PS/HI Note 91-07
September 1991

FITTING OF POLYNOMIAL DISTRIBUTIONS TO TARGET (OR BEAMSCOPE) MEASUREMENTS TO OBTAIN THE R.M.S. EMITTANCE OF THE PROJECTED DENSITY

E. Ivanov*) and H. Schönauer

ABSTRACT

In order to compare emittances measured with flip targets or Beamscope with those obtained from Profile detectors (wire scaners, SEM grids) one can use the Abel transform only if there is no dispersion folded in and if the beam centre can be determined accurately. Fitting a polynomial distribution to the measured data allows reconstruction of the beam centre, and computation of R.M. S. emittances of the projected density by simple analytic formulae.

*) Institute for Nuclear Research and Nuclear Energy, Sofia, Bulgaria

1. Introduction

Measuring transverse beam dimensions or emittances has gained importance over the last years with the operation of proton (or p, resp.) colliders where emittances determine the luminosity. Future proton colliders like LHC requiring very bright beams will impose even more stringent conditions on the beam emittances produced by the injector complex [1].

As the occasions for beam blow-up are multiple, monitoring and comparing emittances in the different machines involved in the LHC injector becomes crucial for commissioning and trouble-shooting.

Devices to measure emittances or beam dimensions can be divided into two groups according to the measured quantities:

(i) Devices measuring betatron amplitude distributions

Ex.: Flip targets, Beamscope [2] in the PSB

(ii) Devices measuring the projected density (profile detctors)

Ex.: SEM grids, flying wire, wire scanner, ionisation beam monitors

Accordingly, emittances quoted are defined differently:

Devices of group (i) define dimensions containing 95 % of the beam, say, while group (ii) devices quote one or two sigmas (r.m.s.) of the *projected density* distribution, FWHH's, tangent footwidths etc.

Obviously, measured emittances of different definition should be compared with care and, if possible, transformed to a common type. Of course, for a rotationally symmetric phase space population, amplitude distribution and projected density are linked through a well-known formal relationship, the Abel transform [3].

For analytically defined distributions, the transform can in many cases be executed analytically, and conversion is particularly simple for Gaussian beams. As a result, emittance measurements of different type are compared as if the beams were indeed Gaussian, a fairly misleading practice for e.g. collimated beams.

In order to compare the emittances of type (i) and type (ii) measurements one needs to measure the complete amplitude or projected density distribution, respectively, and subsequently perform the Abel transform numerically. Computer codes for this task exist for a while [4],[5], but are rarely used in practice.

Note that type (i) devices measure the circulating beam current i(a) as a function of the betatron amplitude a and the amplitude distribution F(a) has to be found by differentiation. This is done numerically in the computer program [4] and electronically by Beamscope [2].

Both methods introduce some errors by the mandatory filtering of the noise generated in the differentiation process, which add to the errors inherent in target or Beamscope measurement.

Nonetheless these transforms are useful in comparing measurements of vertical emittances, where dispersion effects are vanishing in practically all machines. The contrary applies to the horizontal plane; here in many machines, e.g. PS and PSB, the lattice dispersion never vanishes and consequently one always deals with a two-dimensional amplitude distribution with contributions from both betatron amplitudes and momentum deviations. Unless one is able to disentangle the distributions - which is

theoretically possible by independent measurements at locations of different lattice dispersion, but far too complicated in operation - straightforward application of the Abel transform is strictly speaking meaningless and can only be considered as a first approximation.

Even more problematic is the determination of the beam centre as particles of different momentum turn on different closed orbits. Any error in this procedure entails even more severe errors in the r.m.s. emittances of the transform. Beamscope uses an automated procedure based on a tangent fitted to the small amplitude slope of the amplitude distribution, which should increase linearly for smooth phase space density around the origin. While this works well in the vertical plane, the fit has to be done manually on the screen display in the horizontal plane for quantitative measurements and even then it is not always evident. It should be noted that for straightforward measurement of the beam radii containing 95% of the beam particles the beam centre is not required: it is derived from a measurement of the "diameter" of the beam by consecutive scraping from both sides.

Instruments of type (ii) have other problems, like the determination of the baseline and noise in the tails rendering this way the computation of second moments unreliable and sometimes systematically wrong. A frequently used way out is fitting a Gaussian through the data measured and computing the variance analytically.

Again the error is the more signifcant the more the true distribution deviates from a Gaussian.

In view of this facts it appears desirable to have a method at hand that fits a more flexible analytical distribution and yields the beam centre as a byproduct, if possible directly from the beam current i(a) to avoid the noisy differentiation.

