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SOME ISSUES OF THE ELECTRON BEAM NEUTRALISATION AND OF THE MAGNETIC EXPANSION ON THE TRANSVERSE VELOCITY

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1. INTRODUCTION

Electrons with small transverse velocities $\vec{\vartheta}_{\perp}$ and small longitudinal velocity spread play a role in the electron cooling and in positive ion capture processes of electrons by cooled ions. It is not claimed that small transverse velocities are essential for transverse cooling purpose since, as for LEAR, magnetized electron cooling is less dependent of such a parameter [1].

In order to obtain small transverse velocities one can foresee to neutralize the electron beam [2,3] and/or to implement a longitudinal adiabatic beam radial expansion with the help of a longitudinal decreasing magnetic field [4,5]. These two possibilities will be briefly investigated in this paper.

Throughout this note it must be kept in mind that the transverse velocity $\vartheta_{\perp}$ is much smaller than the longitudinal velocity $\vartheta_s = \beta c$.

2. SYMBOLS AND DATA

2.1 Physical Constants

e = 1.602×10^{-19} [C], elementary charge,
m = 9.109×10^{-31} [kg], electron mass,
c = 2.997×10^{8} [ms^{-1}], light velocity,
ε_0 = 8.854×10^{-12} [F·m^{-1}], dielectric constant.

2.2 Cooler Parameters

I_e = electron current intensity [A],
B = guiding magnetic field [T],
a = electron beam radius [m],
$\vartheta_z = \vartheta_s = \beta_s c$ = electron longitudinal velocity [m·s^{-1}],
$n_e = I_e / e(\pi a^2)\vartheta_z$ = electron beam density [m^{-3}],
$\gamma = 1/\sqrt{1-(\vartheta_z/c)^2}$ = relativistic factor,
n_i = density of neutralization ions [m^{-3}],
Z_i = ion charge number,
$\eta = Z_i ni/ne$ = neutralization factor,
$\omega_c = eB/\gamma m$ = cyclotron frequency [s^{-1}].

2.3 First Part of Electron Cooling Set-up (Fig. 1)

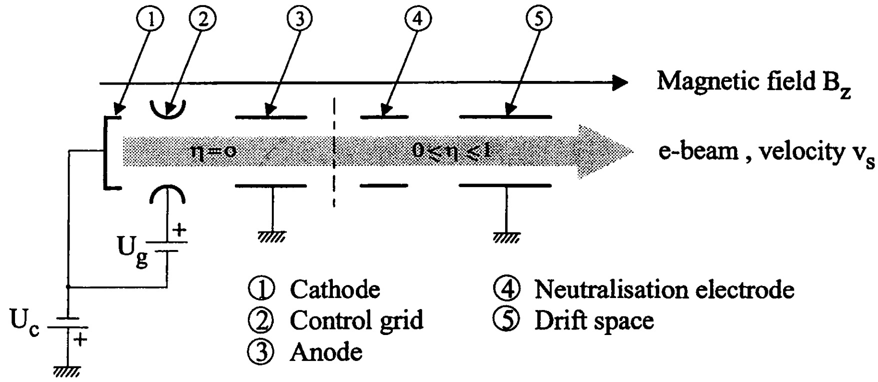
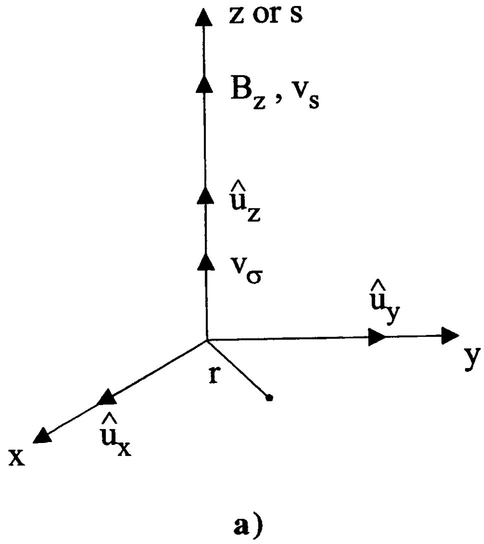


Fig. 1

2.4 System of Coordinates (Fig. 2)



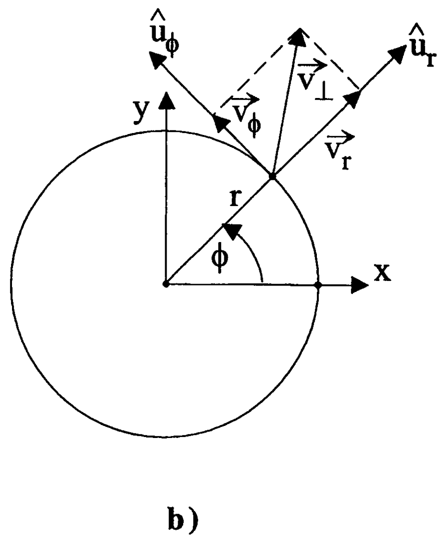


Fig. 2 - Three-dimensional (a) and transverse plane coordinates (b) (Cartesian and cylindrical)

2.5 Nominal Values (Indexed by 0)

$I_{e0} = 0.5$ [A],
$B_{z0} = 634 \times 10^{-4}$ [T],
$a_0 = 2.5 \times 10^{-2}$ [m],
$\omega_{c0} = 1.113 \times 10^{13}$ [rd.s^{-1}],
$n_{e0} = 5.642 \times 10^{13}$ [m^{-3}],
Proton momentum : $p_0 = 88.6 \times 10^6$ eV/c, such that:
$\beta_0 = 0.094$, $\gamma_0 = 1.0045$, $\vartheta_{s0} = \beta_0 c = 2.817 \times 10^7$ ms^{-1}.

2.6 Temperatures

By definition ($k_B \equiv$ Boltzman constant = 1.38×10^{-23} JK^{-1}):

$$\frac{1}{2} m\left[\vartheta_x^2 + \vartheta_y^2 + \vartheta_z^2\right] = \frac{3}{2} k_B\left[T_x + T_y + T_z\right]$$

$$\frac{1}{2} m\left[\vartheta_\perp^2 + \vartheta_s^2\right] = \frac{3}{2} k_B\left[T_\perp + T_s\right]$$

Example : If we state that $k_B T = m\vartheta_p^2$, where p stands for one of the 3 planes, then for an electron $k_B T = 0.1$ eV implies $\vartheta_p = 1.326 \times 10^5$ ms^{-1}.

