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NON-LINEAR SPACE CHARGE EFFECTS IN NON-UNIFORM BEAMS

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Abstract

Phenomena due to non-linear space charge forces in an unbunched, drifting beam of non-uniform charge density in cross section are described in terms of a physical model based on the computed space-charge fields. The qualitative predictions of the model are seen to be in agreement with the results of numerical calculations in 2 and 4 dimensions.

INTRODUCTION

In proton accelerators the charge density across the beam cross-section is usually non-uniform. The effect of this non-uniformity in the presence of space charge is to introduce non-linear Coulomb forces which distort the hypervolume, modify the 2-dimensional projections and affect therefore the maximum diameter and divergence of the beam. In the low energy regions of an accelerator where the intensities are also usually the highest these non-linear effects can be important. In an associated report (Davies, 1969) there is given an example in which the linear approximation predicts a maximum beam diameter which is seen to contain only 39% of the total charge when the non-linear calculation is performed. The present Note is primarily concerned with drifting beams.

THE EFFECTS OF NON-LINEAR FIELDS IN AN UNBUNCHED BEAM

There are two main consequences of the field variations across a non-uniform beam. It is clear from the diagram of the fields in a Gaussian cross-section (Fig. 1, which shows also for comparison the fields in a uniform beam) that :

1) there is a non-linear variation of the horizontal field E_x with the horizontal displacement x,

2) there is a non-linear variation of this horizontal field E_x at a given x with vertical displacement y.

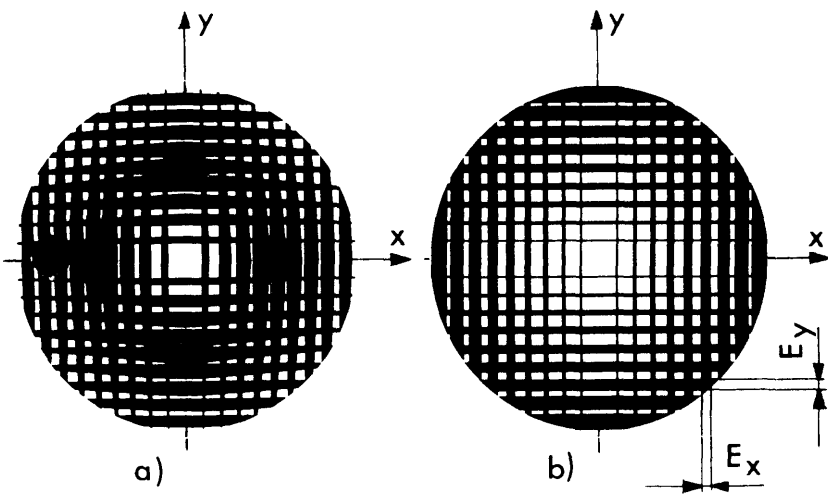


Fig. 1

Diagram of horizontal and vertical fields E_x and E_y in a) Gaussian distribution
b) uniform distribution

The form of $E_x(x)$ along the x axis for a Gaussian distribution ($\hat{x} = 2.5\ \sigma$) is shown in Fig. 2. The change in horizontal momentum in an increment of time will follow the same law so that in the displacement-momentum phase plane (x,x') an ellipse for y = 0 will become distorted as shown (the figure includes the effect of simultaneous drift). Phase space density will be conserved in (x,x') but the distribution of charge projected on to the x axis will become modified (see also Discussion below).

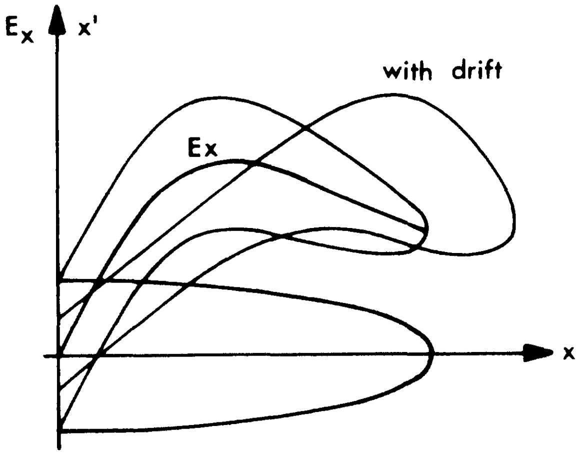


Fig. 2
Effect of a non-linear $E_x(x)$

The second property 2), expressed as $E_x = f(x,y)$, Fig. 3, implies a force coupling between the transverse motions. In order to discuss this coupling we consider an unbunched beam, having initially a probability distribution function $p(x,x',y,y')$. We can obtain the conditional distributions in (x,x') for successive values of y