The method described in the following sections offers exactly the features mentioned above. The class of analytical distributions fitted is the well-known family of "Binomial Distributions" employed by a number of authors, e.g. [6],[7] and described in the following section. The choise of this family of distributions is determined by:

- it has two fitting parameters m, x_L and finite range (x_L), which allow us to obtain the location of the beam centre and sigmas of the projected distributions directly from the beam current.
- The waterbag model (uniform density in phase space) and Gaussian distributions are limiting cases of the binomial distributions (m=0 and m= respectively).
- they include the Kapchinsky-Vladimirsky and the frequently used parabolic distribution.
- there are relatively simple analytical relations for the projected distribution densities and sigmas of the beam.

2.Theoretical analysis

Let assume that our experimental distribution density has a binomial form in each phase plane. In this case the amplitude function will have the form:

$$F(a)= m\ (1-a^2)^{m-1}/\pi \qquad (1)$$

where m= .5 , 1 , 1.5 , 2 ..... and $a= (u^2+v^2)^{1/2}$.

The dimensionless variables u and v are connected to phase space variables x and x' with relations:

$$x= x_L\, u$$
$$x'= x'_L\, a \sin(\phi+\chi)$$

where x_L and x'_L are limiting amplitude and divergence, respectively.

Consequently the corresponding function $i_a(u)$ describing beam current as a function of the target position will have the following form:

$$i_a(u)= 2\pi \int_0^u a\, F(a)\, da= 2m \int_0^u a(1-a^2)^{m-1}\, da= 1-(1-u^2)^m \qquad (1)$$

The projected distribution function (onto the u-axis) g(u) is obtained by integrating $F[a= (u^2+v^2)^{1/2}]$ over all slopes v:

$$g(u)= \int_{-(1-u^2)^{1/2}}^{(1-u^2)^{1/2}} F[a= (u^2+v^2)^{1/2}]\, da= (2m/\pi) \int_0^{(1-u^2)^{1/2}} (1-u^2-v^2)^{m-1}\, dv$$

With the substitution $v= (1-u^2)^{1/2} \sin\alpha$ we find:

$$g(u)=(2m/\pi)\,(1-u^2)^{m-1/2} \int_0^{\pi/2} (\cos\alpha)^{2m-1}\, d\alpha= m\,(1-u^2)^{m-1/2}\,\Gamma(m)\,[\pi^{1/2}\,\Gamma(m+1/2)]^{-1}$$

where $\Gamma(m)$ is the gamma function [8].

One can easily check that g(u) and $i_a(u)$ are normalized to 1 by solving the integrals $(2\int_0^1 g(u)\, du)$ and $(2\pi \int_0^1 a\, F(a)\, da)$, respectively.

From the definition of σ of the projected distribution density we have:

$$\sigma^2= x_L^2 \int_{-1}^{1} u^2\, g(u)\, du= 2m\, x_L^2\, \Gamma(m)\,[\pi^{1/2}\,\Gamma(m+1/2)]^{-1} \int_0^1 u^2\,(1-u^2)^{m-1/2}\, du=$$
$$=x_L^2/[2(m+1)] \qquad (2)$$

Obviously, if we select the analytical fitting curve i_a described by (1), in such a way that differences between i_a and the experimental measured curve i_d will be a minimum, we will immediately obtain the beam parameters x_L and m. On top of that one knows to which precision the real beam is approximated by the fitted binomial distribution.

The conversion from amplitudes and projected density is then reduce to the selection of the fitting parameters m and x_L in (1).

3. Selection of m and x_L: Procedure.

As a result from beam size measurements we have the normalized circulating current as a function of the distance x_e (see Fig.1 where is shown a simple scheme of the measurement).Making the substitutions:

$$nx(n_d - \ell + 1) = x_e(\ell) \quad ; \quad x^*(\ell) = nx(1) - nx(\ell) \text{ for every } \ell \text{ from 1 to } n_d ,$$
where n_d is number of measured points. (3)

we obtain beam losses current i_d as a function of the intercepting target position (see i_d curve in Fig.3).

Let us write once again expression (1) in the form:

$$i_a(i) = 1 - (1-c^2)^{j/2} \qquad (i=1,2.....n_a \; ; \; j= 2\,m=1,2.....)$$
where: $c = [x(i) - (k-1)\,dx]/x_L(k) < 1$; $x_L(k) = -(k-1)\,dx + x_L^{in}$; $dx = x_L^{in}/(n_a-1)$; $k=1,2.....$; x_L^{in} is the initial limiting amplitude (see Fig.4) and j and k are numbers of iteration. (4)

We are now going to compare functions i_a and i_d at points x(i) (i=1,2.....n_a) equidistant one to another and not equivalent with measured points $x^*(\ell)$ (ℓ=1,2.....n_d). Taking into account that function i_d is defined in points $x^*(\ell)$ and performing a three points Lagrange interpolation we obtain:

$$\begin{aligned} i_d(i) = {} & [x(i)-x^*(\ell)][x(i)-x^*(\ell+1)]\{[x^*(\ell-1)-x^*(\ell)][x^*(\ell-1)-x^*(\ell+1)]\}^{-1} i_d[x^*(\ell-1)] + \\ & + [x(i)-x^*(\ell-1)][x(i)-x^*(\ell+1)]\{[x^*(\ell)-x^*(\ell-1)][x^*(\ell)-x^*(\ell-1)]\}^{-1} i_d[x^*(\ell)] + \\ & + [x(i)-x^*(\ell-1)][x(i)-x^*(\ell)]\{[x^*(\ell+1)-x^*(\ell-1)][x^*(\ell+1)-x^*(\ell)]\}^{-1} i_d[x^*(\ell+1)] \end{aligned}$$
where $x^*(\ell-1) < x(i) < x^*(\ell+1)$.