3. RECALL ON TRANSVERSE VELOCITY

It is well known [2, 4] that forces due to the electron-beam space charge will induce on an electron at radius r an azimuthal "drift velocity: ϑ_d" expressed by:

$$\vec{\vartheta}_d(r) = \frac{F_r(r)}{\omega_c} \vec{u}_\phi \quad 0 \le r \le a \tag{1}$$

where $\vec{F}_r(r) = \left[e^2 n_e (1-\eta)(1-\beta^2)/2\varepsilon_0 \gamma^3 m\right] r \vec{u}_r$ is the radial electrical force. (In the following $\beta^2 \ll 1$ will be neglected and it will be considered that $\gamma \equiv 1$.)

Therefore, except in the case of full neutralization ($\eta = 1$) any electron at radius r has an azimuthal drift velocity given by Eq. (1), which velocity must be added to the natural transverse velocity $\vec{\vartheta}_{\perp 0}$ to obtain the total transverse velocity $\vec{\vartheta}_\perp = \vec{\vartheta}_{\perp 0} + \vec{\vartheta}_d$. The variance of the transverse velocity is expressed by: [2]

$$\left\langle \vartheta_\perp^2 \right\rangle = \vartheta_{\perp 0}^2 + 2\vartheta_d^2(r)$$

The natural transverse velocity variance $\vartheta_{\perp 0}^2$ is the sum of $\vartheta_{cathode}^2$, the velocity variance at the cathode output and the velocity variance ϑ_{gun}^2 induced by imperfections of the cooling device mainly at the gun level

$$\vartheta_{\perp 0}^2 = \vartheta_{cathode}^2 + \vartheta_{gun}^2$$

$m\vartheta_{\perp 0}^2$ is of the order of few tenth of an eV.

<u>*Remarks:*</u> (Refer to symbols given in Section 2)

1) Since $e\, n_e (1-\eta) = I(1-\eta)/\left(\pi a^2\right)\vartheta_s$,

$$\boxed{\vartheta_d = C_1 \frac{r}{a^2 B}} \quad C_1 = \frac{I(1-\eta)}{2\pi \varepsilon_0 \gamma^2 \vartheta_s} \tag{3}$$

Therefore, for constant C_1 and r, an increase of the electron-beam radius a by a factor $\sqrt{k}$ together with a decrease of B by k will <u>not change</u> $\vartheta_d(r)$ since $a^2 B$ remains constant.

2) Using the nominal values given in Section 2.5 we find that $\vartheta_d(r{=}a) = 2 \times 10^5$ ms^{-1} while the "natural" velocity is:

for $k_B T_{\perp 0} = 0.1$ eV $\quad \vartheta_{\perp 0} = 1.326 \times 10^5$ ms^{-1}
for $k_B T_{\perp 0} = 0.2$ eV $\quad \vartheta_{\perp 0} = 1.87 \times 10^5$ ms^{-1}

Natural and drift velocities are of the same order.

3) The drift velocity depends on the actual electron radius. Equality between $\vartheta_{\perp 0}$ and $\vartheta_d(r) > 0$ occurs at radius r_1 given by:

$$r_1 = \frac{a}{\sqrt{2}} \frac{\vartheta_{\perp 0}}{\vartheta_d(a)} \quad (r_1 \text{ is supposed to be } \leq a)$$

such as for $r_1 < r < a$ the drift velocity is predominant ($\theta_d > \theta_{\perp 0}$) and conversally for $0 < r < r_1$.

3. ADIABATIC BEAM EXPANSION

In this section the electron beam is supposed to be non-neutralized ($\eta = 0$). The magnetic field $\vec{B} = Bx\,\hat{u}_x + By\,\hat{u}_y + Bz\,\hat{u}_z$ decreases longitudinally ($dBz/dz < 0$) over a given interval where Bx and By are not equal to zero.

3.1 Adiabaticity

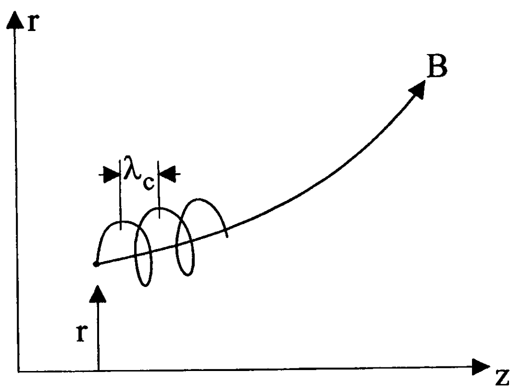


Fig. 3

It is well known that in very homogeneous magnetic fields the electron trajectory follows the magnetic field lines. The projection of the trajectory in the transverse plane (0,x,y) is a circle described with the cyclotron period $t_c = 2\pi/\omega_c$. The longitudinal "cyclotron wavelength", λc, is given by $\lambda c = \vartheta_s t_c = 2\pi\vartheta_s/\omega_c$.

The process is adiabatic if

$$\xi = \frac{\lambda c}{B}\left|\frac{dB}{dz}\right| << 1.$$

In other words adiabaticity supposes that the electron describes many circles around the field line [Fig. 3] before the magnetic field changes significantly. As a consequence it is proven [6] that

$$Br^2 \text{ and } m\vartheta_\perp^2/B \text{ are invariant} \tag{4}$$

Of course, the total energy $m(\vartheta_s^2 + \vartheta_\perp^2)$ remains constant. Moreover, if $\vartheta_\perp$ is reduced by a factor $\sqrt{k}$ the longitudinal velocity ϑ_s is slightly increased by $\Delta\vartheta_s$, which can be approximated by:

$$\Delta\vartheta_s = \frac{\vartheta_\perp^2}{2\vartheta_\sigma}\left(1 - \frac{1}{k}\right) << \vartheta_\perp$$

Now let us consider the effect of a decreasing magnetic field on an electron moving in a region without (Section 3.2) and with (Section 3.3) significant space-charge forces F_r.