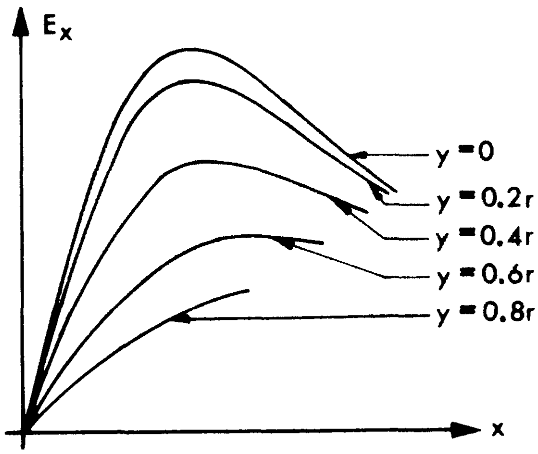


Fig. 3
Effect of $E_x(x,y)$

$$p(x,x')_y = \int_{y'} p(x,x',y,y')\ dy'$$

and allow these to be submitted to impulses $\delta\ x'$ due to the Coulomb forces acting in a drift space δl. We then superimpose these or integrate over y to obtain the marginal distribution or projection on x x'

$$p(x,x') = \int_y \int_{y'} p(x,x',y,y')\ dy\ dy'.$$

If the fields E_x and the resulting impulses $\delta x'$ are independent of y as in a uniform distribution, each distribution will move upwards concentrically in the phase plane (ignoring the effect of drift for simplicity) and the marginal distribution will be unchanged. When $E_x = f(x,y)$ as in Fig. 3 then the conditional distributions will become distorted and at the same time lose their concentricity (Fig. 4) i.e. become displaced relative to each other *. In the case of a Gaussian distribution in (x,x',y,y') originally, with Gaussian distributions in the 2-dimensional projections, one can see that the summation of a number of displaced Gaussians in say x' will result in a projection which is spread out and has an increased standard deviation $\sigma_{x'}$, is non-Gaussian, and has a lower maximum (density in x' in this case).

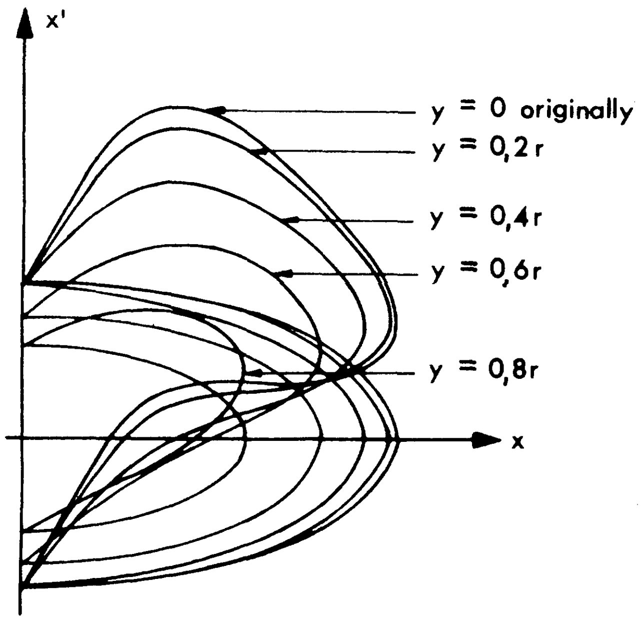


Fig. 4
Superposition of conditional distributions

For large relative displacements of the conditional distributions, the outside envelope may be overlapped by the inner ones, which corresponds to the threshold of total emittance increase analysed by Promé (1969).

It is possible to express the space charge impulses which produce these relative displacements † in terms of the intensity I, the distance through which the forces act δl, the inverse of the radius r, the relativistic factor $\beta^{-2}\gamma^{-2}$ (β^{-1} for the current, β^{-1} for the time and γ^{-2} for the magnetic force correction) and a dimensionless factor F which takes account of the non-uniformity of the cross-section (Taylor, 1968)

$$\text{i.e.} \quad \delta x'_{y_0} - \delta x'_{y_1} \sim \frac{I\,\delta l}{r\,\beta^2\gamma^2}\,F$$

