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SOME COMMENTS ON THE EQUATIONS OF MOTION FOR A PARTICLE IN AN EXTERNAL ELECTROMAGNETIC FIELD

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ABSTRACT

We review the equations of motion for a particle in an external electromagnetic field by using a tensor equation valid in any coordinate system. It is then an easy matter to get the equations of motion in the curvilinear coordinate system normally used for accelerators when the metric tensor $g_{\mu\nu}$ has been calculated. By using the equation for the longitudinal motion, it is possible to write the equations for the transverse motion in a form different from that normally found.

It is shown that when these equations are expanded to second order in the coordinates, they differ slightly from the similar equations found by K. Brown. However, to obtain the final equations of motion, one also has to expand in the momentum deviation. It then emerges that the difference is only of higher order. It is also shown that one gets the same equations of motion by using the Hamiltonian formulation as with the tensor equation.

When one is using perturbation theory the Hamiltonian has to be divided into two parts. It is then customary to expand the Hamiltonian. The Hamiltonian is expanded to the third order, allowing for variation of the curvature. The second-order equations so obtained are shown to be equivalent to the expansions of the tensor equation.

Finally, we present a manifestly covariant Hamiltonian which one does not have to expand.

1. EQUATIONS OF MOTION IN A CURVILINEAR COORDINATE SYSTEM

The equations of motion in a curvilinear system are given by the tensor equation [1, 2]

$$\frac{Du^\mu}{D\tau} = \frac{f^\mu}{m_0} , \qquad \mu = 0, 1, 2, 3, 4, \tag{1}$$

where u^μ is the four-velocity, $u^\mu = dx^\mu/d\tau$, m_0 the rest mass, and τ the proper time. The covariant derivative along the curve $x^\mu(\tau)$ of a contravariant four-vector A^μ is defined by

$$\frac{DA^\mu}{D\tau} \equiv \frac{dA^\mu}{d\tau} + \Gamma^\mu_{\nu\lambda} \frac{dx^\lambda}{d\tau} A^\nu , \tag{2}$$

where $\Gamma^\mu_{\nu\lambda}$ is the affinity given by the metric tensor $g_{\mu\nu}$

$$\Gamma^\mu_{\nu\lambda} \equiv \frac{1}{2} g^{\alpha\mu} \left[\frac{\partial g_{\lambda\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^\alpha}\right] . \tag{3}$$

The metric tensor $g_{\mu\nu}$ defines the differential distance in a particular coordinate system

$$c^2 d\tau^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu . \tag{4}$$

It obeys the following relation

$$g_{\mu\lambda} g^{\lambda\nu} = \delta^\nu_\mu . \tag{5}$$

The transformation from a contravariant vector A^μ to a covariant vector A_μ is given by

$$A_\mu = g_{\mu\nu} A^\nu , \qquad A^\mu = g^{\mu\nu} A_\nu . \tag{6}$$

The four-force f^μ for an electromagnetic field is given by

$$f^\mu = \frac{q}{c} F^\mu{}_\nu \frac{dx^\nu}{d\tau} , \tag{7}$$

where q is the charge of the particle and $F^{\mu}{}_{\nu}$ is the electromagnetic field tensor which, in a Cartesian inertial system, takes the form

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix} \longrightarrow \tilde{F}^{\mu}{}_{\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \tag{8}$$

since the metric tensor is in this case

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9}$$

The form of the field tensor in a general coordinate system may then be calculated from the transformation rule of a contravariant tensor

$$A'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} A^{\alpha\beta} \ . \tag{10}$$

By combining Eqs. (1) and (7) we get the equations of motion for a particle in an electromagnetic field

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = \frac{q}{m_0 c} F^{\mu}{}_{\nu} \frac{dx^{\nu}}{d\tau} \ . \tag{11}$$

This reduces, in an inertial system, to the well-known Lorentz-force law

$$\frac{d\bar{p}}{dt} = q(\bar{E} + \bar{v} \times \bar{B}) \ , \tag{12}$$

and from the time-like part of Eq. (11) assumes the form

$$\frac{dE}{dt} = q\bar{v} \cdot \bar{E} \ , \tag{13}$$

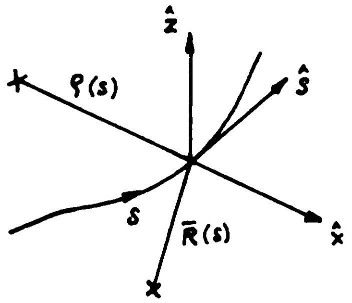
since the four momentum is given by

$$p^\mu = m_0 \frac{dx^\mu}{d\tau} = m_0 \frac{dt}{d\tau}\frac{dx^\mu}{dt} = m_0\gamma \frac{dx^\mu}{dt} = m_0\gamma(c,\bar{v}) = \left(\frac{E}{c}, \bar{p}\right) . \qquad (14)$$

It is seen that if the electric field $\bar{E}$ is orthogonal to the velocity or zero then the energy E of the particle is constant.

