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A BEAMSCOPE POSTPROCESSOR FOR PCs TO COMPUTE R.M.S. EMITTANCES OF THE PROJECTED DENSITY

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ABSTRACT

This note is a companion paper to the PS/HI Note 91-07 dealing with an alternative to the Abel transform relating amplitude distribution and projected density in form of a fit of polynomial distributions and its application to the conversion of emittance results from beamscope or flip targets to profile detector emittance measurements.

The note describes the algorithms used in the FORTRAN code implemented for PCs and, recently, in the real-time environment of the PSB beamscope processing software. Results of both methods are compared for the vertical plane. Agreement is satisfactory and justifies the application of the fitting method to measurements of radial emittances in presence of momentum dispersion where the Abel transform is not strictly meaningful.

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1. Introduction

Measuring transverse beam dimensions or emittances has gained importance over the last years with the operation of proton (or p-pbar, resp.) colliders where emittances determine the luminosity. Future proton colliders like LHC requiring very bright beams will impose even more stringent conditions on the beam emittances produced by the injector complex[1].

Devices measure emittances or beam dimensions can be divided into two groups according to the measured quantities.

(i) - Devices measuring betatron amplitude distributions.

Ex. Flip targets, Beamscope[2] in the PSB.

(ii) - Devices measuring the projected density (profile detectors).

Ex. SEM grids, flying wire, wire scanner, ionisation beam monitors.

We will look at devices of type (i) which measure the circulating beam current i(a) as a function of the betatron amplitude a. In this case the amplitude distribution F(a) has to be found by differentiation, which is done numerically in the computer code and electronically by Beamscope[2].

Existing computer codes use the numerical Abel transform together with a procedure to determine the beam centre[3,4] for calculation of the r.m.s. emittances of the projected density. Beamscope uses an automated procedure based on a tangent fitted to the small amplitude slope of the amplitude distribution, which should increase linearly for smooth phase space density around the origin. This procedure is useful for calculation of vertical emittances, where dispersion effects are vanishing. The contrary applies to the horizontal plane: here in many machines, e.g. PS and PSB, the lattice dispersion never vanishes and consequently one always deals with a two-dimensional amplitude distribution with contributions from both betatron amplitudes and momentum deviations.

Another code[5] uses analytical distributions to fit the real distribution function and obtains the r.m.s. emittances directly by fitting of the circulating current to avoid the noisy differentiation.This method gives good results especially for the horizontal emittances for which the beamscope procedure of beam center determination is not very precise. A pitfall of this method is that the best fitting distribution can be still different from the real one which leads to errors in computed emittances.

In view of this facts it appears desirable to have a computer code which allows to combine the possibilities of the two methods described above.

The present note describes a computer code for computing the r.m.s. emittances of the projected density based on the numerical Abel transform and fitting of the well known family of binomial distributions[6,7], which offers exactly the feature mentioned above.

2. Analytical Description of the Code

Let us consider the betatron oscillations of the particles in the beam like a set of harmonic oscillators.

The one-dimensional density distribution $\rho(x)$ produced by a set of harmonic oscillators, oscillating around x=0 with amplitudes given by an amplitude distribution F(a) is[8]:

$$\rho(x)= (1/\pi) \int_0^R F(a)\,(a^2 - x^2)^{-1/2}\,da \qquad (1)$$

where R denotes the upper limit of the amplitude $a^2= x^2 + y^2$ and x,y are dimensionless variables.