3.2 Effects of a Decreasing Magnetic Field in a Region of Negligible Space Charge ($n_e \cong 0$)

Let us consider a magnetic field $B(z)$ which decreases adiabatically by a factor k over a distance L such that (Fig. 4)

$$B(L) = \frac{1}{k} B(0), \quad k > 1$$

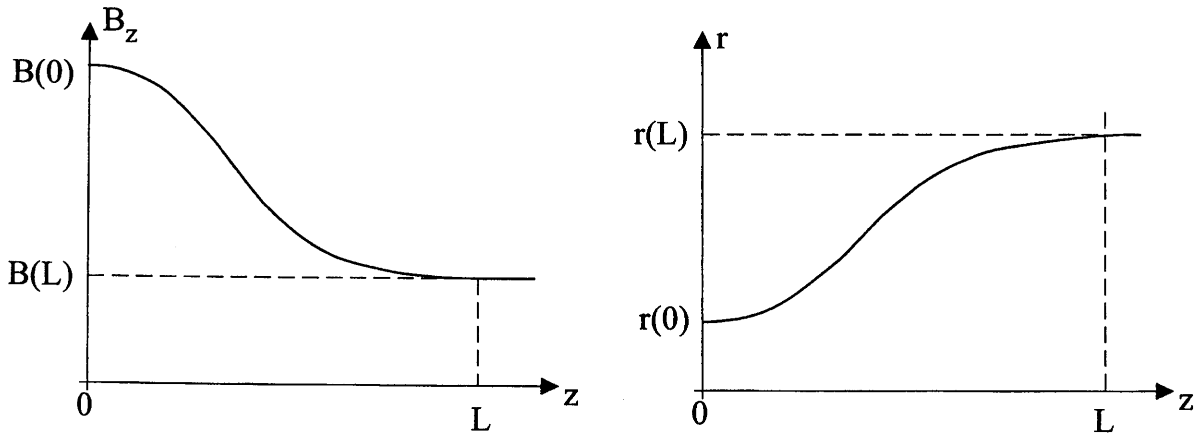


Fig. 4

According to the invariants given in (4):

$$B(0) r_0^2 = B(L) r^2(L)$$

$$r(L) = \sqrt{k}\, r(0)$$

So: - the final radius is increased by $\sqrt{k}$,
- the final transverse velocity is decreased since:

$$\vartheta_\perp(L) = \frac{1}{\sqrt{k}} \vartheta_\perp(0) \tag{4}$$

3.3 Influence of the Space Charge ($n_e \cong 10^{13}$ m^{-3})

Equation (4) does not take into account space-charge effects. In reality the space-charge force responsible of the drift velocity must be considered since $\eta = 0$. For simplicity, let us consider an electron beam without "natural" transverse velocity, i.e.: $\vartheta_{\perp 0} = 0$.

Since $B_L = B(z{=}L) = B_0/k$ (see Fig. 5), the initial radius r_0 becomes at $z = L$:

$$r_L = r_0 \sqrt{k}$$

The transverse drift velocity, as given by Eq. (3) is:

$$\vartheta_d(r,z) = C_1 \frac{r}{r^2(z)B(z)} \tag{5}$$

where C_1 is constant (since I and ϑ_s are practically constant) and, according to Eq. (4), the product $r^2(z)B(z)$ remains constant all along the electron trajectory. The drift velocity is constant whatever z is.

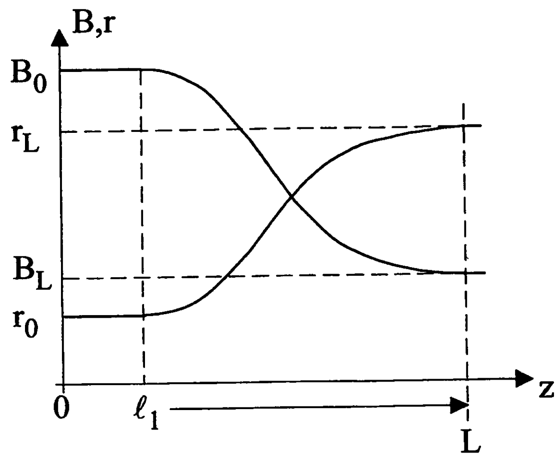


Fig. 5

3.3.1. Important conclusion

In the frame of electron cooling let us consider an electron beam of given radius a, intensity I, longitudinal velocity ϑ_z moving along a magnetic field *line*. One can consider:

- Case 1: The use of a cathode with radius a under magnetic field B. The drift velocity all along the electron beam trajectory will always be (Eq. (3)):

$$\vartheta_d(\text{case 1}) = C_1 \frac{r}{a^2 B}, \quad 0 \le r \le a$$

- Case 2: The use of a cathode with radius $a/\sqrt{k}$ under a magnetic field kB such as at the cathode level

$$\vartheta_d(\text{case 2}) = C_1 \frac{r}{\left(a/\sqrt{k}\right)^2 kB} = C_1 \frac{r}{a^2 B}, \quad 0 \le r \le a/\sqrt{k}$$

Followed by an adiabatic expansion which brings the electron-beam radius from $a/\sqrt{k}$ to a after the magnetic field has been decreased from kB to B.

For $z \ge L$, i.e. at the level of the drift space, the drift velocity ϑ_d(case 1) = ϑ_d(case 2).

<u>Beam expansion does not provide any reduction of the drift velocity</u>

3.4 Reduction of the Total Transverse Velocity

The transverse velocity, without adiabatic magnetic field decrease, is expressed by:

$$\vartheta_\perp^2(r) = \vartheta_{\perp 0}^2 + 2\vartheta_d^2(r) = \vartheta_{\perp 0}^2 + 2\vartheta_d^2(a)\left(\frac{r}{a}\right)^2$$

where a is the actual beam radius.

With adiabatic expansion as defined before, for $z \ge L$, we get:

$$\vartheta_{\perp ad}^2 = \frac{\vartheta_{\perp 0}^2}{k} + 2\vartheta_d^2(a)\left(\frac{r}{a}\right)^2$$

since only the natural velocity is reduced by the adiabatic expansion.

