EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

MPS/Int.BR/75-10
12 November 1975

LONGITUDINAL STABILITY VIA MAGNANI ROTATION

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GENEVA
1975

1. INTRODUCTION

Magnani has found two methods for preventing the longitudinal bunch-to-bunch instability that occurs in the Booster[1]. The first is to shake the bunches at the synchrotron frequency (Magnani shaking); this modifies the particle distribution within the bunch, lowering the central density, and results in a bunch shape that is more stable. The details are given in Ref. 1. The second method is more subtle. The normal phase loop keeps the centre of mass of the five Booster bunches from moving with respect to the RF wave, but does not directly affect the relative motion of the bunches with respect to one another. Magnani has modified this loop so that the input is no longer the centre of mass of the five bunches, but rather the location of a single bunch. Thus the loop locks the position of a single bunch to the RF wave, but since the other four bunches are not controlled they are quickly lost. This is overcome by switching control to bunch 2 after time T, then to bunch 3 after another time T, and so on. This technique works and is informally known as Magnani rotation, although the reason why it works is still a mystery.

It is easy to show that no switching technique can damp the relative motion of identical rigid bunches (Section 2). However, if the rigid bunches oscillate with slightly different synchrotron frequencies, a small damping of the relative motion occurs even with the normal phase loop (Section 3), but the damping time of several thousand synchrotron periods is certainly too long to be effective. With Magnani rotation, this damping rate can be increased by typically a factor of 3 for certain switching speeds (Section 4). This is interesting, but again probably too slow to be effective. An unexpected result is that a parametric resonance occurs that excites the relative motion for certain switching times (Section 5). The resonances are traversed in about 25 msec and produce approximately five e-foldings of the relative bunch-to-bunch motion.

It is also possible that Magnani rotation works by shaking the bunches, since some shaking is unavoidable while rotating from bunch-to-bunch, and also while passing through a parametric resonance.

2. EQUAL BUNCH FREQUENCIES

The equations of motion for two identical rigid bunches are

$$\ddot{\phi}_1 + \omega_s^2 \phi_1 = \dot{\Omega} \tag{1}$$

$$\ddot{\phi}_2 + \omega_s^2 \phi_2 = \dot{\Omega} \; , \tag{2}$$

where ϕ is the phase difference between RF wave and bunch centre, and Ω is the frequency error (the difference between the actual applied frequency of the RF wave and its ideal value, which is twice the revolution frequency for two bunches).

The feedback system controls the value of Ω, based on information about ϕ_1, ϕ_2, $\dot{\phi}_1$, $\dot{\phi}_2$, which can be sampled in various ways and at various times. The important point is that the relative motion of the two bunches, $\phi_- = \phi_1 - \phi_2$ obeys the equation

$$\ddot{\phi}_- + \omega_s^2 \phi_- = 0 \tag{3}$$

which does not contain Ω. So for equal bunch frequencies no feedback system that controls Ω can affect the relative motion. This is true for any number of bunches.

If one could vary the amplitude or phase of the RF voltage quickly, during the time between two bunches passing the RF cavity, then one could act on each bunch separately. In this case Ω becomes Ω_1 for Eq. (1) and Ω_2 for Eq. (2). However, the bandwidth of the phase loop plus cavity is too narrow for this to occur.

Another possibility is that the frequencies of the two bunches are slightly different. A difference of a few percent has been observed before, and would result from differences in bunch length or bunch population. Then the relative mode ϕ_- is coupled to the centre of mass or sum mode ϕ_+, and the feedback system can work on both modes. This is investigated in the next Section.

3. NORMAL PHASE LOOP WITH UNEQUAL BUNCH FREQUENCIES

Let

$$\begin{aligned} \omega_{s1}^2 &= \omega_s^2 + \delta \\ \omega_{s2}^2 &= \omega_s^2 - \delta \end{aligned} \tag{4}$$

be the radian frequencies for the two bunches ($f = \omega/2\pi$ is the frequency in Hz). Since the difference in bunch frequencies $\Delta\omega = \omega_{s1} - \omega_{s2}$ is small, $\delta \simeq \omega_s \Delta\omega$. For the phase loop it is sufficient to take the simple model

$$\Omega = -\alpha(\phi_1 + \phi_2) \ , \tag{5}$$

where α is a constant related to the loop gain.