Repeating this procedure for every i we will find the function i_d in all points x(i).

Note that n_a is usualy choosen between 3 and 10 n_d because the precision of the comparison between i_d and i_a depends on n_a.

Let us introduce the following functions:

$$S(k,j) = S_+(j) + S_-(j) \qquad S_+(k,j) = \sum_{i=1}^{n_a} C_{ad}(i)\,[i_d(i) - i_a(i)]$$

$$S_-(k,j)= \sum_{i=1}^{n_s} C_{da}(i)\,[\dot{i}_d(i) - \dot{i}_s(i)]$$

where the coefficients $C_{ad}(i)$ and $C_{da}(i)$ are defined by:

$C_{ad}(i)=1$ $\qquad$ $C_{ad}(i)=0$

$C_{da}(i)=0$ when $\dot{i}_d(i) > \dot{i}_s(i)$ and $C_{da}(i)=1$ when $\dot{i}_d(i) < \dot{i}_s(i)$

and $H_s(k)= S_+(k,j_{fix}) + |S_-(k,j_{fix})|$ where j_{fix} is determined by $S(k,j_{fix})=0$, for every k.

The procedure of obtaining the best fitting m and x_L consists in looking for a minimum of the $H_s(k)$ function. Because this function is defined only when S(k,j)=0 we have to find this j for which the above condition is realized.

Starting with j=1 (m=.5) we are looking for that j=j_- for which S(k,j) just turned negative.Then j_{fix} is determined by the expression

$$j_{fix}= [-B + (B^2 - 4\,A\,C)^{1/2}]/(2\,A),$$

where: $A= 2\,S(k,j_- - 1) - S(k,j_- - 2) - S(k,j_-)$,

$B= (2\,j_- - 1)\,S(k,j_- - 2) + 4\,(1 - j_-)\,S(k,j_- - 1) + (2\,j_- - 3)\,S(k,j_-)$,

$C=j_-\,(1 - j_-)\,S(k,j_- - 2) + 2\,j_-\,(j_- - 1)\,S(k,j_- - 1) + (3\,j_- - j_-^2 - 2)$.

The typical behaviour of the function S(k,j) for fixed k is shown in Fig.2a.

Repeating this procedure for k=1,2, etc. we are looking for an extremum of the function $H_s(k)$ [see Fig.2b where Hs is shown as a function of x_L defined by (4)].

Assume that for $k=k_+$ we have $H_s(k_+) > H_s(k_+-1)$. Then using $[H_s(k)]'=0$ and a three point Lagrange interpolation we obtain:

$$k_{fix}= [(1-2\,k_+)\,H_s(k_+-2) + 4\,(k_+-1)\,H_s(k_+-1) + (3-2\,k_+)\,H_s(k_+)] *$$
$$* [2\,H_s(k_+-2) + 4\,H_s(k_+-1) - 2\,H_s(k_+)]^{-1}$$

where k_{fix} is the value of k for which $[H_s(k)]'=0$.

The procedure of fitting m and x_L is shown graphically in Fig.3 and Fig.4. They also illustrate the behaviour of the functions S(k,j) and $H_s(k)$ given by Fig.2a and Fig.2b.

From (3) and (4) we obtain for the fitting m, x_L and σ,

$$m= j_{fix}/2 \;\; ; \;\; x_L= x_L^{in} - (k_{fix}-1)\,dx \;\; ; \;\; \sigma= x_L/[2(m+1)]^{1/2}$$

Taking into account relations (3) for the location of the beam centre we obtain.

$$(x_e)_{centre}= nx(1) - (k_{fix}-1)\,dx$$

4. Results

The formulae and the procedure described in the preceding analysis were used for creating a computer code which calculates the quantities:

- fitting m and x_L parameters in expression (4)
- σ of the projected density
- beam centre location and relative error between i_d and i_a curves for comparison into n_a points.

Fig.5 and Fig.6 show experimental and fitting curves for the vertical and horizontal phase plane, respectively. One can see a good coincidence between them.This indicates that the real beam density distribution has approximately a binomial form, which allows us to obtain the σ of the projected density and the location of the beam center with sufficient accuracy.

