Examples of embedded defects (in particle physics and condensed matter)

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We present a series of examples designed to clarify the formalism of the previous paper. After summarizing this formalism in a prescriptive sense, we run through several examples: first, deriving the embedded defect spectrum for the Weinberg-Salam theory, then discussing several examples designed to illustrate facets of the formalism. We then calculate the embedded defect spectrum for three physical grand unified theories and conclude with a discussion of vortices formed in the superfluid ^{3}He-A phase transition.

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I. INTRODUCTION

Embedded defects have received an impressive amount of interest over the last couple of years. Principally this is because the Z string of the Weinberg-Salam theory was recently realized to be stable for part of the parameter space [1], although it proves to be unstable in the physical regime [2,3], though there may be other stabilizing effects [4,5]. However, be it stable or unstable, it may still have important cosmological consequences—as indicated by its connection to baryon number violations [6].

The standard model also admits a one-parameter family of unstable, gauge equivalent vortices, the W strings [7]. Together, with the Z string, these constitute a very nontrivial spectrum of vortices arising from the vacuum structure of the Weinberg-Salam theory, two gauge-inequivalent families of vortices, with one family *invariant* under the residual electromagnetic gauge group and the other a one-parameter family of gauge equivalent vortices. Furthermore, only one of these families has the potential to be stable.

Embedded defects have also been specifically studied in another symmetry breaking scheme, grand unified theory (GUT) flipped SU(5) [8]. One finds an 11-parameter family of gauge equivalent, unstable vortices plus another globally gauge invariant, potentially stable vortex (the V string).

The general formalism for describing embedded defects was derived by Barriola *et al.* [9]. Here the construction of embedded defect solutions for general Yang-Mills theories was described: one defines a suitable *embedded* subtheory of the Yang-Mills theory upon which a topological defect solution may be defined. In extending the embedded subtheory back to the full theory one loses the stabilizing topological nature of the defect, but retains it as solution to the theory.

In the previous paper [10], the underlying group theory behind the formalism of [9] is exploited to determine the properties and spectrum of embedded defects. The purpose of this paper is to provide a list of several examples to illustrate the formalism of the previous paper. First, however, we summarize the formalism derived in [10], so that it may be used prescriptively to determine the spectrum and stability of embedded defects.

II. SUMMARY OF FORMALISM

For a Yang-Mills theory, the embedded defects are determined by the symmetry breaking $G \to H$. The symmetry breaking depends upon a scalar field Φ, lying in a vector space $\mathcal{V}$, acted on by the D representation of G. Denoting the Lie algebra of G by $\mathcal{G}$, the natural action of $\mathcal{G}$ upon Φ is by the derived representation d, defined by $D(e^X) = e^{d(X)}$.

The natural $Gl(\mathcal{V})$ invariant inner product on $\mathcal{V}$ is the real form

$$\langle \Phi, \Psi \rangle = \mathrm{Re}(\Phi^\dagger \Psi), \quad \Phi, \Psi \in \mathcal{V}. \tag{1}$$

The general G-invariant inner product on $\mathcal{G}$ is defined by the decomposition of G into mutually commuting subalgebras $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_n$ and is of the form

$$\langle .,. \rangle = \frac{1}{q_1^2} \{.,.\}_1 + \cdots + \frac{1}{q_n^2} \{.,.\}_n, \tag{2}$$

with $\{.,.\}_i$ the inner product $\{X,Y\} = -p \ \mathrm{Re}(\mathrm{Tr}(X^\dagger Y))$, restricted to $\mathcal{G}_i$. This has n scales characterizing all possible G-invariant inner products on $\mathcal{G}$. In a gauge theory context these scales correspond to the gauge coupling constants.

Note that the same symbol is used to denote the inner product on $\mathcal{G}$ and $\mathcal{V}$; we hope it should be clear from the context which we are using. Corresponding norms for these two inner products are denoted by $\|.\|$. We discuss these inner products more fully in the previous paper [10].

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A reference point $\Phi_0 \in \mathcal{V}$ is arbitrary because of the degeneracy given by the vacuum manifold $M = D(G)\Phi_0 \cong G/H$. Where here H is the residual symmetry group, defined by the reference point Φ_0 to be $H = \{g \in G : D(g)\Phi_0 = \Phi_0\}$. Then H determines a reductive decomposition of $\mathcal{G}$:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \tag{3}$$

with $\mathcal{H}$ the Lie algebra of H, such that

$$[\mathcal{H},\mathcal{H}] \subseteq \mathcal{H} \quad \text{and} \quad [\mathcal{H},\mathcal{M}] \subseteq \mathcal{M}. \tag{4}$$

Under the adjoint action of H, defined $\mathrm{Ad}(h)X = hXh^{-1}$, $\mathcal{M}$ decomposes into irreducible subspaces

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_N . \tag{5}$$

These irreducible spaces describe how the group acts on the vacuum manifold, yielding the family structure for embedded defects.

Finally, recall that the center $\mathcal{C}$ of $\mathcal{G}$ is the set of elements that commute with $\mathcal{G}$. Then the stability of vortices is related to the projection of $\mathcal{C}$ onto $\mathcal{M}$,

$$pr_{\mathcal{M}}(X) = X + X_h , \tag{6}$$

with $X_h \in \mathcal{H}$, the unique element, such that $pr_{\mathcal{M}}(X) \in \mathcal{M}$. One should note $pr_{\mathcal{M}}(\mathcal{C})$ consists of one-dimensional irreducible $\mathcal{M}_i$'s.

This structure is enough to categorize all the topological and nontopological embedded domain walls, embedded vortices, and embedded monopole solutions of a Yang-Mills theory.

A. Domain walls

Embedded domain walls are defined elements $\Phi_0 \in \mathcal{V}$:

$$\Phi(z) = f_{\mathrm{DOM}}(z)\Phi_0 , \tag{7a}$$

$$A^{\mu} = 0, \tag{7b}$$

where f_{DOM} is a real function such that $f_{\mathrm{DOM}}(+\infty) = 1$, and $f_{\mathrm{DOM}}(-\infty)\Phi_0 \neq \Phi_0$ belongs to the vacuum manifold.

Providing the vacuum manifold is connected this solution is unstable, suffering from a short range instability in the scalar field. Solutions within connected parts of the vacuum manifold are gauge equivalent.

B. Vortices

Embedded vortices are defined by pairs $(\Phi_0, X) \in \mathcal{V} \times \mathcal{M}$:

$$\Phi(r,\theta) = f_{\mathrm{NO}}(X;r) D(e^{\theta X})\Phi_0 , \tag{8a}$$

$$\underline{A}(r,\theta) = \frac{g_{\mathrm{NO}}(X;r)}{r} X \hat{\underline{\theta}}. \tag{8b}$$

Here f_{NO} and g_{NO} are the Nielsen-Olesen profile functions for the vortex [11] and we describe their dependence upon X in the Appendix of the previous paper [10]. The vortex generator X has the constraints[1]

$$X \in \mathcal{M}_i , \tag{9a}$$

$$D(e^{2\pi X})\Phi_0 = \Phi_0 . \tag{9b}$$

The winding number of such a vortex is given by $\|X\|/\|X^{\min}\|$, where $X^{\min}$ is a nontrivial minimal generator in the same $\mathcal{M}_i$ as X obeying the above two conditions. Family structure originates from the gauge equivalence of vortices defined by equal norm generators in the same $\mathcal{M}_i$.