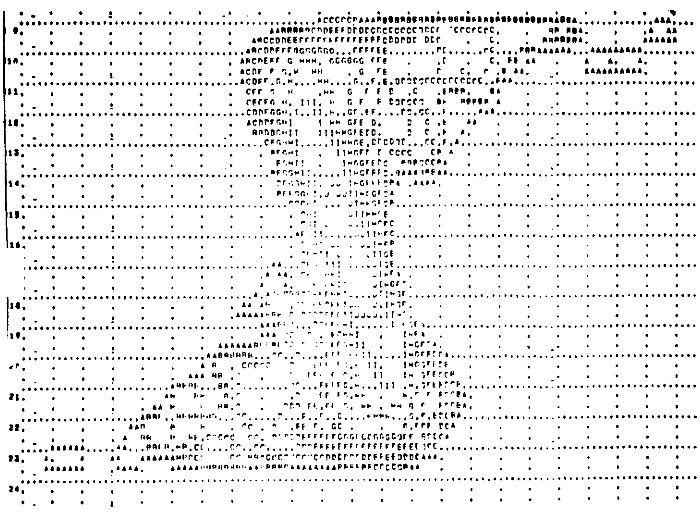
but it does not appear feasible to derive any generally applicable expression for the $\sigma_{x'}$ increase (and the σ_x increase which arises when the effect of the simultaneous drift is included), so that the increase in area in projections has to be calculated for each particular case by a method such as that described by Tanguy (1969).

* We note that the original sectionsy = 0, y = 0.2 r etc. will no longer be planesbecause of simultaneous impulses in the vertical direction.

† This can be applied also to the departure from linearity of the xx' extension.

DISCUSSION

It is seen that there are two distinct non-linear effects produced by the space charge forces in a non-uniform beam. The first, the distortion of the phase space projection, is sometimes referred to as "space charge aberration". Density in the projection is conserved, but the charge density in real space is redistributed. The distorted envelope requires a larger circumscribed elliptical acceptance to accomodate it, leading to an "effective increase" in emittance (Prelec and Passner, 1968). This may be thought of as a particular case of mismatching. The effect is adequately described by a zero emittance calculation (Davies, 1969). The second effect, a coupling which can produce an increase in the area enclosing a given quantity of charge in projection, can be computed by a Lagrangian calculation in 4 dimensions for an unbunched beam, as described by Tanguy (1969). The magnitude of the area increase, in a particular case, can be seen from Figs. 5 a and b due to Warner (1969) which shows the xx' equi-density plot and the resulting density curve (current v emittance) when the Tanguy programme is applied to an initially converging 100 mA beam at 750 keV drifting 1.5 m.



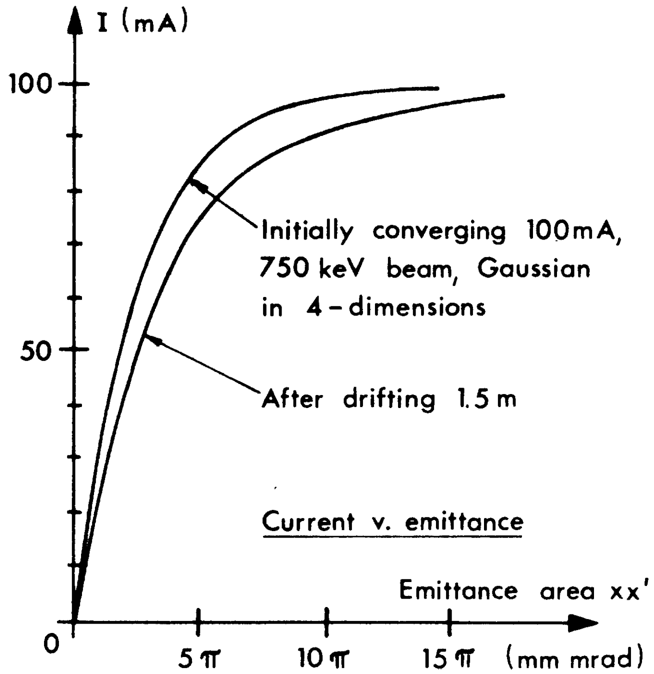


Fig. 5 a : Equi-density plot derived from Tanguy programme output (transformed back into principal axes for computing convenience).

Fig. 5 b : Density curve derived from Fig. 5 a.

A rotationally symmetric Gaussian distribution in which both effects are important was chosen in the previous description, but it is possible to select other distributions where the two effects are unequal. For example a rectangular beam which has a uniform density horizontally and a Gaussian distribution vertically has field variations as shown in Fig. 6, from which one see that the horizontal field E_x is somewhat less aberrant than the vertical field E_y but has a wide variation with the orthogonal dimension whereas the vertical field suffers from both effects.

