Thick Wall from Thin Walls

J.R. Espinosa

Instituto de Física Teórica, IFT-UAM/CSIC,
C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049, Madrid, Spain
and
Theoretical Physics Department, CERN, CH-1211 Geneva 23, Switzerland

Abstract

An Euclidean bounce describing vacuum decay can be considered as an infinite stack of concentric thin shells to which a thin-wall action can be assigned. The integral over all shells produces then a tunneling action that is precisely the action functional in field space of the so-called tunneling potential formalism. This procedure, which works also when gravity is included, gives the simplest derivation of such actions.

1 Introduction

In the problem of false vacuum decay in quantum field theory, there is one particular situation that can be treated semi-analytically and simply: when the energy difference between the false and true vacua is small compared to other scales. In this case, the scalar field $\phi(r)$ of the Euclidean bounce that describes the decay makes a sharp transition between the vacua at a well-defined bounce radius (thus the name "thin-wall case" [1]). However, in general, the bounce has a thick wall, and no analytical treatment is possible.

A generic bounce can be thought of as an infinite stack of concentric thin-wall shells of infinitesimal width (from ϕ to $\phi + d\phi$) across which the (Euclidean) energy changes infinitesimally. As shown in this paper, one can associate an infinitesimal action to such slices, given by the simple thin-wall expression, generalized in the appropriate way. Integrating over all slices one recovers the tunneling action for the general case, albeit written in a particular form: the one recently proposed in the tunneling potential formalism [2,3]. The next section shows how this is done (also with gravity) and the appendix extends the derivation to general spacetime dimension $d > 2$.

The tunneling potential formalism [2,3] for the calculation of tunneling actions does not rely on Euclidean bounces and reformulates the calculation as a simple variational problem in field space. Instead of a bounce one finds a "tunneling potential" function, $V_t(\phi)$, that connects the false vacuum and (the basin of) the true vacuum and minimizes an action functional $S[V_t]$, an integral in field space of a simple action density. The resulting action reproduces the Euclidean result and the formalism has a number of appealing properties that have been discussed elsewhere.

The action functional $S[V_t]$ was first derived starting from the Euclidean approach, using the relation $V_t = V - \dot{\phi}^2/2$ to get rid of all Euclidean quantities in favor of $V_t(\phi)$ and its derivatives, and arriving at a particular second-order differential equation that V_t should satisfy. Then $S[V_t]$ is obtained so that its minimization leads to that differential equation for V_t [2,3] and its normalization is right to reproduce the Euclidean action. An alternative (simpler) derivation of the V_t action uses a canonical transformation between Euclidean and V_t formalisms [4]. The derivation of $S[V_t]$ presented now in this paper is the simplest.

2 Thick Wall as Infinite Stack of Thin Walls

2.1 No Gravity

The thin-wall tunneling action ($d = 4$ spacetime dimensions, no gravity) is

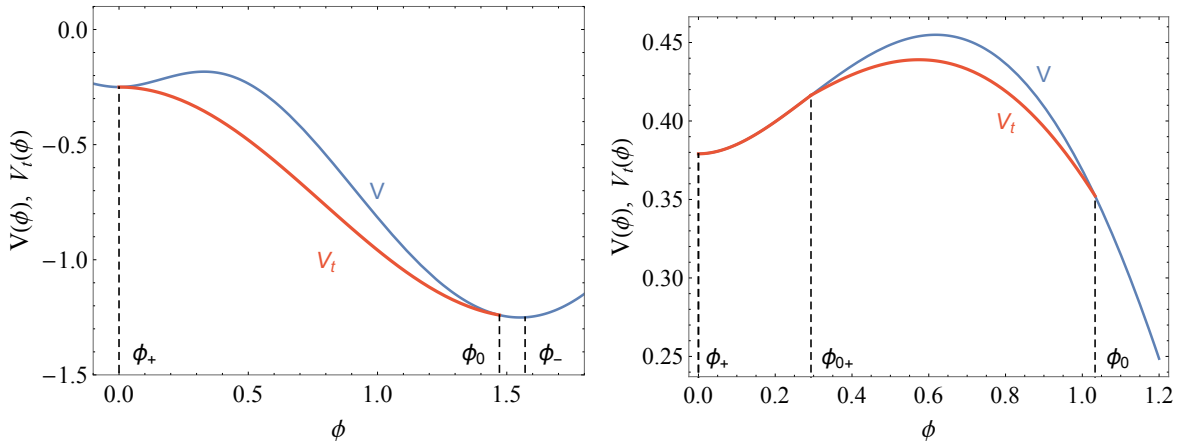$$S_{\text{thin}} = \frac{27\pi^2 \sigma^4}{2\epsilon^3} \ , \tag{1}$$

