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PULSE DURATION MEASUREMENT (FWHM) BY AUTOCORRELATION

P. Joly

Abstract:

In this report the autocorrelation function, which provides FWHM measurement, of a pulse is defined, and the main properties described. Two means of measuring the FWHM are then compared.

A set of numerical examples provides the basis of the study, and a special emphasis is placed on the effects of the asymmetry and the shape of the pulse.

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1. Introduction

When detectors with a response faster than the laser pulse to be measured are scarce, as it is the case in the picosecond range, measuring the autocorrelation function of the pulse provides an alternative solution. The drawback of this indirect method is that the autocorrelation (function) obtained may substantially differ from the original pulse. Nevertheless, assuming a shape for the pulse its full width at half maximum (FWHM) can be easily derived.

In this report the autocorrelation function of a pulse is defined and main properties described. A number of numerical examples which reveal the effect of the pulse shape (sharpness of the rising and falling parts, and non-symmetry) on the FWHM ratio are then presented.

2. Definitions and properties

2.1. Definition by an integral

The autocorrelation function $X_I(\tau)$ of a pulse I(t) is defined by:

$$X_I(\tau) = \int_{-\infty}^{+\infty} I(t)I(t+\tau)dt$$

Simple variable substitutions lead to the properties tabulated below:

	Pulse	Autocorrelation
i.	I(t)	$X_I(\tau) = \int_{-\infty}^{+\infty} I(t)I(t+\tau)dt$ $\int_{-\infty}^{+\infty} I(t')I(t'-\tau)dt' = X_I(-\tau)$
ii.	A I(t)	$\int_{-\infty}^{+\infty} AI(t).AI(-t+\tau).dt = A^2X_I(\tau)$
iii.	I(t/a)	$\int_{-\infty}^{+\infty} I(\frac{t}{a}).AI(\frac{t+\tau}{a}).dt = \lvert a\rvert.X_I(\tau/a)$
iv.	I(-t)	$\int_{-\infty}^{+\infty} I(-t)I(-t+\tau)dt = X_I(\tau)$

The first property (i.), $X_I(\tau) = X_I(-\tau)$, indicates that the autocorrelation function is always symmetrical with respect to the vertical axis, whatever the pulse.

The second (ii.) shows that the effect of multiplying the intensity of the pulse by a constant A, is to multiply the autocorrelation by the square of A.

The third (iii.), shows that, when the pulse is expanded in time by a factor a, the autocorrelation is expanded by exactly the same factor both in intensity and duration. This the keypoint, as it proves that for a given shape of pulse the FWHM of the pulse and of the autocorrelation are proportional.

The last property (iv.) is simply a consequence of the previous one but stresses that two different pulses (a pulse and its copy for which time direction is reversed) have exactly the same autocorrelation function. This indicates that there are problems in reconstructing the pulse from its autocorrelation. To understand why the autocorrelation does not contain the full information on the pulse, it is useful to think in terms of frequencies.

2.2. Definition in frequency space

As time and frequency form a Fourier transform pair, and as the autocorrelation function of a pulse is equal to the convolution of the pulse by its time reversed copy, the Fourier transform of the autocorrelation is the product of the Fourier transform of the pulse by the conjugated:

Time		Frequency
I(t)	↔	$\tilde{I}(\omega) = \lvert\tilde{I}(\omega)\rvert e^{i\varphi(\omega)}$
$X_I(\tau) = \int_{-\infty}^{+\infty} I(t)I(t+\tau)dt$	↔	$\tilde{X}_I(\omega) = \tilde{I}(\omega).\tilde{I}^*(\omega) = \lvert\tilde{I}(\omega)\rvert^2$

It should be noted that $\tilde{X}_I(\omega)$ does not include $\varphi(\omega)$, and herefore, all phase information is lost. Nevertheless it is always possible to construct a symmetrical function $Y_I(\tau)$ which has the same autocorrelation function as I(t).