2. LOCAL COORDINATES FOR A PARTICLE IN AN ACCELERATOR

In accelerator theory it is convenient to use local coordinates for a particle of the following type [3]



where s is the distance along a reference curve $\bar{R}(s)$. The only assumption for this curve is that it should lie in a horizontal plane and have the local curvature

$$h(s) = \frac{1}{\varrho(s)} . \qquad (15)$$

A general vector $\bar{r}$ may then be written as

$$\bar{r}(x,s,z) = \bar{R}(s) + x\hat{x}(s) + z\hat{z}(s) . \qquad (16)$$

It follows that

$$d\bar{r}(x,s,z) = dx\,\hat{x} + (1 + hx)\,ds\,\hat{s} + dz\,\hat{z} \qquad (17)$$

so that

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 = c^2 dt^2 - dx^2 - (1 + hx)^2\, ds^2 - dz^2 . \qquad (18)$$

From Eqs. (5) and (18) we find for the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -(1+hx)^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\dfrac{1}{(1+hx)^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(19)

From Eqs. (8) and (10) we may calculate the field tensor to be

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -\frac{E_s}{1+hx} & -E_z \\ E_x & 0 & -\frac{cB_z}{1+hx} & cB_s \\ \frac{E_s}{1+hx} & \frac{cB_z}{1+hx} & 0 & -\frac{cB_x}{1+hx} \\ E_z & -cB_s & \frac{cB_x}{1+hx} & 0 \end{pmatrix},$$

(20)

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & E_x & E_s(1+hx) & E_z \\ E_x & 0 & cB_z(1+hx) & -cB_s \\ \frac{E_z}{1+hx} & -\frac{cB_z}{1+hx} & 0 & \frac{cB_x}{1+hx} \\ E_z & cB_s & -cB_x(1+hx) & 0 \end{pmatrix}$$

where we have used E_s, B_s instead of E_y, B_y.

The only affinities different from zero are from Eq. (3):

$$\Gamma^{1}_{22} = -h(1 + hx) \ , \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{h}{1 + hx} \ , \quad \Gamma^{2}_{22} = \frac{h'x}{1 + hx} \ ,$$

where a prime denotes a derivative with respect to s. Equation (11) gives the equations of motion:

$$\frac{dE}{d\tau} = q\left[\frac{dx}{d\tau} E_x + (1 + hx) \frac{ds}{d\tau} E_s + \frac{dz}{d\tau} E_z\right]$$

$$\frac{d^2x}{d\tau^2} - h(1 + hx) \left(\frac{ds}{d\tau}\right)^2 = \frac{q}{m_0} \left[\frac{dt}{d\tau} E_x + (1 + hx) \frac{ds}{d\tau} B_z - \frac{dz}{d\tau} B_s\right]$$

(21)

$$\frac{d^2s}{d\tau^2} + \frac{2h}{1 + hx} \frac{dx}{d\tau} \frac{ds}{d\tau} + \frac{h'x}{1 + hx} \left(\frac{ds}{d\tau}\right)^2 = \frac{q}{m_0} \left[\frac{dt}{d\tau} \frac{E_s}{1 + hx} - \frac{dx}{d\tau} \frac{B_z}{1 + hx} + \frac{dz}{d\tau} \frac{B_x}{1 + hx}\right]$$

$$\frac{d^2z}{d\tau^2} = \frac{q}{m_0} \left[\frac{dt}{d\tau} E_z + \frac{dx}{d\tau} B_s - (1 + hx) \frac{ds}{d\tau} B_x\right] .$$