Figure 1: Examples of tunneling potentials V_t for AdS decay (left) and dS decay (right). For Minkowski or no-gravity cases, V_t is monotonic and qualitatively similar to the AdS case.

where σ is the wall tension and $\epsilon > 0$ the difference in energy between the two vacua. In the Euclidean formalism [1] one has [false vacuum at ϕ_+, true vacuum at ϕ_-, with $V_\pm \equiv V(\phi_\pm)$]

$$\sigma \simeq \int_{\phi_+}^{\phi_-} \sqrt{2(V - V_-)}d\phi \,, \qquad \epsilon \simeq -\Delta V \equiv -V_- + V_+ > 0 \,. \tag{2}$$

In the V_t formalism, σ and ϵ are defined more precisely, as

$$\sigma = \int_{\phi_+}^{\phi_0} \sqrt{2(V - V_t)}d\phi \,, \qquad \epsilon = -\Delta V_t \equiv -V_t(\phi_0) + V_t(\phi_+) > 0 \,, \tag{3}$$

where ϕ_0 is the "exit" point for tunneling [$\phi_0 = \phi(0)$ in the Euclidean formalism] with $\phi_0 \simeq \phi_-$ in the thin-wall case. In (3), ϵ is written in terms of V_t rather than V, using the fact that $V_t = V$ at ϕ_+ and ϕ_0 (points at which $\dot{\phi} = 0$). In σ, $V_t(\phi)$ is the solution of the corresponding differential equation. The left plot in Fig. 1 shows an example of V_t. In the thin-wall case, however, V_t can be approximated well by a nearly flat monotonic function that connects the two vacua [2].

Now consider any (generally thick-wall) bounce as a collection of concentric thin slices from ϕ_+ to ϕ_0. Through each of them the field and V_t are changing a little and we assign to them a differential wall tension and differential change in (Euclidean) energy as

$$\boxed{d\sigma = \sqrt{2(V - V_t)}d\phi \,, \qquad d\epsilon = -V_t' d\phi \,.} \tag{4}$$

The relation $V_t = V - \dot{\phi}^2/2$ tells us that V_t can also be interpreted as (minus) the Euclidean energy, so that $d\epsilon$ is the infinitesimal change in Euclidean energy across the wall. To get the thick wall action we integrate the thin-wall action for all slices:

$$S = \int_{\phi_+}^{\phi_0} dS_{\text{thin}} = \int_{\phi_+}^{\phi_0} \frac{27\pi^2 d\sigma^4}{2d\epsilon^3} = 54\pi^2 \int_{\phi_+}^{\phi_0} \frac{(V - V_t)^2}{(-V_t')^3}d\phi \,, \tag{5}$$

which reproduces exactly the tunneling action in the V_t formalism [2].

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2.2 Gravity On: AdS/Minkowski Vacua

If gravity is included, the thin-wall action for the decay of a Minkowski or AdS false vacuum can be written as [5,6]

$$S_{\text{thin}} = \frac{12\pi^2}{\kappa^2 V}\left(\sqrt{1 - 3\kappa V/C^2} - 1\right)\Big|_{\phi_+}^{\phi_0} = \frac{12\pi^2}{\kappa^2 V_t}\left(\sqrt{1 - 3\kappa V_t/C^2} - 1\right)\Big|_{\phi_+}^{\phi_0} , \qquad (6)$$

where $\kappa \equiv 1/m_P^2$ and C^2 is

$$C^2 = 3\kappa V_{t+} + \frac{1}{\sigma^2}\left(\epsilon - \frac{3}{4}\kappa\sigma^2\right)^2 = 3\kappa V_- + \frac{1}{\sigma^2}\left(\epsilon + \frac{3}{4}\kappa\sigma^2\right)^2 . \qquad (7)$$

In the literature, this formula is written in terms of V rather than V_t but we use the fact that $V(\phi_+) = V_t(\phi_+) \equiv V_{t+}$ and $V(\phi_0) = V_t(\phi_0)$ to rewrite it in terms of V_t, which is the natural choice to make the thick-from-thin connection. Similarly, σ and ϵ are defined as in (3), although now V_t solves an equation of motion that includes gravitational effects [3].