Time		Frequency
I(t)	↔	$\tilde{I}(\omega) = \lvert\tilde{I}(\omega)\rvert e^{i\varphi(\omega)}$
$Y_I(\tau)$	↔	$\tilde{Y}_I(\omega) = \lvert\tilde{I}(\omega)\rvert$

The function $Y_I(\tau)$ can be fairly easily computed from $X_I(\tau)$ as it is the inverse Fourier transform of the square root of $\tilde{X}_I$. If I(t) is symmetrical, then $\varphi(\omega) = 0$, and $Y_I(\tau)$ equals I(t).

3. Which method should be used to measure the FWHM?

In the previous section we have seen that the FWHM of the pulse is directly proportional to that of the auto-correlation (the ratio will depend on the shape of the pulse), and that the pulse can be exactly reconstructed if it is symmetrical. To decide between the two approaches let us try a numerical example.

The pulses are defined by the formula:

$$I(t) = \begin{cases} e^{-(t/a)^2} ; t < 0 \\ e^{-(t/b)^2} ; t \geq 0 \end{cases} \quad \text{with } a + b = 1$$

They all have the same width, and as the parameter a ranges from 0.5 to 0.1, the pulse shapes vary from a

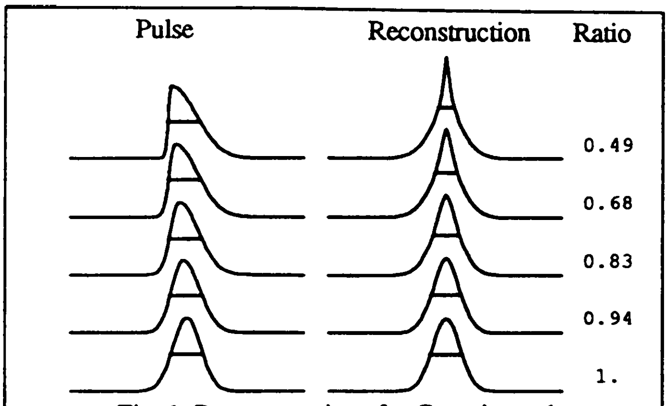

Fig. 1: Reconstructions for Gaussian pulses

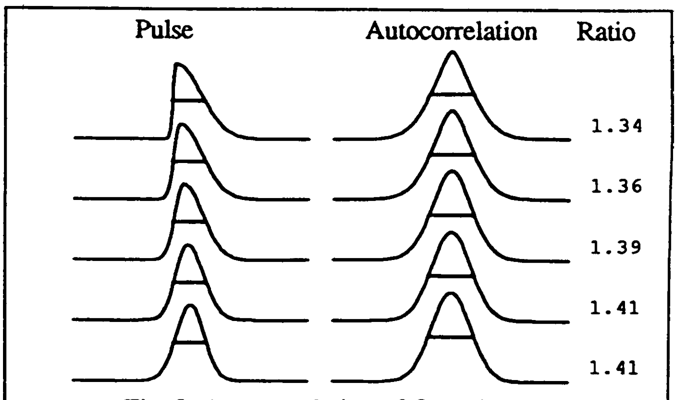

Fig. 2: Autocorrelation of Gaussian pulses

classical Gaussian to an non-symmetrical Gaussian for which the fall time is found to be 9 times the rise time. The results of the computations, which were made by double Fourier transform of a set of 128 samples for each pulse, are given in figures 1 and 2.

In figure 1 reconstructed symmetrical pulses (Y_I) are presented. For the symmetrical pulse the FWHM ratio is 1 as expected but it decreases drastically to less than 0.5 when the pulse becomes non-symmetrical.

In figure 2 the autocorrelation functions (X_I) are presented. Here the ratio for the symmetrical case, as it could be calculated analytically, is $\sqrt{2}$ and it is found to decrease by only 5% when the pulse gets non-symmetrical.