Vortex stability subdivides into two types: dynamical and topological. Furthermore, there are two types of topological stability: Abelian, from $\mathrm{U}(1) \to \mathbf{1}$ symmetry breaking; and non-Abelian, which is otherwise. In [10] we show that Abelian topological and dynamical stability relate to $pr_{\mathcal{M}}(\mathcal{C})$: Abelian topological stability corresponds to a trivial projection, while dynamical stability corresponds to a nontrivial projection.

Generally, only generators $X \in \mathcal{M}_i$ define embedded vortices. However, if the coupling constants $\{q_k\}$ take critical values, such that between, say, $\mathcal{M}_i$ and $\mathcal{M}_j$

$$\frac{\|d(X_i)\Phi_0\|}{\|X_i\|} = \frac{\|d(X_j)\Phi_0\|}{\|X_j\|}, \quad X_i \in \mathcal{M}_i , \quad X_j \in \mathcal{M}_j , \tag{10}$$

then one has extra *combination* embedded vortices defined by generators in $\mathcal{M}_i \oplus \mathcal{M}_j$.

C. Monopoles

Embedded monopoles are defined by triplets $(\Phi_0, X_1, X_2) \in \mathcal{V} \times \mathcal{M} \times \mathcal{M}$:

$$\underline{\Phi}(\underline{r}) = f_{\mathrm{mon}}(r)\hat{\underline{r}}, \tag{11a}$$

$$A_a^{\mu}(\underline{r}) = \frac{g_{\mathrm{mon}}(r)}{r} \epsilon_{\mu ab} X_b , \tag{11b}$$

where $X_3 = [X_1, X_2]$, and we are treating $\underline{\Phi}$ as a vector within in its embedded subtheory.

Monopole generators have the following restrictions [10].

(i) The pair $(X_1, X_2) \in \mathcal{M}_i \times \mathcal{M}_i$, and are properly normalized so that, for $i = \{1,2\}$,

$$\exp(2\pi X_i)\Phi_0 = \Phi_0 . \tag{12}$$

(ii) The pair (X_1, X_2) consists of two members of an orthogonal basis of an $\mathrm{su}(2) \subset \mathcal{G}$, thus

$$\|X_1\| = \|X_2\|, \quad \langle X_1, X_2 \rangle = 0, \tag{13}$$

[1]There are some complications when the rank (see prenote of [10]) of $\mathcal{M}_i$ is greater than one—we shall generally indicate when such happens in the text.

and

$$\{X_1,[X_1,X_2]\}\propto X_2, \quad \{X_2,[X_1,X_2]\}\propto X_1. \tag{14}$$

(iii) The embedded SU(2) is such that $\mathrm{SU}(2)\cap H=\mathrm{U}(1)$, thus

$$[X_1,X_2]\in\mathcal{H}. \tag{15}$$

The winding number of the monopole is given by $\|X_1\|/\|X_1^{\min}\|$, where $X_1^{\min}$ is the minimal generator in the same $\mathcal{M}_i$ as X_1 obeying the above conditions. Monopoles also have a family structure, depending upon which $\mathcal{M}_i$ they are defined from.

III. DEFECTS IN THE WEINBERG-SALAM THEORY

To illustrate our results we rederive the existence and properties of the W and Z strings [1,7] for the Weinberg-Salam theory. One should note that it is the simplest example that illustrates our formalism.

The isospin-hypercharge gauge symmetry $G=\mathrm{SU}(2)_I\times\mathrm{U}(1)_Y$, acts fundamentally on a two-dimensional complex scalar field Φ. As a basis we take the SU(2)-isospin generators to be $X^a=(i/2)\sigma^a$, with σ^a the Pauli spin matrices, and the $\mathrm{U}(1)_Y$-hypercharge generator to be $X^0=(i/2)\mathbf{1}_2$. Then these generators act fundamentally upon the scalar field Φ:

$$d(\alpha^i X^i+\alpha^0X^0)=\alpha^iX^i+\alpha^0X^0. \tag{16}$$

The inner product on $\mathrm{su}(2)_I\oplus\mathrm{U}(1)_Y$ may be written

$$\langle X,Y\rangle=-\frac{1}{g^2}\{2\,\mathrm{Tr}\,XY+(\cot^2\,\theta_w-1)\mathrm{Tr}\,X\;\mathrm{Tr}\,Y\}, \tag{17}$$

with g and g' the isospin and hypercharge gauge coupling constants. The Weinberg angle $\theta_w=\tan^{-1}(g'/g)$.

Choosing a suitable reference point in the vacuum manifold

$$\Phi_0=\frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}, \tag{18}$$

the gauge groups breaks to

$$H=\mathrm{U}(1)_Q=\begin{pmatrix}e^{i\omega} & 0\\ 0 & 1\end{pmatrix}, \tag{19}$$

with $\omega\in[0,2\pi)$. Then H defines the decomposition $\mathcal{G}=\mathcal{H}\oplus\mathcal{M}$, where

$$\mathcal{H}=\begin{pmatrix}i\alpha & 0\\ 0 & 0\end{pmatrix} \quad\text{and}\quad \mathcal{M}=\begin{pmatrix}-i\beta\,\cos 2\theta_w & \gamma\\ -\gamma^* & i\beta\end{pmatrix}, \tag{20}$$

with α, β real, and γ complex. The star denotes a complex conjugation.

Under $\mathrm{Ad}(H)$, $\mathcal{M}$ is reducible to $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$, where

$$\mathcal{M}_1=\begin{pmatrix}-i\beta\,\cos 2\theta_w & 0\\ 0 & i\beta\end{pmatrix} \quad\text{and}\quad \mathcal{M}_2=\begin{pmatrix}0 & \gamma\\ -\gamma^* & 0\end{pmatrix}. \tag{21}$$

The center of $\mathrm{su}(2)_I\oplus\mathrm{u}(1)_Y$, which is $\mathcal{C}=\mathrm{u}(1)_Y$, projects nontrivially onto $\mathcal{M}_1$ under the inner product (17).

The first class of embedded vortices are defined from elements $X\in\mathcal{M}_1$ such that $e^{2\pi X}=1$. Since $\mathcal{M}_1=pr_{\mathcal{M}}(\mathrm{u}(1)_Y)$ these vortices are stable in the coupling constant limit $g\to 0$. From Eq. (8) one immediately writes down the solution as

$$\Phi(r,\theta)=\frac{v}{\sqrt{2}}\,f^Z_{\mathrm{NO}}(r)\begin{pmatrix}0\\ e^{in\theta}\end{pmatrix}, \tag{22a}$$

$$\underline{A}(r,\theta)=\frac{g^Z_{\mathrm{NO}}(r)}{r}\begin{pmatrix}-in\,\cos 2\theta_w & 0\\ 0 & in\end{pmatrix}\hat{\underline{\theta}}, \tag{22b}$$

where n is the winding number of the vortex. Note that this vortex is also invariant under global transformations of the residual gauge symmetry. These solutions are Z strings.