We slice as before a thick wall in an infinite collection of concentric thin walls with $d\sigma$ and $d\epsilon$ given by the same formulas (4). For a given thin-wall slice with field between ϕ and $\phi + d\phi$, C^2 takes now the value

$$C^2(\phi) = 3\kappa V_t + \frac{d\epsilon^2}{d\sigma^2} = \frac{D^2}{2(V - V_t)} , \qquad (8)$$

where

$$D^2 \equiv V_t'^2 + 6\kappa(V - V_t)V_t . \qquad (9)$$

From this it follows $\sqrt{1 - 3\kappa V_t/C^2(\phi)} = -V_t'/D$. For such slice we obtain the infinitesimal action

$$dS_{\text{thin}} = \frac{12\pi^2}{\kappa^2 V_t}\left[\sqrt{1 - \frac{3\kappa V_t}{C^2(\phi)}} - 1\right]\Bigg|_{V_t(\phi)}^{V_t(\phi+d\phi)} = \frac{6\pi^2}{\kappa^2 V_t^2}\left[2 - \sqrt{1 - \frac{3\kappa V_t}{C^2(\phi)}} - \frac{1}{\sqrt{1 - \frac{3\kappa V_t}{C^2(\phi)}}}\right] V_t' d\phi . \qquad (10)$$

Integrating over all slices and using the previous relations, the thick-wall action is

$$S = \int_{\phi_+}^{\phi_0} dS_{\text{thin}} = \frac{6\pi^2}{\kappa^2}\int_{\phi_+}^{\phi_0} \frac{(D + V_t')^2}{DV_t^2} d\phi , \qquad (11)$$

which reproduces the tunneling action of the V_t formalism [3]. Note that $dS_{\text{thin}} > 0$.

2.3 Gravity On: dS Vacua

For the dS case, as is well known [3,6,7], the vacuum decay can be considered as composed of two parts, a Hawking-Moss type transition from the false vacuum field value ϕ_+ to a field value ϕ_{0+} with higher energy and then a Coleman-De Luccia (CdL) type of transition from ϕ_{0+} to ϕ_0. In the V_t formalism, the first part has $V_t = V$ as depicted in Fig. 1, right plot.

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For the Hawking-Moss part, we first have to derive the "thin-wall" approximation for that case. The Hawking-Moss rate is [8]

$$S_{\text{HM}} = \frac{24\pi^2}{\kappa^2} \left(\frac{1}{V_+} - \frac{1}{V_{\text{top}}} \right) , \tag{12}$$

where V_{top} is the value of the potential at the top of the barrier that makes the vacuum classically stable. We can talk of a thin-wall HM transition (thin in field space) from ϕ to $\phi + d\phi$ simply setting $V_+ = V(\phi)$ and $V_{\text{top}} = V(\phi + d\phi)$ to get

$$dS_{\text{HM}} = \frac{24\pi^2 V'}{\kappa^2 V^2} d\phi = \frac{24\pi^2 V_t'}{\kappa^2 V_t^2} d\phi , \tag{13}$$

where, for the last equality, we have used that $V_t = V$ in the HM region of the decay.

For the CdL part of the transition the thin-wall result [6,7] can be written as

$$S_{\text{thin}} = \frac{12\pi^2}{\kappa^2} \left\{ \frac{1}{V_{t+}} \left[1 - \frac{1}{C\sigma} \left(\epsilon - \frac{3\kappa\sigma^2}{4} \right) \right] - \frac{1}{V_{t0}} \left[1 - \frac{1}{C\sigma} \left(\epsilon + \frac{3\kappa\sigma^2}{4} \right) \right] \right\} , \tag{14}$$

with C^2 as in (7). For the slice between ϕ and $\phi + d\phi$ we get $C^2(\phi)$ as in (8) for the AdS/Minkowski case. Setting $V_{t+} = V_t(\phi)$, $V_{t0} = V_t(\phi + d\phi)$, and using $d\sigma$ and $d\epsilon$ as in (4)

$$dS_{\text{thin}} = \frac{6\pi^2}{\kappa^2} \frac{(D + V_t')^2}{D V_t^2} d\phi , \tag{15}$$

which is, again, the correct action density [3]. Adding up all infinitesimal actions from HM and CdL parts we finally get

$$S = \frac{24\pi^2}{\kappa^2} \int_{\phi_+}^{\phi_{0+}} \frac{V_t'}{V_t^2} d\phi + \frac{6\pi^2}{\kappa^2} \int_{\phi_{0+}}^{\phi_0} \frac{(D + V_t')^2}{D V_t^2} = \frac{6\pi^2}{\kappa^2} \int_{\phi_+}^{\phi_0} \frac{(D + V_t')^2}{D V_t^2} , \tag{16}$$

in agreement with the action for dS decay in the V_t formalism [3].

