sqrt{(x_j)^2 + (y_j)^2}$
0	0	0	0	0.01	0.083	0
0.026	0	0.002	0.026	0.01	0.083	0.002
0.053	0.001	0.004	0.053	0.01	0.083	0.004
0.079	0.001	0.007	0.079	0.01	0.083	0.007
0.106	0.001	0.009	0.106	0.01	0.083	0.009
0.132	0.001	0.011	0.132	0.01	0.083	0.011
0.159	0.002	0.013	0.158	0.01	0.083	0.013
0.185	0.002	0.015	0.185	0.01	0.083	0.015
0.212	0.002	0.018	0.211	0.01	0.083	0.018
0.238	0.002	0.02	0.238	0.009	0.076	0.02
0.265	0.003	0.022	0.264	0.008	0.068	0.022
0.291	0.003	0.023	0.291	0.007	0.06	0.023
0.318	0.003	0.025	0.317	0.006	0.052	0.025
0.344	0.003	0.026	0.343	0.005	0.045	0.026
0.371	0.003	0.027	0.37	0.005	0.038	0.027
0.397	0.003	0.028	0.396	0.004	0.032	0.028
0.424	0.003	0.029	0.423	0.003	0.026	0.029
0.45	0.004	0.03	0.449	0.002	0.019	0.03
0.477	0.004	0.03	0.476	0.002	0.013	0.03
0.503	0.004	0.03	0.502	0.001	0.007	0.031
0.53	0.004	0.03	0.529	0	0.002	0.031

$f = 0.362$ $\qquad \frac{x_{(2 \cdot N)}}{Vx0} = 0.368$ $\qquad \frac{y_{(2 \cdot N)}}{Vy0} = 0.38$

The focal length is approximately constant (within 5% for the larger angles<θ).

Traversal length in a biconical magnetic horn

This program caculates the traversal length of a particle issued from the focus through the material of a biconical magnetic horn. The reabsorption and the multiple Coulomb scattering depend of the material traversal length of the particles focused by this magnetic horn. Therefore this traversal length settle the efficiency of the magnetic horn.
All units in SI except otherwise stated,
Enter:

$I := 400000$ $\quad Rmax := 0.03$ $\quad Rmin := 0.006$

$B\rho := 11.675$ $\quad emax := 0.003$ $\quad emin := 0.001$ $\quad n := 30$

****

$\mu 0 := 4\cdot\pi\cdot 10^{-7}$

Characteristic angle:
$$\theta := \sqrt{\left[\frac{\mu 0\cdot I}{(2\cdot\pi\cdot B\rho)}\right]} \qquad \theta = 0.082778$$

Focal length:
$$f := \frac{Rmax}{\theta} \qquad f = 0.362414$$

Range:
$$\alpha := TOL, \frac{\theta}{20} .. \theta$$

Parametric equations:

1st cone:
$$RRH(\alpha) := f\cdot\alpha\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$
$$RH(\alpha) := if(RRH(\alpha)>Rmin, RRH(\alpha), Rmin)$$
$$ZH(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right)$$

2nd cone:
$$RRS(\alpha) := f\cdot\alpha$$
$$RS(\alpha) := if(RRS(\alpha)>Rmin, RRS(\alpha), Rmin)$$
$$ZS(\alpha) := f\cdot\exp\left(-\frac{\alpha^2}{2\cdot\theta^2}\right) + f\cdot\left(\frac{\alpha}{\theta}\right)\cdot\sqrt{\frac{\pi}{2}}\cdot erf\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)$$

Thickness definition:
$$eeH(\alpha) := \frac{emin\cdot Rmax}{RH(\alpha)} \qquad eeS(\alpha) := \frac{emin\cdot Rmax}{RS(\alpha)}$$
$$eH(\alpha) := if(eeH(\alpha)<emax, eeH(\alpha), emax)$$
$$eS(\alpha) := if(eeS(\alpha)<emax, eeS(\alpha), emax)$$

The slope p of the curve defining the horn shape is the derivative of R: dR/dZ which is also (dR/dα)/(dZ/dα). This yield for the first part of the horn:

The derivative of RRH(α) is:
$$f\cdot\exp\left(\frac{-1}{2}\cdot\frac{\alpha^2}{\theta^2}\right) - f\cdot\frac{\alpha^2}{\theta^2}\cdot\exp\left(\frac{-1}{2}\cdot\frac{\alpha^2}{\theta^2}\right)$$

The derivative of ZH(α) is:
$$-f\cdot\frac{\alpha}{\theta^2}\cdot\exp\left(\frac{-1}{2}\cdot\frac{\alpha^2}{\theta^2}\right)$$

Therefore the slope is: $$p1(\alpha) := \frac{\alpha^2 - \theta^2}{\alpha}$$

Similarly for the second part of the horn:

The derivative of RRS(α) is: f

The derivative of ZS(α) is: $$\frac{1}{2}\cdot\frac{f}{\theta}\cdot\sqrt{2}\cdot\sqrt{\pi}\cdot \mathrm{erf}\left(\frac{1}{2}\cdot\alpha\cdot\frac{\sqrt{2}}{\theta}\right)$$

Therefore the slope is: $$p2(\alpha) := \frac{\theta}{\sqrt{\frac{\pi}{2}}\cdot \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}\cdot\theta}\right)}$$

The traversal paths d1(α) and d2(α) are then for R>Rmin:

$$d1(\alpha) := \frac{eH(\alpha)}{\alpha - p1(\alpha)} \qquad d2(\alpha) := \frac{eS(\alpha)}{p2(\alpha)}$$