Finally the Table presents the results from processing of experimental data measured at the PSB. Here σ_f and σ_a denote the σ's of the projected density obtained by fitting procedure and numerical Abel transform, respectively, b.c. and $(b.c.)_f$ are the beam centre locations as determined by Beamscope and by fitting, respectively, and the limiting amplitude x_L and the distribution parameter m are defined by the expression (4). Columns 3 to 5 of the table show the results of the fit if the beam centre is taken from the Beamscope measurement , while columns 6 and 7 display the beam centre reconstructed in the fit and the corresponding r.m.s beam radius σ_f.

One can see a fairly good agreement between the results obtained with both methods, which inspires some confidence into the use of the fitting procedure for the horizontal betatron motion, where dispersion effects spoil the Abel transform method and the beam centre found by Beamscope is not very accurate.

5. References:

1. R. Cappi et al., CERN PS/91/07 and 1991 PAC San Francisco.
2. H.Schönauer, CERN PS/BR/82-8 and 1979 PAC San Francisco.
3. P.Krempl, MPS/Int. BR/74-1
4. P.Krempl, MPS/BR Note 74-16
5. M. Martini, CERN/PS/OP 82-19
6. C. Bovet, CERN SI/Note DL/71-13
7. W.Joho, *Representation of Beam Ellipses for Transport Calculations*, SIN-Report TM-11-14 (1980)
8. M. Abramowich and I. A. Stegun, *Pocketbook of Mathematical Functions*, Verlag Harri Deutsch - Thun - Frankfurt/Main, 1984

Table

Comparison of fitting procedure with Abel transform
for the vertical betatron motion

(b.c.)	σ_a	σ_f	m	x_L	$(b.c.)_f$	σ_f	Polarity
2.67	2.20	2.27	27.21	17.03	2.33	2.0	up
2.88	2.38	2.40	16.40	14.16	3.23	2.12	down
2.68	2.36	2.21	28.62	17.03	2.41	2.0	up
2.86	2.46	2.42	16.38	14.28	3.18	2.16	down
2.73	5.93	5.86	15.68	33.83	1.97	5.23	up
2.65	6.18	6.09	9.82	28.31	3.23	5.63	down
1.96	5.59	5.27	17.91	32.42	2.13	5.36	up
2.72	5.68	5.62	11.44	28.03	2.97	5.37	down
3.22	4.43	4.49	09.25	20.31	2.39	3.84	up
3.13	4.41	4.51	05.81	16.63	3.96	3.87	down