For solutions with the time dependence $e^{j\omega t}$, one finds coupled equations for the sum mode $\phi_+ = \phi_1 + \phi_2$ and the difference mode $\phi_- = \phi_1 - \phi_2$,

$$\begin{aligned} (\omega_s^2 - \omega^2)\phi_+ + \delta\phi_- &= -2j\alpha\omega\phi_- \\ (\omega_s^2 - \omega^2)\phi_- + \delta\phi_+ &= 0 \ , \end{aligned} \tag{6}$$

which have a unique solution provided

$$(\omega_s^2 - \omega^2)(\omega_s^2 - \omega^2 + 2j\alpha\omega) = \delta^2 \ . \tag{7}$$

In the equal frequency limit, $\delta \to 0$ and the two undamped roots

$$\omega_1,\ \omega_2 = \pm\omega_s \tag{8}$$

belong to the difference mode, while the two damped roots

$$\omega_3,\ \omega_4 = -j\alpha \pm \sqrt{\omega^2 - \alpha^2} \tag{9}$$

are associated with the sum mode. Usually $\alpha >> \omega_s$, so

$$\begin{aligned} \omega_3 &\simeq 2j\alpha \\ \omega_4 &\simeq j\,\frac{\omega_s^2}{2\alpha} \ . \end{aligned} \tag{10}$$

A typical value for the Booster is $\alpha \simeq 5\omega_s$, which results in a fast damping $\omega_3^{-1} \simeq 0.016T_s$ for phase transients and a slower damping $\omega_4^{-1} \simeq 1.6T_s$ for radial transients. Here T_s is the synchrotron period $2\pi/\omega_s$.

For small δ, the frequencies are shifted,

$$\omega = \omega_i + a_i\delta^2 \ , \tag{11}$$

where the coefficient a_i is found by substituting (11) into

$$(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)(\omega - \omega_4) = \delta^2 \tag{12}$$

and keeping only terms of order δ^2. For example,

$$\omega = \omega_1 + a_1\delta^2$$

yields

$$a_1 = \frac{1}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_1 - \omega_4)} \simeq \frac{j}{4\alpha\omega_s^2} \ . \tag{13}$$

In this way, we find that a small damping term is added to the difference mode,

$$\omega_1,\ \omega_2 = \pm\omega_s + j\,\frac{\delta^2}{4\alpha\omega_s^2} \ , \tag{14}$$

while the other two roots are essentially unchanged.

The time constant for this damping is

$$\tau = \frac{4\alpha}{\Delta\omega^2} \ . \tag{15}$$

For example, if the bunches differ in frequency by 3% ($\Delta\omega/\omega_s = 0.03$) and $\alpha = 5\omega_s$, the damping time for the difference mode is about $3500T_s$, or several thousand synchrotron periods. This is much too slow to be useful, which is consistent with the fact that the normal phase loop does not prevent bunch-to-bunch motion.

4. MAGNANI ROTATION WITH UNEQUAL BUNCH FREQUENCIES

It is convenient to write the equations of motion as

$$\begin{aligned} \dot{\phi}_1 &= p_1 + \Omega \\ \dot{p}_1 &= -(\omega_s^2 + \delta)\phi_1 \\ \dot{\phi}_2 &= p_2 + \Omega \\ \dot{p}_2 &= -(\omega_s^2 - \delta)\phi_2 \ , \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Omega &= -2\alpha\phi_1 \quad \text{for} \quad 0 \le t < T \\ &= -2\alpha\phi_2 \quad \text{for} \quad T \le t < 2T \ , \\ &\text{etc.,} \end{aligned} \tag{17}$$

and where the auxiliary variable p is related to radial position. This is a fourth-order system of equations with periodic coefficients. Because the coefficients in (17) are piecewise constant, the standard matrix technique can be employed.

