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EXAMPLES OF EMBEDDED DEFECTS (IN PARTICLE PHYSICS AND CONDENSED MATTER)

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Abstract

We present a series of examples designed to clarify the formalism of the companion paper 'Embedded Vortices': where we showed how the family structure and stability of embedded defects is related to group theoretic considerations. After summarising this formalism in a prescriptive sense, we run through several examples: firstly, deriving the embedded defect spectrum for Weinberg-Salam theory, then discussing several examples designed to illustrate facets of the formalism. We then calculate the embedded defect spectrum for three physical Grand Unified Theories and conclude with a discussion of vortices formed in the superfluid ^{3}He-A phase transition.

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1 Introduction.

Embedded defects have received an impressive amount of interest over the last couple of years. This is principally because the Z-string, of the Weinberg-Salam model, was recently discovered to be stable for part of the parameter space [1]. Unfortunately (or fortunately, maybe?), it proves to be unstable in the physical regime [2] [3]; though there may be other stabilising effects [4] [5]. However, be it stable or unstable, it may still have important cosmological consequences — as indicated by its connection to baryon number violation [6].

In addition, the standard model also admits a one-parameter family of unstable, gauge equivalent vortices called W-strings [7]. These W-strings are not gauge equivalent to Z-strings. Thus, a very non-trivial spectrum of vortices arises from the vacuum structure of the standard model: two families of gauge inequivalent vortices, with one family globally gauge *invariant* (under the residual symmetry group) and the other a one parameter family of gauge equivalent vortices. Furthermore, it is only one of these families which has the potential to be stable.

As well as the Weinberg-Salam model, embedded defects have been studied specifically in another symmetry breaking scheme — namely the GUT flipped-$SU(5)$ [8]. One finds an eleven parameter family of gauge equivalent, unstable vortices plus another globally gauge invariant, potentially stable vortex (the V-string). Furthermore, it seems likely (or at least an open question) that the V-string may be stable for physical parameters.

The general formalism for describing embedded defects was derived by Vachaspati, et. al. [9]. They described how to construct an embedded defect solution in a general Yang-Mills theory. The general idea of this formalism was that one defined a *subtheory* (the embedded subtheory) of the full theory upon which one may define a topological defect solution (domain wall, vortex or monopole). In extending the embedded subtheory back to the full theory one loses the topological nature of the defect (which guarantees the stability), but the defect still remains a solution providing certain restrictions are obeyed.

Recently, in a companion paper to this [10], we showed how the group theory which lies behind the formalism of Vachaspati, et. al. is instrumental in determining the properties and spectrum of embedded defects for a general Yang-Mills theory.

By nature, the calculations in that paper are rather technical; although we think the results are fairly simple. The group theory gives one a handle on family structure and how this relates to stability.

The purpose of this paper is to provide a list of several examples to illustrate certain facets of the formalism of that companion paper. Whereas that paper derives the formalism, we intend this paper to describe how to apply the formalism.

We firstly summarise the formalism derived in [10], so that it can be used as a prescriptive tool to find the spectrum of embedded defects, and to determine which defects may be stable.

Then, for our first example, we rederive the existence and properties of the W and Z-strings in the Weinberg-Salam model. We use this example for two reasons. Firstly, it has been exhaustively examined already [1], [7]. Secondly, it is the simplest gauge theory that illustrates our formalism.

As our second example we consider the symmetry breaking $SU(3) \rightarrow SU(2)$. This model serves to illustrate the family structure of embedded defects in more depth than the Weinberg-Salam theory. It also admits embedded (non-topological) monopoles as solutions.

Our next example is to consider the model $U(1) \times U(1) \rightarrow 1$ as an example of a theory which admits 'combination vortices'. These 'combination vortices' are vortices which lie *between* the families of vortices. In general they are only solutions for certain representation-dependant critical values of the coupling constants.

Most of these examples are fairly unphysical and cannot be realistically expected to describe nature. Hence we then discuss defects in three Grand Unified Theories. Namely, Georgi-Glashow $SU(5)$; flipped-$SU(5)$; and Pati-Salam $SU^4(4)$.

We conclude this paper by showing how the techniques used are relevant to a condensed matter system: namely that of vortices formed in the superfluid ^{3}He-A phase transition. This example also conveniently illustrates some properties of combination vortices.

2 Summary of Formalism.

For a Yang-Mills theory, the embedded defect spectrum and properties are entirely determined by the symmetry breaking $G \to H$. The symmetry breaking is determined by a Higgs field Φ and the representation of G, which describes how G acts on the Higgs field Φ. Suppose Φ lies in the vector space $\mathcal{V}$. The inner products that we use are the (real-) inner products defined from the Euclidean inner product for $\mathcal{V}$, and $\mathcal{G}$ has a (real-) inner product defined from the Maxwell term in the Lagrangian. The definition of the inner products is discussed in more detail in the companion paper [10]. We denote the corresponding norms by $\|\,.\,\|$.

It is notationally useful to include the gauge coupling constants in the representation, since this is the quantity that appears in the Lagrangian. Write $G = G_1 \times \cdots \times G_N$, with the G_i's simple or $U(1)$. For each $\mathcal{G}_i$ the derived representation is a map $d_i : \mathcal{G}_i \to \mathrm{aut}(\mathcal{V})$; $\mathrm{aut}(\mathcal{V})$ being automorphisms over $\mathcal{V}$ — *i.e.* actions upon the Higgs representation space. We then *scale* each d_i by the relevant coupling constant q_i for that part of the group. Hence, the *scaled* derived representation (the quantity which appears in the Lagrangian) is $d = \sum_i q_i d_i$ [‡]. The scaled representation, D, of the group is the exponential of this. It should be noted that, except where explicitly stated, we shall always use the scaled representation.

Choose a reference point Φ_c in the vacuum manifold. This reference point is arbitrary because of the degeneracy of choice $D(G)\Phi_c$, which form the vacuum manifold $\mathcal{M}_0$. The residual symmetry group H is determined by Φ_c to be $H = \{g \in G : D(g)\Phi_c = \Phi_c\}$. Then one writes $\mathcal{G}$ as the reductive decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \tag{1}$$

such that

$$[\mathcal{H}, \mathcal{H}] \subseteq \mathcal{H}, \qquad \text{and} \qquad [\mathcal{H}, \mathcal{M}] \subseteq \mathcal{M}. \tag{2}$$

Under the adjoint action of H ($\mathrm{Ad}(h)X = hXh^{-1}$), $\mathcal{M}$ decomposes into irreducible subspaces

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_N. \tag{3}$$

[‡] It should be noted that if d is to be a representation then the non-Abelian scales are fixed by $[d(X_i), d(X_j)] = d([X_i, X_j])$.

These irreducible spaces describe how the group acts on the vacuum manifold; additionally yielding the family structure for embedded defects.

The stability of embedded defects is also related to the family structure above. Recall that the centre $\mathcal{C}$ of $\mathcal{G}$ is the set of elements which commute with $\mathcal{G}$. In general it is an Abelian algebra. The stability of vortices is related to how the centre of $\mathcal{G}$ projects onto $\mathcal{M}$, which is defined by the projection mapping $pr : \mathcal{C} \to \mathcal{M}$, where for $X_c \in \mathcal{C}$,

$$pr(X_c) = X_c + X_h \in \mathcal{M}, \tag{4}$$

where $X_h \in \mathcal{H}$ is the unique element such that this is true. Such a projection of $\mathcal{C}$ onto $\mathcal{M}$ consists of one-dimensional irreducible $\mathcal{M}_i$'s.