As before, we see from the first equation of (21) that if the electric field is zero then the energy E of the particle is constant. We have in this case

$$E = \gamma m_0 c^2 = m_0 c^2 \frac{dt}{d\tau} \longrightarrow$$

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma \frac{d}{dt} = \gamma \frac{ds}{dt}\frac{d}{ds} = \gamma \dot{s} \frac{d}{ds} , \qquad \frac{d^2}{d\tau^2} = \gamma^2 \dot{s}^2 \frac{d^2}{ds^2} + \gamma^2 \ddot{s} \frac{d}{ds} , \tag{22}$$

so that Eqs. (21) may be simplified to

$$x'' + \frac{\ddot{s}}{\dot{s}^2} x' - h(1 + hx) = \frac{v}{\dot{s}} \frac{q}{p} \left[(1 + hx)B_z - z'B_s\right]$$

$$\frac{\ddot{s}}{\dot{s}^2} (1 + hx) + 2hx' + h'x = \frac{v}{\dot{s}} \frac{q}{p} \left[z'B_x - x'B_z\right] \tag{23}$$

$$z'' + \frac{\ddot{s}}{\dot{s}^2} z' = \frac{v}{\dot{s}} \frac{q}{p} \left[x'B_s - (1 + hx)B_x\right] ,$$

where a prime denotes the derivative with respect to s and we have used the fact that $p = \gamma m_0 v = mv$, where m is the relativistic mass. The second equation gives

$$\frac{\ddot{s}}{\dot{s}^2} = - \frac{2hx' + h'x}{1 + hx} + \frac{v}{\dot{s}} \frac{q}{p} \left[\frac{z'}{1 + hx} B_x - \frac{x'}{1 + hx} B_z\right] . \tag{24}$$

From Eq. (17) it follows that

$$v = \frac{dr}{dt} = \sqrt{\dot{x}^2 + (1 + hx)^2 \dot{s}^2 + \dot{z}^2} = \dot{s} \sqrt{x'^2 + (1 + hx)^2 + z'^2} , \tag{25}$$

where we have used

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \dot{s} \frac{d}{ds} \tag{26}$$

so that

$$\frac{v}{\dot{s}} = \sqrt{x'^2 + (1 + hx)^2 + z'^2} . \tag{27}$$

The motion along s may then be described by the two transverse coordinates x and z only by Eqs. (23), (24), and (27)

$$x'' - \frac{x'}{1 + hx}(2hx' + h'x) - h(1 + hx) = \frac{q}{p}\sqrt{x'^2 + (1 + hx)^2 + z'^2}$$

$$\times\left\{(1 + hx)\left[1 + \frac{x'^2}{(1+hx)^2}\right]B_z - z'B_s - \frac{x'z'}{1 + hx}B_x\right\}$$

$$z'' - \frac{z'}{1 + hx}(2hx' + h'x) = \frac{q}{p}\sqrt{x'^2 + (1 + hx) + z'^2}$$

$$\times\left\{x'B_s - (1 + hx)\left[1 + \frac{z'^2}{(1+hx)^2}\right]B_x + \frac{x'z'}{1 + hx}B_z\right\} . \tag{28}$$

The number of degrees of freedom is two since the energy E is conserved and from

$$E = \sqrt{(pc)^2 + (m_0c^2)^2} \tag{29}$$

the total momentum is also conserved. In the accelerator literature [3-5] one normally finds another equation than (23) for the longitudinal motion. It is derived from Eq. (27) and written as

$$\frac{1}{2}\frac{d}{ds}\left(\frac{v}{\dot{s}}\right)^2 = \frac{1}{2}\frac{dt}{ds}\frac{d}{dt}\left(\frac{v}{\dot{s}}\right)^2 = \frac{1}{2}\frac{1}{\dot{s}}\frac{d}{dt}\left(\frac{v}{\dot{s}}\right)^2 = -\left(\frac{v}{\dot{s}}\right)^2\frac{\ddot{s}}{\dot{s}^2}$$

$$= x'x'' + (1 + hx)(hx' + h'x) + z'z'' \tag{30}$$

since v is constant. This equation for $\ddot{s}$ may be verified by solving for x" and z" in (23), using this in (30) and comparing the result with the second equation there.

3. EXPANSION OF THE EQUATIONS OF MOTION TO SECOND ORDER

If we use the expansions

$$\frac{1}{1 + \alpha} = 1 - \alpha + \alpha^2 - \alpha^3 + O(4)$$

$$\sqrt{1 + \alpha} = 1 + \frac{1}{2}\alpha - \frac{1}{8}\alpha^2 + \frac{1}{16}\alpha^3 + O(4) \tag{31}$$

in Eqs. (28) and only keep terms to second order in the coordinates we find

$$x'' - x'(2hx' + h'x) - h(1 + hx)$$

$$= \frac{q}{p}\left[(1 + 2hx + h^2x^2 + \frac{3}{2}x'^2 + \frac{1}{2}z'^2)B_z - (1 + hx)z'B_s - x'z'B_x\right]$$

$$z'' - z'(2hx' + h'x)$$

$$= \frac{q}{p}\left[(1 + hx)x'B_s - (1 + 2hx + h^2x^2 + \frac{1}{2}x'^2 + \frac{3}{2}z'^2)B_x + x'z'B_z\right] . \qquad (32)$$

If we compare with the equations derived by K. Brown, (2.6) in Ref. [6], we find that they differ by some field terms on the right-hand side and also on the left-hand side where we have a term $-2hx'^2$, $-2hx'z'$ instead of $-hx'^2$, $-hx'z'$, in the horizontal and vertical planes respectively.