The solution from 0 to T can be written as

$$X(t) = M_1 X(0) \ , \tag{18}$$

where M is a 4 × 4 matrix (see Appendix) and X is the vector

$$X = \begin{pmatrix} \phi_1 \\ p_1 \\ \phi_2 \\ p_2 \end{pmatrix} . \tag{19}$$

Similarly, the transfer matrix for the second time interval $T \to 2T$ is M_2 and for the complete period $0 \to 2T$,

$$M = M_2 M_1 \tag{20}$$

and after n periods M^n. The stability of the system is governed by the eigenvalues of M. Let Λ_i, X_i with i = 1, 4 be the eigenvalues and eigenvectors of M. Since any initial vector X(0) can be decomposed into a sum of the four eigenvectors, and since after n periods each eigenvector transforms into itself times Λ_i^n, stability requires that the magnitude of Λ_i be less than unity,

$$|\Lambda_i| < 1 \quad \text{for stability.} \tag{21}$$

If we define

$$\Lambda_i \equiv e^{2j\lambda_i T} , \tag{22}$$

then the quantities λ_i are analogous to the frequencies ω_i used before for the system with constant coefficients.

The eigenvalues are determined by computer using the matrix derived in the Appendix. We are interested in the damping rate for the difference mode, which we designate as $S(\alpha, T, \delta)$. This is to be compared with the damping rate $\delta^2/4\alpha\omega_s^2$ found for the normal phase loop.

For small δ, S must have the form

$$S = S(\alpha, T)\delta^2 ,$$

since i) it was shown in Section 2 that the damping rate is zero for $\delta = 0$, and ii) a first-order term in δ implies that the sign of the frequency difference between bunches is important, which is not possible since the bunch labelling in Eq. (16) is arbitrary. One could also guess that in the limit of short switching times $T \ll T_s$, the bunches could not distinguish between Magnani rotation and the normal phase loop, and that in the limit of long switching times $T \gg T_s$, the damping rate should approach zero since the loop would be locked to one bunch in this case. This is verified by the computer.

The result is plotted in Fig. 1, namely the ratio of the two damping rates, Magnani rotation divided by $\Delta\omega^2/4\alpha$ as a function of the switching time T and for various loop gains α. One sees that for T less than $\frac{1}{2}T_s$, Magnani rotation does produce a larger damping rate, especially for large values of α. For somewhat longer switching times, regions of weak antidamping are encountered. In the Booster, the synchrotron frequency varies from 5 kHz near injection to 2 kHz near the end of the acceleration cycle. Therefore, if the switching frequency is $\sim$ 10 kHz or larger, the difference mode remains in the region of enhanced damping for the whole cycle, while for somewhat lower switching frequencies, weak antidamping is encountered during the early part of the cycle.

Occasionally, the computer generated sharp spikes at certain values of T/T_s. At first this was thought to be a computational error, but closer inspection showed that it was a parametric resonance excited by the Magnani rotation. These sharp resonances are not shown in Fig. 1.

5. PARAMETRIC RESONANCE EXCITED BY MAGNANI ROTATION

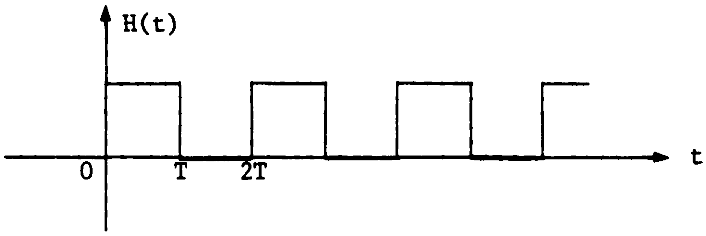
We write the equations of motion in the form

$$\ddot{\phi}_1 + (\omega_s^2 + \delta)\phi_1 = \dot{\Omega}$$
$$\ddot{\phi}_2 + (\omega_s^2 - \delta)\phi_2 = \dot{\Omega} \tag{23}$$

with

$$\dot{\Omega} = -2\alpha\left[H\dot{\phi}_1 + (1 - H)\dot{\phi}_2\right] ,$$

where H(t) is the periodic switching function



They can be combined into one fourth-order equation for ϕ_-,

$$\left[D^2 + 2\alpha D + (\omega_s^2 - \delta)\right]\left[D^2 + (\omega_s^2 + \delta)\right]\phi_- = 4\alpha\delta H(t)D\phi_- , \tag{24}$$