This structure is enough to categorise all the topological and non-topological embedded domain wall, embedded vortex and embedded monopole solutions of a Yang-Mills theory.

2.1 Domain Walls

A domain wall solution is defined only by its reference point, Φ_c. The solution is

$$\Phi(z) = f_{\rm DOM}(z)\Phi_c, \tag{5a}$$

$$A^\mu = 0, \tag{5b}$$

where $f_{\rm DOM}$ is a real function such that $f_{\rm DOM}(+\infty) = 1$, and $f_{\rm DOM}(-\infty)\Phi_c \neq \Phi_c$ belongs to the vacuum manifold.

Providing the vacuum manifold is connected this solution is unstable; suffering from a short range instability in the Higgs field. Solutions within connected parts of the vacuum manifold are gauge equivalent.

2.2 Vortices

An embedded vortex solution is defined by the pair $(\Phi_c \in \mathcal{V}, X_s \in \mathcal{M})$. The solution is [13]

$$\Phi(r,\theta) = f_{\rm NO}(X_s;r)D(e^{\theta X_s})\Phi_c\ ,\ \theta \in [0,2\pi), r \in [0,\infty), \tag{6a}$$

$$\underline{A}(r,\theta) = \frac{g_{\rm NO}(X_s;r)}{r}X_s\underline{\hat{\theta}}. \tag{6b}$$

Here $f_{\rm NO}$ and $g_{\rm NO}$ are the Nielsen-Olesen profile functions for the vortex; their dependence upon X_s is described in the appendix of the companion paper [10]. The vortex generator X_s has the constraints that [§]

$$X_s \in \mathcal{M}_i, \tag{7a}$$

$$D(e^{2\pi X_s})\Phi_c = \Phi_c, \tag{7b}$$

and the winding number of the vortex is given by $n = \| X_s \| \,/\, \| X_s^{\rm min} \|$, where $X_s^{\rm min}$ is the minimal generator in the same $\mathcal{M}_i$ as X_s obeying the above two conditions.

Family structure originates from the result that vortices of the same winding number that are defined by generators in the same $\mathcal{M}_i$ are gauge equivalent.

The stability of vortices subdivides into two types: dynamical stability and topological stability. Furthermore, there are two types of topological stability: Abelian (from broken $U(1)$ parts of the symmetry breaking) and non-Abelian (from quotients by discrete factors). The result is that dynamical and Abelian topological stability originate from the projection of the centre of $\mathcal{G}$ onto $\mathcal{M}$: Abelian topologically stable vortices are generated by elements in the *intersection*; whilst dynamically stable vortices are generated by elements in the non-trivial *projection*.

If the coupling constants are at a critical point where, between $\mathcal{M}_i$ and $\mathcal{M}_j$ say, the scaled representations satisfy:

$$\frac{\| d(X_i)\Phi_c \|}{\| X_i \|} = \frac{\| d(X_j)\Phi_c \|}{\| X_j \|}, \quad X_i \in \mathcal{M}_i, \;\; X_j \in \mathcal{M}_j, \tag{8}$$

then one has vortex solutions defined by generators in $\mathcal{M}_i \oplus \mathcal{M}_j$. Their stability properties are described by the above results.

2.3 Monopoles

An embedded monopole solution is defined by the triplet $(\Phi_c \in \mathcal{V}, X_s, X_s' \in \mathcal{M})$. The solution (with winding number $n = 1$) is

$$\underline{\Phi}(\underline{r}) = f_{\rm mon}(r)\hat{\underline{r}}, \tag{9a}$$

$$A_a^\mu(\underline{r}) = \frac{g_{\rm mon}(r)}{r}\epsilon_{\mu ab}X_b, \tag{9b}$$

[§]there are some complications when the rank (see prenote of [10]) of $\mathcal{M}_i$ is greater than one — we shall generally indicate when such happens in the text.

This solution may be generalised to higher windings, and then the form of the Higgs field corresponds to spherical harmonics. Notationally, we are treating Φ to be a vector within its corresponding embedded subtheory, and we are using $X_3 = [X_1, X_2]$.

The monopole generators have the following restrictions [10]:

$$X_1, X_2 \in \mathcal{M}_i \quad \text{with} \quad \langle X_1, X_2 \rangle = 0, \tag{10a}$$

$$[X_1, X_2] \quad \in \quad \mathcal{H}, \tag{10b}$$

$$\| X_1 \| = \| X_2 \| \quad \text{with} \quad D(e^{2\pi X_1})\Phi_c = \Phi_c. \tag{10c}$$

and the winding number of the monopole is given by $n = \| X_s \| \ / \ \| X_s^{\min} \|$, where $X_s^{\min}$ is the minimal generator in the same $\mathcal{M}_i$ as X_s obeying condition eq. (10).

Monopoles also have a family structure, depending upon which $\mathcal{M}_i$ they are defined from.

3 Defects in the Weinberg-Salam Theory

To illustrate our results we rederive the existence and properties of the W and Z-strings in the Weinberg-Salam model. This model seems to be a good example for two reasons. Firstly, it has been exhaustively examined already [1], [7]. Secondly, it is the simplest gauge theory that illustrates our formalism.

The Weinberg-Salam theory has full gauge symmetry $G = SU(2)_I \times U(1)_Y$ (isospin and hypercharge) acting on a two-dimensional complex Higgs field Φ (i.e $\mathcal{V} = C^2$) by the fundamental representation. The generators of $SU(2)_I$ are $X^a = \frac{i}{2}\sigma^a$ and the $U(1)_Y$ generator is $X^0 = \frac{i}{2}\mathbf{1}_2$, with the scaled derived representation acting as

$$d(\alpha^i X^i + \alpha^0 X^0) = g\alpha^i X^i + g'\alpha^0 X^0, \tag{11}$$

where g and g' are the $SU(2)_I$ and $U(1)_Y$ gauge coupling constants respectively.

The Higgs potential is a Landau potential, $\lambda(\Phi^\dagger\Phi - \frac{1}{2}v^2)^2$. Hence, the vacuum manifold, which is the minimum of the potential, is a three-sphere

We shall take the reference point in the vacuum manifold to be:

$$\Phi_c = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{12}$$

Then the gauge groups breaks to the subgroup that acts trivially upon Φ_c, namely

$$H = U(1)_Q = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & 1 \end{pmatrix}, \tag{13}$$

with $\omega \in [0, 2\pi)$. Then, at this reference point, the Lie algebra of the gauge group decomposes into $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$, where

$$\mathcal{H} = \begin{pmatrix} i\alpha & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{M} = \begin{pmatrix} 0 & \gamma \\ -\gamma^* & i\beta \end{pmatrix}, \tag{14}$$

with α, β are real and γ is complex. The star denoting complex conjugation.