When R<Rmin the traversal length is either 0 below α0 (i.e. through the central hole):

$$\alpha 0 := \frac{Rmin - emax}{ZS\left(\frac{Rmin}{f}\right)} \qquad \alpha 0 = 0.008116$$

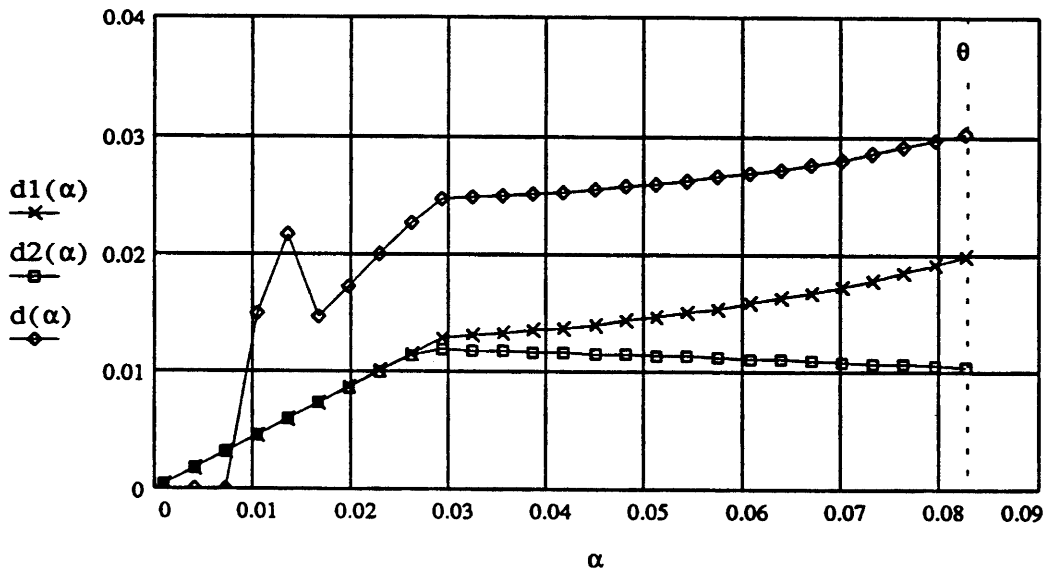
or approximately

$$lneck(\alpha) := d1(\alpha) + (ZS(\alpha) - ZH(\alpha)) + \frac{eS(\alpha)}{|\alpha - p2(\alpha)|}$$

above α0 (but for α<Rmin/f).

The total traversal length is finally:

$$d(\alpha) := \mathrm{if}\left(\alpha<\alpha 0, 0, \mathrm{if}\left(\alpha<\frac{Rmin}{f}, lneck(\alpha), d1(\alpha) + d2(\alpha)\right)\right)$$



Material traversal length for particles coming from the focus

System of differential equations of first order

These programs show a collection of routines to solve a set of differential equations of first order. Some of them are the common Runge-Kutta routines of second or fourth order applied to a set of six differential equations taken as an example: it solves without approximation the problem of the trajectory of a particle in a magnetic field decreasing as the inverse distance to the axis, and draws as a test the front view of a trajectory which is known to stay on a constant radius helix. Another program is an interleaved algorithm of second order applied to the same set of equations for the sake of comparison.

They apply to any set of differential equations of first order, coupled or not, the derivatives being any function of the other variables and/or of the common parameter. They apply also to a differential equation of higher order by considering the different derivatives as variables.

These routines are elementary i.e. they are not implemented with any means to solve stiff differential equations. Any routine can eventually provide inaccurate or even wrong results in cases presenting a singularity or a fractal-like behavior. The eventual user should take them as examples and use them with care. It is advisable in any case to run the program with different numbers of intervals and check that the results stay the same. In another chapter an example of adaptative step solution will be given.

SRI2:interleaved routine of 2nd order(fast), 6 equations.
SRK4:common Runge-Kutta of 4th order, 6 equations.
SRK40:modified Runge-Kutta of 4th order, 6 equations.
SRK2:Runge-Kutta of 2nd order, 6 equations.
S2RK4:Runge-Kutta of 4th order, 2 equations.

System of differential equations
of first order

This program shows a interleaved routine of second order: in each interval the derivative is calculated at the middle of the interval. To start it is necessary to calculate a first auxilliary set of derivatives from which the first middle derivatives are calculated. Initialization of the parameter s, the variables x,y,z and their derivatives Vx, Vy, Vz, sf is the end point of the s range:

$s0 := 0 \qquad sf := 80 \qquad x0 := 1 \qquad y0 := 0 \qquad z0 := 0$

$\theta := 0.08 \qquad Vx0 := 0 \qquad Vy0 := \theta \qquad Vz0 := \sqrt{1 - Vx0^2 - Vy0^2}$

Number of intervals: $N := 100$

Number of points: $n := 20$

The six right-hand sides of the differential equations of the first order are written in the following vector below 1:

$$F(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ -\theta^2\cdot Vz\cdot\left(\frac{x}{x^2+y^2}\right) \\ -\theta^2\cdot Vz\cdot\left(\frac{y}{x^2+y^2}\right) \\ \theta^2\cdot\frac{Vx\cdot x + Vy\cdot y}{x^2+y^2} \end{bmatrix}$$

****

Interval size: $h := \frac{sf - s0}{N} \qquad h2 := \frac{h}{2} \qquad h4 := \frac{h}{4} \qquad h = 0.8$

Range: $i := 0..2\cdot N$

Initializations:

$s_0 := s0 \qquad sf := sf \qquad x_0 := x0 \qquad y_0 := y0 \qquad z_0 := z0$

$Vx_0 := Vx0 \qquad Vy_0 := Vy0 \qquad Vz_0 := Vz0$

First auxiliary point:

$$\begin{bmatrix} sa \\ xa \\ ya \\ za \\ Vxa \\ Vya \\ Vza \end{bmatrix} := \begin{bmatrix} s0 \\ x0 \\ y0 \\ z0 \\ Vx0 \\ Vy0 \\ Vz0 \end{bmatrix} + h4\cdot F(s0,x0,y0,z0,Vx0,Vy0,Vz0)$$

First middle point:

$$\begin{bmatrix} s_1 \\ x_1 \\ y_1 \\ z_1 \\ Vx_1 \\ Vy_1 \\ Vz_1 \end{bmatrix} := \begin{bmatrix} s_0 \\ x_0 \\ y_0 \\ z_0 \\ Vx_0 \\ Vy_0 \\ Vz_0 \end{bmatrix} + h2 \cdot F(sa, xa, ya, za, Vxa, Vya, Vza)$$

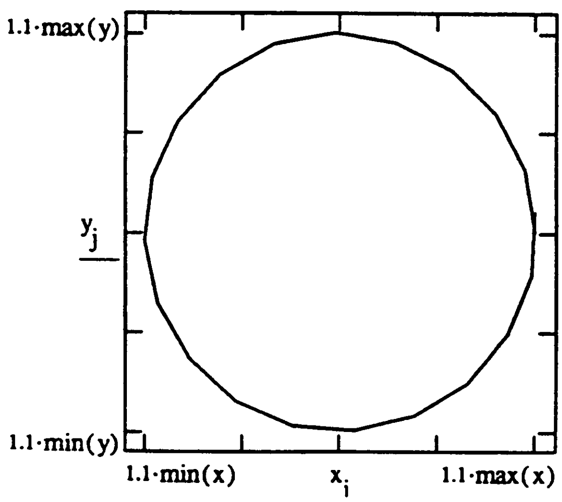
Interleaved iteration:

$$\begin{bmatrix} s_{i+2} \\ x_{i+2} \\ y_{i+2} \\ z_{i+2} \\ Vx_{i+2} \\ Vy_{i+2} \\ Vz_{i+2} \end{bmatrix} := \begin{bmatrix} s_i \\ x_i \\ y_i \\ z_i \\ Vx_i \\ Vy_i \\ Vz_i \end{bmatrix} + h \cdot F\left(s_{i+1}, x_{i+1}, y_{i+1}, z_{i+1}, Vx_{i+1}, Vy_{i+1}, Vz_{i+1}\right)$$

Results:

$$\begin{bmatrix} sa \\ xa \\ ya \\ za \\ Vxa \\ Vya \\ Vza \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1 \\ 0.016 \\ 0.199 \\ -0.001 \\ 0.08 \\ 0.997 \end{bmatrix} \qquad \begin{bmatrix} s_1 \\ x_1 \\ y_1 \\ z_1 \\ Vx_1 \\ Vy_1 \\ Vz_1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.999 \\ 0.032 \\ 0.399 \\ -0.003 \\ 0.08 \\ 0.997 \end{bmatrix}$$

Reduced number of points: $j := 0, \mathrm{floor}\left(2 \cdot \frac{N}{n}\right) .. 2 \cdot N$



Curve of y as a function of x

$s_{2 \cdot N} = 80$

$s_{last(s)} = 80.8$

$h = 0.8$

Table of all results

s_j	x_j	y_j	z_j	Vx_j	Vy_j	$\sqrt{(x_j)^2 + (y_j)^2}$
0	1	0	0	0	0.08	1
4	0.949	0.315	3.987	-0.025	0.076	1
8	0.803	0.597	7.974	-0.048	0.064	1.001
12	0.575	0.82	11.962	-0.065	0.046	1.001
16	0.289	0.959	15.949	-0.076	0.023	1.002
20	-0.026	1.002	19.936	-0.08	-0.002	1.003
24	-0.339	0.944	23.923	-0.075	-0.027	1.003
28	-0.617	0.791	27.911	-0.063	-0.049	1.003
32	-0.833	0.558	31.898	-0.044	-0.066	1.003
36	-0.966	0.269	35.885	-0.021	-0.077	1.003
40	-1.001	-0.047	39.872	0.004	-0.08	1.002
44	-0.935	-0.358	43.86	0.029	-0.075	1.001
48	-0.775	-0.633	47.847	0.051	-0.062	1.001
52	-0.536	-0.844	51.834	0.068	-0.043	1
56	-0.243	-0.97	55.821	0.078	-0.019	1
60	0.074	-0.997	59.808	0.08	0.006	1
64	0.384	-0.924	63.795	0.074	0.031	1.001
68	0.655	-0.757	67.783	0.061	0.052	1.001
72	0.86	-0.514	71.77	0.041	0.068	1.002
76	0.979	-0.219	75.757	0.018	0.078	1.003
80	0.998	0.098	79.744	-0.008	0.079	1.003

System of differential equations
of first order

This program shows a Runge-Kutta routine of fourth order: in each interval the derivatives are calculated at four points of the interval.
Initializations of the parameter s, the variables x,y,z and their derivatives Vx, Vy, Vz, sf is the end point:

$s0 := 0$ $sf := 80$ $x0 := 1$ $y0 := 0$ $z0 := 0$

$\theta := 0.08$ $Vx0 := 0$ $Vy0 := \theta$ $Vz0 := \sqrt{1 - Vx0^2 - Vy0^2}$

Number of intervals: $N := 100$

Number of points: $n := 20$

The six second members of the differential equations of the first order are written in the following vector below 1:

$$F(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ -\theta^2 \cdot Vz \cdot \left(\frac{x}{x^2+y^2}\right) \\ -\theta^2 \cdot Vz \cdot \left(\frac{y}{x^2+y^2}\right) \\ \theta^2 \cdot \frac{Vx \cdot x + Vy \cdot y}{x^2+y^2} \end{bmatrix}$$

****

Interval size: $h := \frac{sf - s0}{N}$ $h2 := \frac{h}{2}$ $h = 0.8$

Range: $i := 0..N$

Initializations:

$$u := \begin{bmatrix} s0 \\ x0 \\ y0 \\ z0 \\ Vx0 \\ Vy0 \\ Vz0 \end{bmatrix}$$

$K1(u) := F(u_0, u_1, u_2, u_3, u_4, u_5, u_6)$

$u1(u) := u + h2 \cdot K1(u)$

$$K1(u) = \begin{bmatrix} 1 \\ 0 \\ 0.08 \\ 0.996795 \\ -0.006379 \\ 0 \\ 0 \end{bmatrix}$$

$K2(u) := F\left(u1(u)_0, u1(u)_1, u1(u)_2, u1(u)_3, u1(u)_4, u1(u)_5, u1(u)_6\right)$

$u2(u) := u + h2 \cdot K2(u)$

$$K2(u) = \begin{bmatrix} 1 \\ -0.002552 \\ 0.08 \\ 0.996795 \\ -0.006373 \\ -0.000204 \\ 0 \end{bmatrix}$$

$K3(u) := F\left(u2(u)_0, u2(u)_1, u2(u)_2, u2(u)_3, u2(u)_4, u2(u)_5, u2(u)_6\right)$

$u3(u) := u + h \cdot K3(u)$

$$K3(u) = \begin{bmatrix} 1 \\ -0.002549 \\ 0.079918 \\ 0.996795 \\ -0.006379 \\ -0.000204 \\ 0 \end{bmatrix}$$

$K4(u) := F\left(u3(u)_0, u3(u)_1, u3(u)_2, u3(u)_3, u3(u)_4, u3(u)_5, u3(u)_6\right)$

$$RK(u) := \frac{K1(u) + 2 \cdot K2(u) + 2 \cdot K3(u) + K4(u)}{6}$$

$$RK(u) = \begin{bmatrix} 1 \\ -0.002551 \\ 0.079946 \\ 0.996795 \\ -0.006375 \\ -0.000204 \\ 0 \end{bmatrix}$$

Initialization of the matrix of the results:

$U^{<0>} := u$

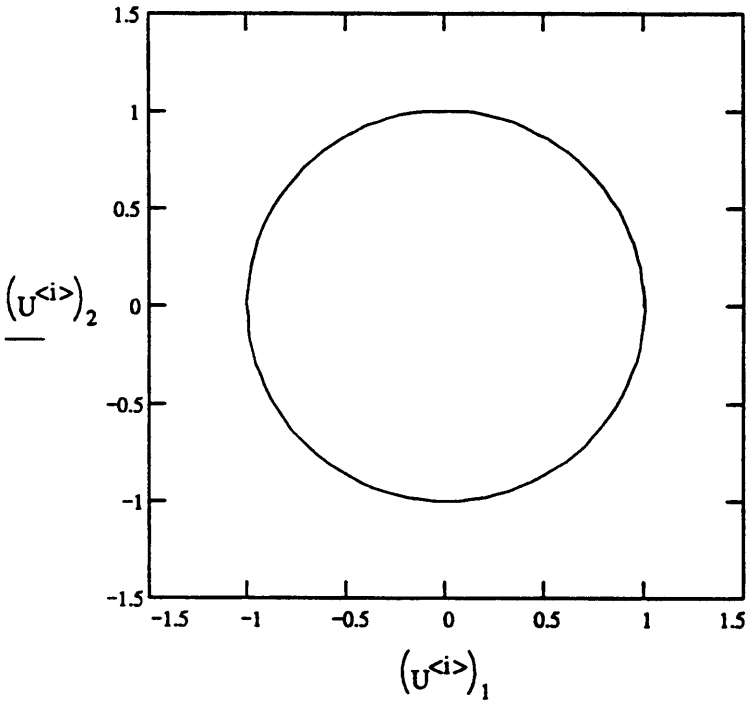
Iteration:

$U^{<i+1>} := U^{<i>} + RK\left(U^{<i>}\right) \cdot h$

Results:

$N = 100$ $\qquad h = 0.8$

$j := 0, \text{floor}\left(\frac{N}{n}\right) .. N$



Curve of y as a function of x

Table of all results

$(U^{<j>})_0$	$(U^{<j>})_1$	$(U^{<j>})_2$	$(U^{<j>})_3$	$(U^{<j>})_4$	$(U^{<j>})_5$	$\sqrt{[(U^{<j>})_1]^2+[(U^{<j>})_2]^2}$
0	1	0	0	0	0.08	1
4	0.949	0.315	3.987	-0.025	0.076	1
8	0.803	0.597	7.974	-0.048	0.064	1.001
12	0.575	0.82	11.962	-0.065	0.046	1.001
16	0.289	0.959	15.949	-0.076	0.023	1.002
20	-0.026	1.002	19.936	-0.08	-0.002	1.003
24	-0.338	0.944	23.923	-0.075	-0.027	1.003
28	-0.617	0.791	27.911	-0.063	-0.049	1.003
32	-0.833	0.559	31.898	-0.044	-0.066	1.003
36	-0.966	0.27	35.885	-0.021	-0.077	1.003
40	-1.001	-0.046	39.872	0.004	-0.08	1.002
44	-0.935	-0.357	43.86	0.029	-0.075	1.001
48	-0.775	-0.633	47.847	0.051	-0.062	1.001
52	-0.537	-0.844	51.834	0.068	-0.043	1
56	-0.244	-0.97	55.821	0.078	-0.02	1
60	0.073	-0.998	59.808	0.08	0.006	1
64	0.383	-0.924	63.795	0.074	0.03	1.001
68	0.655	-0.758	67.783	0.061	0.052	1.001
72	0.86	-0.515	71.77	0.041	0.068	1.002
76	0.978	-0.22	75.757	0.018	0.078	1.003
80	0.998	0.097	79.744	-0.008	0.079	1.003

SRK40

System of six differential equations
of first order

This program shows a modified Runge-Kutta routine of fourth order: in each interval the derivative is calculated at four points of the interval.

Initializations of the parameter s, the variables x,y,z and their derivatives Vx, Vy, Vz, sf is the end point:

$s0 := 0 \quad sf := 80 \quad x0 := 1 \quad y0 := 0 \quad z0 := 0$

$\theta := 0.08 \quad Vx0 := 0 \quad Vy0 := \theta \quad Vz0 := \sqrt{1 - Vx0^2 - Vy0^2}$

Number of intervals: $N := 100$

Number of points: $n := 20$

The six right-hand sides of the differential equations of the first order are written in the following vector below 1:

$$F(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ -\theta^2 \cdot Vz \cdot \left(\frac{x}{x^2 + y^2}\right) \\ -\theta^2 \cdot Vz \cdot \left(\frac{y}{x^2 + y^2}\right) \\ \theta^2 \cdot \frac{Vx \cdot x + Vy \cdot y}{x^2 + y^2} \end{bmatrix}$$

****

Interval size: $h := \frac{sf - s0}{N} \quad h2 := \frac{h}{2} \quad h3 := \frac{h}{3} \quad h = 0.8$

Range: $i := 0..N$

Initializations:

$$u := \begin{bmatrix} s0 \\ x0 \\ y0 \\ z0 \\ Vx0 \\ Vy0 \\ Vz0 \end{bmatrix}$$

$K1(u) := F(u_0, u_1, u_2, u_3, u_4, u_5, u_6)$

$u1(u) := u + h3 \cdot K1(u)$

$$K1(u) = \begin{bmatrix} 1 \\ 0 \\ 0.08 \\ 0.997 \\ -0.006 \\ 0 \\ 0 \end{bmatrix}$$

$$K2(u) := F\left(u1(u)_0, u1(u)_1, u1(u)_2, u1(u)_3, u1(u)_4, u1(u)_5, u1(u)_6\right)$$

$$u2(u) := u + h \cdot K2(u) - h3 \cdot K1(u)$$

$$K2(u) = \begin{bmatrix} 1 \\ -0.002 \\ 0.08 \\ 0.997 \\ -0.006 \\ 0 \\ 0 \end{bmatrix}$$

$$K3(u) := F\left(u2(u)_0, u2(u)_1, u2(u)_2, u2(u)_3, u2(u)_4, u2(u)_5, u2(u)_6\right)$$

$$u3(u) := u + h \cdot K1(u) - h \cdot K2(u) + h \cdot K3(u)$$

$$K3(u) = \begin{bmatrix} 1 \\ -0.003 \\ 0.08 \\ 0.997 \\ -0.006 \\ 0 \\ 0 \end{bmatrix}$$

$$K4(u) := F\left(u3(u)_0, u3(u)_1, u3(u)_2, u3(u)_3, u3(u)_4, u3(u)_5, u3(u)_6\right)$$

$$RK(u) := \frac{K1(u) + 3 \cdot K2(u) + 3 \cdot K3(u) + K4(u)}{8}$$

$$RK(u) = \begin{bmatrix} 1 \\ -0.003 \\ 0.08 \\ 0.997 \\ -0.006 \\ 0 \\ 0 \end{bmatrix}$$

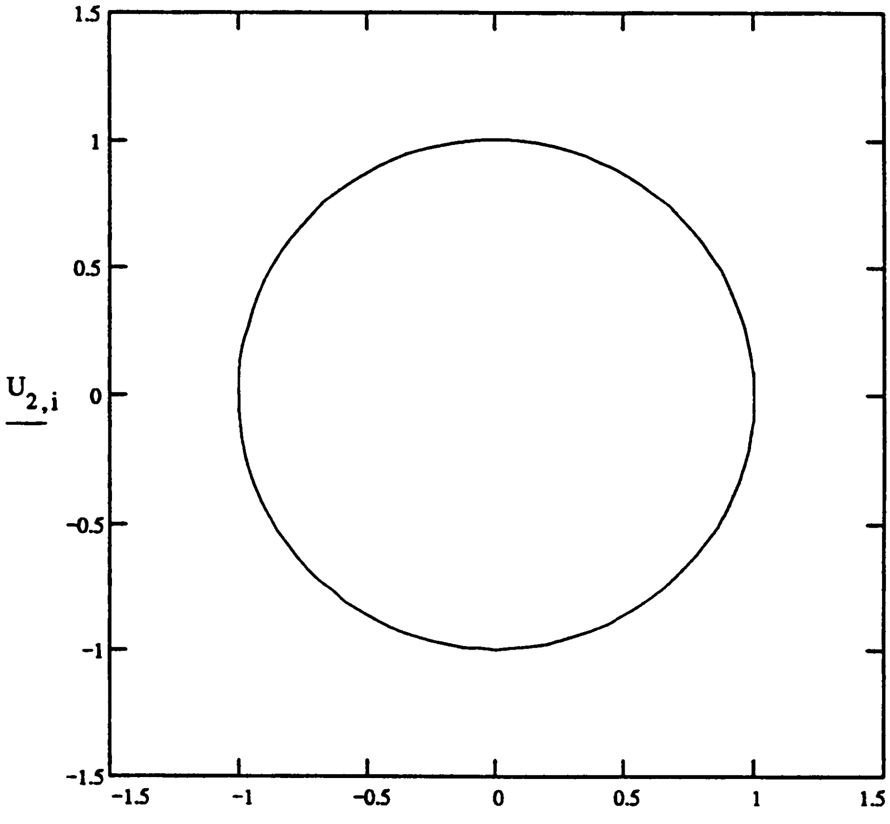
Initialization:

$$U^{<0>} := u$$

Iteration:

$$U^{<i+1>} := U^{<i>} + h \cdot RK\left(U^{<i>}\right)$$

Results:



Curve of y as a function of x

$N = 100$ $h = 0.8$ $\sqrt{(U_{1,N})^2 + (U_{2,N})^2} = 1.003$

$j := 0, \text{floor}\left(\frac{N}{n}\right) .. N$

Table of all results

$U_{(0,j)}$	$U_{1,j}$	$U_{(2,j)}$	$U_{2,j}$	$U_{4,j}$	$U_{5,j}$	$U_{6,j}$	$\sqrt{(U_{1,j})^2 + (U_{2,j})^2}$
0	1	0	0	0	0.08	0.997	1
4	0.949	0.315	0.315	-0.025	0.076	0.997	1.000161
8	0.803	0.597	0.597	-0.048	0.064	0.997	1.000612
12	0.575	0.82	0.82	-0.065	0.046	0.997	1.001263
16	0.289	0.959	0.959	-0.076	0.023	0.997	1.001981
20	-0.026	1.002	1.002	-0.08	-0.002	0.997	1.002623
24	-0.338	0.944	0.944	-0.075	-0.027	0.997	1.00306
28	-0.617	0.791	0.791	-0.063	-0.049	0.997	1.003206
32	-0.833	0.559	0.559	-0.044	-0.066	0.997	1.00303
36	-0.966	0.27	0.27	-0.021	-0.077	0.997	1.002569
40	-1.001	-0.046	-0.046	0.004	-0.08	0.997	1.001914
44	-0.935	-0.357	-0.357	0.029	-0.075	0.997	1.001196
48	-0.775	-0.633	-0.633	0.051	-0.062	0.997	1.00056
52	-0.537	-0.844	-0.844	0.068	-0.043	0.997	1.000133
56	-0.244	-0.97	-0.97	0.078	-0.02	0.997	1.000001
60	0.073	-0.998	-0.998	0.08	0.006	0.997	1.000192
64	0.383	-0.924	-0.924	0.074	0.03	0.997	1.000667
68	0.655	-0.758	-0.758	0.061	0.052	0.997	1.001329
72	0.86	-0.515	-0.515	0.041	0.068	0.997	1.002046
76	0.978	-0.22	-0.22	0.018	0.078	0.997	1.002674
80	0.998	0.097	0.097	-0.008	0.079	0.997	1.003087

SRK2

System of differential equations of first order

This program shows a Runge-Kutta routine of second order: in each interval the derivative is calculated at two points of the interval.

Initializations of the parameter s, the variables x,y,z and their derivatives Vx, Vy, Vz, sf is the end point :

$s0 := 0 \qquad sf := 80 \qquad x0 := 1 \qquad y0 := 0 \qquad z0 := 0$

$\theta := 0.08 \qquad Vx0 := 0 \qquad Vy0 := \theta \qquad Vz0 := \sqrt{1 - Vx0^2 - Vy0^2}$

Number of intervals: $N := 100$

Number of points: $n := 20$

The six second members of the differential equations of the first order are written in the following vector below 1:

$$F(s,x,y,z,Vx,Vy,Vz) := \begin{bmatrix} 1 \\ Vx \\ Vy \\ Vz \\ -\theta^2 \cdot Vz \cdot \left(\dfrac{x}{x^2 + y^2}\right) \\ -\theta^2 \cdot Vz \cdot \left(\dfrac{y}{x^2 + y^2}\right) \\ \theta^2 \cdot \dfrac{Vx \cdot x + Vy \cdot y}{x^2 + y^2} \end{bmatrix}$$

****

Interval size: $h := \dfrac{sf - s0}{N} \qquad h2 := \dfrac{h}{2} \qquad h = 0.8$

Range: $i := 0..N$

Initializations:

$$u := \begin{bmatrix} s0 \\ x0 \\ y0 \\ z0 \\ Vx0 \\ Vy0 \\ Vz0 \end{bmatrix}$$

~~Four~~ Two estimates of the derivatives are made:

$$K1(u) := F\left(u_0, u_1, u_2, u_3, u_4, u_5, u_6\right)$$

$$u1(u) := u + h2 \cdot K1(u)$$

$$K1(u) = \begin{bmatrix} 1 \\ 0 \\ 0.08 \\ 0.996795 \\ -0.006379 \\ 0 \\ 0 \end{bmatrix}$$

$$K2(u) := F\left(u1(u)_0, u1(u)_1, u1(u)_2, u1(u)_3, u1(u)_4, u1(u)_5, u1(u)_6\right)$$

$$K2(u) = \begin{bmatrix} 1 \\ -0.002552 \\ 0.08 \\ 0.996795 \\ -0.006373 \\ -0.000204 \\ 0 \end{bmatrix}$$