where $D = d/dt$. The RHS can be regarded as a periodic forcing term, which is small because it contains the small parameter δ. Resonance occurs if this forcing term contains frequencies near to the roots of the LHS, mainly $\pm\omega_s$ since the roots of the first bracket are too strongly damped to be excited. Near this resonance, $D^2 \simeq -\omega_s^2$ and the first bracket in (24) can be replaced by $2\alpha D$, so the equation reduces to the Hill equation

$$\ddot{\phi}_- + \omega_s^2\phi_- = 2\delta H(t)\phi_- , \tag{25}$$

where the small δ terms have been neglected on the LHS. This can be further simplified by keeping only the k^{th} term in the Fourier expansion of H(t):

$$\ddot{\phi}_- + \omega_s^2\left[1 - h_k \sin \omega_k t\right]\phi_- = 0 , \tag{26}$$

where

$$\left.\begin{aligned} h_k &= \frac{4}{\pi}\,\frac{1}{2k - 1}\,\frac{\Delta\omega}{\omega_s} \\ \omega_k &= \pi\,\frac{2k - 1}{T} \end{aligned}\right\} k = 1, 2, 3, \ldots .$$

This is the standard form for the Mathieu-Hill equation[2]. Resonance occurs when the forcing frequency ω_k is twice the natural frequency ω_s, which corresponds to the switching times

$$\frac{T}{T_s} = \frac{2k - 1}{4} = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots . \qquad (27)$$

The full width of the resonance is

$$\Delta\omega_k = \omega_s h_k = \frac{4}{\pi} \frac{\Delta\omega}{2k - 1} , \qquad (28)$$

and the maximum e-folding rate at the centre is $\frac{1}{4}\Delta\omega_k$. Neither parameter depends on the switching time T, loop gain α, or synchrotron frequency, but only on the difference in bunch frequencies $\Delta\omega$.

The growth and damping rates for the difference mode, including parametric resonances, is shown in Fig. 2 for the case $\alpha = 5\omega_s$ and 3% difference in bunch frequencies. Small secondary resonances occur at $T/T_s = 1, 1.5, 2, \ldots$, which can also be derived from Eq. (26) [2]. A close-up of the first resonance region is shown in Fig. 3. Agreement between the matrix method and the analytic approximation [Eqs. (25) and following] is extremely good: the differences are too small to be seen in Figs. 2 and 3.

For the usual example of 3% difference in bunch frequencies, the width of the first resonance region is 120 Hz, and since the synchrotron frequency in the Booster changes by 3 kHz in 600 msec, the resonance region will be crossed in 24 msec. The e-folding time is about 4 msec, so we expect about five e-foldings to occur during this time.

REFERENCES

1) J. Gareyte et al., Beam dynamics experiments in the CERN PS Booster, Proc. Particle Accelerator Conf., Washington, DC, 1975, IEEE Trans. Nuclear Sci. NS-22, 1855 (1975).

2) N. Minorsky, Non-linear oscillations (D. Van Nostrand Company Inc., Princeton, New Jersey, 1962), p. 509.