We can determine the radius at which the natural and drift velocity contributions are equal

$$\vartheta_{0\perp}^2 = 2\vartheta_d^2(r_1) \Rightarrow \vartheta_\perp^2(r_1) = 2\vartheta_{\perp 0}^2$$

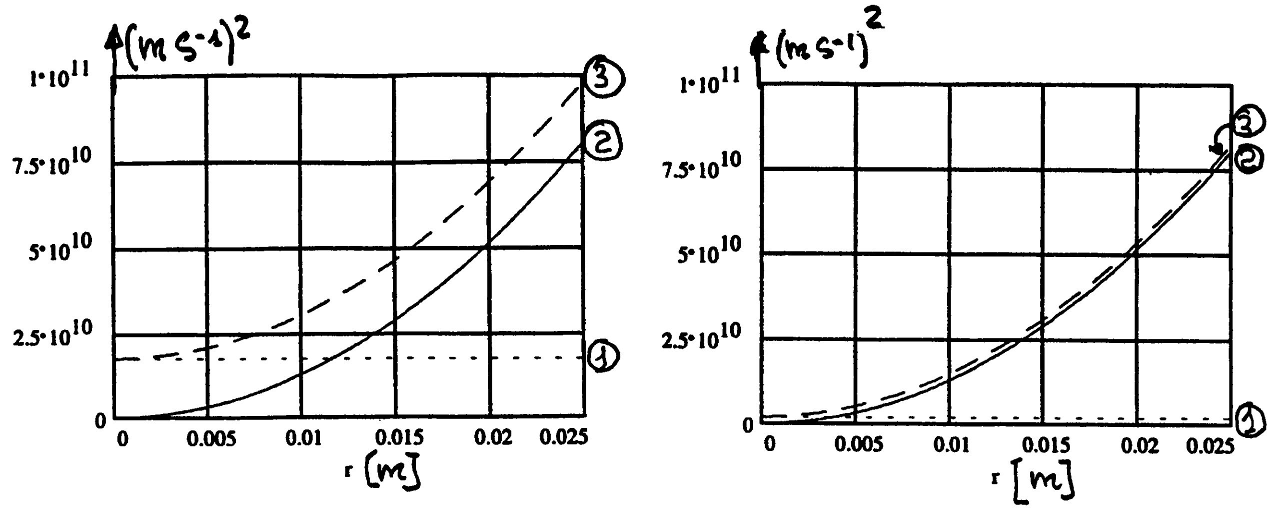
$$r_1 = \frac{a}{\sqrt{2}}\frac{\vartheta_{0\perp}}{\vartheta_d(a)},$$

When

$$k_B T_{\perp 0} = 0.1 \text{ eV} \qquad r_1 = 11.7 \times 10^{-3} \text{ m}$$

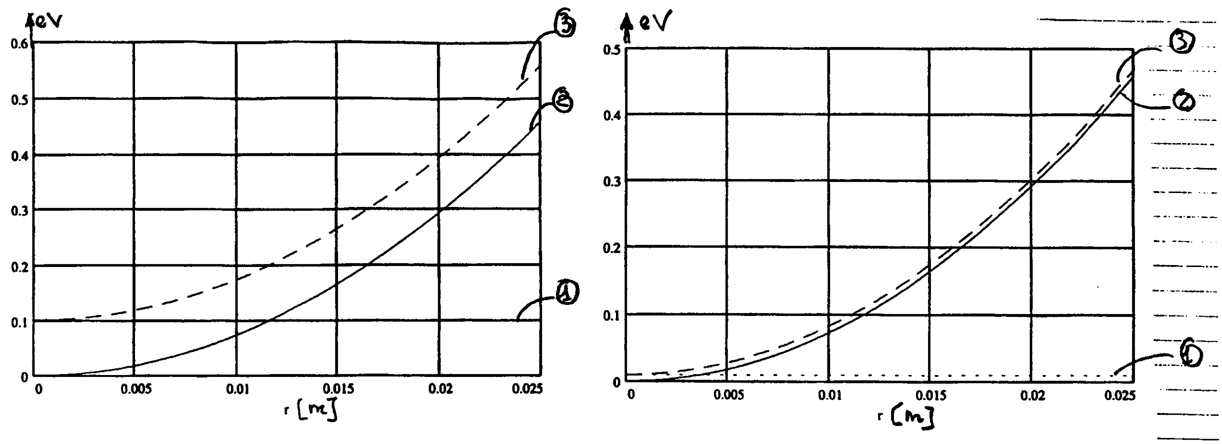
$$k^B T_{\perp 0} = 0.1 \text{ eV/9} \qquad r_1 = 3.9 \times 10^{-3} \text{ m}$$

Examples for $m\vartheta_{\perp 0}^2 = k_B T_{\perp 0} = 0.1$ eV and 0.1 eV/k = 0.1 eV/9 are given in Figs. 6a and b.



a b

Fig. 6 - a) $k_B T_{\perp 0}$ = 0.1 eV, b) $k_B T_{\perp 0}$ = (0.1/9) eV.
(1) Square of thermal velocity $\vartheta_{\perp 0}^2$. *(2)* $2\vartheta_d^2$, ϑ_d *is the drift velocity.*
(3) Square of transverse velocity: $\vartheta_\perp^2 = \vartheta_{\perp 0}^2 + 2\vartheta_d^2$.



a b

Fig. 7 - Same as in Fig. 6, but with energies expressed in eV and k = 9

4. EFFECTS OF NEUTRALISATION ON TRANSVERSE VELOCITY

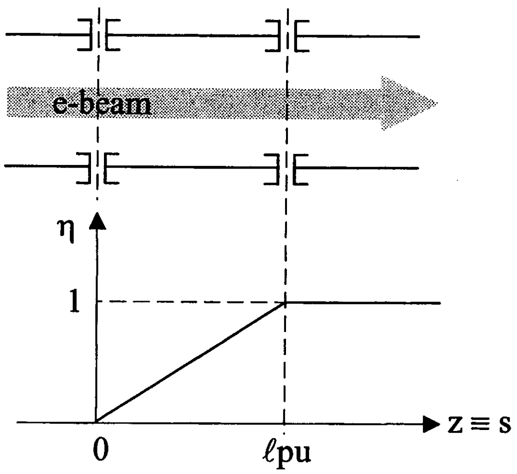


Fig. 8

In this paragraph the longitudinal magnetic field is constant.

