Integrability of Large-Charge Sectors in Generic 2D EFTs

Matthew Dodelson[1,2], Simeon Hellerman[2], Masataka Watanabe[3,4,5], and Masahito Yamazaki[2,6]

[1] *CERN, Theoretical Physics Department, CH-1211 Geneva 23, Switzerland*

[2] *Kavli Institute for the Physics and Mathematics of the Universe* (WPI), *The University of Tokyo, Kashiwa, Chiba 277-8582, Japan*

[3] *Graduate School of Informatics, Nagoya University, Nagoya 464-8601, Japan*

[4] *Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan*

[5] *Department of Particle Physics and Astrophysics, Weizmann Institute of Science, Rehovot 7610001, Israel*

[6] *Trans-Scale Quantum Science Institute, The University of Tokyo, Tokyo 113-0033, Japan*

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Abstract

It is shown that integrability is an accidental property of generic two-dimensional $O(2)$-symmetric asymptotically-free theories in the regime where the charge density is much larger than the dynamical scale. We show this by constructing an infinite tower of higher-spin conserved currents in the most generic effective Lagrangian at large chemical potential to all orders in perturbative expansion in the renormalization-group invariant coupling constant.

1 Introduction

In physics it is notoriously difficult to solve general quantum field theories. One of the rare exceptions is a special class of theories called integrable field theories in two spacetime dimensions, theories which have infinitely-many conserved changes and are "exactly solvable." We can extract a large amount of quantitative information about such theories thanks to their exact solvability. Moreover, they often serve as useful stepping stones for analyzing more general theories, and for extracting general physics lessons. One should quickly add, however, that we expect a generic quantum field theory to be non-integrable.

The goal of this short note is to point out the existence of classical integrability for a large class of effective field theories (EFTs) in two dimensions. Our EFTs are *generic*, the only requirement being that the theory arises from the large-charge sector [1–11] of an $O(2)$-symmetric asymptotically free theory.[1] Since our discussion is within EFT, our analysis automatically incorporates many examples of EFT arising via Renormalization-Group (RG) flows from well-defined UV theories, such as non-linear sigma models.

The rest of this paper is organized as follows. After a summary of the large-charge EFT with $O(2)$ symmetry in section 2, we present a proof of existence of higher-spin conserved currents in section 3. We end in section 4 with comments and discussions.

2 Large-charge EFT for asymptotically free theories

Let us consider an asymptotically-free QFT $\mathcal{T}_{\text{UV}}$ with an $O(2)$ global symmetry in D spacetime dimensions. (We will restrict to $D = 2$ in the next section.) This theory can be regarded as the "UV theory" in our discussion. Note that we do *not* need to assume that this theory is integrable.

Since the theory is asymptotically-free, we expect the theory to generate a dynamical scale, which we denote by M. For our considerations we also introduce an IR regulator $1/R$, where R can be regarded as the size of the spatial sphere S^{D-1}. We need a hierarchy $M \gg 1/R$. In general these are the only mass scales we expect in the theory. In general we expect the theory to be strongly coupled in the IR, where we do not expect perturbative techniques to be applicable.

Let us now consider the setup with a very large $O(2)$ charge $Q \gg 1$. The crucial difference from our previous discussion is that we now have a new mass scale μ, as determined by the charge density ρ on the sphere: $\mu^{D-1} \sim \rho \sim Q/R^{D-1}$. In practice we can regard this scale as the chemical potential for the $O(2)$ symmetry, i.e. the Vacuum Expectation Value (VEV) $\langle A_0 \rangle$ of the time component of the background $O(2)$ gauge field A_μ.

Let us consider the situation where Q is very large such that μ is above the dynamical

[1] This paper extends the analysis of the large-charge sectors, previously done in three and higher dimensions, to two spacetime dimensions.

scale:

$$\mu \gg M \gg 1/R \,. \tag{2.1}$$

Due to the introduction of the chemical potential, the fields charged under the $O(2)$ symmetry will have a mass of $O(\mu)$, and hence can be integrated out below the scale μ. In particular, we expect that the running of the coupling constant stops at the energy scale μ, and when $\mu \gg M$ we expect the theory to have a weakly-coupled EFT description. We call this theory $\mathcal{T}_{\rm EFT}$.[2]

Which degrees of freedom should be kept in the EFT? A natural candidate is obtained by considering the complex "order parameter" $\Phi = |\Phi| e^{i\theta}$: its phase is shifted by the $O(2)$ symmetry transformation $A_\mu \to A_\mu + \partial_\mu \chi$ as $\theta \to \theta + \chi$, and θ represents the flat direction respecting the $O(2)$ symmetry.