We now choose a trajectory for a particle with some momentum p_0 as the reference curve (e.g. the closed orbit in a circular machine). We then define the momentum deviation δ by

$$\delta \equiv \frac{p - p_0}{p_0} . \qquad (33)$$

where p is the momentum for an arbitrary particle. The curvature $h(s)$ is then given by the vertical field $B_z(s) = -h(s)p_0/q$. Since the reference curve was assumed to lie in a horizontal plane it follows that B_x is zero at this curve. What remains is then only the terms $x'^2\delta/(1+\delta)$ and $x'z'\delta/(1+\delta)$, in the horizontal and vertical planes respectively. Since one should also expand in δ these terms are of higher (third) order.

The well-known linear equations are obtained by using the field expansions [3, 4]

$$\frac{q}{p_0} B_x = 0 , \quad \frac{q}{p_0} B_s = 0 , \quad \frac{q}{p_0} B_z = -h + kx \qquad (34)$$

and by only keeping linear terms in the coordinates and δ in Eqs. (32), which then give

$$x'' + (h^2 - k)x = \delta h$$
$$z'' + kz = 0 . \qquad (35)$$

4. HAMILTONIAN FORMALISM

There are two ways of carrying out a Hamiltonian formulation of a relativistic particle in an external field [7, 8]. Either one works in a specific Lorentz frame (non-covariant formulation) or one attempts a fully covariant description. We will start with the first one since it is the one normally applied to accelerators [5, 9-15]. We will see, however, that it has some drawbacks when one applies perturbation theory so we will therefore investigate if they may be solved by using the other formulation.

4.1 Hamiltonian for a specific Lorentz frame

The Hamiltonian is given by [8]

$$H = q\Phi + c\sqrt{(\bar{p} - q\bar{A})^2 + m_0^2c^2} \tag{36}$$

where $\bar{A}$ is the vector potential, Φ is the scalar potential for the external electromagnetic field,

$$\bar{B} = \nabla \times \bar{A}$$
$$\bar{E} = -\frac{\partial \bar{A}}{\partial t} - \nabla\Phi \tag{37}$$

and $\bar{p}$ the conjugate momenta. Hamilton's equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial x_i}$$
$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial p_i} \tag{38}$$

lead to the Lorentz force law (12) [12, 14].

The Hamiltonian for the curvilinear system is obtained by a canonical transformation with the generating function [13]

$$F_2(\bar{r},\bar{p}) = \bar{p}\cdot\bar{r} = \bar{p}\cdot[\bar{R}(s) + x\hat{x}(s) + z\hat{z}(s)] , \tag{39}$$

where $\bar{R}(s)$ is the reference curve,

$$\left\{\begin{aligned} P_x &= \frac{\partial F_2}{\partial x} = \bar{p}\cdot\hat{x} \\ P_s &= \frac{\partial F_2}{\partial s} = \bar{p}\cdot\hat{s}(1 + hx) \\ P_z &= \frac{\partial F_2}{\partial z} = \bar{p}\cdot\hat{z} \\ X &= \frac{\partial F_2}{\partial p_x} = x \\ S &= \frac{\partial F_2}{\partial p_s} = s \\ Z &= \frac{\partial F_2}{\partial p_z} = z \end{aligned}\right. \tag{40}$$

$$H_1 = H + \frac{\partial F_2}{\partial t} .$$

The old Hamiltonian may now be transformed and if we again use small letters for the new coordinates we have

$$H_1 = e\Phi + c\sqrt{m_0c^2 + (p_x - eA_x)^2 + \left(\frac{p_s}{1 + hx} - eA_s\right)^2 + (p_z - eA_z)^2} , \tag{41}$$

where we have put $q = e$ and A_x, A_s, A_z are the components of the vector potential in the curvilinear system*). If Φ and A_x, A_s, A_z are time independent, then H_1 is a constant of motion which we may identify as the energy E. To change from t to s as the independent variable, we take $-p_s$ as a new Hamiltonian [2, 9, 12-14]. If we take $\Phi = 0$ we have

$$H_1 = E = \sqrt{(pc)^2 + (m_0c^2)} , \tag{42}$$

where p is the total momentum.