Let an electron cross a neutralization electrode. We consider the idealized case where the neutralization factor η varies linearly through the electrode of length ℓpu such that (Fig. 8):

$$\eta = \begin{cases} 0 & \text{for } z \le 0 \\ \dfrac{z}{\ell pu} & \text{for } 0 \le z \le \ell pu \\ 1 & \text{for } z \ge \ell pu \end{cases} \tag{6}$$

From previous studies [7] it is known that for $z \le 0$ where the space charge is not compensated at all

$$\vartheta_r(t) = \vartheta_d \sin(\omega_c t) + \vartheta_{\perp 0} \cos(\omega_c t)$$

$$\vartheta_\phi(t) = \vartheta_d\big((1-\cos(\omega_c t)) + \vartheta_{\perp 0} \sin(\omega_c t)\big)$$

$$|\vartheta_\perp(t)| = \sqrt{\vartheta_r^2(t) + \vartheta_\phi^2(t)}$$

$$\left\langle \vartheta_\perp^2 \right\rangle_t = \vartheta_{\perp 0}^2 + 2\vartheta_d^2(r)$$

With the definition given in Eq. (6) it is shown [Appendix 1, Eq. 1] that for $0 \le z \le \ell pu$ we obtain

$$\begin{aligned} \vartheta_r(\ell) &= \vartheta_r(0) + \vartheta_d\left[\frac{1}{\arg}(\cos(\arg) - 1) + \sin(\arg)\right] \\ \vartheta_\phi(\ell) &= \vartheta_\phi(0) + \vartheta_d\left[\frac{1}{\arg}\sin(\arg) - \cos(\arg)\right] \end{aligned} \tag{7}$$

with $\arg = \omega_c(\ell/\vartheta_s)$.

Of course, $\vartheta_\perp(z \ge \ell pu) = \vartheta_\perp(z = \ell pu)$ since the electron beam is then fully neutralized and no forces will modify the electron transverse velocities once the electrode is crossed.

Considering the case where $\vartheta_{\perp 0} = \sqrt{\vartheta_r^2(0) + \vartheta_\phi^2(0)}$ is neglected and the process is fully adiabatic, i.e. $\arg = \omega_c(\ell pu/\vartheta_s) >> 1$, then from Eq. (7):

$$\boxed{\vartheta_\perp(\ell pu) = \vartheta_d}$$

Under these conditions the drift velocity is reduced by √2. Neutralization induces a relative small reduction of the transverse velocity amplitude.

5. JOINT USE OF NEUTRALIZATION AND ADIABATIC MAGNETIC EXPANSION

One can foresee the following arrangement (Fig. 9):

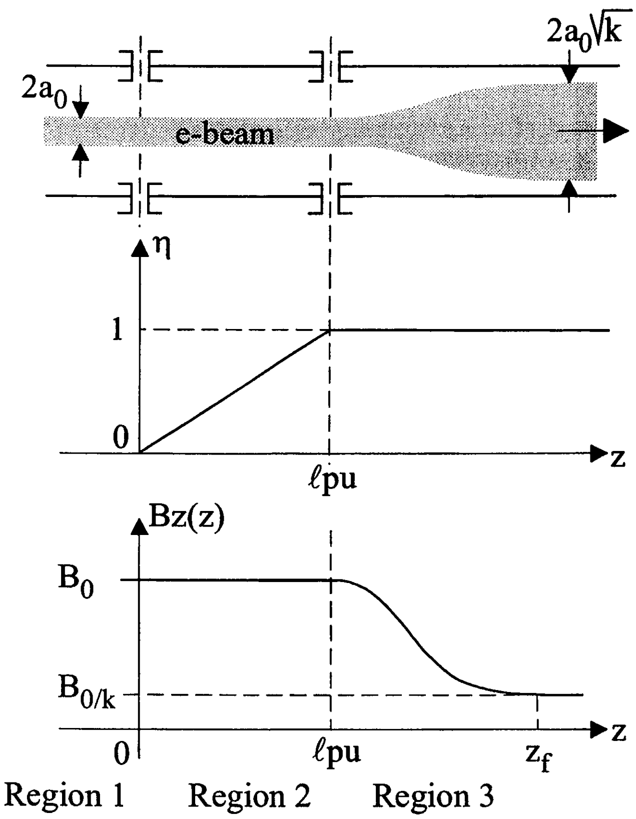


Fig. 9

- In region 1 the transverse velocity is:

$$\vartheta_\perp^2(r) = \vartheta_{\perp 0}^2 + 2\vartheta_d^2(r)$$

- In region 2 the beam is progressively neutralized, $\eta(z{=}0) = 0$, $\eta(z{=}\ell pu) = 1$. At $z = \ell pu$

$$\theta_{\perp \ell pu}^2(r) = \vartheta_{\perp 0}^2 + \vartheta_d^2(r)$$

- In region 3 the beam is submitted to an adiabatic magnetic expansion such as, for $z \geq z_f$

$$\vartheta_\perp^2\left(\sqrt{k}\, r\right) = \frac{\vartheta_{\perp \ell pu}^2(r)}{k}$$

Example:

If we consider the velocities at the border of the electron beam, the electron-beam intensity is still $Ie = 0.5$ A, $k_B T_{0\perp} = 0.1$ eV and $k = 9$.

Region 1: Beam radius $a = 1$ cm

$$k_B T_{\perp 0} = 0.1 \text{ eV} \Rightarrow \vartheta_{\perp 0} = 1.326 \times 10^5 \text{ ms}^{-1}$$

$$\vartheta_d = 2 \times 10^5 \text{ ms}^{-1} \text{ at } r = a,\ B = 9B_0 = 0.57 \text{ T}$$

$$\vartheta_\perp^2 = 9.76 \times 10^{10} \text{ [ms}^{-1}]^2 \Rightarrow k_B T_\perp = 5.55 \times 10^{-1} \text{ eV}$$

End of Region 2: Beam radius $a = 1$ cm

$$kT_{\perp 0} = 0.1 \text{ eV} \Rightarrow \vartheta_{\perp 0} = 1.326 \times 10^5 \text{ ms}^{-1}$$

$$\vartheta_d = 2 \times 10^5 \text{ ms}^{-1} \text{ at } r = 0,\ B = 9B_0 = 0.57 \text{ T}$$

$$\vartheta_\perp^2 = 5.758 \times 10^{10} \text{ [ms}^{-1}]^2 \Rightarrow k_B T_\perp = 0.33 \text{ eV}$$

End of Region 3: $B = B_0 = 0.0634$ [T], $a = 3$ cm

$$\vartheta_\perp^2(r = a) = \frac{5.758 \times 10^{10}}{9} \text{ [ms}^{-1}] \Rightarrow k_B T_\perp = 0.036 \text{ eV}$$

The reduction of the transverse energy between Region 1 and the output of Region 3 is significant.

