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SECOND ORDER TUNE SHIFT IN A COMPENSATED SUPER CELL

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1. INTRODUCTION

In most modern storage rings, the dominant non linear fields limiting the single particle stability come from chromaticity correction sextupoles. In order to minimize the reduction of acceptance due to these non linearities, the phase advance per FODO cell in the arcs where the sextupoles are placed is chosen to be a rational fraction of 2π whenever this is possible and $\pi/2$ or $\pi/3$ are the favourite values. However it has been recently proposed for HERA [1,2] many phase advances between 60 and 90 degrees

$$\mu = \frac{2k+1}{n} 2\pi \qquad (1)$$

The repetitive element of the lattice including the sextupoles is a *super cell* with the same phase advance $(2k + 1)2\pi$ in each plane. Under these conditions, the first order sextupolar non linear terms are cancelled. The influence of the second order terms remains to be analyzed. The test parameter we have chosen to study is the horizontal betatron tune shift in the special case of a pure horizontal motion.

2. TUNE SHIFT EXPRESSION

In the first order perturbation theory, it can be shown that sextupolar fields excite modes whose characteristic phases are [3,4]

$$\mu_x \, , \; \mu_x + 2\mu_y \, , \; \mu_x - 2\mu_y \, , \; 3\mu_x$$

The virtue of the compensation is to annul integrals of the type

$$\int_0^L f(s) \cos(m\,\mu_x(s) + n\,\mu_y(s))\, ds$$

over the length L of a super cell. However, there is a cross-talk between the sextupoles of a super cell and, as a consequence, a net deformation of the particle trajectory which provokes a tune shift. The effect is quite similar to the variation of chromaticity induced by the alteration of the β-function on off-momentum orbits. Several formalisms have been used to derive the expression of the tune shift [5,6,7,8] that we write

$$\Delta_2 Q_x = \frac{J_x}{64\pi}\int_0^C ds \int_s^{s+L} k'(s)\,k'(s')\,\beta_x^{3/2}(s)\,\beta_x^{3/2}(s')\left[-3\frac{\cos(-\pi Q_x + \mu_x(s') - \mu_x(s))}{\sin \pi Q_x} - \frac{\cos 3(-\pi Q_x + \mu_x(s') - \mu_x(s))}{\sin 3\pi Q_x}\right] ds' \quad (2)$$

J is the action variable, k' the sextupolar focusing strength, Q the betatron tune, β the β-fu ion and μ the betatron phase advance. The expression can be simplifie.. by taking the compensation assumption into account

$$\int_0^L k'(s)\,\beta_x^{3/2}(s) \exp im\mu_x(s)\, ds = 0 \qquad m = 1 \text{ or } 3 \quad (3)$$

Let us note that

$$\int_L^{s+L} k'(s')\,\beta_x^{3/2}(s') \cos m(-\pi Q_x + \mu_x(s') - \mu_x(s))ds' = \int_0^s k'(s')\beta_x^{3/2}(s') \cos m(\pi Q_x + \mu_x(s') - \mu_x(s))\,ds' \quad (4)$$

By performing the substitution

$$\int_s^{s+L} = \int_0^L - \int_0^s + \int_L^{s+L}$$

and applying the rules (3) and (4), one finds

$$\Delta_2 Q_x = \frac{3J_x}{32\pi} \sum_{m=1,3} \int_0^C k'(s)\,\beta_x^{3/2}(s)\,ds \int_0^s k'(s')\,\beta_x^{3/2}(s') \frac{\sin m(\mu_x(s') - \mu_x(s))}{m} ds' \quad (5)$$

3. CORRELATIONS

A super cell is a string of *n* FODO cells each with a phase advance μ_0. The self-correlation terms which represent the interaction of a sextupole with itself can only occur for thick elements; they are not treated here. For the cross-correlation terms there is no loss of generality assuming that the sextupoles are thin lenses of integrated strength $k'\ell$ and, for a more precise model, it is always possible to perform an analytical integration over each lumped element [9]. The double ∫ can then be replaced by a double Σ

$$\Delta_2 Q_x = \frac{3J_x}{32\pi} \sum_{m=1,3} \sum_{i=1}^{n} (k'l)_i \, \beta_{xi}^{3/2} \sum_{j=1}^{i} (k'l)_j \, \beta_{xj}^{3/2} \frac{\sin m(\mu_{xj} - \mu_{xi})}{m} \qquad (6)$$