These results show that the FWHM of the autocorrelation is much less sensitive to the non-symmetry of the pulse compared to the one of the reconstructed

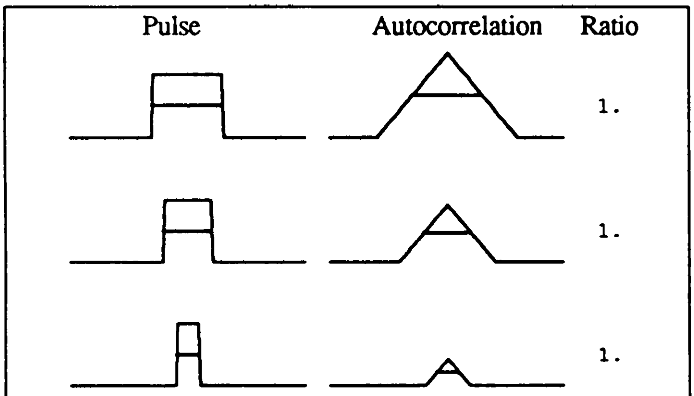

Fig. 3: Autocorrelation of square pulses

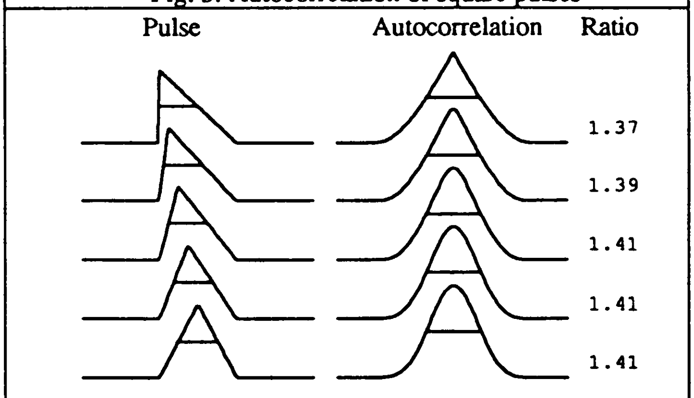

Fig. 4: Autocorrelation of triangle pulses

symmetrical pulse. However, as the ratio should depend on the pulse shape, the point has to be studied on a variety of examples.

4. More examples of FWHM ratios

All the following numerical computations where made using Fourier transforms on sets of 128 samples.

4.1. Square pulses

As can be seen from figure 3, the ratio is always 1.

4.2. Triangular pulses

All the triangles in figure 4 have the same width, the duration of the ascent varying from zero to half the base time. The ratios are very close to those found for the Gaussians (same analytical value for the symmetrical case), and the reduction is less than 3%.

4.3. Sech^2 pulses

Presented in figure 5, sech² pulses are defined by the formula:

$$I(t)=\begin{cases}\operatorname{sech}^2(t/a);t<0\\ \operatorname{sech}^2(t/b);t\geq 0\end{cases} \quad \text{with } a+b=1$$

They all have the same width, and the parameter a ranges from 0.5 to 0.1.

These pulse are very close to the Gaussian ones. The difference is that the beginning of the ascent, as well as