The new Hamiltonian is then

$$H_2 = -p_s = -(1 + hx)\left[eA_s + \sqrt{p^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2}\right] . \tag{43}$$

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*) They are defined by $\bar{A} = A_x\hat{x} + A_s\hat{s} + A_z\hat{z}$, and p_x, p_s, p_z are in fact the covariant components of the vector $\bar{p}$ in the curvilinear system.

Hamilton's equation gives

$$
\left[\begin{aligned}
x' &= \frac{dx}{ds} = \frac{\partial H_2}{\partial p_x} = \frac{(1 + hx)(p_x - eA_x)}{\sqrt{\quad}} \\
z' &= \frac{dz}{ds} = \frac{\partial H_2}{\partial p_z} = \frac{(1 + hx)(p_z - eA_z)}{\sqrt{\quad}} \\
p_x' &= \frac{dp_x}{ds} = -\frac{\partial H_2}{\partial x} = (1 + hx)\, e \frac{\partial A_s}{\partial x} + heA_s + h\sqrt{\quad} \\
&\qquad + (1 + hx)\frac{(p_x - eA_x)e\frac{\partial A_x}{\partial x} + (p_z - eA_z)e\frac{\partial A_z}{\partial x}}{\sqrt{\quad}} \\
p_z' &= \frac{dp_z}{ds} = -\frac{\partial H_2}{\partial x} = (1 + hx)\, e \frac{\partial A_s}{\partial z} \\
&\qquad + (1 + hx)\frac{(p_x - eA_x)e\frac{\partial A_x}{\partial z} + (p_z - eA_z)e\frac{\partial A_z}{\partial z}}{\sqrt{\quad}} \\
\frac{1}{\dot{s}} &= \frac{dt}{ds} = \frac{\partial H_2}{(-\partial H_1)} = \frac{(1 + hx)E}{c^2\sqrt{\quad}} = \frac{m(1 + hx)}{\sqrt{\quad}} \\
\frac{d(-H_1)}{ds} &= -\frac{\partial H_2}{\partial t} = 0
\end{aligned}\right. \tag{44}
$$

where

$$\sqrt{\quad} \equiv \sqrt{p^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2} \; . \tag{45}$$

From these equations it follows that

$$\sqrt{x'^2 + (1 + hx)^2 + z'^2} = \frac{p(1 + hx)}{\sqrt{\quad}} = \frac{p}{m\dot{s}} = \frac{mv}{m\dot{s}} = \frac{v}{\dot{s}} \; . \tag{46}$$

We may solve for p_x and p_z in the first two equations of (44) and then take the derivative with respect to s

$$
\begin{aligned}
\frac{d}{ds} p_x &= \frac{d}{ds}\left(\frac{\dot{s}}{v} x' + eA_x\right) = \frac{\dot{s}}{v}\left(x'' + \frac{\ddot{s}}{\dot{s}^2} x'\right) + eA_x' \\
\frac{d}{ds} p_z &= \frac{d}{ds}\left(\frac{\dot{s}}{v} z' + eA_z\right) = \frac{\dot{s}}{v}\left(z'' + \frac{\ddot{s}}{\dot{s}^2} z'\right) + eA_z' \; ,
\end{aligned} \tag{47}
$$

where we have used

$$\frac{d}{ds}\dot{s} = \frac{dt}{ds}\frac{d}{dt}\dot{s} = \frac{\ddot{s}}{\dot{s}} . \qquad (48)$$

Since we assumed that $\partial A_x/\partial t = \partial A_z/\partial t = 0$ we find

$$A'_x = \frac{dA_x}{ds} = x'\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial s} + z'\frac{\partial A_x}{\partial z}$$

$$A'_z = \frac{dA_z}{ds} = x'\frac{\partial A_z}{\partial x} + \frac{\partial A_z}{\partial s} + z'\frac{\partial A_z}{\partial z} . \qquad (49)$$