It is simple to verify that $\mathcal{M}$ is reducible under the adjoint action of H and decomposes to the irreducible subspaces $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where

$$\mathcal{M}_1 = \begin{pmatrix} 0 & 0 \\ 0 & i\beta \end{pmatrix} \quad \text{and} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & \gamma \\ -\gamma^* & 0 \end{pmatrix}. \tag{15}$$

We also need to find the projection of the centre of $\mathcal{G}$ onto $\mathcal{M}$. Clearly the centre of $\mathcal{G}$ is $\mathcal{C} = u(1)_Y$. Then it is simple to see that $\mathrm{pr}(u(1)_Y) = \mathcal{M}_1$.

The first class of embedded vortices are defined from elements $X_s \in \mathcal{M}_1$ such that $e^{2\pi X_s} = 1$. Furthermore, since $\mathcal{M}_1 = \mathrm{pr}(u(1)_Y)$ these vortices are stable in a limit of the coupling constants — namely $g \to 0$. From eq.(6) one immediately writes down the solution as:

$$\Phi(r,\theta) = \frac{v}{\sqrt{2}} f^Z_{\mathrm{NO}}(r) \begin{pmatrix} 0 \\ e^{in\theta} \end{pmatrix}, \tag{16a}$$

$$d(\underline{A}(r,\theta)) = \frac{g^Z_{\mathrm{NO}}(r)}{r} \begin{pmatrix} 0 & 0 \\ 0 & in \end{pmatrix} \underline{\hat{\theta}}, \tag{16b}$$

where n is the winding number of the vortex. It should be noted that this vortex is also gauge invariant under global gauge transformations of the residual gauge symmetry. Clearly these solutions are the familiar Z-strings.

The second class of embedded vortices are defined from elements $X_s \in \mathcal{M}_2$ such that $e^{2\pi X_s} = 1$. Furthermore, since $\mathcal{M}_2$ is not the projection of the centre of $\mathcal{G}$ there is no limit of the coupling constants in which the vortex is stable. From eq.(6) one immediately writes down the solution as:

$$\Phi(r,\theta) = \frac{v}{\sqrt{2}} f^W_{\rm NO}(r) \begin{pmatrix} e^{i\delta}\sin n\theta \\ \cos n\theta \end{pmatrix}, \qquad (17a)$$

$$d(\underline{A}(r,\theta)) = \frac{g^W_{\rm NO}(r)}{r} \begin{pmatrix} 0 & ne^{i\delta} \\ -ne^{-i\delta} & 0 \end{pmatrix} \underline{\hat{\theta}}, \qquad (17b)$$

with $e^{i\delta} = \gamma/|\gamma|$ and n the winding number of the vortex. All the isolated solutions of the same winding number in this one-parameter family are gauge equivalent. Furthermore, the anti-string is gauge equivalent to the string. Thus, isolated solutions are parameterised by the positive winding number only. Clearly these solutions are the familiar W-strings.

For the case of Weinberg-Salam theory, generators in $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy the condition $\| d(X)\Phi_c \| \,/\, \| \Phi_c \| = n$ of the appendix in the companion paper [10]. Thus, the profile functions for the Z and W-strings are related to those of the Abelian-Higgs model and, additionally, we have the relation between vortices of equal winding (first stated by MacDowell and Tornkvist [11])

$$f^Z_{\rm NO}(\lambda; r) = f^W_{\rm NO}(\frac{\lambda}{\kappa^2}; \kappa r), \qquad (18a)$$

$$g^Z_{\rm NO}(\lambda; r) = g^W_{\rm NO}(\frac{\lambda}{\kappa^2}; \kappa r), \qquad (18b)$$

where $\kappa = \sqrt{\frac{g^2+g'^2}{g^2}}$ and λ is the quartic scalar self coupling.

4 The Model $SU(3) \to SU(2)$.

From the discussion in section (2), where we showed that an embedded vortex which may be stable is always gauge invariant under global gauge transformations of the residual symmetry group, one might think that, perhaps, all globally gauge invariant vortices may be stable — which would be a very strong result. We give the above model as a counterexample to this hypothesis; as a solution it admits an unstable globally gauge invariant vortex. In addition it is a nice example of a model admitting embedded (non-topological) monopoles.

The (original) gauge group is $G = SU(3)$, which acts on a three-dimensional complex Higgs field by the fundamental representation. Denoting the generators by

$\{X^a : a = 1 \cdots 8\}$, the scaled derived representation acts upon the Higgs field as:

$$d(\alpha^i X^i) = g\alpha^i X^i, \tag{19}$$

where g is the $SU(3)$ coupling constant.

A Landau potential is sufficient to break the symmetry. This is because $\mathcal{M}$ is of the same dimension as the highest dimensional sphere to be contained within $\mathbf{C}^3$. Hence, the vacuum manifold is isomorphic to a five-sphere, with G transitive over it.

We shall take the reference point in the vacuum manifold to be

$$\Phi_c = v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{20}$$

Then the gauge group breaks to $H = SU(2)$, with H nestled in G as

$$H = \begin{pmatrix} SU(2) & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix} \subset G. \tag{21}$$

At the reference point Φ_c, $\mathcal{G}$ decomposes under the adjoint action of H into irreducible subspaces of the form $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$, where

$$\mathcal{M}_1 = \begin{pmatrix} i\gamma & 0 & 0 \\ 0 & i\gamma & 0 \\ 0 & 0 & -2i\gamma \end{pmatrix}, \quad \text{and} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a^* & -b^* & 0 \end{pmatrix}, \tag{22}$$

with γ real and a, b complex.

Since the centre of $su(3)$ is trivial, $\mathcal{M}_1$ cannot be its projection onto $\mathcal{M}$

The first class of vortex solutions are classified by $X_s \in \mathcal{M}_1$. They are given by, in the temporal gauge,

$$\Phi(r,\theta) = v f_{\mathrm{NO}}(X_1; r) \begin{pmatrix} 0 \\ 0 \\ e^{in\theta} \end{pmatrix}, \tag{23a}$$

$$\underline{A}(r,\theta) = \frac{g_{\mathrm{NO}}(X_1; r)}{r} \begin{pmatrix} -in/2 & 0 & 0 \\ 0 & -in/2 & 0 \\ 0 & 0 & in \end{pmatrix} \underline{\hat{\theta}}. \tag{23b}$$

The integer n is the winding number of the vortex. These solutions have no semi-local limit and are therefore always unstable.

The second class of vortex solutions are those classified by $X_s \in \mathcal{M}_2$. They are a three-parameter family of gauge equivalent, unstable solutions.

It is easily verified that $\| d(X)\Phi_c \| \, / \, \| \Phi_c \|$ is the winding number of the defect for both equivalence classes. Therefore the profile functions for both classes coincide with each other and the Abelian-Higgs model.

There are embedded monopole solutions in this model. These solutions are not topologically stable — so therefore they are unstable [12]. The solutions are specified by a gauge equivalent class of generators $X_m, X'_m \in \mathcal{M}_2$, such that $\langle X_m, X'_m \rangle = 0$ and $[X_m, X'_m] \in \mathcal{H}$. A class of such generators is

$$X_m = \mathrm{Ad}(h) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X'_m = \mathrm{Ad}(h) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{24}$$

with

$$[X_m, X'_m] = \mathrm{Ad}(h) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{H}, \tag{25}$$

where h is some element in H. There is a one-to-one correspondence between elements in H and the choice of embedded monopole. It should be noted that elements of the form

$$X = \mathrm{Ad}(h) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X' = \mathrm{Ad}(h) \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{26}$$

do not define monopole solutions because $[X, X'] \notin \mathcal{H}$. Anti-monopoles are defined in the above form but with one of the generators negative.