If all the sextupoles are different, the above expression has to be calculated numerically. When the chromaticity is corrected with two families of sextupoles, interesting simplifications appear. The cross-correlations belong to four classes: F-F, D-D, F-D, D-F. In (6), the part which depends on the summation indices is

$$S_m = \sum_{i=1}^{2n} \beta_{xi}^{3/2} \, (k'l)_i \sum_{j=1}^{i} \beta_{xj}^{3/2} \, (k'l)_j \sin m(\mu_{xj} - \mu_{xi}) \qquad (7)$$

It can be split into the four contributions

$$S_m = [\, \beta_{xF}^3 \, (k'l)_F^2 + \beta_{xD}^3 \, (k'l)_D^2 \,] \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sin j(m\mu)$$

$$+ (\beta_{xF}\beta_{xD})^{3/2} (k'l)_F (k'l)_D \, [\, \sum_{i=1}^{n-1} \sum_{j=1}^{i} \sin (2j-1)(m\mu/2) + \sum_{i=1}^{n} \sum_{j=1}^{i} \sin (2j-1)(m\mu/2) \,] \qquad (8)$$

The sum of the trigonometric series has a closed form which is derived from the expression

$$\sum_{j=j_1}^{j_2-j_3} e^{i(j-j_4)\mu} = \frac{\sin(1-j_1+j_2-j_3)(\mu/2)}{\sin(\mu/2)} \, e^{i(\frac{j_1+j_2-j_3}{2} - j_4)\mu} \qquad (9)$$

which becomes

$$\sum_{j=j_1}^{n-j_3} e^{i(j-j_4)\mu} = \frac{\sin(1-j_1-j_3)(\mu/2)}{\sin(\mu/2)} \, e^{i(\frac{j_1-j_3}{2} - j_4)\mu} \qquad (10)$$

when the compensation condition

$$e^{in\mu} = 1$$

is applied. It can thus be shown that

$$S_m = \frac{n}{\sin(m\mu/2)} [\, (\beta_{xF}\beta_{xD})^{3/2} (k'l)_F (k'l)_D - \frac{1}{2} (\beta_{xF}^3 \, (k'l)_F^2 + \beta_{xD}^3 \, (k'l)_D^2) \cos(m\mu/2) \,] \qquad (11)$$

4. ROLE OF CELL PHASE ADVANCE

The discussion of the correlations shows that the tune shift in a super cell of *n* cells is

$$\Delta_2 Q_x = \frac{3J_x}{32\,\pi} \sum_{m=1,3} \frac{S_m}{m} \qquad (12)$$

The analysis can be continued assuming that the sextupoles are superimposed to the quadrupoles and that they correct the cell chromaticity only

$$k' = k/D \qquad (13)$$

where k is the quadrupole component of the integrated focusing strength kl related to μ and to the cell length L_c through the expression

$$kl = \pm (4/L_c)\sin(\mu/2) \tag{14}$$

+ or - signs standing for F- and D- elements respectively and D is the orbit dispersion

$$D = (\Phi L_c/2)\frac{1 \pm (1/2)\sin(\mu/2)}{\sin^2(\mu/2)} \tag{15}$$

with Φ, the bending angle per magnet. The β-function is given by

$$L_c \frac{1 \pm \sin(\mu/2)}{\sin\mu} \tag{16}$$

After substitution of $k'\ell$ and β into (11), we get

$$\Delta_2 Q_x = \frac{3 J_x n}{4\pi L_c \Phi^2} \sum_{m=1,3} F_m(\mu) \tag{17}$$

with

$$F_m(\mu) = \frac{\tan^3\frac{\mu}{2}}{m\sin\frac{m\mu}{2}} \left\{ -\frac{\cos^3\frac{\mu}{2}}{1-\frac{1}{4}\sin^2\frac{\mu}{2}} + \frac{1}{2}\cos\frac{m\mu}{2}\left[\frac{(1+\sin\frac{\mu}{2})^3}{(1+\frac{1}{2}\sin\frac{\mu}{2})^2} + \frac{(1-\sin\frac{\mu}{2})^3}{(1-\frac{1}{2}\sin\frac{\mu}{2})^2} \right] \right\} \tag{18}$$

