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Kira and the block-triangular form

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For many state-of-the-art cross section computations the standard approach of Feynman integral reduction with the Laporta algorithm is the main bottleneck of the computation. We study a new approach of Feynman integral reduction by introducing a block-triangular form, which is a smaller system of equations compared to the system of equations which is generated with the Laporta algorithm. The construction of the block-triangular form and its implementation in the program Kira is the main interest of this report.

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1. Introduction

Within the Large Hadron Collider era the high energy physics regime of the Standard Model is probed with ever increasing accuracy. Many next-to-next-to-leading order, or, in some cases, even higher [1] accuracy goals have to be reached for the calculation of cross sections for many scattering processes.

To aid these goals we have implemented a tool named `Kira` [2] to manage very demanding integration-by-parts (IBP) reductions. Later we increased `Kira`'s toolbox [3] by including the finite field reconstruction methods [4–9] with `FireFly` [7, 9]. This helps in addition to study many user defined problems which involve as well non-trivial linear relations [10–12].

The best adopted strategy in the calculation of scattering amplitudes is to express first Feynman integrals in terms of the so called master integrals. The same relations can be used in many methods to compute the master integrals themselves [13–20].

The only known general purpose strategy to perform the IBP reductions is the Laporta algorithm [21] and its implementation is realized in many publicly available tools [2, 22–24]. This algorithm involves a treatment of a big number of redundant integrals, a feature which leads to high consumption of computing resources. Thus some special purpose strategies exist to circumvent several draw backs of the Laporta algorithm e.g: syzygy equations to reduce the size of involved integrals in the reduction process [25–30], the employment of algebraic geometry techniques which help to reduce the swell of intermediate expressions grow [31–33], new algorithms with intersection numbers are explored [34–38], and finite field and interpolation techniques are most of the time mandatory [4–9].

In this report we focus on the improvement of general IBP reductions by adopting a new strategy, which goes by the name of block-triangular form [39, 40]. This strategy aims to construct small-sized IBP systems in block-triangular form and furthermore as demonstrated in [40] leads to reduced main memory usage and to reduced overall run-time.

In this report after a brief introduction of notations we describe our plans on how to deploy the block-triangular form algorithm in `Kira`.

2. Feynman integral reductions

2.1 Preliminaries

The primary object of interest is the Feynman integral in the loop-momenta representation

$$T(a_1,\ldots,a_N) = \int \left(\prod_{i=1}^{L} \mathrm{d}^d\ell_i\right) \frac{1}{P_1^{a_1} P_2^{a_2}\cdots P_N^{a_N}}, \tag{1}$$

where $P_j = q_j^2 - m_j^2$, $j = 1,\ldots,N$, are the inverse propagators (omitting the Feynman prescription). The momenta q_j are linear combinations of the loop momenta ℓ_i, $i = 1,\ldots,L$ for an L-loop integral, and external momenta p_k, $k = 1,\ldots,E$ for $E+1$ external legs (or $E = 0$ for vacuum integrals), and m_j are the propagator masses. The a_j are the (integer) propagator powers. The set of inverse propagators must be complete and independent in the sense that every scalar product of momenta can be uniquely expressed as a linear combination of the P_j, squared masses m_j^2, and external kinematical invariants. The number of propagators is thus $N = \frac{L}{2}(L + 2E + 1)$ including auxiliary propagators that only appear with $a_j \leq 0$.

Integrals of the form (1) for different values of a_j are in general not independent. *Integration-by-parts* (IBP) identities [41, 42] and *Lorentz-invariance* (LI) identities [43], as well as symmetry relations lead to linear relations between them. These identities can be used to express all integrals through linear combinations of master integrals, which serve as a basis.

`Kira` employs a variant of the Laporta algorithm [21]: IBP, LI, and symmetry relations are generated for different values for the a_j, resulting in a linear system of equations. This system of equations is then systematically solved with a Gauss-type elimination algorithm to express integrals which are regarded more complicated in terms of simpler integrals.

As a measure of complexity it is useful to define the number

$$t = \sum_{j=1}^{N} \theta(a_j - \tfrac{1}{2}), \tag{2}$$

of propagators with positive power, the sum r of all positive powers, and the negative sum of all non-positive powers s,

$$r = \sum_{j=1}^{N} a_j \theta(a_j - \tfrac{1}{2}), \qquad s = -\sum_{j=1}^{N} a_j \theta(\tfrac{1}{2} - a_j), \tag{3}$$

where $\theta(x)$ is the Heaviside step function. These values are used as limits in the choice of the sets a_j for which the IBP, LI, and symmetry relations are generated.

With $\frac{\hat{s}}{t}$ we denote a set of all Feynman integrals which can be constructed with the parameters $s = 0, \ldots, \hat{s}$ and t.