3   Conclusions

Although the thin wall action is a particular limit of the general tunneling action, this paper shows how the general (thick) action can be recovered from the particular (thin) case: consider any tunneling bounce as an infinite collection of concentric thin shells, apply to them the thin-wall action (defined in a concrete way) and integrate over the full collection.

Obviously, the thin-to-thick connection discussed in this paper does not lead to an algorithm useful to calculate the action or V_t (e.g. numerically) as the thin-wall infinitesimal actions involve the global V_t.

The result is nevertheless interesting because the general action obtained is naturally a functional in field space and corresponds precisely to the action used in the tunneling potential formalism. The derivation in this paper can thus be considered as the simplest derivation of such actions.

Alternatively, the result can be interpreted as validating the picture of the slices as thin-wall shells between nearby Euclidean energies carrying an infinitesimal wall tension in a sense made precise by equation (4). This also lends support to the definition

$$\sigma = \int_{\phi_+}^{\phi_0} \sqrt{2(V - V_t)}d\phi \ , \tag{17}$$

for general walls. In terms of the field slope, this corresponds to $\sigma = \int |\dot{\phi}|d\phi$, which has been advocated elsewhere (e.g. in [11]).

Acknowledgments

A   General d

A.1   No Gravity

The derivation goes through for $d > 2$ spacetime dimensions. Without gravity, the starting point is the thin-wall action (see e.g. [9, 10])

$$S_{\text{thin}} = \frac{(d-1)^{d-1}\pi^{d/2}\sigma^d}{\Gamma(1 + d/2)\epsilon^{d-1}} \ , \tag{18}$$

where σ is the wall tension and $\epsilon > 0$ the difference in energy between the two vacua, as in (3). The total thick-wall action is then, using (4):

$$S = \int_{\phi_+}^{\phi_0} dS_{\text{thin}} = \int_{\phi_+}^{\phi_0} \frac{(d-1)^{d-1}\pi^{d/2}(d\sigma)^d}{\Gamma(1 + d/2)(d\epsilon)^{d-1}} = \frac{(d-1)^{d-1}}{\Gamma(1 + d/2)} \int_{\phi_+}^{\phi_0} \frac{[2\pi(V - V_t)]^{d/2}}{(-V_t')^{d-1}}d\phi \ , \tag{19}$$

which reproduces exactly the tunneling action in the V_t formalism.

A.2   Gravity On: Minkowski/AdS Vacua

Including gravity, the thin-wall action for AdS or Minkowski decay reads

$$S_{\text{thin}} = -\frac{\pi^{d/2}(d-1)^{d-2}}{\kappa\Gamma(1 + d/2)C^{d-2}} \left[d\sqrt{1 - z} + z(d-1)\,_2F_1(1/2, d/2; 1 + d/2; z) \right]\Big|_{\phi_+}^{\phi_0} \ , \tag{20}$$

where $z = 2\kappa_d V_t / C^2$, $\kappa_d = \kappa(d-1)/(d-2)$, and

$$C^2 = 2\kappa_d V_{t+} + \frac{1}{\sigma^2}\left(\epsilon - \frac{1}{2}\kappa_d\sigma^2\right)^2 = 2\kappa_d V_{t-} + \frac{1}{\sigma^2}\left(\epsilon + \frac{1}{2}\kappa_d\sigma^2\right)^2 . \tag{21}$$

For the infinitesimal slices we have

$$C^2(\phi) = 2\kappa_d V_t + \frac{\epsilon^2}{\sigma^2} = \frac{D_d^2}{2(V - V_t)} , \tag{22}$$

with $D_d^2 = V_t'^2 + 4\kappa_d(V - V_t)V_t$, and the infinitesimal action is

$$dS_{\text{thin}} = \frac{(d-1)^{d-1}\pi^{d/2}}{C^d\Gamma(1+d/2)}\left[-\frac{d}{\sqrt{1-z}} + (d-1)\,_2F_1(1/2, d/2; 1+d/2; z)\right] V_t' d\phi . \tag{23}$$

Using the identity

$$_2F_1(1/2, d/2; 1+d/2; z) = \frac{1}{d-1}\left[\frac{d}{\sqrt{1-z}} - (1-z)^{-d/2}\,_2F_1\left(\frac{d-1}{2}, \frac{d}{2}; \frac{d}{2}+1; \frac{z}{z-1}\right)\right] , \tag{24}$$

we arrive at the action

$$S = \frac{(d-1)^{d-1}}{\Gamma(1+d/2)}\int_{\phi_+}^{\phi_0}\frac{[2\pi(V-V_t)]^{d/2}}{|V_t'|^{d-1}}\,_2F_1\left(\frac{d-1}{2}, \frac{d}{2}; \frac{d+2}{2}; 1 - \frac{D_d^2}{V_t'^2}\right) d\phi , \tag{25}$$

which reproduces the result obtained in the V_t formalism [10].