The second class of embedded vortices are defined from elements $X\in\mathcal{M}_2$ such that $e^{2\pi X}=1$. From Eq. (8) one immediately writes down the solution as

$$\Phi(r,\theta)=\frac{v}{\sqrt{2}}\,f^W_{\mathrm{NO}}(r)\begin{pmatrix}e^{i\delta}\sin n\theta\\ \cos n\theta\end{pmatrix}, \tag{23a}$$

$$\underline{A}(r,\theta)=\frac{g^W_{\mathrm{NO}}(r)}{r}\begin{pmatrix}0 & ne^{i\delta}\\ -ne^{-i\delta} & 0\end{pmatrix}\hat{\underline{\theta}}, \tag{23b}$$

with $e^{i\delta}=\gamma/|\gamma|$ and n the winding number of the vortex. All the isolated solutions of the same winding number in this one-parameter family are gauge equivalent. Furthermore, the antivortex is gauge equivalent to the vortex, so isolated solutions are parametrized by the positive winding number only. These solutions are W strings.

The above generators in $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the condition $\|d(X)\Phi_0\|/\|\Phi_0\|=n$ of the Appendix in the previous paper [10]. Thus, profile functions for the Z and W strings are related (first stated in [12]);

$$f^Z_{\mathrm{NO}}(\lambda;r)=f^W_{\mathrm{NO}}\left(\frac{\lambda}{\kappa^2};\kappa r\right), \tag{24a}$$

$$g^Z_{\mathrm{NO}}(\lambda;r)=g^W_{\mathrm{NO}}\left(\frac{\lambda}{\kappa^2};\kappa r\right), \tag{24b}$$

where $\kappa=\sqrt{(g^2+g'^2/g^2)}$ and λ is the quartic scalar self-coupling.

IV. THE MODEL SU(3)→SU(2)

We give here an example of a model that admits as a solution an unstable globally gauge invariant vortex. In addition it is a nice example of a model admitting nontopological embedded monopoles.

The gauge group is $G=\mathrm{SU}(3)$, acting fundamentally on a three-dimensional complex scalar field. Denoting the generators by $\{X^a : a=1\cdots 8\}$, the derived representation acts as

$$d(\alpha^i X^i)=\alpha^i X^i. \tag{25}$$

A Landau potential is sufficient to break the symmetry, because $\mathcal{M}$ is of the same dimension as the maximal sphere contained within $\mathbf{C}^3$. Hence, the vacuum manifold is isomorphic to a five-sphere, with G transitive over it.

Taking the reference point in the vacuum manifold to be

$$\Phi_0=v\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{26}$$

the gauge group breaks to $H=\mathrm{SU}(2)$,

$$H=\begin{pmatrix}\mathrm{SU}(2) & \vdots & 0\\ \cdots & \cdots & \cdots\\ 0 & \vdots & 1\end{pmatrix}\subset G. \tag{27}$$

At the reference point Φ_0, $\mathcal{G}$ decomposes under $\mathrm{Ad}_G(H)$ into irreducible subspaces of the form $\mathcal{G}=\mathcal{H}\oplus\mathcal{M}_1\oplus\mathcal{M}_2$, where

$$\mathcal{M}_1=\begin{pmatrix}i\gamma & 0 & 0\\ 0 & i\gamma & 0\\ 0 & 0 & -2i\gamma\end{pmatrix}$$

and

$$\mathcal{M}_2=\begin{pmatrix}0 & 0 & a\\ 0 & 0 & b\\ -a^* & -b^* & 0\end{pmatrix}, \tag{28}$$

with γ real and a, b complex.

The first class of vortex solutions are classified by $X\in\mathcal{M}_1$. They are given by

$$\Phi(r,\theta)=v f_{\mathrm{NO}}(X_1;r)\begin{pmatrix}0\\0\\e^{in\theta}\end{pmatrix}, \tag{29a}$$

$$\underline{A}(r,\theta)=\frac{g_{\mathrm{NO}}(X_1;r)}{r}\begin{pmatrix}-in/2 & 0 & 0\\ 0 & -in/2 & 0\\ 0 & 0 & in\end{pmatrix}\underline{\hat{\theta}}. \tag{29b}$$

The integer n is the winding number of the vortex. These solutions have no semilocal limit and are therefore always unstable. The second class of vortex solutions are those classified by $X\in\mathcal{M}_2$. They are a three-parameter family of gauge equivalent, unstable solutions.

The vortex winding number in both classes $\mathcal{M}_1$ and $\mathcal{M}_2$ is $\|d(X)\Phi_0\|/\|\Phi_0\|$. From the Appendix of the previous paper [10], profile functions for both classes coincide with each other and the Abelian-Higgs model.

Nontopological embedded monopole solutions are present in this model. The solutions are specified by a gauge-equivalent class of generators $(X,Y)\in\mathcal{M}_2\times\mathcal{M}_2$, such that $\langle X,Y\rangle=0$ and $[X,Y]\in\mathcal{H}$. A class of such generators is

$$X=\mathrm{Ad}(h)\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad Y=\mathrm{Ad}(h)\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}, \tag{30}$$

with

$$[X,Y]=\mathrm{Ad}(h)\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\in\mathcal{H}, \tag{31}$$

where h is some element in H. There is a one-to-one correspondence between elements in H and the choice of the embedded monopole. It should be noted that elements of the form

$$X'=\mathrm{Ad}(h)\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\quad Y'=\mathrm{Ad}(h)\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix} \tag{32}$$

do not define monopole solutions because $[X',Y']\notin\mathcal{H}$. Antimonopoles are defined in the above form but with one of the generators negative.

In conclusion, there is a two-parameter family of unstable embedded monopole solutions of the form defined in Eq. (11).

V. THE MODEL U(1)×U(1)→1

This model is presented to illustrate combination vortices. By ‘‘combination vortices’’ we mean vortices that are generated by elements that are not in any of the irreducible spaces $\mathcal{M}_i$, the vortex generators being instead *between* the spaces.

In Sec. II, we said that such combination vortices are solutions, providing the coupling constants take a critical set of values. We illustrate this principle by explicitly finding such solutions in the model U(1)×U(1)→1.

The gauge group is $G=\mathrm{U}(1)_X\times\mathrm{U}(1)_Y$, with elements

$$g(\theta,\varphi)=\begin{pmatrix}e^{i\theta} & 0\\ 0 & e^{i\varphi}\end{pmatrix}\in G, \tag{33}$$

and $\theta,\varphi\in[0,2\pi)$. Generators of $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_Y$ are

$$X=\begin{pmatrix}i & 0\\ 0 & 0\end{pmatrix},\quad Y=\begin{pmatrix}0 & 0\\ 0 & i\end{pmatrix}. \tag{34}$$

The group G acts fundamentally on a two-dimensional complex scalar field $\Phi=(\phi_1,\phi_2)^T$

$$D(g(\theta,\varphi))=\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}. \tag{35}$$

The inner product on $\mathcal{G}$ is of the form

$$\langle X,Y\rangle=-\mathrm{Tr}(XQ^{-1}Y),$$

with

$$Q=\begin{pmatrix} q_1^2 & 0 \\ 0 & q_2^2 \end{pmatrix}, \tag{36}$$

where q_1 and q_2 are the coupling constants for the respective parts of G.