In conclusion, there is a two-parameter family of embedded monopole solutions of the form defined in eq. (9).

5 The Model $U(1) \times U(1) \to 1$.

This model is presented to illustrate combination vortices. By 'combination vortices' we mean vortices that are generated by elements that are not in any of the irreducible spaces $\mathcal{M}_i$; the vortex generators being, instead, *between* the spaces.

In section (2), we said that such combination vortices are solutions if the representation allows them. This corresponds to the coupling constants taking a critical set of values. We illustrate this principle by explicitly finding such solutions in the model $U(1) \times U(1) \to 1$.

The gauge group is $G = U(1)_X \times U(1)_Y$, and we represent the general group element $g(\theta, \varphi)$ by

$$g(\theta, \varphi) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \in G, \tag{27}$$

with $\theta, \varphi \in [0, 2\pi)$. The generators of $U(1)_X$ and $U(1)_Y$ are, respectively,

$$X = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}. \tag{28}$$

The group G acts on a two-dimensional complex Higgs field $\Phi = (\phi_1, \phi_2)^\top$ by the fundamental representation, such that the coupling constants scale the derived representation:

$$D(g(\theta, \varphi)) = \begin{pmatrix} e^{iq_1\theta} & 0 \\ 0 & e^{iq_2\varphi} \end{pmatrix} \tag{29}$$

where q_1 and q_2 are the coupling constants for the respective parts of G.

To obtain the required symmetry-breaking, i.e G breaks to triviality, we must choose the parameters of the Higgs potential correctly. The most general, renormalisable, gauge invariant Higgs potential for this theory is:

$$V(\phi_1, \phi_2) = \lambda_1(\phi_1^*\phi_1 - v_1^2)^2 + \lambda_2(\phi_2^*\phi_2 - v_2^2)^2 + \lambda_3\phi_1^*\phi_1\phi_2^*\phi_2. \tag{30}$$

For some range of $(\lambda_1, \lambda_2, \lambda_3, v_1, v_2)$ (the range being unimportant to our arguments) this is minimised by a two-torus of values and G breaks to triviality.

Then, we may take the Higgs reference point to be

$$\Phi_c = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}, \tag{31}$$

where unless $v_1^2 = v_2^2$, the primed vevs v_1', v_2' are unequal to v_1 and v_2. Then the group G breaks to the trivial group $H = \{1\}$. Under the adjoint action of H, the Lie algebra of G splits into

$$\mathcal{G} = \mathcal{M}_1 \oplus \mathcal{M}_2, \tag{32}$$

with

$$\mathcal{M}_1 = \begin{pmatrix} ia & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 \\ 0 & ib \end{pmatrix}, \tag{33}$$

with a, b real.

The topology of the vacuum manifold is non-trivial, hence vortex solutions that are generated by elements in $\mathcal{M}_1$ or $\mathcal{M}_2$ are topologically stable. These vortices are well defined and are stationary solutions of the Lagrangian.

It is interesting to consider the existence of vortices generated by elements in the whole of $\mathcal{M}_1 \oplus \mathcal{M}_2$, and not just vortices generated in either of these two spaces separately. Combination vortices may exist when the coupling constants are such that eq. (8) is satisfied. Substitution of the generators X and Y into it yields the condition, that combination vortices exist for:

$$\frac{\| d(X) \|}{\| X \|} = \frac{\| d(Y) \|}{\| Y \|} \Rightarrow q_1^2 = q_2^2. \tag{34}$$

When $q_1 = q_2$, the elements of the Lie algebra that generate closed geodesics are of the form

$$X_s = \delta X + \epsilon Y, \tag{35}$$

such that there exists an $\omega > 0$ with $D(e^{X_s \omega})\Phi_c = \Phi_c$. Since the coupling constants are equal, this says that X_s generates a $U(1)$-sub-group of G. Relating this back to the geometry of a torus, we see that the constraint on ϵ and δ is, providing both ϵ and δ are non-zero,

$$\frac{\epsilon}{\delta} \text{ are rational.} \tag{36}$$

One can interpret the effect of scaling the Higgs representation as 'twisting' directions in the Higgs representation space relative to directions in the Lie algebra. This twisting only happens between the irreducible subspaces of $\mathcal{M}$.

However, not all of these geodesics define embedded vortices. One also needs to satisfy cond. (2) in the companion paper [10], which is

$$\langle \Psi, \frac{\partial V}{\partial \Phi} \rangle = 0, \tag{37}$$

where $\Psi \in \mathcal{V}_{\rm emb}^{\perp}$ and $\Phi \in \mathcal{V}_{\rm emb}$. Trivial substitution yields

$$\lambda_1 = \lambda_2 = \lambda,\ v_1^2 = v_2^2 = v^2,\ \text{and } \epsilon = \delta. \tag{38}$$

This is the only combination vortex.

6 Embedded Defects in Realistic GUT models

We now gives some examples of the embedded defect spectrum in some realistic GUT models. The examples here are certainly not meant to be exhaustive, merely just a few of the simplest examples.

6.1 Georgi-Glashow $SU(5)$

The gauge group is $G = SU(5)$ [14], which acts on a twenty-four dimensional Higgs field Φ by the adjoint action. For a Higgs vacuum,

$$\Phi_c = v \begin{pmatrix} \frac{2}{3}\mathbf{1_3} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & -\mathbf{1_2} \end{pmatrix}, \tag{39}$$

the gauge group G breaks to $H = SU(3)_c \times SU(2)_I \times U(1)_Y$, which is contained in G as:

$$\begin{pmatrix} SU(3)_c & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & SU(2)_I \end{pmatrix} \times \begin{pmatrix} e^{\frac{2}{3}i\theta}\mathbf{1_3} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & e^{-i\theta}\mathbf{1}_2 \end{pmatrix} \subset SU(5). \tag{40}$$

Then, to find the embedded defect spectrum one determines the reduction of $\mathcal{G}$ into $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$ and finds the irreducible spaces of $\mathcal{M}$ under the adjoint action of H. The space $\mathcal{M}$ is

$$\mathcal{M} = \begin{pmatrix} \mathbf{0_3} & \vdots & \underline{A} \\ \cdots & \cdots & \cdots \\ -\underline{A}^\dagger & \vdots & \mathbf{0_2} \end{pmatrix}, \tag{41}$$

which is irreducible under the adjoint action of H.

Thus the defect spectrum of the model is: monopoles, which can be confirmed to be topologically stable; and a family of unstable Lepto-quark strings. The family of lepto-quark strings is complicated by $\mathcal{M}$ containing *two* distinct (non-proportional) commuting generators.

6.2 Flipped-$SU(5)$

For a more detailed discussion of embedded defects and their properties in flipped-$SU(5)$, see [8].

The gauge group is $G = SU(5) \times \widetilde{U(1)}$ [15], which acts on a ten dimensional, complex Higgs field (conveniently represented as a five by five, complex antisymmetric matrix) by the **10**-antisymmetric representation. Denoting the generators of $SU(5)$ as X^a and $\widetilde{U(1)}$ as $\widetilde{X}$, the scaled derived representation acts on the Higgs field as:

$$d(\alpha^i X^i + \alpha^0 \widetilde{X}) = g\alpha^i (X^i \Phi + \Phi X^{i\top}) + \tilde{g}\alpha^0 \widetilde{X}\Phi. \tag{42}$$

Here g and $\tilde{g}$ are the $SU(5)$ and $\widetilde{U(1)}$ coupling constants, respectively.