A.3 Gravity On: dS vacua

For the dS case, the thin-wall approximation gives the action (for the CdL part) [10]

$$S_{\text{thin}} = \frac{\pi^{d/2}(d-1)^{d-2}}{\kappa C^{d-2}\Gamma(1+d/2)}\left[2F(z_T) - F(z_+) - F(z_0) + G(z_+) - G(z_T)\right] , \tag{26}$$

where

$$\begin{aligned} F(z) &= d\sqrt{1-z} + z(d-1)\,_2F_1(1/2, d/2; d/2+1, z) , \\ G(z) &= 2(d-1)z^{1-d/2}\,_2F_1(1/2, d/2; d/2+1, 1) , \end{aligned} \tag{27}$$

with $z = 2\kappa_d V_t / C^2$ and C^2 is as given in (21). The qualitative shape of V_t can be of three different types [10], controlled by the value of z_T. When the maximum of V_t happens at ϕ_+ (ϕ_0), then $z_T = z_+$ ($z_T = z_0$). If the maximum occurs at some point in between, then $z_T = 1$.

For the infinitesimal slices we just need to consider the two former cases, $z_T = z_+$ and $z_T = z_-$ (corresponding to $V_t' < 0$ and $V_t' > 0$ respectively). The case $z_T = 1$ would occur only for the infinitesimal slice right at the maximum of V_t and is therefore irrelevant for the final result. Using $C(\phi)$ as given in (22) and replacing $z_+ = 2\kappa_d V_t(\phi)/C(\phi)^2$, $z_0 = 2\kappa_d V_t(\phi + d\phi)/C(\phi)^2$ we get, for the case $z_T = z_+$

$$dS_{\text{thin}} = \frac{\pi^{d/2}(d-1)^{d-2}}{\kappa C^{d-2}\Gamma(1+d/2)}[-F'(z)]\frac{2\kappa_d V_t'}{C^2}d\phi , \tag{28}$$

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and for the case $z_T = z_0$

$$dS_{\text{thin}} = \frac{\pi^{d/2}(d-1)^{d-2}}{\kappa C^{d-2}\Gamma(1+d/2)}\left[F'(z) - G'(z)\right]\frac{2\kappa_d V_t'}{C^2}d\phi \, , \tag{29}$$

where $z = 2\kappa_d V_t(\phi)/C(\phi)^2$. In both cases, using the identity (24) and paying attention to the sign of V_t', one arrives at

$$dS_{\text{thin}} = \frac{\pi^{(d+1)/2}R_t^d}{\Gamma[(d+1)/2]}(V_t' + |V_t'|)d\phi + {}_2F_1\left(\frac{d-1}{2}, \frac{d}{2}; \frac{d}{2}+1; 1-\frac{D^2}{V_t'^2}\right)s_0 d\phi \, , \tag{30}$$

where

$$R_t^2 = \frac{(d-1)(d-2)}{\kappa|V_t|} \, , \qquad s_0 = \frac{(d-1)^{d-1}[2\pi(V-V_t)]^{d/2}}{\Gamma(1+d/2)|V_t'|^{d-1}} \, . \tag{31}$$

This result agrees with the action density obtained in [10].

For the Hawking-Moss part of the dS action one has [10]

$$S_{\text{HM}} = \frac{-4\sqrt{\pi}V}{(d-2)\Gamma\left(\frac{d+1}{2}\right)}\left[\frac{\pi(d-1)(d-2)}{2\kappa V}\right]^{d/2}\Bigg|_{V_+}^{V_{\text{top}}} \, . \tag{32}$$

The infinitesimal action for a HM transition from ϕ to $\phi+d\phi$, obtained taking the ϕ derivative of the action above, should then be integrated in the HM region of the transition, between ϕ_+ and some ϕ_{0+}. One therefore gets

$$\delta_{\text{HM}}S = \frac{-4\sqrt{\pi}V_t}{(d-2)\Gamma\left(\frac{d+1}{2}\right)}\left[\frac{\pi(d-1)(d-2)}{2\kappa V_t}\right]^{d/2}\Bigg|_{V_{t+}}^{V_{t,0+}} \, . \tag{33}$$

This reproduces the corresponding result in the V_t formalism [10].

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