To break G to triviality, the parameters of the scalar potential must be chosen correctly. The general, renormalizable, gauge-invariant scalar potential for this theory is

$$V(\phi_1,\phi_2)=\lambda_1(\phi_1^*\phi_1-v_1^2)^2+\lambda_2(\phi_2^*\phi_2-v_2^2)^2 + \lambda_3\phi_1^*\phi_1\phi_2^*\phi_2. \tag{37}$$

For some range of $(\lambda_1,\lambda_2,\lambda_3,v_1,v_2)$ (the range being unimportant to our arguments), this is minimized by a two torus of values, then G breaks to triviality.

Without loss of generality the scalar field reference point is chosen to be

$$\Phi_0=\begin{pmatrix} v_1' \\ v_2' \end{pmatrix}, \tag{38}$$

where unless $v_1^2=v_2^2$, the primed vacuum expectation value (VEV) v_1', v_2' are unequal to v_1 and v_2. Then the group G breaks to the trivial group $H=\mathbf{1}$. Under the adjoint action of H, the Lie algebra of G splits into

$$\mathcal{G}=\mathcal{M}_1\oplus\mathcal{M}_2, \tag{39}$$

with

$$\mathcal{M}_1=\begin{pmatrix} ia & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_2=\begin{pmatrix} 0 & 0 \\ 0 & ib \end{pmatrix}, \tag{40}$$

and a, b real.

The topology of the vacuum manifold is nontrivial, hence vortex solutions that are generated by elements in $\mathcal{M}_1$ or $\mathcal{M}_2$ are topologically stable. These vortices are well defined and are stationary solutions of the Lagrangian.

It is interesting to consider the existence of vortices generated by elements in the whole of $\mathcal{M}_1\oplus\mathcal{M}_2$, and not just vortices generated in either of these two spaces separately. Combination vortices may exist when the coupling constants are such that Eq. (10) is satisfied. Substitution of the generators X and Y into Eq. (10) yields the condition that combination vortices exist for

$$\frac{\|d(X)\|}{\|X\|}=\frac{\|d(Y)\|}{\|Y\|}\Rightarrow q_1^2=q_2^2. \tag{41}$$

When $q_1=q_2$, the Lie algebra elements that generate closed geodesics are of the form

$$Z=\delta X+\epsilon Y, \tag{42}$$

providing there exists $\omega>0$ with $D(e^{Z\omega})\Phi_0=\Phi_0$. Since the coupling constants are equal Z generates a U(1) subgroup of G. Relating this back to the geometry of a torus, the constraint on nonzero ϵ and δ is

$$\frac{\epsilon}{\delta}\in Q, \tag{43}$$

the rational numbers. One can interpret the effect of the scaling as ‘‘twisting’’ directions in the tangent space to the vacuum manifold relative to directions in the Lie algebra. This twisting only happens between the irreducible subspaces of $\mathcal{M}$.

However, not all of these geodesics define embedded vortices. One also needs to satisfy condition (2) in the previous paper [10]:

$$\left\langle \Psi, \frac{\partial V}{\partial\Phi}\right\rangle=0, \tag{44}$$

where $\Psi\in\mathcal{V}_{\mathrm{emb}}^{\perp}$ and $\Phi\in\mathcal{V}_{\mathrm{emb}}$. A trivial substitution yields

$$\lambda_1=\lambda_2=\lambda, \quad v_1^2=v_2^2=v^2, \quad \text{and} \quad \epsilon=\delta. \tag{45}$$

This is the only combination vortex.

VI. EMBEDDED DEFECTS IN REALISTIC GUT MODELS

We now gives some examples of the embedded defect spectrum in some realistic GUT models. The examples here are certainly not meant to be exhaustive, merely just a few of the simplest examples.

A. Georgi-Glashow SU(5)

The gauge group is $G=\mathrm{SU}(5)$ [13], acting on a 24-dimensional scalar field Φ by the adjoint action. For scalar vacuum

$$\Phi_0=v\begin{pmatrix} \frac{2}{3}\mathbf{1}_3 & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & -\mathbf{1}_2 \end{pmatrix}, \tag{46}$$

G breaks to $H=\mathrm{SU}(3)_C\times\mathrm{SU}(2)_I\times\mathrm{U}(1)_Y$,

$$\begin{pmatrix} \mathrm{SU}(3)_0 & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathrm{SU}(2)_I \end{pmatrix} \times\begin{pmatrix} e^{(2/3)i\theta}\mathbf{1}_3 & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & c^{-i\theta}\mathbf{1}_2 \end{pmatrix}\subset\mathrm{SU}(5). \tag{47}$$

To find the embedded defect spectrum one determines the reduction of $\mathcal{G}$ into $\mathcal{G}=\mathcal{H}\oplus\mathcal{M}$ and finds the irreducible spaces of $\mathcal{M}$ under the adjoint action of H. The space $\mathcal{M}$ is

$$\mathcal{M}=\begin{pmatrix} \mathbf{0}_3 & \vdots & \underline{A} \\ \cdots & \cdots & \cdots \\ -\underline{A}^\dagger & \vdots & \mathbf{0}_2 \end{pmatrix}, \tag{48}$$

with $\underline{A}$ a two-by-three complex matrix. This is irreducible under the adjoint action of H.

Thus the defect spectrum of the model is monopoles, which can be confirmed to be topologically stable, and a family of unstable leptoquark strings. The family of leptoquark strings is complicated by $\mathcal{M}$ containing *two* distinct (nonproportional) commuting generators.

B. Flipped SU(5)

For a more detailed discussion of embedded defects and their properties in flipped SU(5), see [8]. The gauge group is $G=\mathrm{SU}(5)\times\widetilde{U(1)}$ [14], and acts upon a complex ten-dimensional scalar field (which we conveniently represent as a five-by-five complex antisymmetric matrix) by the **10**-antisymmetric representation. Denoting the generators of SU(5) as X^a and $\widetilde{U(1)}$ as $\widetilde{X}$, the derived representation acts upon the scalar field as

$$d(\alpha^i X^i+\alpha^0\widetilde{X})=\alpha^i(X^i\Phi+\Phi X^{i^T})+\alpha^0\widetilde{X}\Phi. \tag{49}$$

The inner product upon su(5)⊕u(1) is of the form

$$\langle X,Y\rangle=-\frac{1}{g^2}\left\{\mathrm{Tr}\,XY+\frac{1}{5}(\cot^2\Theta-1)\mathrm{Tr}\,X\;\mathrm{Tr}\,Y\right\}, \tag{50}$$

where g and $\widetilde{g}$ are the SU(5) and $\widetilde{U(1)}$ coupling constants. The GUT mixing angle is $\tan\Theta=\widetilde{g}/g$. For the following discussion it is necessary to explicitly know the following generators

$$X^{15}=ig\sqrt{\frac{3}{2}}\begin{pmatrix} \frac{2}{3}\mathbf{1}_3 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & -\mathbf{1}_2 \end{pmatrix}, \quad \widetilde{X}=i\widetilde{g}\mathbf{1}_5. \tag{51}$$

These generators are normalized with respect to Eq. (50).