Since we are introducing the chemical potential, we have the shift $A_0 \to A_0 + \mu$, which can be traded for a shift $\theta \to \theta + \mu t$. This means that we need to re-expand the theory around a new vacuum as

$$\theta = \mu t + \theta' \,, \tag{2.2}$$

where θ' represents a fluctuation around the new vacuum. The newly-chosen vacuum breaks the $O(2)$ symmetry, and θ can be light degrees of freedom (Nambu-Goldstone mode) of the broken $O(2)$ symmetry.

For the compatibility with the $O(2)$ shift symmetry, we only expect derivatives of θ to appear in the Lagrangian. Moreover, terms with higher derivatives will be suppressed by the scale separation of the theory, which is $1/\mu$ in this case. Finally, the Lagrangian should be compatible with the underlying Lorentz symmetry. We can thus write down the Lagrangian for the EFT $\mathcal{T}_{\rm EFT}$ to be the "free" (indeed free when $D = 2$) Lagrangian, up to corrections suppressed by $\mu^{-1} \propto Q^{-1/(D-1)}$:

$$\mathcal{L} = \frac{1}{g_{\rm EFT}^2} \left|\partial\theta\right|^D + O\left(Q^{-\frac{1}{D-1}}\right) = \frac{2}{g_{\rm EFT}^2} K + O\left(Q^{-\frac{1}{D-1}}\right) \,, \quad K \equiv \frac{1}{2} \left|\partial\theta\right|^D \,, \tag{2.3}$$

where $|\partial\theta| \equiv \sqrt{\partial_\mu\theta\partial^\mu\theta}$ and θ is to be expanded as in (2.2).

The EFT coupling constant $g_{\rm EFT}$ does not run under the RG flow inside the EFT, since we have already integrated out all the degrees of freedom other than θ. Its value can be matched with the gauge coupling $g_{\rm UV}$ of the UV theory—for example, with the coupling constant of the UV $O(N)$ theory. Even though our theories are weakly-coupled, $g_{\rm UV}$ has RG running, which can be accounted by the one-loop and higher-loop computations, and we can match the value of $g_{\rm EFT}$ with that of $g_{\rm UV}$ RG-evolved down

[2] In the literature one often takes the small radius limit $R \to 0$, to create the hierarchy $1/R \gg M$, to appeal to weakly-coupled descriptions. One might then hope to extrapolate back to the large-radius limit $R \to \infty$. Here the radius R is kept large, and we instead generate a new scale μ by focusing on the large-charge sector.

to the energy scale μ. Since the details will depend on the choice of the UV theory, let us here express this as

$$\mathcal{L} = f(\ell)K + (\text{subleading in } 1/Q), \quad \text{where} \quad \ell \equiv \log\left(\frac{2K}{M^2}\right), \tag{2.4}$$

where the coupling constant has been replaced by a function $f(\ell)$. We can write this more concisely as

$$\mathcal{L} = \mathcal{F}(K) + (\text{subleading in } 1/Q) \ . \tag{2.5}$$

where we make the dependence on the dynamical scale implicit.

3 Higher-spin currents in generic large-charge EFT

3.1 Construction of the higher-spin currents

In the rest of this paper we specialize to $D = 2$ spacetime dimensions. We will work with the EFT Lagrangian at (exactly) *leading order* in the large charge expansion,

$$\mathcal{L} = \mathcal{F}(K) \ . \tag{3.1}$$

This corresponds to taking all the possibilities for the perturbative corrections while ignoring all non-perturbative corrections in g_{UV}. We will use the light-cone coordinates $ds^2 = 2dx^+dx^-$, with metric $g_{+-} = g^{+-} = 1, g_{++} = g_{--} = g^{++} = g^{--} = 0$. This means, for example, $K = \frac{1}{2}(\partial\theta)^2 = (\partial_+\theta)(\partial_-\theta)$. The equation of motion of this theory is given by

$$\theta_{,+-} = \mathcal{G}(K)\left[\frac{\theta_{,-}}{\theta_{,+}}\theta_{,++} + \frac{\theta_{,+}}{\theta_{,-}}\theta_{,--}\right] \ , \tag{3.2}$$

where we defined

$$\mathcal{G}(K) \equiv -\frac{1}{2}\frac{K\mathcal{F}''(K)}{\mathcal{F}'(K) + K\mathcal{F}''(K)} \ . \tag{3.3}$$

We claim that the most generic form of the effective Lagrangian (3.1) allows for an infinite tower of higher-spin conserved currents: for any positive integer k we find a spin-$2k$ current, which is conserved on-shell.[3] Since these currents have different spins it immediately follows that these charges are linearly independent, and we have an infinite

[3] While these currents exist as formal objects for any complex value of k, single-valuedness of correlation functions in the EFT restricts k to be integer or half-integer. More generally objects with arbitrary fractional powers of the product $\partial_+\theta\partial_-\theta$ exist in the EFT but generically do not come from operators in the UV theory.

family of independent conserved charges, making the EFT integrable.[4] Since this is a statement about the EFT, this holds irrespective of the choice of the UV theory.