2.2 Block-triangular form

Compared to the integral reduction with the standard Laporta algorithm, the block-triangular form involves ideally several orders of magnitude less equations and integrals in the reduction process. Furthermore one may impose a requirement that the block-triangular form contains relations with polynomial coefficients of low degree. The main equation to construct the block-triangular form is

$$\sum_j c_j(d, \vec{s}) I_j = 0, \tag{4}$$

and in general the coefficients are

$$c_j(d, \vec{s}) = \sum_{i=0}^{d_{\max}} d^i \sum_{\vec{l} \in \Omega_{k_j}}^{k_{\max}} \hat{c}_j^{i,l_1,\ldots,l_M} s_1^{l_1} \cdots s_M^{l_M}, \tag{5}$$

where $\hat{c}_j^{i,l_1,\ldots,l_M}$ are free numeric parameters left to be fixed and

$$\Omega_{k_j} = \{\vec{l} \in \mathbb{N}^M | \sum_{i=0}^{M} l_i = k_j\}. \tag{6}$$

Here I_j are Feynman integrals of different complexity and with different mass dimension. The coefficients $c_j(d, \vec{s})$ are polynomials of mass dimension k_j. The least possible value of k_{max} is the biggest mass dimension difference between all the integrals I_j. In practice a bigger value for k_{max} is required. The value of d_{max} dictates the polynomial degree of the coefficients in the dimensional regularization parameter d.

For demonstration purpose we discuss the following problem, the construction of a block-triangular form consisting of N_J equations for a set of Feynman integrals $\{J_j\}$ with the same integral complexity $s = \hat{s}$ and t, where N_J is the number of J_j integrals. We make an Ansatz for all Feynman integrals which should appear in the equation (4)

$$I_j \in \{\frac{\hat{s}}{t}, \frac{\hat{s}-1}{t-1}, \frac{\hat{s}-2}{t-2}\}. \tag{7}$$

This Ansatz works for many applications where the Feynman integrals J_j are of complexity $s > 2$. In the next step let us assume that we are able to perform the reduction of all Feynman integrals I_j in equation (7) in terms of the master integrals. First we plug in the IBP reductions into the equation (4) and secondly we group all coefficients in front of the master integrals. All coefficients in front of the master integrals have to vanish individually because the master integrals are linearly independent. This gives us N_M relations between the undetermined coefficients $\hat{c}_j^{i,l_1,\dots,l_M}$, where N_M is the number of master integrals. Many of the coefficients $\hat{c}_j^{i,l_1,\dots,l_M}$ are redundant and can be reduced to a linearly independent set by evaluating each of the N_M relations for several numerical points in $(d, \vec{s})$ and solving all relations for the $\hat{c}_j^{i,l_1,\dots,l_M}$. This gives a reduction of the coefficients

$$\hat{c}_j^{i,l_1,\dots,l_M} = \sum_m x_m C_m, \tag{8}$$

in terms of the master coefficients C_m. With equation (8) we are equipped to construct the block-triangular form. This time instead of using the IBP reductions for the integrals I_j we plug in the results for the coefficients $\hat{c}_j^{i,l_1,\dots,l_M}$ in terms of the master coefficients in to the equation (4). This equations is now used to construct the final block-triangular form: now the equation (4) only depends on the master coefficients C_m and we may set arbitrary numeric numbers for these master coefficients C_m and the equation (4) will be valid by construction. We choose N_J different numeric configurations for the master coefficients in such a way that we get N_J relations which involve linearly independently all J_j integrals. This is what we call a block-triangular form. In most cases a block-triangular form is more desirable than the final reduction of the Feynman integrals, because the numeric evaluation of the block-triangular form is faster, and numerically more stable due to smaller and shorter equations in the block-triangular form compare to the numerical evaluation of the IBP reduction coefficients.

In `Kira` we aim to construct the block-triangular form without imposing the knowledge of the full reduction of the Feynman integrals. Note that the IBP reduction coefficients are much bigger than the coefficients we aim to find in the block-triangular form. Thus we will perform the reduction of the integrals with the finite field methods and skip the knowledge of the IBP reduction coefficients. We use `Kira` to accomplish this task, because the finite field methods are supported within this tool. When using the finite field methods the implementation of the block-triangular form is a straight forward task. But because the implementation of the block-triangular form in `Kira` is still work in progress we do not go into the rich implementation details.

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This time we evaluate the IBP reductions numerically for $(d, \vec{s})$ modulo large prime number, and insert the numerical IBP reductions into the equation (4). From this for each new numerical point N_M relations follow which we solve instantly on-the-fly within Kira framework for the coefficients $\hat{c}_j^{i,l_1,\ldots,l_M}$. Thanks to the on-the-fly implementation in Kira we know when no new information is generated from sampling in $(d, \vec{s})$. We stop and do not proceed with the next prime number, yet. As before we get relations as in equation (8), but the numeric parameters x_m are modulo the first prime number. We manage the master coefficients C_m in the following way: for each master coefficient we generate one relation by setting it to unity and the rest of master coefficients to zero. This way we can easily find N_J relations which are sufficient to build a block-triangular form for the J_j. Before we proceed with the next prime number to reconstruct the numeric parameters x_m we set all master coefficients to zero, which we did not set to unity to generate N_J relations. By setting the master coefficients to zero we minimize the initial Ansatz in equation (5). One important benefit of this trick is, we apply the finite field reconstruction only to the relevant numeric parameters x_m in equation (8). The criteria to minimize the possible terms in the Ansatz of equation (5) and truncating the computation to the finite field reconstruction to only relevant parameters x_m is the major feature, which we aim to implement in Kira.

3. Conclusions

We have reviewed the concept of a block-triangular form within the program Kira. The public release of the new feature is work in progress and will be discussed somewhere else in more detail. We believe that the new emerging algorithms to construct the block-triangular form will allow us to study many demanding reduction problems.

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