The inverse relation has the form:

$$F(a)= -2\,a \int_0^R \rho'(x)\,(x^2 - a^2)^{-1/2}\,dx$$

The beam current i(x) not intercepted by a target located at the distance x from the reference orbit is given by:

$$i(x)= \int_0^x F(a)\,da \qquad (2)$$

Using relation (1) and (2) we can easy obtain the following relation for the projected density:

$$\rho(x)= (1/\pi) \int_x^R i(a)\,(a^2 - x^2)^{-1/2}\,da$$

Transverse emittance is frequently defined by[9]:

$$\varepsilon_{x,y}= \sigma_{x,y}{}^2/\beta_{x,y} \qquad (3)$$

where $\beta_{x,y}$ is the lattice function and

$$\sigma_x{}^2= [\int_0^R x^2\,\rho(x)\,dx\,]\,[\int_0^R \rho(x)\,dx\,]^{-1}\ ;\ \sigma_y{}^2= [\int_0^R y^2\,\rho(y)\,dy\,]\,[\int_0^R \rho(y)\,dy\,]^{-1}\,. \qquad (4)$$

If we assume that our real two-dimensional density distribution has a binomial form:

$$F(a)= m\,(1-a^2)^{m-1}/\pi\,, \qquad (5)$$

we will have the following relations for the quantities i(x), $\rho(x)$ and $\sigma(x)$:

$$i(x) = 1 - (1 - x^2)^m \qquad (6)$$

$$\rho(x) = m\,(1 - x^2)^{m-1/2}\,\Gamma(m)\,[\pi^{1/2}\,\Gamma(m+1/2)]^{-1}$$

$$\sigma_x^2 = x_L^2/[2(m+1)] \qquad (7)$$

Here $\Gamma(m)$ denotes the gamma function, x_L the limiting amplitude and x is a dimensionless variable[7] $0 < x < 1$.
The same relations hold in the (y,y') phase plane.

Obviously if we select the analytical fitting curve $i_a(x)$ described by (6) in such a way that the differences between $i_a(x)$ and measured curve $i_d(x)$ will be a minimum, we will be immediately in position to calculate the transverse emittance. We remember that the precision of emittances obtained in such a way depends on the differences between the fitting distribution and the real one.

3. Numerical Treatment

a) Input Data

The BEAMSCOPE measurement is based on perturbing the reference orbit in the horizontal or vertical direction and scraping the beam at a fixed aperture, which is done during two consecutive machine cycles to both sides of the reference orbit (left-right in the horizontal direction and up - down in the vertical one). As a result of this procedure we have circulating beam currents i_{left}, i_{right} or i_{up}, i_{down} as a function of bump amplitude, which is equivalent to a function of position of a moving target. As mentioned BEAMSCOPE determines the beam centre location for both directions with respect to the reference orbit using an automated procedure based on a tangent fitted to the small amplitude slope of the amplitude distribution.

b) Calculation of the Reference Orbit Position

As was mentioned in the introduction the procedure of the beam center determination used by BEAMSCOPE gives results which are not precise for the horizontal measurements. In this case we use the procedure described below.

Doing the transformations

$$x_g(ngk-i+1) = x_{left}(i) - AP \qquad y_g(ngk-i+1) = i_{left}(i)$$

$$x_d(ndk-j+1) = AP - x_{right}(j) \qquad y_d(ndk-j+1) = i_{right}(j)$$

for every i=1 . . . ngk and j=1 . . . ndk we obtain the beam current as a function of the equivalent target position. Here AP is the half-aperture of the fixed scraper.

Performing the calculations

$$x_{gt}(i) = x_d(min)+[y_g(i)-y_d(min)]\,[x_d(max)-x_d(min)]\,[y_d(max)-y_d(min)]^{-1}$$

$$x_{sg} = \sum_{i=1}^{ngk} 0.5\,[x_g(i)+x_{gt}(i)] \qquad (8)$$

$$x_{dt}(i)= x_g(min)+[y_d(i)-y_g(min)]\ [x_g(max)-x_g(min)]\ [y_g(max)-y_g(min)]^{-1}$$

$$x_{sd}= \sum_{i=1}^{ndk} 0.5\ [x_d(i)+x_{dt}(i)]$$

then the calculated reference orbit position x_{of} will be:

$$x_{of}= (x_{sg}+x_{sd})/(ngk+ndk)$$

The Figure shows beam currents $y_g(i)$, $y_d(j)$ as a functions of the target positions $x_g(i)$, $x_d(j)$, respectively, and gives the geometric interpretation of calculations (8).