For a vacuum given by

$$\Phi_0=\frac{v}{\sqrt{2}}\begin{pmatrix} \mathbf{0}_3 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & I \end{pmatrix}, \quad \text{where } I=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{52}$$

one breaks SU(5)×U(1) to the standard model $H=\mathrm{SU}(3)_C\times\mathrm{SU}(2)_I\times\mathrm{U}(1)_Y$, provided that the parameters of the potential satisfy η^2, $\lambda_1>0$ and $(2\lambda_1+\lambda_2)>0$. The V and hypercharge fields are given by

$$V_\mu=\cos\Theta A_\mu^{15}-\sin\Theta\widetilde{A}_\mu, \quad X_V=\cos\Theta X^{15}-\sin\Theta\widetilde{X}, \tag{53a}$$

$$Y_\mu=\sin\Theta A_\mu^{15}+\cos\Theta\widetilde{A}_\mu, \quad X_Y=\sin\Theta X^{15}+\cos\Theta\widetilde{X}. \tag{53b}$$

Then $d(\mathcal{H})\Phi_{\mathrm{vac}}=0$. The isospin and color symmetry groups are

$$\begin{pmatrix} \mathrm{SU}(3)_C & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathrm{SU}(2)_I \end{pmatrix}\subset\mathrm{SU}(5). \tag{54}$$

To find the embedded defect spectrum one determines the reduction of $\mathcal{G}$ into $\mathcal{G}=\mathcal{H}\oplus\mathcal{M}$ and finds the irreducible spaces of $\mathcal{M}$ under $\mathrm{Ad}(H)$, which is $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ such that

$$\mathcal{M}_1=\mathbf{R}X_V, \quad \mathcal{M}_2=\begin{pmatrix} \mathbf{0}_3 & \vdots & \underline{A} \\ \cdots & \cdots & \cdots \\ -\underline{A}^\dagger & \vdots & \mathbf{0}_2 \end{pmatrix}. \tag{55}$$

The first space $\mathcal{M}_1$ is the projection of $\widetilde{\mathrm{u}(1)}$ onto $\mathcal{M}$. This is important for the stability of vortex solutions defined from it. Such vortices are stable in the limit $\Theta_{\mathrm{GUT}}\to\pi/2$, then, by continuity, also in a region around $\pi/2$. The second space $\mathcal{M}_2$ generates a family of unstable leptoquark strings and nontopological monopoles. The family of leptoquark strings is complicated by $\mathcal{M}$ containing *two* distinct (nonproportional) commuting generators.

C. Pati-Salam $\mathrm{SU(4)\times SU(4)\to SU(3)_0\times SU(2)_I\times U(1)_Y}$

Pati and Salam emphasized a series of models of the form $G=G^S\times G^W$, where G^S and G^W are identical strong and weak groups related by some discrete symmetry [15]. The above model is the simplest one of this form. The model is actually $[\mathrm{SU}(4)\times\mathrm{SU}(4)]_L\times[\mathrm{SU}(4)\times\mathrm{SU}(4)]_R$ (''L'' and ''R'' denoting the separate couplings to left and right-handed fermions) to accommodate parity violation in weak interactions. For simplicity we shall only consider half of the model.

The gauge group $G=\mathrm{SU}(4)^S\times\mathrm{SU}(4)^W$, breaking to $H=[\mathrm{SU(3)\times U(1)}]^S\times\mathrm{SU}(2)^W$:

$$\begin{pmatrix} \mathrm{SU}(3)_C & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & \mathrm{U}(1)_Y \end{pmatrix}_S \times\begin{pmatrix} \mathrm{SU}(2)_I & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{1}_2 \end{pmatrix}_W \subset\mathrm{SU}(4)^S\times\mathrm{SU}(4)^W. \tag{56}$$

Writing $\mathcal{G}=\mathcal{H}\oplus\mathcal{M}$, the irreducible spaces of $\mathcal{M}$ under $\mathrm{Ad}(H)$ are $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2\oplus\widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is a collection of four irreducible spaces, with

$$\mathcal{M}_1=\begin{pmatrix} \mathbf{0}_3 & \vdots & \underline{A} \\ \cdots & \cdots & \cdots \\ -\underline{A}^\dagger & \vdots & 0 \end{pmatrix}_S, \quad \mathcal{M}_2=\begin{pmatrix} \mathbf{0}_2 & \vdots & B \\ \cdots & \cdots & \cdots \\ -B^\dagger & \vdots & \mathbf{0}_2 \end{pmatrix}_W,$$

and

$$\tilde{\mathcal{M}}=\begin{pmatrix} \mathbf{0}_2 & \vdots & \mathbf{0}_2 \\ \cdots & \cdots & \cdots \\ \mathbf{0}_2 & \vdots & C \end{pmatrix}_W \oplus \begin{pmatrix} i\alpha\mathbf{1}_2 & \vdots & 0_2 \\ \cdots & \cdots & \cdots \\ \mathbf{0}_2 & \vdots & -i\alpha\mathbf{1}_2 \end{pmatrix}_W, \tag{57}$$

where $\underline{A}$ is a complex three-dimensional vector, B and C are complex two-by-two matrices, with C, an anti-Hermitian, and α is a real number.

Each of the above spaces gives rise to their respective embedded defects. First, $\mathcal{M}_1$ gives rise to topologically stable monopoles and a five-parameter family of unstable vortices. Secondly, $\mathcal{M}_2$ gives rise to nontopological unstable monopoles and a seven-parameter family of unstable vortices. Thirdly, $\tilde{\mathcal{M}}$, which is a collection of four irreducible spaces, admits globally gauge invariant unstable vortices. In addition, $\tilde{\mathcal{M}}$ has combination vortex solutions between the four irreducible spaces, of which it consists.

VII. VORTICES IN THE ^{3}He-*A* PHASE TRANSITION

We wish to show here that our results on the classification of vortices for general gauge theories are also relevant for condensed matter systems. As an example we choose the ^{3}He-*A* phase transition, though we expect the general onus of our results to be applicable to other situations having a similar nature.

Superfluid ^{3}He has global symmetries of spin [$\mathrm{SO}(3)_S$ rotations], angular rotations ($\mathrm{SO}(3)_L$), and a phase (associated with particle number conservation). It has several phase transitions corresponding to different patterns of breaking this symmetry. We concentrate here on the *A*-phase transition.

Condensed matter systems, such as ^{3}He, have added complications above that of gauge theories, meaning that we cannot just naively apply the approach used in the rest of this paper. This complication originates through the order parameter being a *vector* under spatial rotations, not a scalar as in conventional gauge theories. The upshot being that extra terms are admitted in the Lagrangian that are not present in a conventional gauge theory. These terms couple derivatives of components with different angular momentum quantum numbers and so are not invariant under $\mathrm{SO}(3)_L$ rotations in the conventional sense—thus spoiling the $\mathrm{SO}(3)_L$ invariance. The general effect of this is to complicate the spectrum of vortex solutions, and their actual form and interaction.

Our tactic to investigate the effect of these extra noninvariant $\mathrm{SO}(3)_L$ terms is to first examine the ^{3}He-*A* phase transitions without the inclusion of these terms so that we may use the techniques of embedded vortices used in the rest of this paper, and then see how these terms affect the solutions.