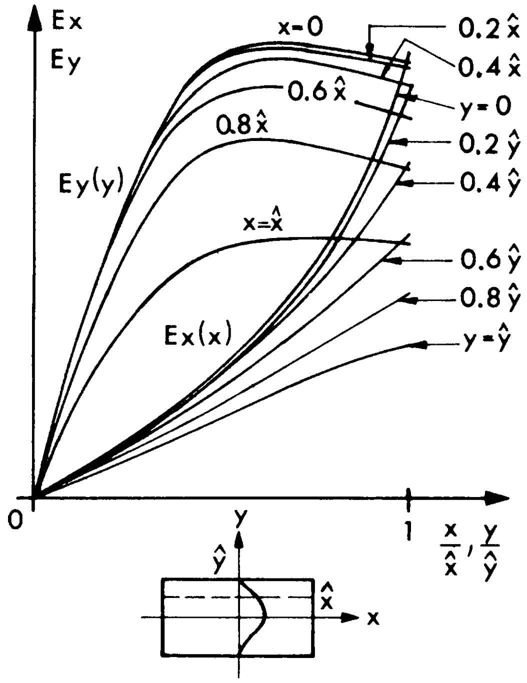


Fig. 6 : Coulomb fields in a rectangular beam having uniform density horizontal and Gaussian ($\hat{y}$=2.5σ) vertically.

We have been concerned so far with the initial changes produced by non-linear fields. The redistribution should continue in the succeeding steps according to the new initial distributions, including any changes in radius produced by linear external focusing, the process continuing at a diminishing rate as the distribution becomes more uniform. The calculations in fact show that eventually the real space distributions become approximately uniform and then develop a crater rim when crossovers occur after which, with continuous focusing the charge begins to regroup itself around the centre.* The fact that the beam passes through the approximately uniform condition brings to one's attention the important point that something approaching a uniform distribution is a necessary but not sufficient condition for a stationary state, and that the initial conditions in all the co-ordinates (x,x',y,y') must be invoked. This is illustrated by the simple example (Fig. 7) of a non-linear line charge in xx' chosen so as to give a uniform projected density in x.

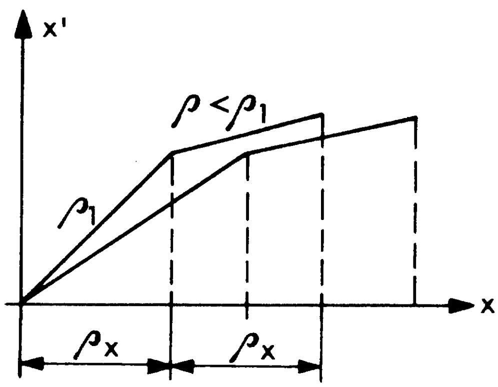


Fig. 7.

* and thereafter oscillates between these extremes. P. Lapostolle (1969) has found that the oscillation can be reduced to a quasi-stationary state by correct choice of continuous external field.

After a drift through a certain distance one sees that the projected density of the outer region will become greater than that of the inner region. This suggests that a distribution which has a non-linear extension in xx' cannot remain stationary with time even if the density in real space is temporarily uniform. The exact requirements of the 4-dimensional distribution for the stationary state have been analysed in detail by Kapchinskij and Vladimirskij (1959) and more recently by Lapostolle (1969) and Promé (1969).

We note from the Davies and Tanguy treatments that a hollow beam develops when cross-overs occur. In the present state of the PS-Linac, the 500 keV pre-injector beam appears to be already in the hollow condition at the entrance to the Linac, whereas from 10 MeV onwards the distributions are still bell-shaped. In order to assess whether the spreading is significant at 50 MeV, where the beam is bunched tightly initially, and therefore could suffer from transverse-longitudinal as well as enhanced transverse-transverse coupling, a 6-dimensional programmed based on the existing programme BUNCH is being prepared.

The theory of these effects, and the quantitative results obtained when the non-linear fields are applied to the equations of motion, are treated in detail in the two associated papers by Davies (1969) and Tanguy (1969).

CONCLUSIONS

A non-uniform distribution of charge density across a beam leads to aberrations in the displacement-momentum projection of the hypervolume, to an increase in area of this projection and to a redistribution of charge across the beam. For low values of $\frac{I}{\beta^2\gamma^2}$ for a rotationally symmetric beam these effects may be ignored, but at the high intensities and low energies encountered in pre-injectors the effects are significant. It remains to be seen whether they become of importance over the long drift distance at 50 MeV, when the beam is roughly Gaussian in all projections measured so far at the Linac output, and in addition is tightly bunched.

ACKNOWLEDGEMENTS

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