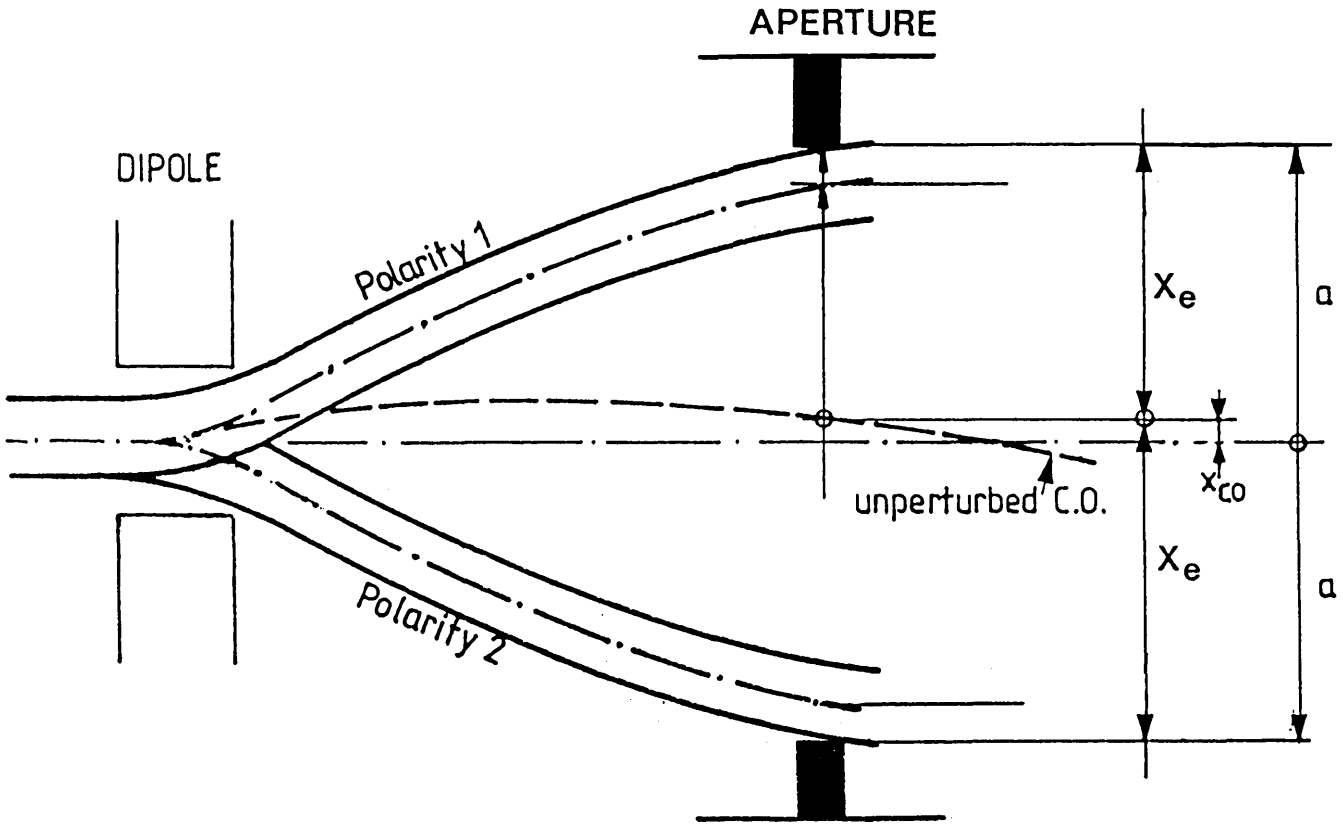

Fig.1 Simple scheme of the measurments

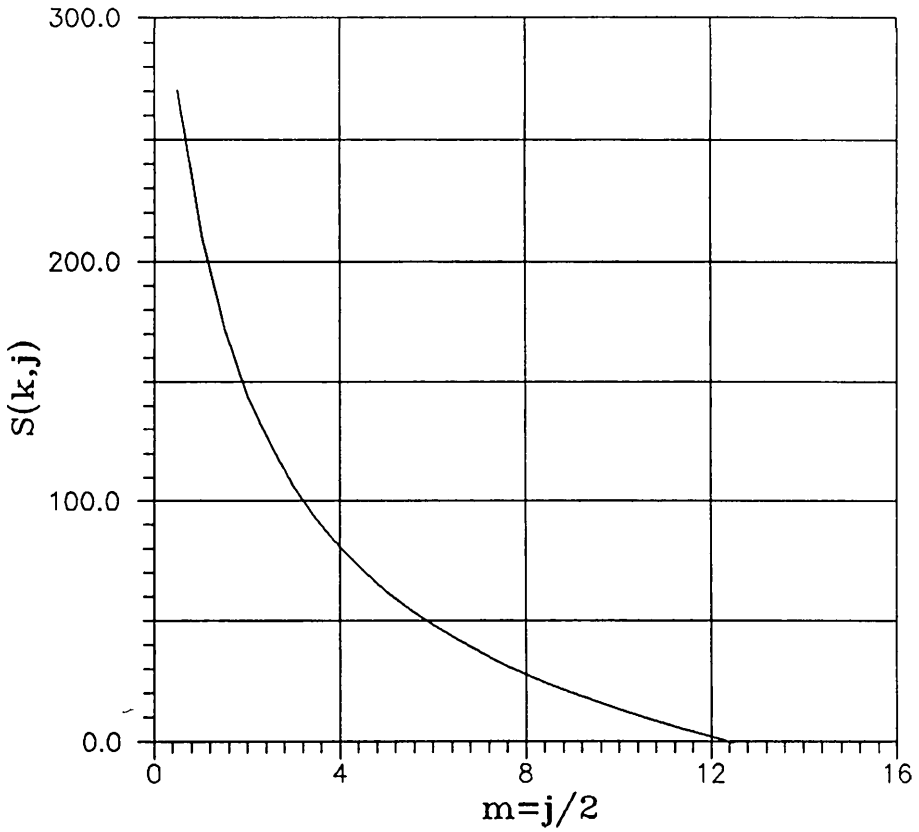

Fig.2a Function S(k,j) vs. parameter m in the distribution function.