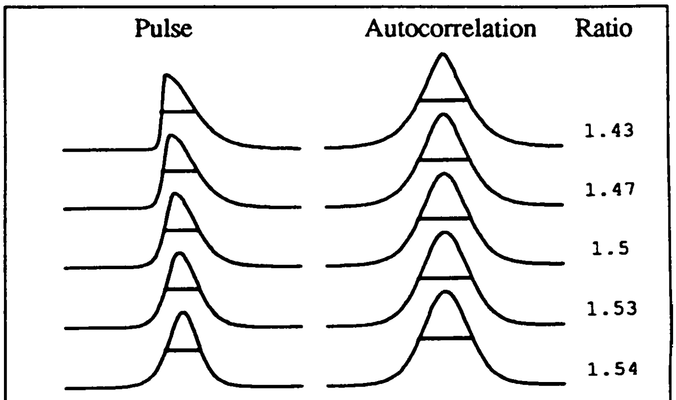

Fig. 5: Autocorrelation of sech² pulses

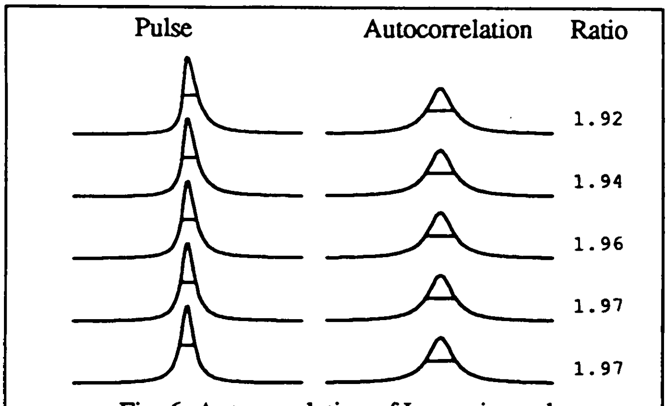

Fig. 6: Autocorrelation of Lorenzian pulses

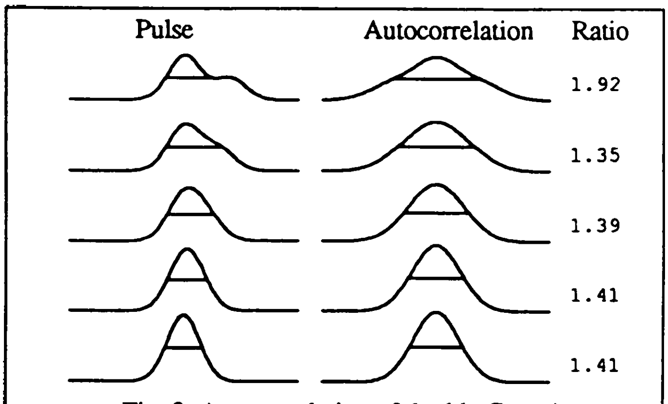

Fig. 9: Autocorrelation of double Gaussian

the end of the descent, are slightly slower. Nevertheless, the ratio is larger: 1.54 compared to 1.41 for symmetrical case.

This larger ratio is perhaps the reason for the popularity of the sech² model, as it yields pulse widths 9% shorter than those obtained from the Gaussian model. Further more, as the Gaussian pulse is the theoretical limit for the product of the pulse duration by the optical bandwidth, "real pulses" might well be quite close to the sech².

Here the reduction due to non-symmetry is about 9%.

4.4. Lorenzian pulses

Lorenzian pulses, presented figure 6, are defined by the formula:

$$I(t)=\begin{cases}\dfrac{1}{1+(t/a)^2};t<0\\ \dfrac{1}{1+(t/b)^2};t\geq 0\end{cases}\quad \text{with } a+b=1$$

They all have the same width, and the range of a is the same as in the previous examples.

The ratios are dramatically larger than in the previous cases, but the reduction due to asymmetry is less than 3%.

It should be noted that the Lorenzian pulses have long wings. These may be linked to the change in the ratio.

4.5. $1/(1+|t|^{\alpha})$ pulses

To see if the assumption made just above may be verified, two sets of pulses with variable wings are examined.

The pulses of the first set, presented figure 7, are defined by: $I(t)=\dfrac{1}{1+|a|^{\alpha}}$. As the parameter α ranges from 2 (Lorenzian) to 4, the ratio decreases drastically from 1.97 to 1.34.

The pulses of the second set (cf figure 8) are defined by:

$$I(t)=\begin{cases}e^{-t^2} & t<0\\ \dfrac{1}{1+t^{\alpha}} & t\geq 0\end{cases}$$

For these pulses having a Gaussian ascent and a descent given by $\dfrac{1}{1+|a|^{\alpha}}$, the ratios are intermediate between 1.41 and the values of their correspondants in the first set.