A. The ^{3}He-*A* phase transition

The full symmetry group of liquid ^{3}He is

$$G_{3_{\mathrm{He}}}=\mathrm{SO}(3)_S\times\mathrm{SO}(3)_L\times\mathrm{U}(1)_N, \tag{58}$$

which acts on the two group-index order parameter $A_{\alpha j}$ by the $\mathit{fund.}_S\otimes\mathit{fund.}_{L,N}$ representation of $G_{3_{\mathrm{He}}}$. Denoting

$$A_{\alpha j}=\Delta_0 d_\alpha \Psi_j, \tag{59}$$

with unit vector $d_\alpha\in\mathbf{R}^3$ and $\Psi_j=(\hat{\mathbf{e}}_1+i\hat{\mathbf{e}}_2)/\sqrt{2}\in\mathbf{C}^3$, where $\hat{\mathbf{e}}_1,\ \hat{\mathbf{e}}_2\in\mathbf{R}^3$ such that $\hat{\mathbf{e}}_1.\hat{\mathbf{e}}_1=\hat{\mathbf{e}}_2.\hat{\mathbf{e}}_2=1$ and $\hat{\mathbf{e}}_1.\hat{\mathbf{e}}_2=0$. The quantity Δ_0 is a real number unimportant for the present discussion.

Then $G_{3_{\mathrm{He}}}$ acts on $A_{\alpha j}$ fundamentally:

$$D((g_S,g_L,g_N))_{\alpha j\beta k}A_{\beta k}=\Delta_0(g_S\mathbf{d})_\alpha(g_Lg_N\mathbf{\Psi})_j. \tag{60}$$

In addition $G_{3_{\mathrm{He}}}$ is a global symmetry of the field theory.

The field theory is described by the Lagrangian

$$\mathcal{L}[A_{\alpha j}]=\mathcal{L}_{\mathrm{sym}}[A_{\alpha j}]+\tilde{\mathcal{L}}[A_{\alpha j}], \tag{61}$$

with $\mathcal{L}_{\mathrm{sym}}$ having $G_{3\mathrm{He}}$ global symmetry and $\tilde{\mathcal{L}}$ representing the extra vector type couplings of the order parameter. We may write

$$\mathcal{L}_{\mathrm{sym}}[A_{\alpha j}]=\gamma\partial_iA^*_{\alpha j}\partial_iA_{\alpha j}-V[A_{\alpha j}], \tag{62}$$

with V some Landau-type potential invariant under $G_{3\mathrm{He}}$. For the vector type couplings we write

$$\tilde{\mathcal{L}}[A_{\alpha j}]=\gamma_1\partial_iA^*_{\alpha i}\partial_jA_{\alpha j}+\gamma_2\partial_iA^*_{\alpha j}\partial_jA_{\alpha i}, \tag{63}$$

which are explicitly not $\mathrm{SO}(3)_L$ invariant. By partial integration of the *action* integral, this may be rewritten as

$$\tilde{\mathcal{L}}[A_{\alpha j}]=(\gamma_1+\gamma_2)\partial_iA^*_{\alpha i}\partial_jA_{\alpha j}=\tilde{\gamma}\partial_iA^*_{\alpha i}\partial_jA_{\alpha j}. \tag{64}$$

The *A* phase is reached through symmetry breaking with a vacuum of the form

$$A_0=\Delta_0\mathbf{d}_0\mathbf{\Psi}_0,$$
$$\text{where}\quad \mathbf{d}_0=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \mathbf{\Psi}_0=\begin{pmatrix}1\\i\\0\end{pmatrix}, \tag{65}$$

so that the residual symmetry group is

$$H_A=\mathrm{U}(1)_{S_3}\times\mathrm{U}(1)_{L-N}\times\mathbf{Z}_2, \tag{66}$$

where

$$\mathrm{U}(1)_{S_3}=\left\{\begin{pmatrix}1&0&0\\0&\cos\alpha&\sin\alpha\\0&-\sin\alpha&\cos\alpha\end{pmatrix}_S :\alpha\in[0,2\pi)\right\}, \tag{67a}$$

$$\mathrm{U}(1)_{L-N}=\left\{ e^{-i\alpha}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}_L : \alpha\in[0,2\pi) \right\}, \tag{67b}$$

$$\mathbf{Z}_2=\left\{ \mathbf{1}_S\times\mathbf{1}_L,\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_S \times\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_L \right\}. \tag{67c}$$

It should be noted that the $\{L,N\}$ part of the group is similar to the Weinberg-Salam theory at $\Theta_w=\pi/4$, but taking the limit in which (both) of the coupling constants become zero. However, note that $\mathrm{SO}(3)_L$ is *not* simply connected, this has important stabilizing effects on the vortices [16].

Writing $\mathcal{G}_{3\mathrm{He}}=\mathcal{H}_A\oplus\mathcal{M}$, the irreducible spaces of $\mathcal{M}$ under the adjoint action of H_A are denoted by $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2\oplus\mathcal{M}_3$, with

$$\mathcal{M}_1\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ -\alpha & -\beta & 0 \end{pmatrix}_L, \quad \mathcal{M}_2=\begin{pmatrix} 0 & \gamma & \delta \\ -\gamma & 0 & 0 \\ -\delta & 0 & 0 \end{pmatrix}_S,$$

$$\text{and} \quad \mathcal{M}_3=\frac{\epsilon}{2}\begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i \end{pmatrix}_L, \tag{68}$$

and α, β, γ, δ, ϵ are real numbers.

B. Vortices in the $\mathrm{SO}(3)_L$ symmetric theory

We firstly analyze the theory when $\tilde{\gamma}=0$, so that the Lagrangian is $\mathrm{SO}(3)_L$ symmetric. In this regime the techniques of embedded vortices are applicable.

1. Embedded vortices

The first class of generators, $\mathcal{M}_1$, give a one-parameter family of gauge-equivalent global vortices, with profiles of the form

$$A(r,\theta)=\Delta_0\overline{f}(n/\sqrt{2};r)\mathbf{d}_0\begin{pmatrix} \cos\alpha/2+i\sin\alpha/2\cos n\theta \\ -\sin\alpha/2+i\cos\alpha/2\cos n\theta \\ -i\sin n\theta \end{pmatrix}. \tag{69}$$

Here n is the winding of the vortex, α labels the family member, and $\overline{f}$ is defined below. These are the disgyration vortices of ^{3}He.

The second class of generators, $\mathcal{M}_2$, give a one-parameter family of gauge-equivalent global vortices, with profiles of the form

$$A(r,\theta)=\Delta_0\overline{f}(n;r)\begin{pmatrix} \cos n\theta \\ -\cos\alpha\sin n\theta \\ \sin\alpha\sin n\theta \end{pmatrix}\mathbf{\Psi}_0. \tag{70}$$

Here n is the winding of the vortex, and α labels the family member. These are the so-called spin vortices.

The third class of generators, $\mathcal{M}_3$, give a gauge-invariant global vortex, with a profile of the form

$$A(r,\theta)=\Delta_0\overline{f}(n;r)\mathbf{d}_0 e^{in\theta}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{71}$$

Here n is the winding of the vortex and α labels the family member. These vortices are the so-called singular-line vortices.

The profile functions depend upon the embedded vortex considered, generated by X_{emb}, say, and are minima of the Lagrangian

$$\mathcal{L}[f]=\frac{\gamma\Delta_0^2}{2}\left(\frac{df}{dr}\right)^2+\frac{\gamma f^2}{2r^2}\|X_{\mathrm{emb}}A_0\|^2-V[f(r)], \tag{72}$$

where V is the potential, which is independent of the defect considered. Writing $\|X_{\mathrm{emb}}A_0\|=n\|A_0\|$ we refer to the solutions as $\overline{f}(n;r)$.