It is necessary for the following discussion to know a couple of generators explicitly, namely:

$$X^{15} = i\sqrt{\frac{3}{10}} \begin{pmatrix} \frac{2}{3}\mathbf{1_3} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & -\mathbf{1_2} \end{pmatrix}, \qquad \widetilde{X} = i\sqrt{\frac{12}{5}}\mathbf{1_5}. \tag{43}$$

These generators are properly normalised with respect to a standard inner product on the Lie algebra.

For a vacuum given by

$$\Phi_c = \frac{v}{\sqrt{2}} \begin{pmatrix} \mathbf{0_3} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & I \end{pmatrix}, \quad \text{where} \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{44}$$

one obtains breaking to the standard model $H = SU(3)_c \times SU(2)_I \times U(1)_Y$ provided the parameters of the potential satisfy $\eta^2, \lambda_1 > 0$ and $(2\lambda_1 + \lambda_2) > 0$. The V and hypercharge fields and generators are given by ¶

$$V_i = \cos\Theta A_i^{15} - \sin\Theta \widetilde{A}_i, \qquad X_V = \cos\Theta X^{15} - \sin\Theta \widetilde{X}, \tag{45a}$$

$$Y_i = \sin\Theta A_i^{15} + \cos\Theta \widetilde{A}_i, \qquad X_Y = \sin\Theta X^{15} + \cos\Theta \widetilde{X}. \tag{45b}$$

¶We are using a slightly different definition from that used in [8] where the GUT mixing angle is $\tan\Theta = \widetilde{g}/g$. Then $d(\mathcal{H})\Phi_{\rm vac} = 0$, with the isospin and colour symmetry groups nestled in $SU(5)$ as

$$\begin{pmatrix} SU(3)_c & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & SU(2)_I \end{pmatrix} \subset SU(5). \tag{46}$$

Then, to find the embedded defect spectrum one determines the reduction of $\mathcal{G}$ into $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$ and finds the irreducible spaces of $\mathcal{M}$ under the adjoint action of H. The space $\mathcal{M}$ reduces into two irreducible spaces under the adjoint action of H, which are:

$$\mathcal{M}_1 = \{\alpha X_V : \alpha \text{ real}\}, \qquad \mathcal{M}_2 = \begin{pmatrix} \mathbf{0_3} & \vdots & \underline{A} \\ \cdots & \cdots & \cdots \\ -\underline{A}^\dagger & \vdots & \mathbf{0_2} \end{pmatrix}. \tag{47}$$

The first space $\mathcal{M}_1$ is the projection of $\widetilde{u(1)}$ onto $\mathcal{M}$. This is important for the stability of vortex solutions defined from it. Such vortices are stable in the limit $\Theta_{\rm GUT} \to \frac{\pi}{2}$, then, by continuity, also in a region around $\frac{\pi}{2}$.

The second space $\mathcal{M}_2$ generates: a family of unstable Lepto-quark strings; and also unstable (not topological) monopoles. The family of lepto-quark strings is complicated by $\mathcal{M}$ containing *two* distinct (non-proportional) commuting generators.

6.3 Pati-Salam $SU(4) \times SU(4) \to SU(3)_c \times SU(2)_I \times U(1)_Y$

Pati and Salam emphasised a series of models of the form $G = G^S \times G^W$, where G^S and G^W are identical strong and weak groups related by some discrete symmetry [16]. The above model is the simplest one of this form. The model is actually $(SU(4) \times SU(4))_L \times (SU(4) \times SU(4))_R$ ('L' and 'R' denoting the separate couplings to left and right-handed fermions) to accommodate parity violation in weak interactions. We shall only consider one half of the model.

The gauge group is $G = SU(4)^S \times SU(4)^W$ which breaks to $H = (SU(3) \times U(1))^S \times SU(2)^W$, where H is nestled in G in the following way:

$$\begin{pmatrix} SU(3)_c & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & U(1)_Y \end{pmatrix}_S \times \begin{pmatrix} SU(2)_I & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{1_2} \end{pmatrix}_W \subset SU(4)^S \times SU(4)^W. \tag{48}$$

Then write $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$. The irreducible spaces of $\mathcal{M}$ under the adjoint action of H are denoted by $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \widetilde{\mathcal{M}}$ (here, $\widetilde{\mathcal{M}}$ is a collection of four irreducible spaces), with

$$\mathcal{M}_1 = \begin{pmatrix} \mathbf{0_3} & \vdots & \underline{A} \\ \cdots & \cdots & \cdots \\ -\underline{A}^\dagger & \vdots & 0 \end{pmatrix}_S, \qquad \mathcal{M}_2 = \begin{pmatrix} \mathbf{0_2} & \vdots & B \\ \cdots & \cdots & \cdots \\ -B^\dagger & \vdots & \mathbf{0_2} \end{pmatrix}_W,$$

$$\text{and} \qquad \widetilde{\mathcal{M}} = \begin{pmatrix} \mathbf{0_2} & \vdots & \mathbf{0_2} \\ \cdots & \cdots & \cdots \\ \mathbf{0_2} & \vdots & C \end{pmatrix}_W \oplus \begin{pmatrix} i\alpha\mathbf{1_2} & \vdots & \mathbf{0_2} \\ \cdots & \cdots & \cdots \\ \mathbf{0_2} & \vdots & -i\alpha\mathbf{1_2} \end{pmatrix}_W, \tag{49}$$

where $\underline{A}$ is a three dimensional complex vector, B and C are two by two complex matrices, with C anti-hermitian, and α is a real number.

Each of the above spaces gives rise to their respective embedded defects. Firstly, $\mathcal{M}_1$ gives rise to topologically stable monopoles and a five parameter family of unstable strings. Secondly, $\mathcal{M}_2$ gives rise to unstable (non-topological) monopoles and an seven parameter family of unstable strings. Thirdly, $\widetilde{\mathcal{M}}$, which is a collection of four irreducible spaces, admits globally gauge invariant unstable string solutions (under the residual symmetry group). In addition, $\widetilde{\mathcal{M}}$ has combination string solutions between the four irreducible spaces that it consists of.

7 Vortices in the ^{3}He-A Phase Transition

We wish to show her that our results on the classification of vortices for general gauge theories are also relevant for condensed matter systems. As an example we choose the ^{3}He-A phase transition, though we expect the general onus of our results to be applicable to other situations having a similar nature.

Superfluid ^{3}He has global symmetries of spin ($SO(3)_S$ rotations), angular rotations ($SO(3)_L$) and a phase (associated with particle number conservation). It has several phase transitions corresponding to different patterns of breaking this symmetry. We concentrate here on the A-phase transition.

A condensed matter system such as ^{3}He has added complication above that of gauge theories, meaning that we cannot just naively apply the approach used in the rest of this paper. This complication originates through the order parameter being a *vector* under spatial rotations, not a scalar as in conventional gauge theories. The upshot being that extra terms are admitted in the Lagrangian that are not present in a conventional gauge theory. These terms couple derivatives of components with different angular momentum quantum numbers and are so not invariant under $SO(3)_L$ rotations in the conventional sense — thus spoiling the $SO(3)_L$ invariance. The general effect of this is to complicate the spectrum of vortex solutions, and their actual form and interaction.