4.6. Double Gaussian pulses

Double Gaussian pulses, presented figure 9, are defined by $I(t)=e^{-t^2}+.5\,e^{-(t-t_0)^2}$. As t_0 ranges from 0 to 1.5 (four lower curves), the ratio decreases slightly from $\sqrt{2}$ to 1.35 as the assymetry of the pulse increases. The deviant ratio of 1.92 for the top curves, is due to the fact that the FWHM of the pulse only measures the first hump; it illustrates the limitions of the FWHM measurement applied to pulses of complex shapes.

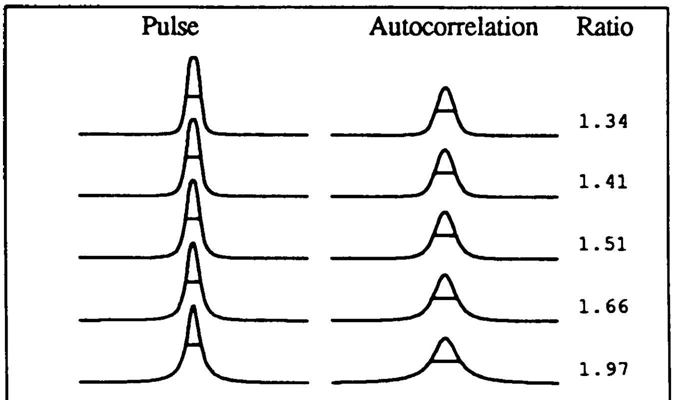

Fig. 7: Autocorrelation of the first set

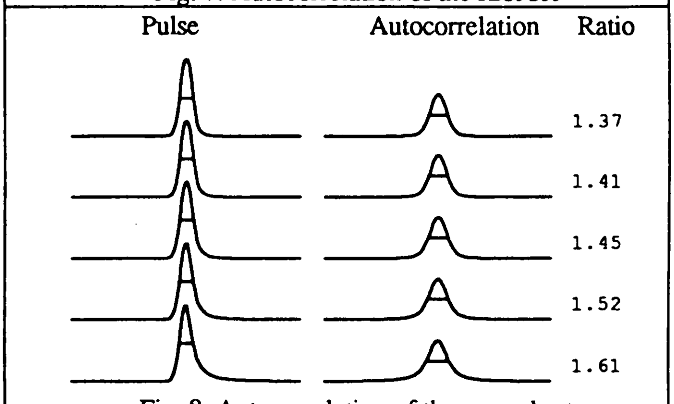

Fig. 8: Autocorrelation of the second set

4.7. Special pulse

To model the pulse delivered by a saturable absorber mode lock laser, a pulse defined by $I(t) = \exp(-\alpha/t - \alpha t + 2\alpha)$ is considered. It has been constructed to have a sharp rise and an exponential fall, the rate of which is controlled by α. Apart from -αt, other terms containing α are present to insure that the pulse always has the peak value I(1) = 1. As showed by figure 10, for α ranging from 1 to 5, the autocorrelation ratio increases from 1.39 to 1.42.

This small change of the ratio with α is an encouraging result.

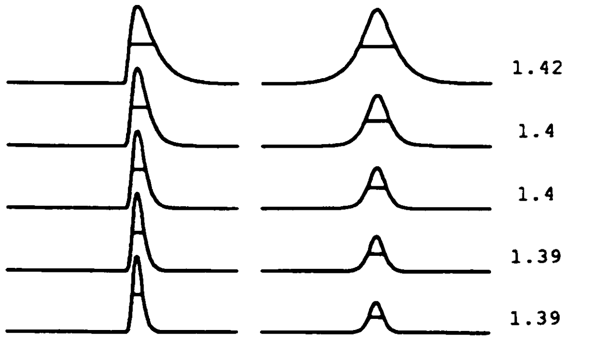

Fig. 10: Autocorrelation of the special pulse

5. Assessment on the FWHM ratios

The ratios, between the FWHM of the autocorrelation functions and corresponding pulses, for all the numerical examples presented above are summarised in figure 11. Black marks correspond to the symmetrical pulses.