The beam centre location x_{oo} and reference orbit position are connected through the relation:

$$x_{oo}= x_{of}+AP$$

c) Calculation of the Amplitude and Projected Density Distributions

Knowing the reference orbit position and doing the transformations:

$$x_{gk}(i)= x_{of} - x_g(i-1) \qquad\qquad x_{dk}(j)= x_d(j-1) - x_{of}$$
$$y_{gk}(i)= y_g(i-1) \qquad\qquad y_{dk}(j)= y_d(j-1)$$

we will be in position to calculate the amplitude distribution $F(a)= F(a_k)= F_a(k)$ given by (2).

For this purpose we calculate the functions $y_{dk}(k)$ and $y_{gk}(k)$ ($k=1 \dots n$) at equidistant points $x(k)$ and take their derivatives using a three point Lagrange interpolation[5].

The next step is to calculate the projected density distribution given by (1).

Let the function $F(a)$ be linear between points a_k and a_{k+1}, for every $k=1 \dots n-1$:

$F(a)= C_k + D_k\, a$, $\qquad a_k < a < a_{k+1}$

Then the integral (1) will have the form:

$$\rho(x_k)= (1/\pi)\ [C_k \int_{x_k}^{R} (a^2 - x^2)^{-1/2}\, da + D_k \int_{x_k}^{R} a\ (a^2 - x^2)^{-1/2}\, da]$$

Dividing the interval $(x_k;R)$ into the intervals $(x_k;x_{k+1})$, $(x_{k+1};x_{k+2})$ .... $(x_{n-1};x_n=R)$ and using Abel transforms tabulated in Ref.[8], where:

$$\int_{x_k}^{x_{k+1}} (a^2 - x^2)^{-1/2}\, da= \mathrm{arccosh}(x_{k+1}/x_k)$$

$$\int_{x_k}^{x_{k+1}} a\,(a^2 - x^2)^{-1/2}\, da = [(x_{k+1})^2 - (x_k)^2]^{-1/2} ,$$

we obtain the following formulae for the projected density:

$$\rho(x_k) = (1/\pi) \sum_{\ell=k}^{n} \{C_\ell \operatorname{arccosh}(x_{\ell+1}/x_\ell) + D_\ell [(x_{\ell+1})^2 - (x_\ell)^2]^{-1/2}\}$$

This formulae is valid for all k= 2 . . . n exept k=1 where x_k=0. In this case the above formulae reduces to:

$$\rho(x_1=0) = (1/\pi) \{D_1 x_2 + \sum_{\ell=2}^{n} [C_\ell \ln(x_{\ell+1}/x_\ell) + D_\ell (x_{\ell+1} - x_\ell)]\}$$

For the calculation of the coefficients C_k and D_k we use the simple relations that connect these coefficients with the function F(a) in the points a_k and a_{k+1}, for every k.

d) Calculation of the Sigmas of the Projected Density and r.m.s. Emittances.

Using a trapezoidal aproximation for the numerical calculation of the integrals (4) we obtain the following formulas for the sigmas of the projected density:

$$\sigma_x = \{\sum_{i=2}^{n} [\rho(x_i)\, x_i^2 + \rho(x_{i-1})\, x^2_{i-1}]\, [\rho(x_i) + \rho(x_{i-1})]^{-1}\}^{1/2} ,$$

$$\sigma_y = \{\sum_{i=2}^{n} [\rho(y_i)\, y_i^2 + \rho(y_{i-1})\, y^2_{i-1}]\, [\rho(y_i) + \rho(y_{i-1})]^{-1}\}^{1/2} .$$

Then the r.m.s. emittances of the beam are determined by (3).

e) Fitting Procedure.

Assuming as in chapter 2 that the real two-dimensional density distribution is close to the binomial (5), we have to find the parameters m and x_L for which the diference between the analytical fitted curve $i_a(x)$ described by (6) and the measured curve $i_d(x)$ will be a minimum.