Ïnitialization of the matrix of all results:

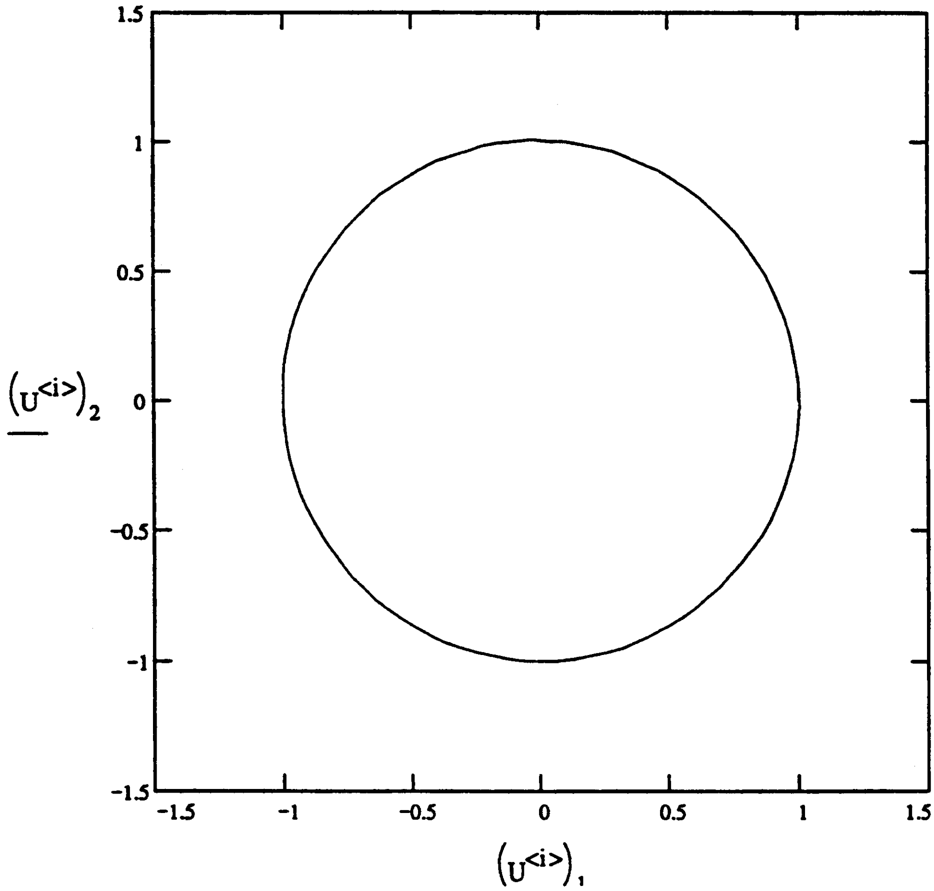
$$U^{<0>} := u$$

Iteration:

$$U^{<i+1>} := U^{<i>} + K2\left(U^{<i>}\right) \cdot h$$

Results:

$N = 100$ $h = 0.8$ $j := 0, \text{floor}\left(\frac{N}{n}\right) .. N$



Curve of y as function of x

Table of all results

$(U^{<j>})_0$	$(U^{<j>})_1$	$(U^{<j>})_2$	$(U^{<j>})_3$	$(U^{<j>})_4$	$(U^{<j>})_5$	$\sqrt{[(U^{<j>})_1]^2+[(U^{<j>})_2]^2}$
0	1	0	0	0	0.08	1
4	0.949	0.315	3.987	-0.025	0.076	1
8	0.803	0.598	7.974	-0.048	0.064	1.001
12	0.575	0.82	11.962	-0.065	0.046	1.002
16	0.289	0.96	15.949	-0.076	0.023	1.003
20	-0.026	1.003	19.936	-0.08	-0.002	1.003
24	-0.338	0.945	23.923	-0.075	-0.027	1.004
28	-0.617	0.793	27.911	-0.063	-0.049	1.004
32	-0.833	0.56	31.898	-0.044	-0.066	1.004
36	-0.966	0.272	35.885	-0.021	-0.077	1.003
40	-1.002	-0.044	39.872	0.004	-0.08	1.003
44	-0.936	-0.356	43.86	0.029	-0.075	1.002
48	-0.777	-0.631	47.847	0.051	-0.062	1.001
52	-0.539	-0.843	51.834	0.067	-0.043	1
56	-0.246	-0.969	55.821	0.078	-0.02	1
60	0.072	-0.998	59.808	0.08	0.006	1
64	0.382	-0.925	63.796	0.074	0.03	1.001
68	0.654	-0.759	67.783	0.061	0.052	1.002
72	0.86	-0.517	71.77	0.041	0.068	1.003
76	0.979	-0.222	75.757	0.018	0.078	1.004
80	1	0.095	79.745	-0.007	0.079	1.004

System of two differential equations of first order

Initializations of the parameter t and variables x and y, tf is the end value:

$t_0 := 0$ $\quad$ $tf := 15$ $\quad$ $x_0 := 1$ $\quad$ $y_0 := 3$

Number of intervals: $\quad n := 100$ $\quad$ intervals

Two differential equations of the first order:

$x'(t,x,y) := 1.5 \cdot x - x \cdot y$

$y'(t,x,y) := -3 \cdot y + 2 \cdot x \cdot y$

****

Interval size: $\quad h := \frac{tf - t_0}{n}$ $\quad$ $h2 := \frac{h}{2}$ $\quad$ $h = 0.15$

$k := 0..n$ $\quad$ $j := 0..n-1$

$$FF(t,x,y) := \begin{pmatrix} x'(t,x,y) \\ y'(t,x,y) \end{pmatrix}$$

4th order Runge-Kutta:

$kk1(t,x,y) := FF(t,x,y)$

$kk2(t,x,y) := FF\left(t + h2, x + h2 \cdot kk1(t,x,y)_0, y + h2 \cdot kk1(t,x,y)_1\right)$

$kk3(t,x,y) := FF\left(t + h2, x + h2 \cdot kk2(t,x,y)_0, y + h2 \cdot kk2(t,x,y)_1\right)$

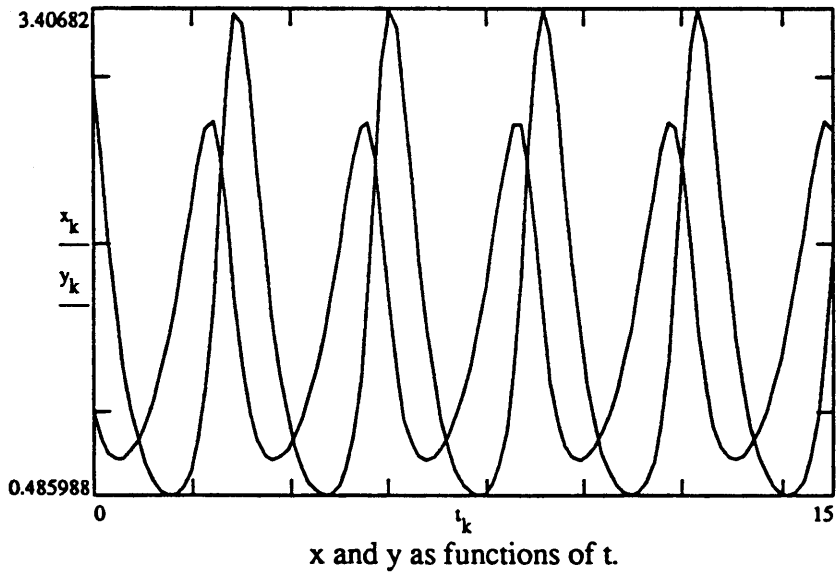
$kk4(t,x,y) := FF\left(t + h, x + h \cdot kk3(t,x,y)_0, y + h \cdot kk3(t,x,y)_1\right)$

$rkk(t,x,y) := \frac{1}{6} \cdot (kk1(t,x,y) + 2 \cdot kk2(t,x,y) + 2 \cdot kk3(t,x,y) + kk4(t,x,y))$

Iteration:

$$\begin{bmatrix} t_{j+1} \\ x_{j+1} \\ y_{j+1} \end{bmatrix} := \begin{bmatrix} t_j \\ x_j \\ y_j \end{bmatrix} + h \cdot \begin{bmatrix} 1 \\ rkk\left(t_j, x_j, y_j\right)_0 \\ rkk\left(t_j, x_j, y_j\right)_1 \end{bmatrix}$$

Results:



$n = 100$

$h = 0.15$

x and y as functions of t.

EASY ACCESS TO THE AAPC PROGRAMS

1) for a PS division user on the PC network two methods are at disposal:

a) In the Program Manager click the CAD/CAE icon,
Open MATHCAD (by double clicking the MATHCAD icon),
click File ,
click Open Document,
then in the following window go to the Drives cell and choose g:\srv1_ps\usr,
then go to the directory: G:\home\s\schnurig\aapc
and open the chosen file.

It is advisable to go first to an introduction file, e.g. Fodo for any Fodo*.mcd file. At this point it is possible to use the programs which don't need to write onto the previous mentioned directory (as it is the case for all introduction file), but to use extensively the programs any user must copy the needed files into his or her own directory in order to enable the computer to write data files related to his or her own work.

b) In the Program Manager,
go first to the File Manager,
go to G:\ drive by clicking on it.
find G:\home\s\schnurig\aapc
open by clicking the chosen file: as the files with .doc and .mcd extensions are live the corresponding programs WORD or MATHCAD will automatically be loaded. The further use is the same as in the previous method but copying the files is here particularly easy as it can be made by the user by simply draging the AAPC directory icon to his or her own C:\ disk.

2) for a CERN user on the PC network:
As the access through Windows is not very simple, it is proposed here to access and copy the AAPC files in the following way:
at the DOS prompt type F:\Login
then type: login srv1_ps/guest
when hooked to the PS server,
load Norton Comander by simply typing nc at the DOS prompt,
open the G:\home\s\schnurig\aapc directory by typing it at the DOS prompt.
The user has now to create a private directory in his or her own disk (aapc for example) by pressing F7 (once hooked to C:\)
Then Norton can do the copying of the files by selecting them, pressing F5...
Any further use of the programs goes through the common procedure of WINDOWS as in 1) but the files are now in C:\aapc.

3) for a stand-alone user:
It is only possible to use the programs if MATHCAD is available. Then it is possible to get the programs on a diskette upon request. It is worthwhile to say that a great majority of these programs have been developed on the DOS version of MATHCAD and can then run on smaller computers. As the WINDOWS version of the programs cannot worked on the DOS version the requested programs have to be specified (WINDOWS or DOS).

any request or advice to : J.-C. Schnuriger CERN/PS-AR
TEL (767) 4169 or SCHNURIG@CERNVM