2. Combination vortices

Because the symmetries $G_{3\mathrm{He}}$ are global there are combination vortex solutions between the three families of generators. The most general combination embedded vortex is generated by a combination of generators from each of the three classes—this is the spin–singular line-disgyration combination vortex. Because of the way we shall determine such vortices we first discuss the singular line-disgyration combination.

One obtains a discrete spectrum of singular line-disgyration combination embedded vortices. Solutions are of the form

$$A(r,\theta)=\Delta_0 f(X;r)\mathbf{d}_0\ \exp(X\theta)\mathbf{\Psi}_0, \tag{73a}$$

$$\text{with}\ X=\frac{a}{2}\left(i\mathbf{1}_3+\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_L\right) +b\begin{pmatrix} 0 & 0 & 1 \\ -0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}_L. \tag{73b}$$

Then some algebra yields

$$A(r,\theta)=\Delta_0\overline{f}(p;r)\mathbf{d}_0 e^{ia\theta/2}\times\begin{pmatrix}\cos\,\theta s+\dfrac{ia}{2s}\sin\,\theta s\\ -\dfrac{a}{2s}\sin\,\theta s+\dfrac{i}{s^2}\left(b^2+\dfrac{a^2}{4}\cos\,\theta s\right)\\ -\dfrac{b}{s}\sin\,\theta s+\dfrac{iab}{2s^2}(\cos\,\theta s-1)\end{pmatrix},\tag{74}$$

where $s=\sqrt{a^2/4+b^2}$ and $p=\sqrt{(7m^2+n^2)/2}$. Using the single valuedness constraint that $A(r,2\pi)=A(r,0)$ gives the following discrete spectrum of values for a and b:

$$a=2m,\quad b=\pm\sqrt{n^2-m^2},\quad m,n\in\mathbf{Z}.\tag{75}$$

It seems that the singular line vortex and the disgyration may not be continuously deformed into one another, since if this was to be the case then the spectrum of combination vortices should be *continuous*. We obtain a *discrete* spectrum. For them to be continuously deformable into one another we need solutions that are not of the embedded type.

The spin–singular line-disgyration combination vortex can be constructed from the above form. Since the generators for spin vortices commute with the generators for singular line-disgyration combination vortices, the form of solution is a spin vortex combined with a singular line-disgyration combination, i.e.,

$$A(r,\theta)=\Delta_0\overline{f}(\sqrt{(7m^2+2j^2+n^2)/2};r)\begin{pmatrix}\cos j\theta\\ -\cos\,\alpha\,\sin j\theta\\ -\sin\,\alpha\,\sin j\theta\end{pmatrix},$$

$$e^{ia\theta/2}\begin{pmatrix}\cos\,\theta s+\dfrac{ia}{2s}\sin\,\theta s\\ -\dfrac{a}{2s}\sin\,\theta s+\dfrac{i}{s^2}\left(b^2+\dfrac{a^2}{4}\cos\,\theta s\right)\\ -\dfrac{b}{s}\sin\,\theta s+\dfrac{iab}{2s^2}(\cos\,\theta s-1)\end{pmatrix},\tag{76}$$

with a and b as above and j an integer. Again the spectrum is discrete.

In particular, we shall need to know the form of the spin-singular line combination embedded vortex, which is

$$A(r,\theta)=\Delta_0 f(\sqrt{(j^2+n^2)};r)\begin{pmatrix}\cos j\theta\\ -\cos\,\alpha\,\sin j\theta\\ -\sin\,\alpha\,\sin j\theta\end{pmatrix}e^{in\theta}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}.\tag{77}$$

3. Stability of the embedded vortices

The topology of the vacuum manifold contains loops, which are incontractible and thus gives classes of stable vortices. With each of the families of embedded (and combination) vortices an element of the homotopy group may be associated,[2] which tells one whether that family of vortices is topologically stable or unstable.

The vacuum manifold looks like

$$\frac{\mathrm{SO}(3)_S\times\mathrm{SO}(3)_L\times\mathrm{U}(1)_N}{\mathrm{U}(1)_{S_3}\times\mathrm{U}(1)_{L-N}\times\mathbf{Z}_2}=\frac{S_S^{(2)}\times S_{L,N}^{(3)}/\mathbf{Z}_2}{\mathbf{Z}_2}.\tag{78}$$

Here $S^{(n)}$ is an n sphere. This vacuum manifold contains *three* inequivalent families of incontractible loops. First, those contained within just $S_{L,N}^{(3)}/\mathbf{Z}_2$. Secondly, those going from the identity, through $S_S^{(2)}$ into $S_{L,N}^{(3)}/\mathbf{Z}_2$ by the $\mathbf{Z}_2$ factor, and then back to the identity. Thirdly, there are combinations of the first two types. The classes of the first homotopy group of the vacuum manifold are thus

$$\pi_1\left(\frac{\mathrm{SO}(3)_S\times\mathrm{SO}(3)_L\times\mathrm{U}(1)_N}{\mathrm{U}(1)_{S_3}\times\mathrm{U}(1)_{L-N}\times\mathbf{Z}_2}\right)=\mathbf{Z}_4.\tag{79}$$

This gives rise to three different topological charges for the vortices, the charge labeling the family from which they originate. Technically, the $\mathbf{Z}_4$ arises from two separate $\mathbf{Z}_2$ contributions, and then we can label the charge (p,q), with p, $q=0,1$; however, a more convenient notation (which will be better contextualized in the conclusions) is to assign a single index to these as in [16], ν: $(0,0)=0,(1,0)=1/2,(0,1)=1,(1,1)=3/2=-1/2$.

The $\nu=1/2$ stable vortices are half-quantum spin (singular line-disgyration) combinations, where one makes use of the $\mathbf{Z}_2$ mixing of the spin and angular groups for stability. Considering the spin-singular line combination above [Eq. (74)], the stable half-quantum spin-singular line combination vortex corresponds to $j=n=1/2$:

$$A(r,\theta)=\Delta_0\overline{f}(1/\sqrt{2};r)\begin{pmatrix}\cos\,\theta/2\\ -\cos\,\alpha\,\sin\,\theta/2\\ -\sin\,\alpha\,\sin\,\theta/2\end{pmatrix}e^{i\theta/2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}.\tag{80}$$

Of course, there are also half-quantum spin-disgyration vortices, and combinations in between. These all have topological charge $\nu=1/2$.

The $\nu=1$ stable vortices are some of the singular line [Eq. (68)] and disgyration embedded vortices [Eq. (66)], also including the combination vortices [Eq. (71)] in between. These all have the form above. The winding number $n=1$ vortices are the only stable solutions. Odd-n vortices may decay to these, also having topological charge $\nu=1$; even-n decays to the vacuum, having topological charge $\nu=0$. Finally, the $\nu=3/2$ vortices are combinations of the $\nu=1/2$ and $\nu=1$ vortices.