Our discussion of integrability uses the equation of motion, and is purely classical. However, it is also the case that our analysis applies to a general EFT, and we can thus incorporate quantum effects from RG flows to the parameters in the classical Lagrangian.

We are interested in constructing a spin-$2k$ conserved current with components (T_{2k}, Θ_{2k-2}), satisfying the conservation condition

$$\partial_+ T_{2k} = \partial_- \Theta_{2k-2} \ . \tag{3.4}$$

This will lead to spin-$(2k-1)$ conserved charges[5]

$$Q_{2k-1} = \int dx^1 \ (T_{2k} + \Theta_{2k-2}) \ . \tag{3.5}$$

For $k = 1$ we have a spin-2 current, which we identify with the stress-energy tensor as expected. In the following we will consider the case $k > 1$, since these are the charges relevant for integrability.

The components T_{2k} and Θ_{2k-2} of the current can be expressed as

$$T_{2k} = A_{2k}(\ell)(\partial_-\theta)^{2k} \ , \tag{3.6}$$

$$\Theta_{2k-2} = B_{2k}(\ell)(\partial_+\theta)(\partial_-\theta)^{2k-1} \ . \tag{3.7}$$

for ℓ-dependent functions $A_{2k}(\ell), B_{2k}(\ell)$. Note that this form is by no means special (for example, we have discarded terms including higher derivatives of θ). We will however see that this special form is enough to construct one set of higher-spin currents by explicitly finding a candidate for A and B which exactly satisfies the conservation law. For $k > 1$, $A_{2k}(\ell)$ is a solution to a second-order ODE

$$\frac{A''_{2k}(\ell)}{h(\ell)} + \left[\frac{2k-1}{h(\ell)} - 1 - \frac{h'(\ell)}{h^2(\ell)}\right] A'_{2k}(\ell) - k(2k-1)A_{2k}(\ell) = 0 \ , \tag{3.8}$$

and $B_{2k}(\ell)$ is determined from $A_{2k}(\ell)$ as

$$B_{2k}(\ell) = \frac{1}{k-1}\left[-kA_{2k}(\ell) + \frac{A'_{2k}(\ell)}{h(\ell)}\right] \ , \tag{3.9}$$

where we defined

$$h(\ell) \equiv \frac{d\log \mathcal{F}'(K)}{d\ell} = \frac{K\mathcal{F}''(K)}{\mathcal{F}'(K)} = -\frac{2\mathcal{G}(K)}{2\mathcal{G}(K)+1} \ . \tag{3.10}$$

This means that a tower of higher-spin currents exists whenever there is a solution to A_{2k} for all values of $k > 1$.

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4 More precisely we find integrability in the massless sector of the EFT. It is an interesting question to extend the discussion to massive Goldstone sectors, see e.g. [12, 13] for related discussion.

5 In our analysis integer and half-integer values of k appear on equal footing.

3.2 Proof of conservation equations

Let us show the current conservation by explicit computations. We compute

$$\partial_+ T_{2k} = [2kA_{2k}(\ell) + A'_{2k}(\ell)](\theta_{,-})^{2k-1}(\theta_{,+-}) + A'_{2k}(\ell)(\theta_{,-})^{2k}\frac{\theta_{,++}}{\theta_{,+}} , \tag{3.11}$$

$$\begin{aligned}\partial_- \Theta_{2k-2} = {} & [(2k-1)B_{2k}(\ell) + B'_{2k}(\ell)](\theta_{,-})^{2k-2}(\theta_{,+})(\theta_{,--}) \\ & + [B_{2k}(\ell) + B'_{2k}(\ell)](\theta_{,-})^{2k-1}(\theta_{,+-}) ,\end{aligned} \tag{3.12}$$

and we obtain the conservation equation (3.4) by equating the two expressions.

Thanks to the equation of motion (3.2), we can eliminate $\theta_{,+-}$ in terms of $\theta_{,++}$ and $\theta_{,--}$, to convert the conservation equations into two linear first-order ordinary differential equations:

$$A'_{2k}(\ell) = -(2k-1)B_{2k}(\ell) - B'_{2k}(\ell) , \tag{3.13}$$

$$\mathcal{G}(K)[2kA_{2k}(\ell) + A'_{2k}(\ell)] + A'_{2k}(\ell) = \mathcal{G}(K)[B_{2k}(\ell) + B'_{2k}(\ell)] . \tag{3.14}$$

Note that the first (second) equation is independent of (depends on) the function $\mathcal{F}[K]$, and hence of the choice of the EFT.