This expression is better appreciated by applying it to a full ring of circumference C (the integer value of Q does not matter in the present discussion!). For N super cells, we have

$$\Delta_2 Q_{xt} = N\,\Delta_2 Q_x \tag{19}$$

$$C = Nn\,L_c \tag{20}$$

$$\pi = Nn\Phi \tag{21}$$

so that

$$\Delta_2 Q_x = \frac{3 J_x (Nn)^3}{4\pi^3 C} G(\mu) \tag{22}$$

The function

$$G(\mu) = \sum_{m=1,3} F_m(\mu) \tag{23}$$

is plotted in Fig. 1, it presents a sharp peak near $\pi/2$ and decays quickly to zero. This behaviour is not intuitive and would deserve further investigations especially in the case of real machines like LEP which have to be retuned to work at high energy. The test would consist of calculating the sextupolar tune shift at various working points near $\pi/2$, finding when its sign changes and determining the dynamic aperture in these conditions.

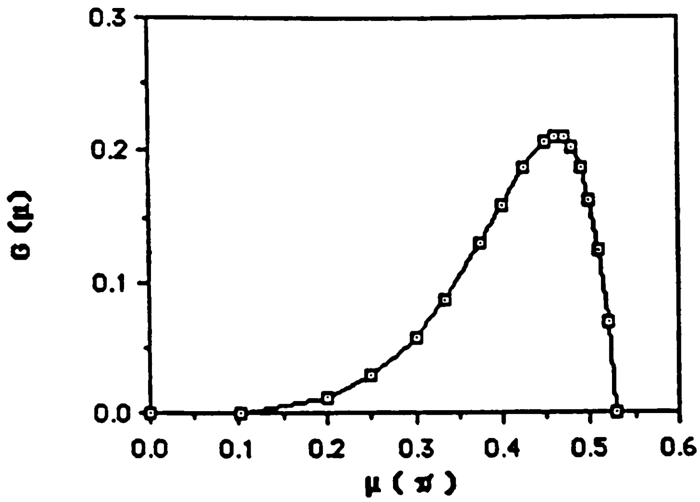

Fig. 1 Variation of the sextupolar tune-shift with the betatron phase advance per cell

REFERENCES

[1,2] R. Brinkmann, F. Willeke, *Chromatic Corrections and Dynamic Aperture in the HERA Electron Ring, Part I and II*, DESY 87-037.

[3] B. Autin, *Non Linear Betatron Oscillations*, AIP Conference Proceedings 153, SLAC Summer School 1985.

[4] T. Collins, *Distortion Functions*, Fermilab 84/114 (1984).

[5] F. Willeke, *Analytic Study of the Tevatron Non Linear Dynamics*, Fermilab FN-422.

[6] J. Bengtsson, *Non Linear Transverse Dynamics for Storage Rings with the Application to the Low Energy Antiproton Ring at CERN*. Ph.D. Thesis.

[7] W. Scandale, *Numerical Results on the Effect of the Parasitic Multipoles in the Superconducting Dipoles ` LHC*. LHC Note 57.

[8] A. Bazzani, P. Mayzanti, G. Servizi, G. Tı ıetti, *Normal Forms for Hamiltonian Maps and Non Linear Effects in a LHC Model*, LHC Note 66.

[9] B. Autin, J. Bengtsson, *Application of Symbolic Computation to the Search of Complicated Primitives: the Example of the Betatron Integrals*, Computer Physics Communications 48, (1988).