Our tactic to investigate the effect of these extra non-invariant $SO(3)_L$ terms is to firstly examine the ^{3}He-A phase transitions without inclusion of these terms so that we may use the techniques of embedded vortices used in the rest of this paper, and then to see how these terms affect the solutions.

7.1 The ^{3}He-A Phase Transition

The full symmetry group of liquid ^{3}He is

$$G_{3He} = SO(3)_S \times SO(3)_L \times U(1)_N, \tag{50}$$

which acts on the two group-index order parameter $A_{\alpha j}$ by the *fund.*$_S\otimes$ *fund.*$_{L,N}$ representation of G_{3He}. Denoting

$$A_{\alpha j} = \Delta_0 d_\alpha \Psi_j, \tag{51}$$

with unit vector $d_\alpha \in \mathbf{R}^3$ and $\Psi_j = (\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2)/\sqrt{2} \in \mathbf{C}^3$, where $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2 \in \mathbf{R}^3$ such that $\hat{\mathbf{e}}_1.\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2.\hat{\mathbf{e}}_2 = 1$ and $\hat{\mathbf{e}}_1.\hat{\mathbf{e}}_2 = 0$. The quantity Δ_0 is a real number unimportant for the present discussion.

Then G_{3He} acts on $A_{\alpha j}$ fundamentally:

$$D((g_S, g_L, g_N))_{\alpha j \beta k} A_{\beta k} = \Delta_0 (g_S \mathbf{d})_\alpha (g_L g_N \mathbf{\Psi})_j. \tag{52}$$

In addition G_{3He} is a global symmetry of the field theory.

The field theory is described by the Lagrangian

$$\mathcal{L}[A_{\alpha j}] = \mathcal{L}_{\rm sym}[A_{\alpha \rm j}] + \widetilde{\mathcal{L}}[A_{\alpha j}], \tag{53}$$

with $\mathcal{L}_{\text{sym}}$ having G_{3He} global symmetry and $\widetilde{\mathcal{L}}$ representing the extra vector type couplings of the order parameter. We may write

$$\mathcal{L}_{\text{sym}}[A_{\alpha\text{j}}] = \gamma \partial_i A^{\star}_{\alpha j} \partial_i A_{\alpha j} - V[A_{\alpha j}], \tag{54}$$

with V some Landau-type potential invariant under G_{3He}. The vector-type couplings we write

$$\widetilde{\mathcal{L}}[A_{\alpha j}] = \gamma_1 \partial_i A^{\star}_{\alpha i} \partial_j A_{\alpha j} + \gamma_2 \partial_i A^{\star}_{\alpha j} \partial_j A_{\alpha i}, \tag{55}$$

which are explicitly not $SO(3)_L$ invariant. By partial integration of the *action* integral, this may be rewritten as

$$\widetilde{\mathcal{L}}[A_{\alpha j}] = (\gamma_1 + \gamma_2) \partial_i A^{\star}_{\alpha i} \partial_j A_{\alpha j} = \tilde{\gamma} \partial_i A^{\star}_{\alpha i} \partial_j A_{\alpha j}. \tag{56}$$

The A-phase is reached through symmetry breaking with a vacuum of the form

$$A_c = \Delta_0 \mathbf{d}_c \mathbf{\Psi}_c, \text{ where } \mathbf{d}_c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{\Psi}_c = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \tag{57}$$

so that the residual symmetry group is

$$H_A = U(1)_{S_3} \times U(1)_{L-N} \times \mathbf{Z}_2, \tag{58}$$

where

$$U(1)_{S_3} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}_S : \alpha \in [0, 2\pi) \right\} \tag{59a}$$

$$U(1)_{L-N} = \left\{ e^{-i\alpha} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}_L : \alpha \in [0, 2\pi) \right\} \tag{59b}$$

$$\mathbf{Z}_2 = \left\{ \mathbf{1}_S \times \mathbf{1}_L, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_S \times \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_L \right\} \tag{59c}$$

It should be noted that the $\{L, N\}$ part of the group is similar to the Weinberg-Salam theory at $\Theta_w = \pi/4$, but taking the limit in which (both) of the coupling constants become zero. However, note that $SO(3)_L$ is *not* simply connected; this has important stabilising effects on the vortices [17].

Writing $\mathcal{G}_{3He} = \mathcal{H}_A \oplus \mathcal{M}$, the irreducible spaces of $\mathcal{M}$ under the adjoint action of H_A are denoted by $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$, with

$$\mathcal{M}_1 = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ -\alpha & -\beta & 0 \end{pmatrix}_L , \qquad \mathcal{M}_2 = \begin{pmatrix} 0 & \gamma & \delta \\ -\gamma & 0 & 0 \\ -\delta & 0 & 0 \end{pmatrix}_S ,$$

$$\text{and} \qquad \mathcal{M}_3 = \frac{\epsilon}{2} \begin{pmatrix} i & 1 & 0 \\ -1 & i & 0 \\ 0 & 0 & i \end{pmatrix}_L , \tag{60}$$

and $\alpha, \beta, \gamma, \delta, \epsilon$ are real numbers.

7.2 Vortices in the $SO(3)_L$ Symmetric Theory

We firstly analyse the theory when $\tilde{\gamma} = 0$, so that the Lagrangian is $SO(3)_L$ symmetric. In this regime the techniques of embedded vortices are applicable.

7.2.1 Embedded Vortices

The first class of generators, $\mathcal{M}_1$, give a one parameter family of gauge equivalent global vortices, with profiles of the form

$$A(r, \theta) = \Delta_0 \bar{f}(n/\sqrt{2}; r)\mathbf{d}_c \begin{pmatrix} \cos\alpha/2 + i\sin\alpha/2\cos n\theta \\ -\sin\alpha/2 + i\cos\alpha/2\cos n\theta \\ -i\sin n\theta \end{pmatrix} . \tag{61}$$

Here n is the winding of the vortex, α labels the family member, and $\bar{f}$ is defined below. These are the disgyration vortices of ^{3}He.

The second class of generators, $\mathcal{M}_2$, give a one parameter family of gauge equivalent global vortices, with profiles of the form

$$A(r, \theta) = \Delta_0 \bar{f}(n; r) \begin{pmatrix} \cos n\theta \\ -\cos\alpha\sin n\theta \\ \sin\alpha\sin n\theta \end{pmatrix} \mathbf{\Psi}_c . \tag{62}$$

Here n is the winding of the vortex, and α labels the family member. These are the, so called, spin vortices.

The third class of generators, $\mathcal{M}_3$, give a gauge invariant global vortex, with a profile of the form

$$A(r,\theta) = \Delta_0 \bar{f}(n;r)\mathbf{d}_c e^{in\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{63}$$

Here n is the winding of the vortex, and α labels the family member. These vortices are the, so called, singular-line vortices.