Combining Eqs. (44), (46), (48), and (49) gives

$$x'' + \frac{\ddot{s}}{\dot{s}^2}x' - h(1 + hx)$$

$$= \frac{v}{\dot{s}}\frac{e}{p}\left[hA_s + (1 + hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s} - z'\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$z'' + \frac{\ddot{s}}{\dot{s}^2}z' = \frac{v}{\dot{s}}\frac{e}{p}\left[-\frac{\partial A_z}{\partial s} + (1 + hx)\frac{\partial A_s}{\partial z} + x'\frac{\partial A_x}{\partial z} - x'\frac{\partial A_z}{\partial x}\right] , \qquad (50)$$

where $\ddot{s}$ is calculated by taking the derivative of the fifth equation in (44)

$$\frac{d}{ds}\dot{s} = \frac{\ddot{s}}{\dot{s}} = \frac{d}{ds}\frac{\sqrt{\quad}}{m(1 + hx)}$$

$$= -(h'x + hx')\frac{\sqrt{\quad}}{m(1 + hx)^2}$$

$$- \frac{(p_x - eA_x)(p'_x - eA'_x) + (p_z - eA_z)(p'_z - eA'_z)}{m(1 + hx)\sqrt{\quad}} = \dots$$

$$= -\dot{s}\frac{2hx' + h'x}{1 + hx} - \frac{e}{m}\frac{1}{(1 + hx)^2}\left\{x'\left[hA_s + (1 + hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right]\right.$$

$$\left. - z'\left[\frac{\partial A_z}{\partial s} - (1 + hx)\frac{\partial \lambda_s}{\partial z}\right]\right\} . \qquad (51)$$

Since

$$\bar{B} = \nabla \times \bar{A} , \qquad (52)$$

and in an orthogonal coordinate system [1]

$$(\nabla \times \bar{A})_i = \frac{h_i}{h_1 h_2 h_3} \sum_{jk} \varepsilon^{ijk} \frac{\partial}{\partial x^j} h_k A_k , \qquad (53)$$

where h_i are the diagonal elements of $g_{\mu\nu}$ in (19), we have

$$\begin{aligned} B_x &= \frac{1}{1 + hx} \frac{\partial A_z}{\partial s} - \frac{\partial A_s}{\partial z} \\ B_s &= \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ B_z &= \frac{h}{1 + hx} A_s + \frac{\partial A_s}{\partial x} - \frac{1}{1 + hx} \frac{\partial A_x}{\partial s} . \end{aligned} \qquad (54)$$

The equations of motion are then

$$\begin{aligned} x'' + \frac{\ddot{s}}{\dot{s}^2} x' - h(1 + hx) &= \frac{e}{p} \frac{v}{\dot{s}} [(1 + hx)B_z - z'B_s] \\ \frac{\ddot{s}}{\dot{s}^2} (1 + hx) + 2hx' + h'x &= \frac{e}{p} \frac{v}{\dot{s}} [z'B_x - x'B_z] \\ z'' + \frac{\ddot{s}}{\dot{s}^2} z' &= \frac{e}{p} \frac{v}{\dot{s}} [x'B_s - (1 + hx)B_x] \end{aligned} \qquad (55)$$

in agreement with Eqs. (23).

4.2 Expansion of the Hamiltonian to third order

To apply perturbation theory, one has to divide the Hamiltonian into two parts

$$H \equiv H_0 + V , \qquad (56)$$

where H_0 is the part of the Hamiltonian for which the equations of motion can be solved. To do this for the Hamiltonian (43) one normally expands the square root. We expand in powers of the deviations from a reference orbit x, p_x, z, p_z, δ defined by a reference particle with momentum p_0. In (33), δ is defined as

$$\delta \equiv \frac{p - p_0}{p_0} .$$

The Hamiltonian (43) may then be written as

$$H_3 = -p_0 (1 + hx) \left[\frac{e}{p_0} A_s + (1 + \delta) \sqrt{1 - \frac{(p_x - eA_x)^2}{p_0^2 (1 + \delta)^2} - \frac{(p_z - eA_z)^2}{p_0^2 (1 + \delta)^2}} \right] . \tag{57}$$

The square root may be expanded from Eqs. (31) as

$$\sqrt{1 + \alpha} = 1 + \frac{1}{2} \alpha - \frac{1}{8} \alpha^2 + \frac{1}{16} \alpha^3 + O(4) .$$

The Hamiltonian to third order is then

$$H_4 = -p_0 (1 + hx) \left[\frac{e}{p_0} A_s + 1 + \delta - \frac{(p_x - eA_x)^2}{2p_0^2 (1 + \delta)} - \frac{(p_z - eA_z)^2}{2p_0^2 (1 + \delta)} \right] . \tag{58}$$