6. CONCLUSIONS

For dense electron beams the use of adiabatic expansion, as a result of the longitudinal magnetic field decrease (by a factor *k*), provides an effective reduction of the transverse velocity of the electrons having a small radius only. Indeed, the "natural" transverse energy $\vartheta_{\perp 0}^{2}$ is decreased by *k* but the "drift" velocity due to the space charge remains unchanged. Therefore the transverse velocity of electrons having a large radius remains unchanged

The use of neutralization alone does not provide a significant reduction of the transverse velocity.

A more promising scheme consists in the use of an adiabatic magnetic expansion in a region where the electron beam is already neutralized. The reduction of the initial transverse velocity occurs at all radii of the electron beam.

As already mentioned in the introduction these conclusions concern only the transverse velocity and do not prejudge on its effects on any electron-cooling time. More precisely if, on the electron-cooling side the "magnetization effects" are experimentally proved, the consequences are different [8].

7. REFERENCES

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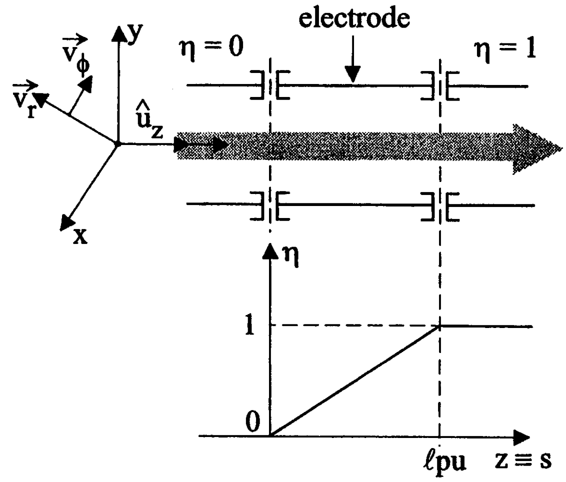
APPENDIX 1

TRANSVERSE VELOCITIES WITH AND WITHOUT NEUTRALIZATION

1. AIM (Fig. 1)

We foresee to compute the evolution of the transverse velocity components ϑ_r and ϑ_Φ ($\vec{\vartheta}_\perp = \vartheta_r \hat{u}_r + \vartheta_\phi \hat{u}_\phi$) of an electron beam of intensity *Ie* moving at a longitudinal velocity ϑ_s ($\vartheta_s >> \vartheta_\perp = \sqrt{\vartheta_r^2 + \vartheta_\phi^2}$.



Region 1 Region 2

Fig. 1

The electron beam of radius a moves from the left to the right side. Up to $z = 0$ the neutralization $\eta = Zn_i/ne = 0$.

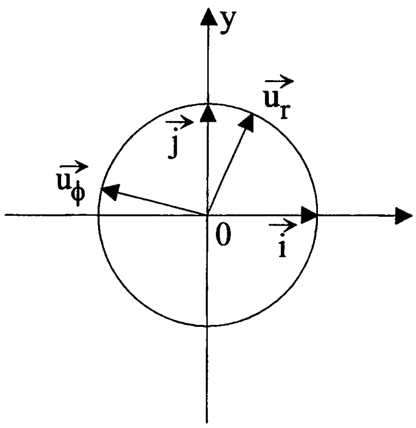
Then when the electron beam passes by the neutralization electrode of length ℓpu, the neutralization factor increases linearly. At the end of the neutralization electrode the electron beam enters the drift space where $\eta = 1$.

The neutralization factor increases linearly such that:

$$\eta = \begin{cases} 0 & \text{for } z \leq 0 \\ z/\ell pu & \text{for } 0 \leq z \leq \ell pu \\ 1 & \text{for } z \geq \ell pu \end{cases}$$

The longitudinal magnetic field $B = B\vec{u}_z$ is constant.

2. VELOCITIES IN REGION 1



From Eq. (10) of Ref. [1] and neglecting $\beta^2 << 1$, the transverse electrical force acting on an electron at radius r is:

$$F_r = \frac{1}{m}\frac{e^2 n_e(1-\eta)}{2\varepsilon_0} r = \frac{e}{m}(1-\eta)\cdot\frac{en_e}{2\varepsilon_0} r$$

with $\eta = 0$ in the present case.

The azimuthal drift velocity ϑ_d induced by F_r and the longitudinal magnetic field B of the cooler is given by:

$$\vec{\vartheta}_d = \frac{F_r}{\omega_c}\vec{u}_\phi \text{ with } \omega_c = \frac{eB}{m\gamma}$$

If, for $t = 0$, the natural velocity is $\vartheta_{\perp 0}$ from Eqs (12) and (13) of Ref. [1] one gets:

$$\boxed{\begin{gathered} \vartheta_r(t) = \vartheta_d \sin(\omega_c t) + \vartheta_{\perp 0}\cos(\omega_c t) \\ \vartheta_\phi(t) = \vartheta_d\left(1 - \cos(\omega_c t)\right) + \vartheta_{\perp 0}\sin(\omega_c t) \\ \left|\vec{\vartheta}_\perp(t)\right| = \sqrt{\vartheta_r^2(t) + \vartheta_\phi^2(t)} \end{gathered}}$$

Velocities are related to temperatures using the following equality:

$$k_B T\ [\mathrm{eV}] = \frac{m}{e}\vartheta^2$$

For $k_B T = 1$ eV:

$$\vartheta = \sqrt{\frac{e}{m}} = 4.193 \times 10^5\ \mathrm{m \cdot s^{-1}}$$

2.1 TRANSVERSE VELOCITY

For simplicity $\vartheta_{\perp 0}$ is neglected. Then from $\vartheta_r = V_d \sin(\omega_c t)$, $\vartheta_\phi = V_d(1\text{-}\cos(\omega_c t))$ we obtain

$$\vartheta_\perp^2 = \vartheta_d^2[2 - 2\cos(\omega_c t)] = 4\vartheta_d^2 \sin\left(\frac{\omega_c t}{2}\right)$$