Let us introduce the following functions:

$$S(k,j) = S_+(j) + S_-(j), \qquad S_+(j) = \sum_{i=1}^{n} C_{ad}(i)\, [i_d(x_i) - i_a(x_i)]$$

$$S_-(j)= \sum_{i=1}^{n} C_{da}(i)\ [i_d(x_i) - i_a(x_i)]\ ,$$

where the coefficients $C_{ad}(i)$ and $C_{da}(i)$ are defined by:

$C_{ad}(i)=1$ $\quad$ $C_{da}(i)=0$ for $i_d(x_i)>i_a(x_i)$, and $C_{ad}(i)=0$ $\quad$ $C_{da}(i)=1$ for $i_d(x_i)<i_a(x_i)$

and $H_s(k)= S_+(k,j_{fix}) + |S_-(k,j_{fix})|$ where j_{fix} is determined by $S(k,j_{fix})=0$ for every k.

Then the procedure of obtaining the best fitting m and x_L consists in looking for a minimum of the function $H_s(k)$. It is then easy to obtain the sigmas of the projected distribution and the r.m.s. emittances of the beam from these parameters m and x_L using relations (3) and(7). This procedure is described in detail in the note [5].

We have to note that for the vertical phase plane this procedure reduces to looking for the optimal m in (5).In this case the parameter x_L is calculated using the beam centre location obtained by BEAMSCOPE.

4. Calculation Possibilities.

The present postprocessor calculates:

- amplitude distribution
- projected density distribution
- sigma of the projected density
- r.m.s. emittance of the beam

using the numerical Abel transform and fitting the real density distribution with a binomial one to obtain the above values.

The precision of the results obtained in this way strongly depends upon the precision of the procedure for the beam centre determination.

We have three ways to determine the location of beam centre:

(i) - from BEAMSCOPE.

(ii) - with the procedure described in chapter 3(b).

(iii) - with the fitting procedure of chapter 3(e).

The choice of one of these methods depends upon the plane of measurement.

Table shows this choice as a function of the method used for calculation of the sigmas of both the projected densitese and the plane of betatron motion.

Table

vertical betatron motion		horizontal betatron motion	
numerical Abel transform	fitting procedure	numerical Abel transform	fitting procedure
(i)	(i)	(i)	(iii)

5. Results.

In the appendix results of the calculation of the r.m.s. PS Booster emittances are shown.

One can see a good agreement between results obtained by numerical Abel transform and fitting procedure. The little difference of the results, especially for the vertical betatron motion, is a consequence of the numerical transform errors or of the fact that our real distribution density is not exactly binomial. Obviosly one can succesfully use any of this two methods to obtain the vertical r.m.s. emittance of the beam.

For the horizontal betatron motion we prefer to use the fitting procedure, where dispersion effects have not so much influence on the results as it is the case for the numerical Abel transform.

References:

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2. H. Schönauer, CERN PS/BR/82-8 and 1979 PAC San Francisco.
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5. E. Ivanov and H. Schönauer, CERN PS/HI/Note 91-7.
6. C. Bovet, CERN SI/Note DL/71-13.
7. W. Joho, Representation of Beam Ellipses for Transport Calculations, SIN-Report TM-11-14(1980).
8. P. Krempl, MPS/Int. BR/74-1.
9. Y. Baconnier, CERN/PS 87-89(PSR).

Appendix

We use the following abbreviations:
HOR;VER - horizontal or vertical betatron motion.
Qh,v - horizontal and vertical betatron tunes.
B - average magnetic field.
signa-a;sigma-f - σ's of the projected density obtained by the numerical Abel transform and the fitting procedure, respectively.
e/nce-a;e/nce-f - r.m.s. emittances obtained by the same methods.
bc-a;bc-f - Beamscope bump amplitude corresponding to beam centre, measured and from fitting procedure, respectively.
XL; m - fitting parameters in (6).
U and D - up and down measurements.
M - results obtained for the current which is averaged between i_{left} and i_{right}.