Hamilton's equations now give

$$\left\{ \begin{aligned} x' &= \frac{\partial H_4}{\partial p_x} = (1 + hx) \frac{p_x - eA_x}{p_0 (1 + \delta)} \\ z' &= \frac{\partial H_4}{\partial p_z} = (1 + hx) \frac{p_z - eA_z}{p_0 (1 + \delta)} \\ p_x' &= - \frac{\partial H_4}{\partial x} = p_0 h \left[\frac{e}{p_0} A_s + 1 + \delta - \frac{(p_x - eA_x)^2}{2p_0^2 (1 + \delta)} - \frac{(p_z - eA_z)^2}{2p_0^2 (1 + \delta)} \right] \\ &\quad + (1 + hx) \frac{e}{p_0} \left[\frac{\partial A_s}{\partial x} + \frac{p_x - eA_x}{p_0 (1 + \delta)} \frac{\partial A_x}{\partial x} + \frac{p_z - eA_z}{p_0 (1 + \delta)} \frac{\partial A_z}{\partial x} \right] \\ p_z' &= - \frac{\partial H_4}{\partial z} = (1 + hx) \frac{e}{p_0} \left[\frac{\partial A_s}{\partial x} + \frac{p_x - eA_x}{p_0 (1 + \delta)} \frac{\partial A_x}{\partial z} + \frac{p_z - eA_z}{p_0 (1 + \delta)} \frac{\partial A_z}{\partial z} \right] \\ \frac{1}{\dot{s}} &= \frac{\partial H_4}{\partial (-H_1)} = \frac{1}{v} (1 + hx) \left[1 + \frac{(p_x - eA_x)^2}{2p_0^2 (1 + \delta)} + \frac{(p_z - eA_z)^2}{2p_0^2 (1 + \delta)} \right] \\ \frac{d(-H_1)}{ds} &= - \frac{\partial H_4}{\partial t} = 0 . \end{aligned} \right. \tag{59}$$

Since the Hamiltonian is expanded to third order, the equations of motion are correct to second order.

Solving for p_x and p_z in Eqs. (59), we have

$$\frac{d}{ds} p_x = \frac{d}{ds}\left[\frac{p_0(1+\delta)}{1+hx} x' + eA_x\right]$$

$$= \frac{p_0(1+\delta)}{1+hx}\left[x'' - \frac{(h'x + hx')x'}{1+hx} + \frac{1+hx}{1+\delta}\frac{e}{p_0} A'_x\right]$$

$$\frac{d}{ds} p_z = \frac{d}{ds}\left[\frac{p_0(1+\delta)}{1+hx} z' + eA_z\right] \tag{60}$$

$$= \frac{p_0(1+\delta)}{1+hx}\left[z'' - \frac{(h'x + hx')z'}{1+hx} + \frac{1+hx}{1+\delta}\frac{e}{p_0} A'_z\right] .$$

Putting this equal to p'_x and p'_z in (59), by using Eqs. (49) and only keeping second-order terms, we find

$$x'' - x'(h'x + hx')$$

$$= h(1+hx)\left[1 - \frac{1}{2}x'^2 - \frac{1}{2}z'^2\right] + (1 - \delta + \delta^2)(1+hx)\frac{e}{p_0}$$

$$\times \left[hA_s + (1+hx)\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s} + z'\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right] \tag{61}$$

$$z'' - z'(h'x + hx')$$

$$= (1 - \delta + \delta^2)\frac{e}{p_0}\left[(1+hx)\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} + x'\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right] ,$$

where we have used Eqs. (31). By using Eqs. (54) we finally have

$$x'' - h'xx' - \frac{h}{2}(x'^2 - z'^2) - h(1+hx)$$

$$= (1 - \delta + \delta^2)\frac{e}{p_0}[(1 + 2hx + h^2x^2)B_z - (1+hx)z'B_s] \tag{62}$$

$$z'' - z'(h'x + hx')$$

$$= (1 - \delta + \delta^2)\frac{e}{p_0}[(1+hx)x'B_s - (1 + 2hx + h^2x^2)B_x] .$$

This differs slightly from Eqs. (32), but by again taking into account that h is given by $B_z = -hp_0/q$, and that the reference curve was assumed to lie in a horizontal plane so that B_x is zero, it is seen that the difference is only in higher-order terms.

As we have seen, one normally has to expand the Hamiltonian we used. This is no problem and may, at least in principle, be extended to any order. However, at least from an aesthetic point of view, it would be nice to use a formalism where one only has to find expansions for the vector potential and not the Hamiltonian itself. That would mean that one does not have to approximate the dynamical part of the system, only the driving sources. Of course, one still has the problem of finding solutions to the equations of motion, which is normally done by applying some sort of perturbation theory which then gives approximate solutions.