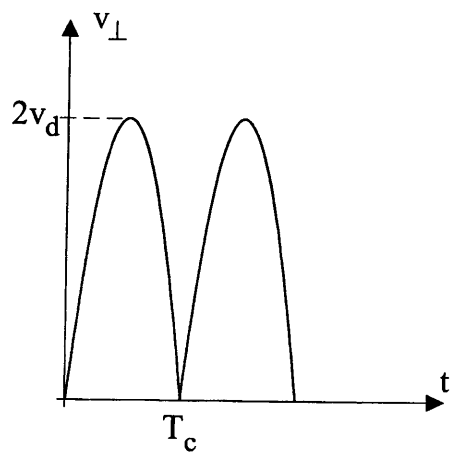


Fig. 2

$$\boxed{\vartheta_\perp = 2\vartheta_d \left|\sin\left(\frac{\omega_c t}{2}\right)\right|}$$

$\vartheta_\perp$ oscillates between 0 and $2V_d$ (Fig. 2) with a period T_c such that

$$\frac{\omega_c T_c}{2} = \pi,$$

$$T_c = \frac{2\pi}{\omega_c}$$

Since

$$\frac{1}{\pi}\int_0^\pi \sin\theta\, d\theta = \frac{2}{\pi}$$

The mean value of ϑ_d is

$$\boxed{\langle\vartheta_d\rangle = \frac{4}{\pi}\vartheta_d}$$

Including $\vartheta_{\perp 0}$ the natural velocity (see Ref. [1], Eq. (13)):

$$\left\langle \vartheta_\perp^2 \right\rangle = \vartheta_{\perp 0}^2 + 2\vartheta_d^2$$

3. VELOCITIES IN REGION 2

The electron beam enters at $z \equiv s = 0$ with a radial velocity $\vartheta_r(z{=}0) = \vartheta_r(0)$, $\vartheta_\phi(z{=}0) = \vartheta_\phi(0)$ that are the values taken at any time before. Then with (see Appendix 2, formula 2) arg = $\omega_c\ (\ell/\vartheta_s)$ $(0 \le l \le lpu)$, one gets:

$$\begin{aligned} \vartheta_r(l) &= \vartheta_r(0) + \vartheta_d\left[\frac{1}{\arg}(\cos(\arg) - 1) + \sin(\arg)\right] \\ \vartheta_\phi(l) &= \vartheta_\phi(0) + \vartheta_d\left[\frac{1}{\arg}\sin(\arg) - \cos(\arg)\right] \end{aligned} \tag{1}$$

3.1 Digression on Adiabaticity

For simplicity $\vartheta_r(0)$ and $\vartheta_\phi(0)$ are neglected. Let us consider the value of ℓ at the end of the electrode; $\ell = \ell pu$.

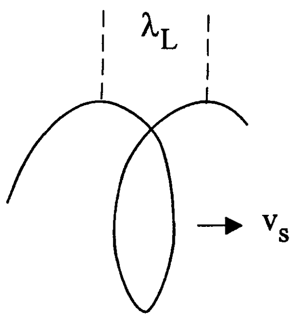


Fig. 3

In the case where: arg = $(\omega_c\ \ell pu)/\vartheta_s \gg 2\pi$, the cyclotron wavelength is given by (Fig. 3):

$$\omega_c\ \Delta t = 2\pi, \quad \Delta t = \frac{2\pi}{\omega_c}$$

and

$$\lambda_L = \vartheta_s\ \Delta t = \frac{\vartheta_s}{\omega_c} 2\pi$$

The condition $\quad \dfrac{\omega_c}{\vartheta_s}\ell pu \gg 2\pi \Rightarrow \dfrac{\ell pu}{\lambda_L} \gg 1.$

The electron executes many transverse oscillations on the way between the input and the output of the electrodes. It sees adiabatically the decrease of the transverse space-charge forces. Under such conditions ($\vartheta_r(0) = 0 = \vartheta_\phi(0)$, $(\omega_c\ \ell pu)/\vartheta_s \gg 2\pi$) the equations simplify to:

$$\vartheta_r(\ell pu) = \vartheta_d \sin(\arg)$$

$$\vartheta_\phi(\ell pu) = -\vartheta_d \cos(\arg)$$

$$\vartheta_\perp(\ell pu) = \sqrt{\vartheta_r^2(\ell pu) + \vartheta_\phi^2(\ell pu)} = \vartheta_d$$

Remark: In region 3, $F_r = 0$ with full neutralization and the electrons continue the Larmor motion with $\vartheta_\phi = \vartheta_d \cos(\omega_c t + \varphi_0)$, $\vartheta_r = \vartheta_d \sin(\omega_c t + \varphi_0)$. For $\omega_c t_{collision} \gg 1$ ("magnetization") this has no influence on the cooling time.

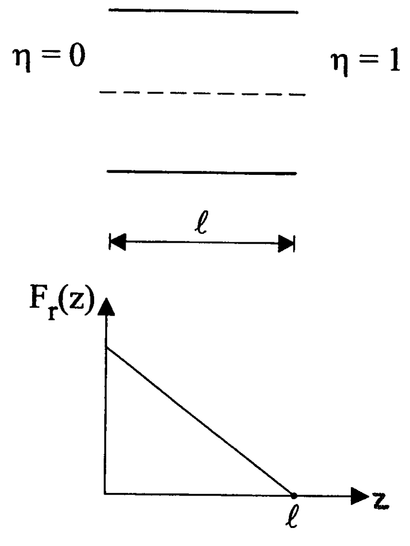
Conclusion: Between the input and the output of the neutralization electrode:

- The maximum amplitude of $\vartheta_\perp$ reduces from $2\vartheta_d$ to ϑ_d,
- The mean amplitude of $\vartheta_\perp$ reduces from $(4/\pi)\vartheta_d$ to ϑ_d (or $V_\perp^2$ from $(16/\pi^2)$ ϑ_d^2 to ϑ_d^2).

Our assumptions do concern the best case. Indeed, for 88.6 MeV/c, Ie = 0.5 A, $r = a$ = 2.5×10^{-2} m, $B = 636 \times 10^{-4}$ T, we find $\vartheta_d = 2 \times 10^5$ ms-1 while for a gun with temperature equivalent to 0.1 eV one has $\vartheta_{\perp 0} = 1.32 \times 10^5$ m/s, which *a priori* cannot be neglected, as we did.