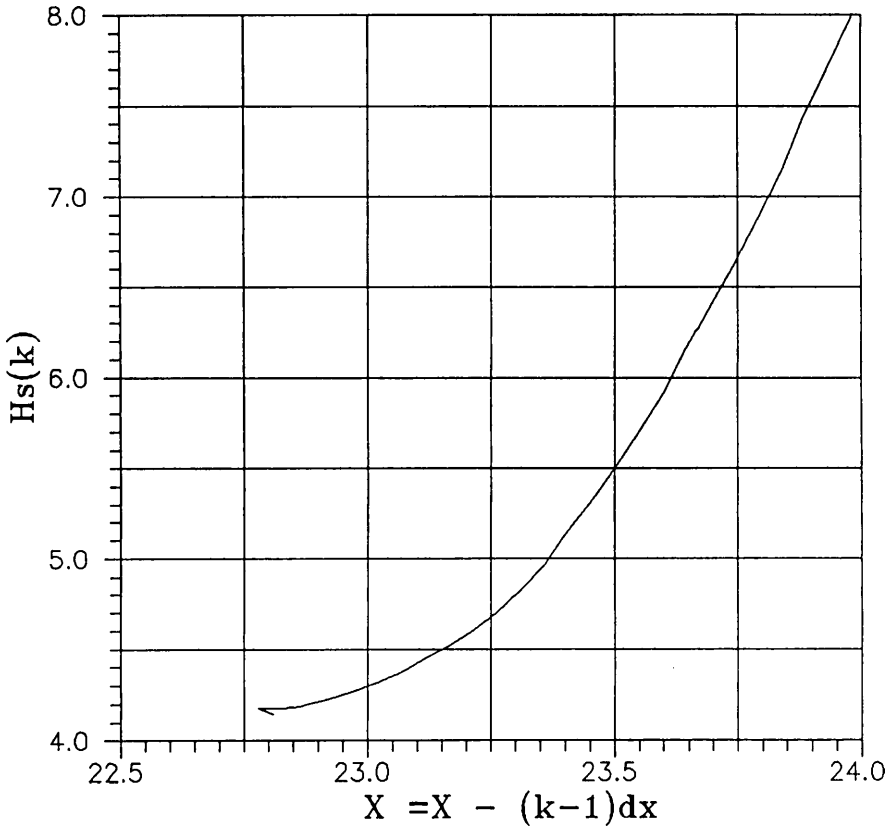

Fig.2b Function Hs(k) vs. limiting amplitude X .

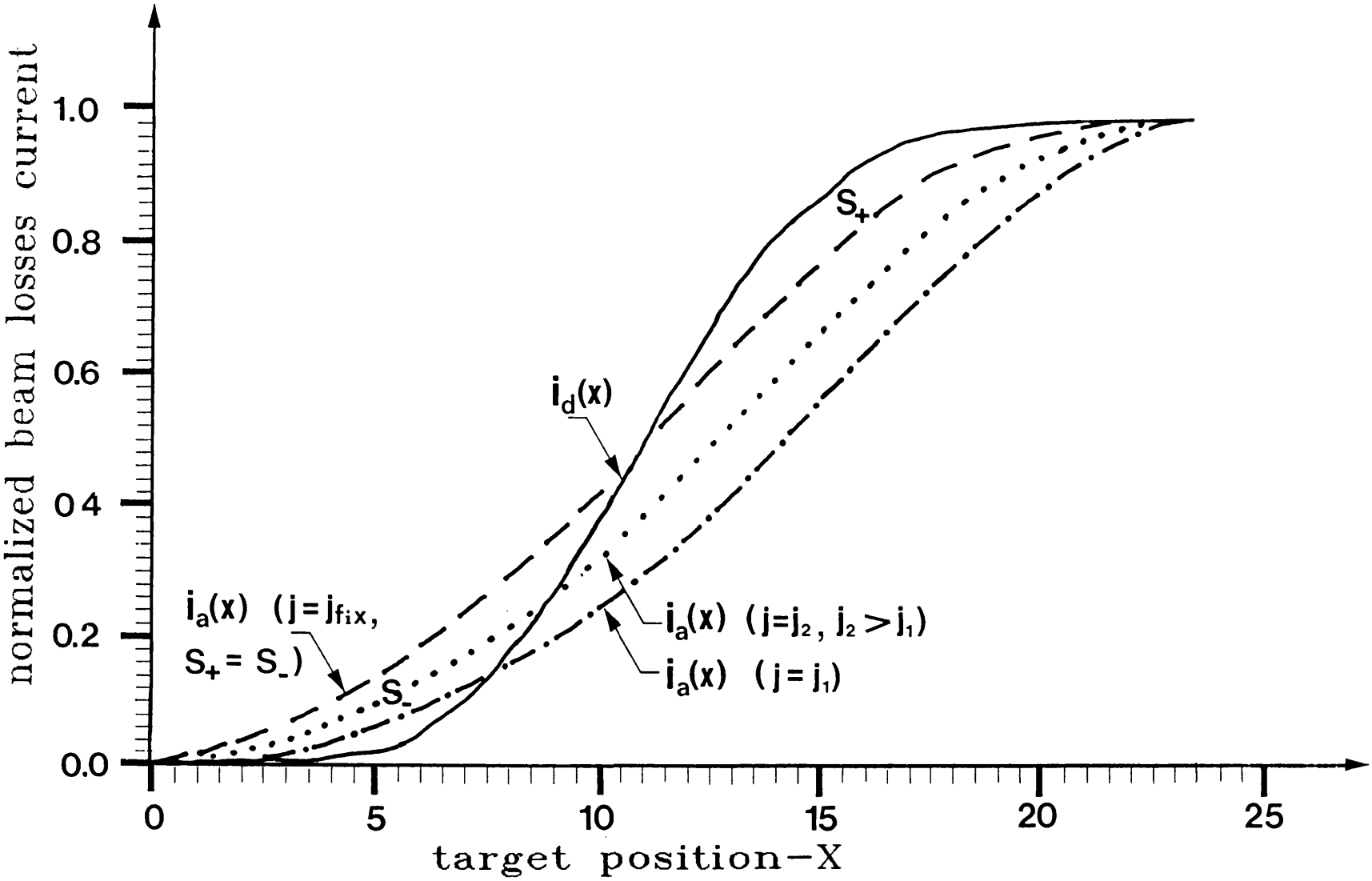

Fig.3 Normalized beam losses current vs. target position.Curves $i_a(x)$ show behaviour of $S=S_+ + S_-$ as a function of j.