C. Vortex spectra of the full ^{3}He theory

We wish to find the embedded vortex spectrum of the full ^{3}He theory, when one is including terms that are not invari-

[2]More precisely, with the family *and* the winding number, but we shall only be considering unit winding number vortices.

ant under spatial rotations of the Lagrangian. Our tactic is to see which of the above embedded vortex solutions remain solutions in the full theory. This is facilitated by investigating how the profile equations are modified by inclusion of terms that are not invariant under $SO(3)_L$—if the profile equations make sense, for instance they must only be radially dependent, then one can say that those embedded vortices remain solutions to the theory.

Providently, it transpires that only those embedded vortices which are *topologically stable* remain solutions to the full ^{3}He Lagrangian, with inclusion of terms that are not rotationally symmetric.

1. Singular-line vortices

The singular-line vortex has a profile of the form [from Eq. (68)]

$$A(r,\theta)=\Delta_0 f(n;r)\mathbf{d}_0 e^{in\theta}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}, \tag{81}$$

where n is the winding number of the vortex. Substitution into the full Lagrangian [Eq. (58)] yields the profile equation to be

$$\mathcal{L}[f]+\tilde{\mathcal{L}}[f]=(2\gamma+\tilde{\gamma})\Delta_0^2\left[\left(\frac{df}{dr}\right)^2+\frac{n^2f^2}{r^2}\right]-2\tilde{\gamma}\Delta_0^2\,\frac{nf}{r}\,\frac{df}{dr} - V[f(r)]. \tag{82}$$

Since the extra term nff'/r is least dominant asymptotically we may conclude the singular line ansatz is still a solution to the full Lagrangian, but with a slightly modified profile function.

2. Spin vortices

Vortices embedded solely in the spin sector [with profiles given by Eq. (67)] are solutions to the full Lagrangian because the embedded defect formalism is applicable to symmetry-invariant parts of the Lagrangian, which the spin sector is.

This observation is backed up within the mathematics; one may show that for the spin vortex Ansatz

$$\partial_i A^\star_{\alpha i}\partial_j A_{\alpha j}=\partial_i A^\star_{\alpha j}\partial_i A_{\alpha j}\,. \tag{83}$$

Thus the terms of $\tilde{\mathcal{L}}$ that are not invariant under spatial rotations become equivalent to the kinetic terms of the symmetric ^{3}He Lagrangian for spin vortices.

3. Disgyration vortices

The embedded disgyration vortex has a profile of the form in Eq. (66); to simplify the matter we shall consider the family member with $\alpha=0$ (without loss of generality):

$$A(r,\theta)=\Delta_0 f(n;r)\mathbf{d}_0\begin{pmatrix}1\\ i\cos n\theta\\ -i\sin n\theta\end{pmatrix}, \tag{84}$$

where n is the winding of the vortex. Substitution into the full Lagrangian [Eq. (58)] yields terms that are not invariant under spatial rotations

$$\tilde{\mathcal{L}}[f]=\tilde{\gamma}\Delta_0^2\left[\left(\cos\theta\,\frac{df}{dr}\right)^2+\left(\cos n\theta\sin\theta\,\frac{df}{dr} - \frac{nf}{r}\cos\theta\sin n\theta\right)^2\right]. \tag{85}$$

Since the profile function $f(r)$ is independent of θ, and the Lagrangian $\mathcal{L}_{\rm sym}[f]+\tilde{\mathcal{L}}[f]$ that describes $f(r)$ is not rotationally symmetric, we conclude that the embedded disgyration vortices do not remain a solution when nonspatially rotationally symmetric terms are added to the Lagrangian.

4. Combination vortices

In general only combinations of embedded vortices that individually remain solutions when nonspatially symmetric terms are added to the Lagrangian remain solutions. Thus the only combination embedded vortices that are solutions to the full Lagrangian $\mathcal{L}_{\rm sym}+\tilde{\mathcal{L}}$ are the *combination spin-singular line vortices*.

D. Conclusions

We conclude, by comparing the results of Sec. VII C3 with Sec. VII B3, that embedded vortices that are solutions when rotationally nonsymmetric terms are added to the Lagrangian,

$$\tilde{\mathcal{L}}[A_{\alpha j}]=(\gamma_1+\gamma_2)\partial_i A^\star_{\alpha i}\partial_j A_{\alpha j}=\tilde{\gamma}\partial_i A^\star_{\alpha i}\partial_j A_{\alpha j}\,, \tag{86}$$

those vortices that are topologically stable, or higher winding number counterparts of those vortices. The topologically stable embedded vortices are labeled by their topological charge ν [16] and take the following forms.

First, the half-quantum spin-singular line combination vortex, has topological charge $\nu=1/2$ and looks like

$$A(r,\theta)=\Delta_0\overline{f}(1/\sqrt{2};r)\begin{pmatrix}\cos\theta/2\\ -\cos\alpha\sin\theta/2\\ -\sin\alpha\sin\theta/2\end{pmatrix}e^{i\theta/2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}. \tag{87}$$

Secondly, the singular line vortex has topological charge $\nu=1$ and looks like

$$A(r,\theta)=\Delta_0\overline{f}(1;r)\mathbf{d}_0 e^{in\theta}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}. \tag{88}$$

Thirdly and finally, the combination of the above two vortices has topological charge $\nu=3/2$ and looks like

$$A(r,\theta)=\Delta_0\overline{f}(\sqrt{5/2};r)\begin{pmatrix}\cos\ \theta/2\\ -\cos\ \alpha\ \sin\ \theta/2\\ -\sin\ \alpha\ \sin\ \theta/2\end{pmatrix}e^{i3\theta/2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}. \tag{89}$$

This vortex winds around the singular line part one and a half times and around the spin part half a time. One should note that from the above spectrum a new meaning for the topological charge ν may be interpreted, as the winding number of the singular line part of the vortex.

Another, final, observation that we would like to make is that upon the addition of spatial nonrotationally symmetric terms to the Lagrangian the only embedded vortices that remain solutions to the theory are those which contain *no angular dependence of those spatially associated components of the order parameter* (i.e., none are generated by any part of $SO(3)_L$). With hindsight, this may be expected to be the case, but it is pleasing to see it coming through in the mathematics. This leads one to wonder (or conjecture, perhaps) if a similar phenomena happens in other cases where the spatial rotation group acts nontrivially upon the order parameter.

VIII. CONCLUSIONS

We conclude by summarizing our main results.

(1) In Sec. II we summarized the formalism of the previous paper.

(2) In Sec. III we rederived the embedded defect spectrum of the Weinberg-Salam model. Our results are in agreement with other methods.

(3) In Sec. IV we derived the embedded defect spectrum of the model SU(3)→SU(2), finding embedded monopoles, gauge invariant unstable vortices, and a family of unstable vortices.

(4) In Sec. V we illustrated ''combination vortices'' by the model U(1)×U(1)→1. This illustrates how such objects may only be solutions in certain limits of the coupling constants, and the form of their spectrum when such solutions have been found.

(5) In Sec. VI we examined the embedded defect spectrum for three realistic GUT models, namely, Georgi-Glashow SU(5), Flipped SU(5), and Pati-Salam $SU^4(4)$.

(6) Finally, in section VII, we illustrated how our formalism may also be used in some condensed matter contexts, using the specific example of vortices in ^{3}He-A. This also illustrated combination vortices and some of their stability properties.

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