Input file : hos.dat

BSC TEST 13/ 5/1991 18:14:22 ser. 964 ring-3 Qh,v=4.160,5.230 B=6860[G] VER

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
102.324	2.42	2.23	03.93	03.35	31.41	31.41	17.13	28.45	U
099.121	2.37	2.41	03.77	03.91	25.76	25.76	14.19	16.27	D

BSC TEST 13/ 5/1991 18:30:58 ser. 968 ring-3 Qh,v=4.160,5.230 B=6860[G] HOR

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
097.995	4.09	3.54	09.04	06.79	52.89	52.31	28.83	32.10	M

BSC TEST 13/ 5/1991 18:41:56 ser. 976 ring-3 Qh,v=4.160,5.230 B=6860[G] HOR

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
092.840	3.76	3.95	07.64	08.42	49.19	49.50	26.61	21.75	M

BSC TEST 13/ 5/1991 18:43:26 ser. 977 ring-3 Qh,v=4.160,5.230 B=6860[G] HOR

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
094.961	4.13	4.02	09.25	08.73	49.52	49.56	26.49	20.73	M

BSC TEST 13/ 5/1991 18:51: 7 ser. 978 ring-3 Qh,v=4.160,5.230 B=6860[G] VER

INTEGR.	sigma-n	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[ma]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
099.270	2.22	2.27	03.31	03.45	31.27	31.27	17.03	27.21	U
099.946	2.38	2.40	03.81	03.87	25.72	25.72	14.16	16.40	D

BSC TEST 13/ 5/1991 18:52:27 ser. 980 ring-3 Qh,v=4.160,5.230 B=6860[G] VER

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
099.870	2.36	2.21	03.74	03.29	31.28	31.28	17.03	28.62	U
099.034	2.46	2.42	04.08	03.94	25.75	25.75	14.28	16.38	D

BSC TEST 27/ 5/1991 18:10:42 ser. 280 ring-3 Qh,v=4.266,5.278 B=1300[G] HOR

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
099.681	8.02	7.75	35.48	33.08	50.41	50.87	44.19	15.26	M

BSC TEST 27/ 5/1991 18:15:53 ser. 283 ring-3 Qh,v=4.266,5.278 B=1300[G] VER

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
098.016	5.59	5.27	20.91	18.62	30.56	30.56	32.42	17.91	U
099.299	5.68	5.62	21.61	21.15	25.88	25.88	28.03	11.44	D

BSC TEST 21/ 6/1991 17: 7:14 ser. 440 ring-3 Qh,v=4.250,5.230 B=1300[G] HOR

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
098.817	7.74	7.88	32.95	34.14	49.06	49.39	46.29	16.25	M

BSC TEST 21/ 6/1991 17:11:26 ser. 444 ring-3 Qh,v=4.250,5.230 B=1300[G] VER

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
101.059	5.93	5.86	23.60	23.04	31.37	31.37	33.83	15.68	U
102.476	6.18	6.09	25.68	24.87	25.96	25.96	28.31	09.82	D

BSC TEST 21/ 6/1991 17:21:43 ser. 993 ring-4 Qh,v=4.160,5.230 B=6860[G] VER

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
100.727	4.41	4.49	13.05	13.52	31.82	31.82	20.31	09.25	U
099.248	4.41	4.51	13.09	13.66	25.47	25.47	16.63	05.81	D

BSC TEST 21/ 6/1991 18:18:27 ser. 992 ring-4 Qh,v=4.160,5.230 B=6860[G] HOR

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
099.193	6.80	6.49	25.02	22.78	56.54	56.08	26.96	07.63	M

BSC TEST 16/ 7/1991 21: 7:30 ser. 487 ring-3 Qh,v=4.270,5.290 B=1260[G] VER

INTEGR.	sigma-a	sigma-f	e/nce-a	e/nce-f	bc-a	bc-f	XL	m .	
--------	[mm]	[mm]	[mm.mrad]	[mm.mrad]	[mm]	[mm]	[mm]	------- .	
100.146	6.55	6.13	28.69	25.13	31.45	31.45	34.80	15.13	U
097.880	6.56	6.17	28.78	25.47	25.66	25.66	28.97	10.03	D