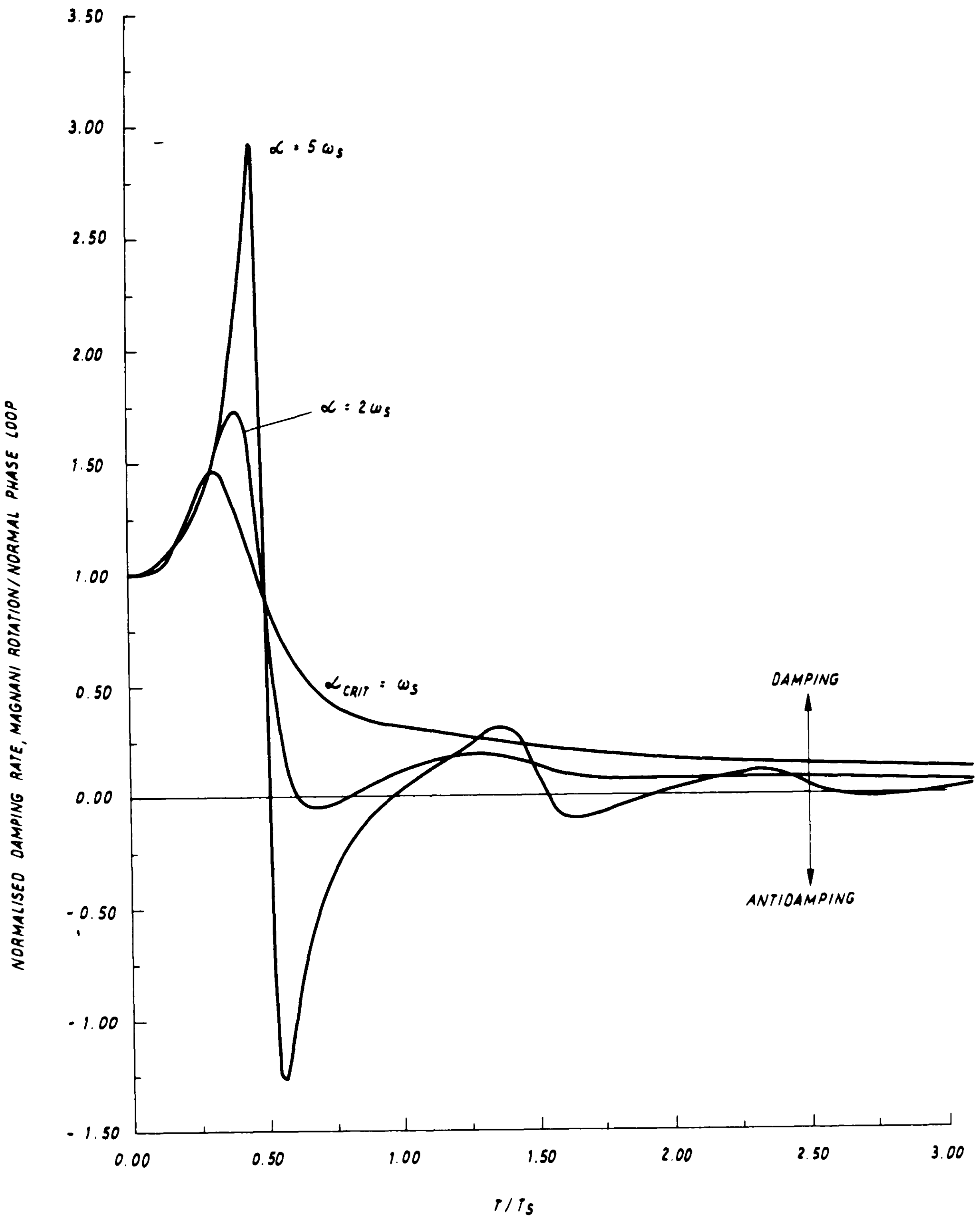


Fig. 1 Damping of the difference mode by Magnani rotation

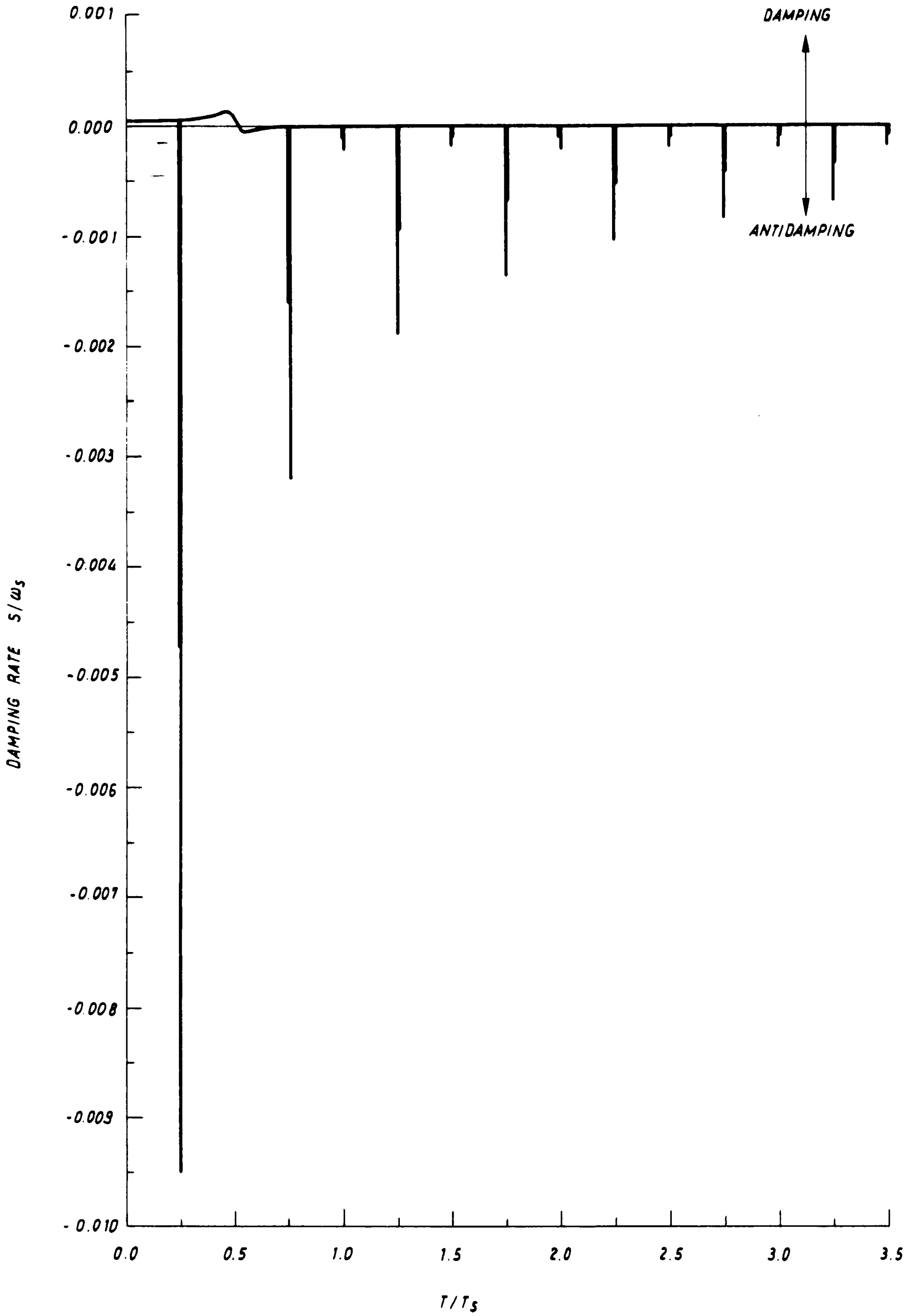


Fig. 2 Growth and damping rates for the difference mode, including parametric resonances, for 3% difference in bunch frequencies and $\alpha = 5\omega_s$.