From these equations we can eliminate the functions $B_{2k}(\ell), B'_{2k}(\ell)$ as

$$B_{2k}(\ell) = \frac{1}{k-1}\left[-kA_{2k}(\ell) + \frac{A'_{2k}(\ell)}{h(\ell)}\right] , \tag{3.15}$$

$$B'_{2k}(\ell) = \frac{1}{k-1}\left[-\left(k + \frac{h'(\ell)}{h^2(\ell)}\right)A'_{2k}(\ell) + \frac{A''_{2k}(\ell)}{h(\ell)}\right] . \tag{3.16}$$

After plugging these expressions back into (3.13), we obtain the ODE for $A_{2k}(\ell)$ as in (3.8).

3.3 Example

While the existence of solutions to the ODE (3.8) is sufficient for the integrability of the EFT, one might be tempted to find analytic expressions for the conserved current. This is possible for the "leading-log" EFT, for which the Lagrangian takes the form

$$\mathcal{L} = \mathcal{F}[K] = \beta\left(\log\frac{2K}{M^2} - 1\right)K , \tag{3.17}$$

which leads to $\mathcal{F}'[K] = \beta\ell$ and $h[\ell] = 1/\ell$.

In this case, the ODE for $A_{2k}(\ell)$ reads

$$\ell A''_{2k}(\ell) + (2k-1)\ell A'_{2k}(\ell) - k(2k-1)A_{2k}(\ell) = 0 . \tag{3.18}$$

One of the solution to this equation is

$$A_{2k}(\ell) = L_k^{(-1)}[-(2k-1)\ell] , \tag{3.19}$$

where $L_k^{(\alpha)}[x]$ is a generalized Laguerre polynomial of order k and argument x (for $\alpha = 0$ this reduces to the ordinary Laguerre polynomial).[6] For example, we have

$$\begin{aligned} A_4(\ell) &= \ell(3\ell + 2) \ , \\ A_6(\ell) &= \ell(25\ell^2 + 30\ell + 6) \ , \\ A_8(\ell) &= \ell(343\ell^3 + 588\ell^2 + 252\ell + 24) \ . \end{aligned} \tag{3.20}$$

4 Discussions

In the standard discussion of integrability, it is often stated that classical integrability implies factorization of the tree-level S-matrix, at least in the integrable subsector. In particular, the theory allows for no particle production, and this severely constrains the form of the Lagrangian, at least order by order in perturbation theory (see [14–17]). This is in apparent contradiction with our statement that integrability exists for a generic large-charge EFT.

One should note, however, that the connection between classical integrability and factorization of the tree-level S-matrix is in general broken for integrable two-dimensional field theories with massless excitations [18]: our EFT applies to a massless mode θ associated with the symmetry breaking. Related to this, our theory is an EFT and its kinetic term is non-canonical, contrary to what is often assumed in the literature. In short, we do not in general expect factorization of the tree-level S-matrix in our theories. In fact, it is not obvious whether the S-matrix in these theories is well-defined, due to the presence of IR divergences.[7] It would be interesting to understand whether one can construct an S-matrix for the EFT, or if there is some other dynamical implication of the infinite family of charges.

There are several directions for future research. First, it is natural to extend our analysis to large-charge EFTs associated with more general theories, such as EFTs with $O(N)$ symmetries with $N \geq 2$. Second, it would be interesting to compare our analysis with detailed analysis of specific UV integrable field theories, such as the $O(N)$ model [20, 21]. It would also be interesting to see how the integrability of EFT fits into the framework of four-dimensional Chern-Simons theory [22, 23].

[6] Since the ODE is second order, there are two independent solutions. Here we have shown solutions that are polynomial in ℓ. The other is given by the Meijer's G-function $G^{2\,0}_{1\,2}\left(-(2k-1)\ell \,\middle|\, \begin{matrix} 1+k \\ 0,\,1 \end{matrix}\right)$, which is expanded as $e^{-(2k-1)\ell}\sum_i 1/\ell^i$.

[7] Here we are referring to the S-matrix of fluctuations around the large charge vacuum. The S-matrix around the trivial vacuum $\theta = 0$ is IR finite (see e.g. [19]). However, the higher spin charges act nonlinearly around the trivial vacuum, so the usual argument [16, 17] for the factorization of the S-matrix does not apply. We thank Sergei Dubovsky for raising this question.

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