The profile functions depend upon the embedded vortex considered, generated by X_{emb} say, and are minima of the Lagrangian

$$\mathcal{L}[f] = \frac{\gamma\Delta_0^2}{2}\left(\frac{df}{dr}\right)^2 + \frac{\gamma f^2}{2r^2} \parallel X_{\text{emb}} A_c \parallel^2 - V[f(r)], \tag{64}$$

where V is the potential, which is independent of the defect considered. Writing $\parallel X_{\text{emb}} A_c \parallel = n \parallel A_c \parallel$ we refer to the solutions as $\bar{f}(n;r)$.

7.2.2 Combination Vortices

Because the symmetries G_{3He} are global there are combination vortex solutions between the three families of generators. The most general combination embedded vortex is generated by a combination of generators from each of the three classes — this is the spin - singular line - disgyration combination vortex. Because of the way we shall determine such vortices we firstly discuss the singular line -disgyration combination.

One obtains a discrete spectrum of singular line-disgyration combination embedded vortices. Solutions are of the form

$$A(r,\theta) = \Delta_0 f(X_s;r)\mathbf{d}_c \exp(X_s\theta)\mathbf{\Psi}_c, \tag{65a}$$

$$\text{with } X_s = \frac{a}{2}\left(i\mathbf{1}_3 + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_L\right) + b\begin{pmatrix} 0 & 0 & 1 \\ -0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}_L. \tag{65b}$$

Then some algebra yields

$$A(r,\theta) = \Delta_0 \bar{f}(p;r)\mathbf{d}_c e^{ia\theta/2} \begin{pmatrix} \cos\theta s + \frac{ia}{2s}\sin\theta s \\ -\frac{a}{2s}\sin\theta s + \frac{i}{s^2}(b^2 + \frac{a^2}{4}\cos\theta s) \\ -\frac{b}{s}\sin\theta s + \frac{iab}{2s^2}(\cos\theta s - 1) \end{pmatrix}, \tag{66}$$

where $s = \sqrt{a^2/4 + b^2}$ and $p = \sqrt{(7m^2+n^2)/2}$. Using the single valuedness constraint that $A(r,2\pi) = A(r,0)$ gives the following discrete spectrum of values for a and b:

$$a = 2m, \qquad b = \pm\sqrt{n^2 - m^2}, \qquad m,n \in \mathbf{Z}. \tag{67}$$

It seems that the singular line vortex and the disgyration may not be continuously deformed into one another, since if this was to be the case then the spectrum of combination vortices should be *continuous*. We obtain a *discrete* spectrum. For them to be continuously deformable into one another we need solutions that are *not* of the embedded type.

The spin - singular line - disgyration combination vortex can be constructed from the above form. Since the generators for spin vortices commute with the generators for singular line - disgyration combination vortices, the form of solution is a spin vortex combined with a singular line - disgyration combination, *i.e.*

$$\begin{aligned} A(r,\theta) = \quad & \Delta_0 \quad \bar{f}(\sqrt{(7m^2+2j^2+n^2)/2};r) \begin{pmatrix} \cos j\theta \\ -\cos\alpha\sin j\theta \\ -\sin\alpha\sin j\theta \end{pmatrix} \\ & e^{ia\theta/2} \begin{pmatrix} \cos\theta s + \frac{ia}{2s}\sin\theta s \\ -\frac{a}{2s}\sin\theta s + \frac{i}{s^2}(b^2 + \frac{a^2}{4}\cos\theta s) \\ -\frac{b}{s}\sin\theta s + \frac{iab}{2s^2}(\cos\theta s - 1) \end{pmatrix}, \end{aligned} \tag{68}$$

with a and b as above and j an integer. Again the spectrum is discrete.

In particular, we shall need to know the form of the spin - singular line combination embedded vortex, which is:

$$A(r,\theta) = \Delta_0 f(\sqrt{(j^2+n^2)};r) \begin{pmatrix} \cos j\theta \\ -\cos\alpha\sin j\theta \\ -\sin\alpha\sin j\theta \end{pmatrix} e^{in\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{69}$$

7.2.3 Stability of the Embedded Vortices

The topology of the vacuum manifold contains loops which are incontractible and thus gives classes of stable vortices. With each of the families of embedded (and combination) vortices an element of the homotopy group may be associated ‖ which tells one whether that family of vortices is topologically stable or unstable.

The vacuum manifold looks like

$$\frac{SO(3)_S \times SO(3)_L \times U(1)_N}{U(1)_{S_3} \times U(1)_{L-N} \times \mathbf{Z}_2} = \frac{S^{(2)}_S \times S^{(3)}_{L,N}/\mathbf{Z}_2}{\mathbf{Z}_2}. \tag{70}$$

Here $S^{(n)}$ is an n-sphere. This vacuum manifold contains *three* inequivalent families of incontractible loops. Firstly, those contained within just $S^{(3)}_{L,N}/\mathbf{Z}_2$. Secondly, those going from the identity, through $S^{(2)}_S$ into $S^{(3)}_{L,N}/\mathbf{Z}_2$ by the $\mathbf{Z}_2$ factor, and then back to the identity. Thirdly, there are combination of the first two types. The classes of the first homotopy group of the vacuum manifold are thus

$$\pi_1\left(\frac{SO(3)_S \times SO(3)_L \times U(1)_N}{U(1)_{S_3} \times U(1)_{L-N} \times \mathbf{Z}_2}\right) = \mathbf{Z}_4. \tag{71}$$

This gives rise to three different topological charges for the vortices, the charge labelling the family from which they originate. Technically, the $\mathbf{Z}_4$ arises from two separate $\mathbf{Z}_2$ contributions, and then we can label the charge (p, q), with $p, q = 0, 1$; however, a more convenient notation (which will be better contextualised in the conclusions) is to assign a single index to these as in [17], ν: $(0,0) = 0, (1,0) = 1/2, (0,1) = 1, (1,1) = 3/2 = -1/2$.

The $\nu = 1/2$ stable vortices are half-quantum spin - (singular line - disgyration) combinations — where one makes use of the $\mathbf{Z}_2$ mixing of the spin and angular groups for stability. Considering the spin - singular line combination above (eq. (69)), the stable half-quantum spin-singular line combination vortex corresponds to $j = n = \frac{1}{2}$:

$$A(r,\theta) = \Delta_0 \bar{f}(1/\sqrt{2}; r) \begin{pmatrix} \cos\theta/2 \\ -\cos\alpha \sin\theta/2 \\ -\sin\alpha\sin\theta/2 \end{pmatrix} e^{i\theta/2} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \tag{72}$$

‖more precisely, with the family *and* the winding number, but we shall only be considering unit winding number vortices

Of course, there are also half-quantum spin - disgyration vortices, and combinations in between. These all have topological charge $\nu = 1/2$.

The $\nu = 1$ stable vortices are some of the singular line (eq. (63)) and disgyration embedded vortices (eq. (61)), also including the combination vortices (eq. (66)) inbetween. These all have the form above. The winding number $n = 1$ vortices are the only stable solutions. Odd-n vortices may decay to these, also having topological charge $\nu = 1$; even-n decay to the vacuum, having topological charge $\nu = 0$.

Finally, the $\nu = 3/2$ vortices are combinations of the $\nu = 1/2$ and $\nu = 1$ vortices.