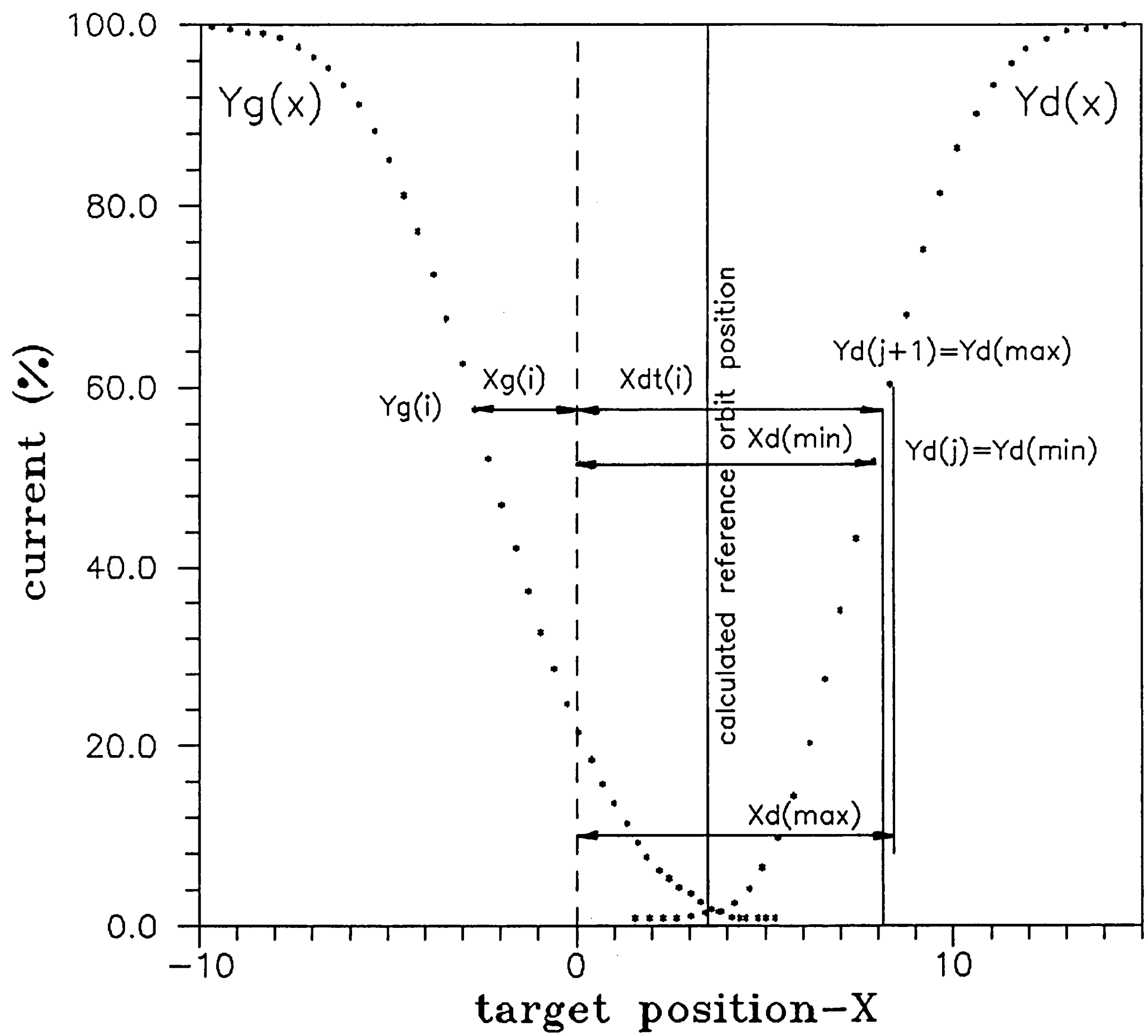

Fig. Currents Yg(i) and Yd(j) vs. target position.