APPENDIX 2

NEUTRALIZATION ELECTRODE



Let us consider the general equation (15) of Ref. [1]:

$$\vartheta_{\perp}(s) = \vartheta_{\perp 0} + \ell^{i\omega_c t}\int_0^t F_r(t_1)\ell^{-i\omega_c t_1}\, dt_1$$

where [1]:

$$F_r(z) = \left(\frac{eE_0}{m}\right)\left[1-\frac{z}{\ell}\right] \qquad E_0 = \frac{en_e}{2\varepsilon_0}r$$

i.e. $F_r(z{=}0) = eE_0/m$ at the entrance of the electrode where the beam is not neutralized and $F_r(z{=}\ell) = 0$ at the output, $\ell \equiv z = \vartheta t$ where the beam is fully neutralized. The integration has to be taken from $t = 0$ to $t = \ell/\vartheta$. We consider also that r and so E_0 are constant.

<u>Integral part</u> of $\vartheta_{\perp}(s)$:

$$A = \int_0^{\ell/\vartheta} \frac{eE_0}{m}\,\ell^{-i\omega_c t}\left[1-\frac{\vartheta t}{\ell}\right]dt$$

We omit eE_0/m such that for $\ell > 0$

$$B = \int_0^{\ell/\vartheta} \ell^{-i\omega_c t}\left[1-\frac{\vartheta t}{\ell}\right]dt$$

$$\ell^{-i\omega_c t}dt = du, \quad u = -\frac{1}{i\omega_c}\ell^{-i\omega_c t}, \quad \left[1-\frac{\vartheta t}{\ell}\right] = \xi, \quad d\xi = -\frac{\vartheta}{\ell}dt$$

$$B = \left\{-\frac{1}{i\omega_c}\ell^{-i\omega_c t}\left(1-\frac{\vartheta t}{\ell}\right)\right\}\Bigg|_0^{\ell/\vartheta} - \frac{\vartheta}{\ell}\frac{1}{i\omega_c}\int_0^{\ell/\vartheta}\ell^{-i\omega_c t}dt$$

The term:
$$\left\{-\frac{1}{i\omega_c}\ell^{-i\omega_c t}\left(1-\frac{\vartheta t}{\ell}\right)\right\}\Bigg|_0^{\ell/\vartheta} = \frac{1}{i\omega_c}$$

The integral term is:

$$\frac{\vartheta}{\ell}\frac{1}{(i\omega_c)^2}\ell^{-i\omega_c t}\Bigg|_0^{\ell/\vartheta} = -\frac{\vartheta}{\ell}\frac{1}{\omega_c^2}\left[\ell^{-i\omega_c(\ell/\vartheta)} - 1\right]$$

$$= \frac{\vartheta}{\ell}\frac{1}{\omega_c^2}\left[1-\ell^{-i\omega_c(\ell/\vartheta)}\right]$$

finally $$\boxed{A\,[\mathrm{m\cdot s^{-1}}] = \frac{eE_0}{m}\left[\frac{\vartheta}{\ell}\frac{1}{\omega_c^2}\left(1-\ell^{-i\omega_c(\ell/\vartheta)}\right)-\frac{i}{\omega_c}\right]}$$

And, at $t = \ell/\vartheta$

$$\vartheta_\perp(\ell) = \vartheta_{\perp 0} + \ell^{i\omega_c(\ell/\vartheta)}\frac{eE_0}{m}\left[\frac{\vartheta}{\ell}\frac{1}{\omega_c^2}\left(1-\ell^{-i\omega_c(\ell/\vartheta)}\right)-\frac{i}{\omega_c}\right]$$

$$\boxed{\vartheta_\perp(\ell) = \vartheta_{\perp 0} + \frac{eE_0}{m\omega_c}\ell^{i\omega_c(\ell/\vartheta)}\left[\frac{\vartheta}{\ell}\frac{1}{\omega_c}\left(1-\ell^{-i\omega_c(\ell/\vartheta)}\right)-i\right]}$$

by definition $eE_0/m\omega_c = V_{d0}$, the nominal drift velocity.

$$\vartheta_\perp(\ell) = \left[\vartheta_r(0) + i\vartheta_\varphi(0)\right] + V_{d0}\ell^{i\omega_c(\ell/\vartheta)}\left[\frac{\vartheta}{\ell}\frac{1}{\omega_c}\left(1-\ell^{-i\omega_c(\ell/\vartheta)}\right)-i\right] \tag{1}$$

Let us improve this expression:

$$V_{d0}\ell^{i\omega_c(\ell/\vartheta)}\left[\frac{\vartheta}{\ell}\frac{1}{\omega_c}\left(1-\ell^{-i\omega_c(\ell/\vartheta)}\right)-i\right] =$$

$$= V_{d0}\left[\frac{\vartheta}{\ell}\frac{1}{\omega_c}\ell^{i\omega_c(\ell/\vartheta)} - \frac{\vartheta}{\ell}\frac{1}{\omega_c} - i\ell^{i\omega_c(\ell/\vartheta)}\right]$$

$$= V_{d0}\left\{\frac{\vartheta}{\ell}\frac{1}{\omega_c}\left[\cos\left(\omega_c\frac{\ell}{\vartheta}\right) + i\sin\omega_c\frac{\ell}{\vartheta}\right] - \frac{\vartheta}{\ell}\frac{1}{\omega_c} - i\left(\cos\omega_c\frac{\ell}{\vartheta} + i\sin\omega_c\frac{\ell}{\vartheta}\right)\right\}$$

We set $\omega_c(\ell/\vartheta) = \arg$, such that the last equation reported in Eq. (1) gives

$$\boxed{\begin{aligned}\vartheta_r(\ell) &= \vartheta_r(0) + V_{d0}\left[\frac{1}{\arg}\left(\cos(\arg)-1\right)+\sin(\arg)\right]\\ \vartheta_\phi(\ell) &= \vartheta_\phi(0) + V_{d0}\left[\frac{1}{\arg}\left(\sin(\arg)-\cos(\arg)\right)\right]\end{aligned}} \tag{2}$$

It can be checked that when $\arg \to 0$ or $\ell \to 0$, the initial conditions are fulfilled.

On the other hand, when $\arg >> 1$ or $\ell \to \ell pu$, then

$$\left|\vartheta_r(\ell) - \vartheta_{r(0)}\right|^2 + \left|\vartheta_\phi(\ell) - \vartheta_\phi(0)\right|^2 = V_{d0}^2$$

Neglecting the natural velocity ($\vartheta_r(0) = 0 = \vartheta_\phi(0)$),

$$\vartheta_\perp(0) = \vartheta_\perp(\ell) = \vartheta_\mathrm{d}$$