7.3 Vortex Spectra of the Full ^{3}He Theory

We wish to find the embedded vortex spectrum of the full ^{3}He theory, when one is including terms which are not invariant under spatial rotations of the Lagrangian. Our tactic is to see which of the above embedded vortex solutions remain solutions in the full theory. This is facilitated by investigating how the profile equations are modified by inclusion of terms that are not invariant under $SO(3)_L$ — if the profile equations make sense, for instance they must only be radially dependent, then one can say that those embedded vortices remain solutions to the theory.

Providently, it transpires that only those embedded vortices which are *topologically stable* remain solutions to the full ^{3}He Lagrangian with inclusion of terms that are not rotationally symmetric.

7.3.1 Singular-Line Vortices

The singular-line vortex has a profile of the form (from eq. (63))

$$A(r,\theta) = \Delta_0 f(n;r)\mathbf{d}_c e^{in\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \tag{73}$$

where n is the winding number of the vortex. Substitution into the full Lagrangian (eq. (53)) yields the profile equation to be

$$\mathcal{L}[f] + \widetilde{\mathcal{L}}[f] = (2\gamma + \tilde{\gamma})\Delta_0^2 \left(\left(\frac{df}{dr} \right)^2 + \frac{n^2 f^2}{r^2} \right) - 2\tilde{\gamma}\Delta_0^2 \frac{nf}{r}\frac{df}{dr} - V[f(r)]. \tag{74}$$

Since the extra term nff'/r is least dominant asymptotically we may conclude the the singular line *Ansatz* is still a solution to the full Lagrangian, but with a slightly modified profile function.

7.3.2 Spin Vortices

Vortices embedded solely in the spin sector (with profiles given by eq. (62)) are solutions to the full Lagrangian because the embedded defect formalism is applicable to symmetry-invariant parts of the Lagrangian — which the spin sector is.

This observation is backed up within the mathematics; one may show that for the spin vortex *Ansatz*

$$\partial_i A^{\star}_{\alpha i} \partial_j A_{\alpha j} = \partial_i A^{\star}_{\alpha j} \partial_i A_{\alpha j}. \tag{75}$$

Thus the terms of $\widetilde{\mathcal{L}}$ that are not invariant under spatial rotations become equivalent to the kinetic terms of the symmetric ^{3}He Lagrangian for spin vortices.

7.3.3 Disgyration Vortices

The embedded disgyration vortex has a profile of the form in eq. (61); to simplify the matter we shall consider the family member with $\alpha = 0$ (without loss of generality)

$$A(r, \theta) = \Delta_0 f(n; r) \mathbf{d}_c \begin{pmatrix} 1 \\ i \cos n\theta \\ -i \sin n\theta \end{pmatrix}. \tag{76}$$

where n is the winding of the vortex. Substitution into the full Lagrangian (eq. (53)) yields terms that are not invariant under spatial rotations

$$\widetilde{\mathcal{L}}[f] = \tilde{\gamma} \Delta_0^2 \left(\left(\cos\theta \frac{df}{dr} \right)^2 + \left(\cos n\theta \sin\theta \frac{df}{dr} - \frac{nf}{r} \cos\theta \sin n\theta \right)^2 \right). \tag{77}$$

Since the profile function $f(r)$ is independent of θ, and the Lagrangian $\mathcal{L}_{\text{sym}}[f] + \widetilde{\mathcal{L}}[f]$ that describes $f(r)$ is not rotationally symmetric, we conclude that the embedded disgyration vortices do not remain a solution when non-spatially rotationally symmetric terms are added to the Lagrangian.

7.3.4 Combination Vortices

In general only combinations of embedded vortices that individually remain solutions when non-spatially symmetric terms are added to the Lagrangian remain solutions. Thus the only combination embedded vortices that are solutions to the full Lagrangian $\mathcal{L}_{\rm sym} + \widetilde{\mathcal{L}}$ are the *combination spin-singular line vortices.*

7.4 Conclusions

We conclude, by comparing the results of sec. (7.3.3) with sec. (7.2.3), that embedded vortices that are solutions when terms rotationally non-symmetric terms are added to the Lagrangian,

$$\widetilde{\mathcal{L}}[A_{\alpha j}] = (\gamma_1 + \gamma_2)\partial_i A^\star_{\alpha i}\partial_j A_{\alpha j} = \tilde{\gamma}\partial_i A^\star_{\alpha i}\partial_j A_{\alpha j}, \tag{78}$$

are those vortices that are topologically stable, or higher winding number counterparts of those vortices. The topologically stable embedded vortices are labelled by their topological charge ν [17] and take the following forms.

Firstly, the half-quantum spin-singular line combination vortex, which has topological charge $\nu = 1/2$ and looks like

$$A(r,\theta) = \Delta_0 \bar{f}(1/\sqrt{2};r)\begin{pmatrix} \cos\theta/2 \\ -\cos\alpha\sin\theta/2 \\ -\sin\alpha\sin\theta/2 \end{pmatrix} e^{i\theta/2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{79}$$

Secondly, the singular line vortex, which has topological charge $\nu = 1$ and looks like

$$A(r,\theta) = \Delta_0 \bar{f}(1;r)\mathbf{d}_c e^{in\theta}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{80}$$

Thirdly and finally, the combination of the above two vortices, which has topological charge $\nu = 3/2$ and looks like

$$A(r,\theta) = \Delta_0 \bar{f}(\sqrt{5/2};r)\begin{pmatrix} \cos\theta/2 \\ -\cos\alpha\sin\theta/2 \\ -\sin\alpha\sin\theta/2 \end{pmatrix} e^{i3\theta/2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}. \tag{81}$$

This vortex winds around the singular line part one and a half times and around the spin part half a time.

One should note that from the above spectrum a new meaning for the topological charge ν may be interpreted: as the winding number of the singular line part of the vortex.

Another, final, observation that we would like to make is that upon addition of spatial non-rotationally symmetric terms to the Lagrangian the only embedded vortices that remain solutions to the theory are those which contain *no angular dependence of those spatially associated components of the order parameter* (*i.e.* non are generated by any part of $SO(3)_L$). With hindsight, this may be expected to be the case, but it is pleasing to see it coming through in the mathematics. This leads one to wonder (or conjecture, perhaps) if a similar phenomena happens in other cases where the spatial rotation group acts non-trivially upon the order parameter.

Conclusions

We conclude by summarising our main results:

1. In section 2 we summarised the formalism of the companion paper 'Embedded Vortices' [10].

2. In section 3 we rederived the embedded defect spectrum of the Weinberg-Salam model. Our results are in agreement with other methods.

3. In section 4 we derived the embedded defect spectrum for the model $SU(3) \rightarrow SU(2)$, finding: embedded monopoles, gauge invariant unstable strings and a family of unstable strings. This illustrates: not all globally gauge invariant vortices are stable.

4. In section 5 we illustrated 'combination vortices' by the model $U(1) \times U(1) \rightarrow 1$. This illustrates how such objects may only be solutions in certain limits of the coupling constants, and the form of their spectrum when such solutions have been found.

5. In section 6, we examined the embedded defect spectrum for three realistic GUT models, namely: Georgi-Glashow $SU(5)$; Flipped-$SU(5)$; and Pati-Salam $SU^4(4)$. This illustrated how our formalism may be used for realistic models.

6. Finally, in section 7, we illustrated how our formalism may also be used in some condensed matter contexts — using the specific example of vortices in ^{3}He-A. This also illustrated combination vortices and some of their stability properties.

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