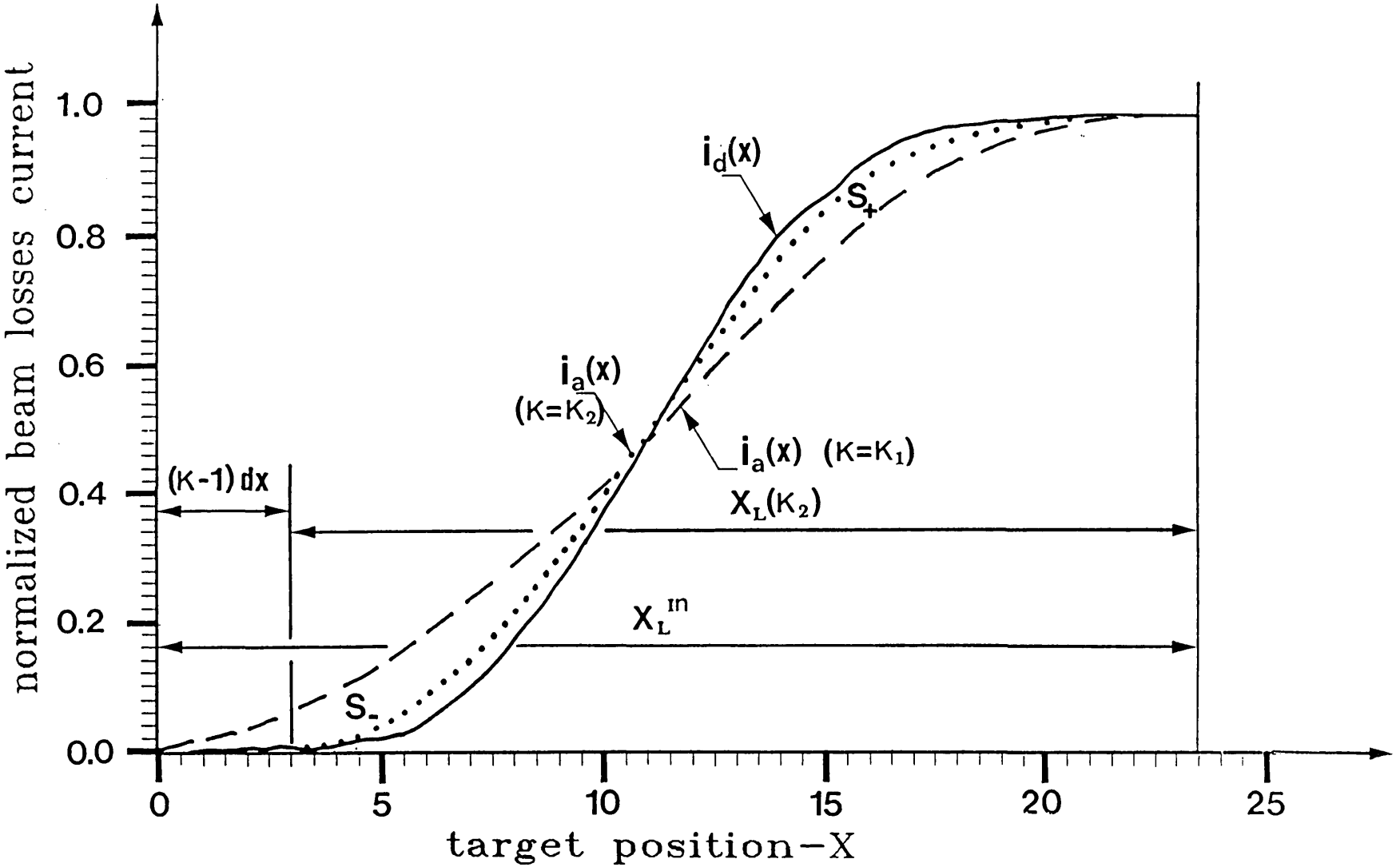

Fig.4 Normalized beam losses current vs. target position.Curves $i_a(x)$ show behaviour of $Hs=S_+ + |S_-|$ as a function of k.