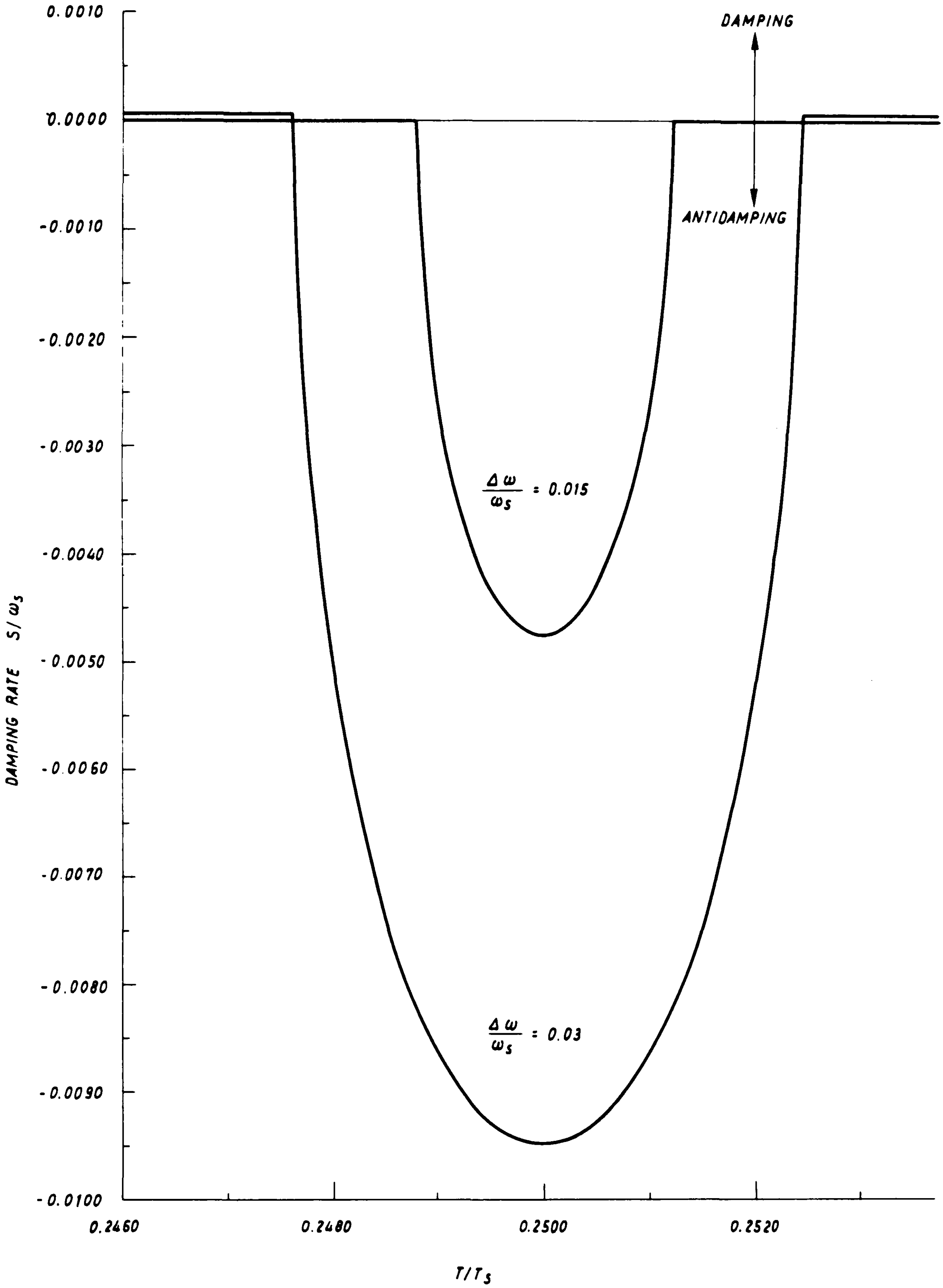


Fig. 3 Close-up of first parametric resonance (k = 1)

APPENDIX

We wish to find the transfer matrix for Eq. (16). For the time interval $0 \rightarrow T$, the solution has the form

$$\begin{aligned} X_1(t) &= A(\alpha, \delta, t)X_1(0) \\ X_2(t) &= B(\alpha, \delta, t)X_1(0) + C(\delta, t)X_2(0) , \end{aligned} \tag{A.1}$$

where

$$X_1 = \begin{pmatrix} \phi_1 \\ p_1 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} \phi_1 \\ p_2 \end{pmatrix},$$

and A, B, C are 2×2 matrices.

A is the transfer matrix for bunch 1, which obeys the equations

$$\begin{aligned} \dot{\phi}_1 &= p_1 - 2\alpha\phi_1 \\ \dot{p}_1 &= -(\omega_s^2 + \delta)\phi_1 , \end{aligned} \tag{A.2}$$

or more compactly

$$\dot{X}_1 = L_1 X_1 . \tag{A.3}$$

One finds

$$A = \frac{1}{s_2 - s_1} \begin{bmatrix} (2\alpha + s_2)e^{s_1 t} - (2\alpha + s_1)e^{s_2 t} & -e^{s_1 t} + e^{s_2 t} \\ (2\alpha + s_1)(2\alpha + s_2)(e^{s_1 t} - e^{s_2 t}) & -(2\alpha + s_1)e^{s_1 t} + (2\alpha + s_2)e^{s_2 t} \end{bmatrix}, \tag{A.4}$$

where s_1 and s_2 are the roots of the characteristic equation

$$s^2 + 2\alpha s + (\omega_s^2 + \delta) = 0 . \tag{A.5}$$

C is the transfer matrix for bunch 2 when bunch 1 is not moving, $X_1 = 0$. Then

$$\begin{aligned} \dot{\phi}_2 &= p_2 \\ \dot{p}_2 &= -(\omega_s^2 - \delta)\phi_2 \end{aligned} \tag{A.6}$$

or more compactly

$$\dot{X}_2 = L_2 X_2 . \tag{A.7}$$

We find

$$C = \begin{bmatrix} \cos gt & \frac{1}{g} \sin gt \\ -g \sin gt & \cos gt \end{bmatrix} \tag{A.8}$$

where $g = \sqrt{\omega_s^2 - \delta}$. We could also write formally

$$A = e^{L_1 t} , \quad C = e^{L_2 t} , \tag{A.9}$$

with $A(0) = \underline{I}$, $C(0) = I$, and $B(0) = 0$.