The difference in span of the sets of marks illustrate that asymmetry has a much smaller effect compared to the way the pulse rises and falls: in the examples the reduction of the ratio, as the pulse loses its symmetry, never exceeds 9%, whereas going from the square pulse to the Lorenzian it is almost doubled. If the shape of the pulse is not known, this change of a factor two in the ratio gives a very important uncertainty on the FWHM of the pulse derived from the one of the autocorrelation. This note should be tempered as the limits of the range correspond to extreme cases. As most of the other examples give ratios between 1.35 and 1.54; we can consider that a value of 1.45 can be applied and gives the FWHM duration of the pulse to within less than 10%.

6. Conclusion

As seen in section 3, using the symmetrical reconstruction of the pulse to directly have its FWHM duration is only valid for symmetrical pulses. Otherwise, it is preferable to use the FWHM of the autocorrelation function as the ratio between the durations is affected much less by asymmetry. In this case the numerical examples treated tend to show that a ratio of 1.45 should give an approximation of the duration, within 10%, for pulses with a "good shape".

Therefore, it is always interesting to supplement the temporal measurement by autocorrelation with another which would indicate if the pulse is symmetrical.

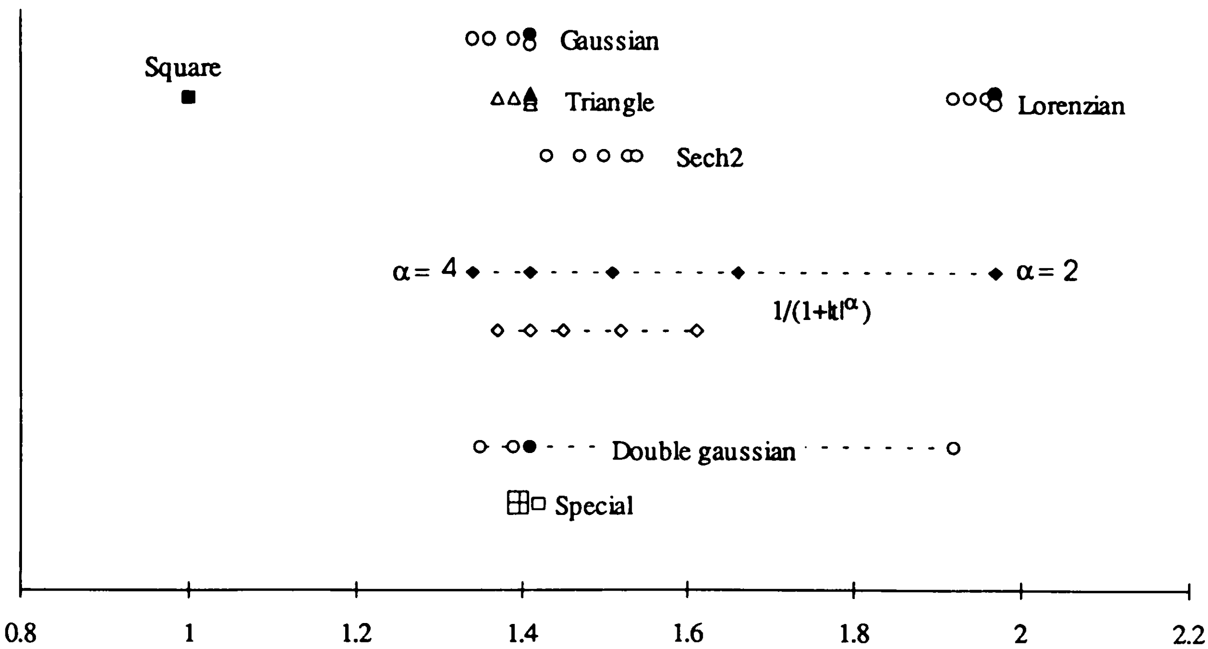

fig. 11: Value of the ratios between the FWHM of autocorrelation and pulses