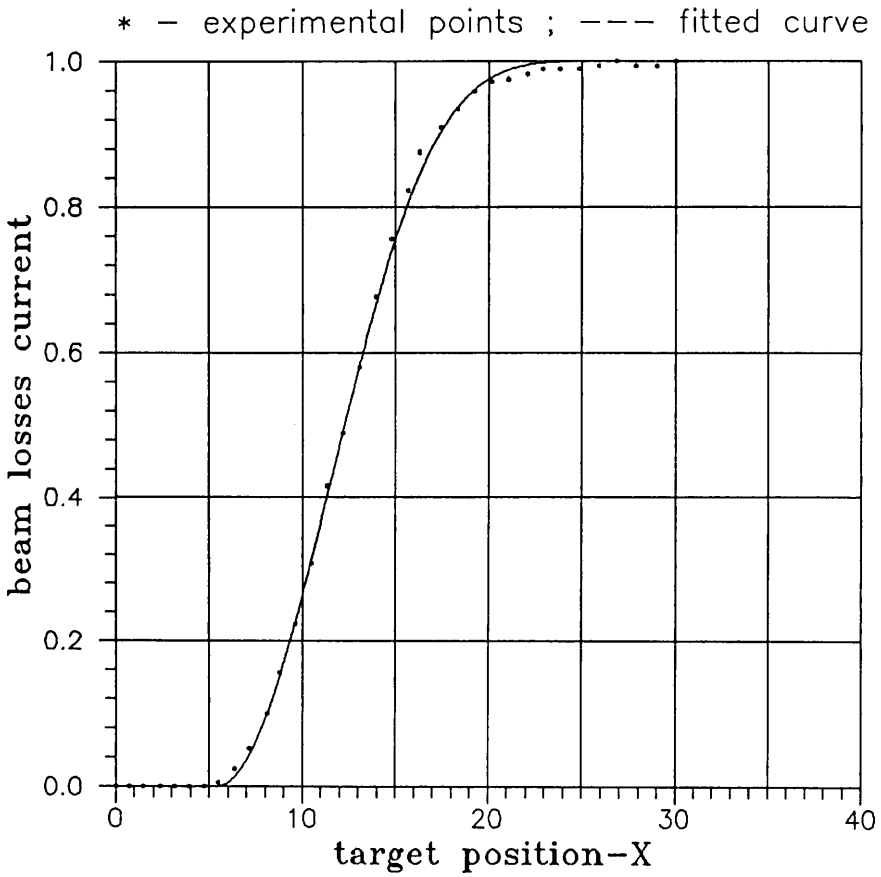

Fig.5 Normalized beam losses current vs. target position (vertical phase plane) m=8.52,X =24.67, =5.65mm

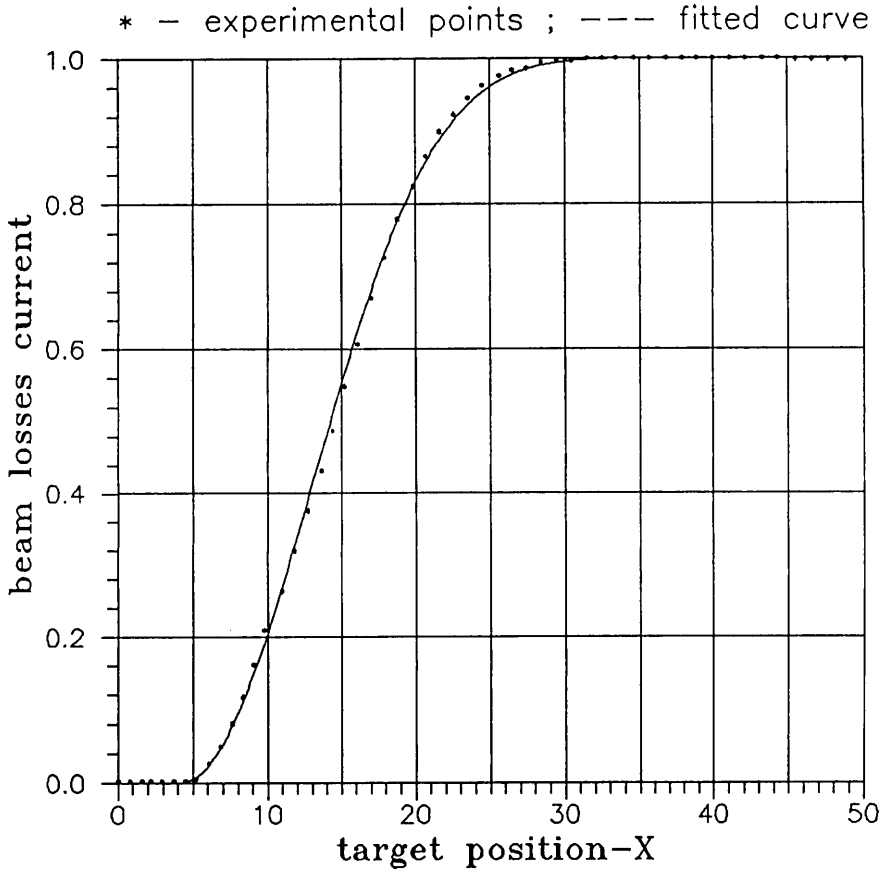

Fig.6 Normalized beam losses current vs. target position (horizontal phase plane) m=13.18,X =44.76, =8.41mm.