B is related to A and C. It is convenient to write Eq. (16) for bunch 2 in the form

$$\dot{X}_2 = L_2 X_2 + L_3 X_1 \tag{A.10}$$

with

$$L_2 = \begin{bmatrix} 0 & 1 \\ -(\omega_s^2 - \delta) & 0 \end{bmatrix} , \quad L_3 = \begin{bmatrix} -2\alpha & 0 \\ 0 & 0 \end{bmatrix} .$$

By substituting (A.1) into (A.10) we find

$$\dot{X}_2 = L_2 B X_1(0) + L_2 C X_2(0) + L_3 A X_1(0) , \tag{A.11}$$

while differentiating (A.1) yields

$$\dot{X}_2 = \dot{B} X_1(0) + \dot{C} X_2(0) , \tag{A.12}$$

and since $\dot{C} = L_2 C$, we find the differential equation for B,

$$\dot{B} = L_2 B + L_3 A . \tag{A.13}$$

This can be integrated with the condition $B(0) = 0$ to give

$$\begin{aligned} B(t) &= e^{L_2 t} \int_0^t e^{-L_2 t'} L_3 A(t') \, dt' \\ &= \int_0^t C(t - t') L_3 A(t') \, dt' , \end{aligned} \tag{A.14}$$

which is the desired relation. The final result for B can be written in terms of the integrals

$$\begin{aligned} I_c(s, \delta, t) &= \int_0^t e^{st'} \cos g(t - t') \, dt' \\ I_s(s, \delta, t) &= \int_0^t e^{st'} \sin g(t - t') \, dt' \end{aligned} \tag{A.15}$$

as

$$B_{11} = -\frac{2\alpha}{s_2 - s_1}\left[(2\alpha + s_2)I_c(s_1, \delta, t) - (2\alpha + s_1)I_c(s_2, \delta, t)\right]$$

$$B_{12} = -\frac{2\alpha}{s_2 - s_1}\left[-I_c(s_1, \delta, t) + I_c(s_2, \delta, t)\right]$$

$$B_{21} = \frac{2\alpha g}{s_2 - s_1}\left[(2\alpha + s_2)I_s(s_1, \delta, t) - (2\alpha + s_1)I_s(s_2, \delta, t)\right]$$

$$B_{22} = \frac{2\alpha g}{s_2 - s_1}\left[-I_s(s_1, \delta, t) + I_s(s_2, \delta, t)\right] ,$$

where

$$I_c(s, \delta, t) = \frac{1}{s^2 + g^2}\left[s\, e^{st} - s \cos gt + g \sin gt\right]$$

$$I_s(s, \delta, t) = \frac{1}{s^2 + g^2}\left[g\, e^{st} - g \cos gt - s \sin gt\right] .$$

The 4×4 matrix M_1 for the time interval $0 \to T$ is therefore

$$M_1 = \begin{bmatrix} A(\alpha, \delta, T) & 0 \\ B(\alpha, \delta, T) & C(\delta, T) \end{bmatrix} \tag{A.16}$$

Because of the symmetry of Eq. (16), the matrix M_2 for the second interval $T \to 2T$ is

$$M_2 = \begin{bmatrix} C(-\delta, T) & B(\alpha, -\delta, T) \\ 0 & A(\alpha, -\delta, T) \end{bmatrix} \tag{A.17}$$

and the desired matrix for the complete period is $M = M_2M_1$.

To check the algebra, the matrix M was also found by numerical integration of Eq. (16). For example, starting with the initial vector $X(0) = (1, 0, 0, 0)$ yields the final vector $X(2T) = (M_{11}, M_{21}, M_{31}, M_{41})$, and so on.