5. MANIFESTLY COVARIANT HAMILTONIAN FORMALISM

5.1 A covariant Hamiltonian

In the case of a particle in an external electromagnetic field it is possible to define a covariant Hamiltonian valid in any coordinate system by [8]

$$H = \frac{1}{2m_0c}(p_\mu - qA_\mu)(p^\mu - qA^\mu) = \frac{1}{2m_0c}\left[p_\mu p^\mu - 2qp_\mu A^\mu + q^2 A_\mu A^\mu\right], \quad (63)$$

where

$$\begin{aligned} A^\mu &\equiv (\frac{\Phi}{c}, \bar{A}) \\ p^\mu &\equiv m_0 u^\mu + qA^\mu \\ u^\mu &\equiv \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma(c,\bar{v}) \ . \end{aligned} \quad (64)$$

Hamilton's equations are in a Cartesian system

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \frac{\partial H}{\partial p_\mu} = \frac{1}{m_0c}(p^\mu - qA^\mu) \\ \frac{dp^\mu}{d\tau} &= -\frac{\partial H}{\partial x_\mu} = \frac{q}{m_0c}(p^\nu - qA^\nu)\,\partial^\mu A_\nu \ . \end{aligned} \quad (65)$$

Since

$$\frac{dA^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\partial_\nu A^\mu \ , \quad (66)$$

and solving in (65) for p^μ

$$p^\mu = m_0 c \frac{dx^\mu}{d\tau} + qA^\mu \ , \tag{67}$$

gives in the second equation of (65)

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{q}{m_0 c} (\partial^\mu A_\nu - \partial_\nu A^\mu) \frac{dx^\nu}{d\tau} = \frac{q}{m_0 c} \tilde{F}^\mu{}_\nu \frac{dx^\nu}{d\tau} \ , \tag{68}$$

where we have used the definition of the electromagnetic field tensor [8]. This may be compared with Eq. (11).

5.2 Hamiltonian for the curvilinear system

The metric tensor is in this case given by (19). The normal components of the vector potential A_x, A_s, A_z, in the curvilinear system (see subsection 4.1) are related to the contravariant and covariant components by

$$\begin{aligned} &A^1 = A_x \ , \quad A^2 = \frac{A_s}{1 + hx} \ , \quad A^3 = A_z \ , \quad A^0 = \frac{\Phi}{c} \\ &A_1 = -A_x \ , \quad A_2 = -(1 + hx)A_s \ , \quad A_3 = -A_z \ , \quad A_0 = \frac{\Phi}{c} \ . \end{aligned} \tag{69}$$

If we then use x, s, z instead of 1, 2, 3 to label the covariant components of the conjugate momenta, we find with (6) and (63) that

$$H = \frac{1}{2m_0} \left[\left(\frac{E}{c} - q \frac{\Phi}{c} \right)^2 - (p_x + qA_x)^2 - \left(\frac{p_s}{1 + hx} + qA_s \right)^2 - (p_z + qA_z)^2 \right] \tag{70}$$

We assume that there is no electric field, $\Phi = 0$, and that the vector potentials are time independent:

$$\frac{\partial A_x}{\partial t} = \frac{\partial A_s}{\partial t} = \frac{\partial A_z}{\partial t} = 0 \ . \tag{71}$$

The Hamiltonian is then time independent and simplified to

$$H = \frac{1}{2m_0} \left[\frac{E^2}{c^2} - (p_x + qA_x)^2 - \left(\frac{p_s}{1 + hx} + qA_s \right)^2 - (p_z + qA_z)^2 \right] \ . \tag{72}$$

Hamilton's equations give

$$
\left\{
\begin{aligned}
& c\,\frac{dt}{d\tau} = c\,\frac{\partial H}{\partial E} = \frac{E}{m_0 c} = \gamma c \\
& \frac{1}{c}\frac{dE}{d\tau} = -\frac{1}{c}\frac{\partial H}{\partial t} = 0 \\
& \frac{dx}{d\tau} = \frac{\partial H}{\partial p_x} = -\frac{1}{m_0}\,(p_x + qA_x) \\
& \frac{dp_x}{d\tau} = -\frac{\partial H}{\partial x} = \frac{q}{m_0}\left[(p_x + qA_x)\,\frac{\partial A_x}{\partial x} + \left(\frac{p_s}{1 + hx} + qA_s\right)\left[\frac{\partial A_s}{\partial x} - \frac{hp_s}{q(1+hx)^2}\right]\right. \\
& \qquad\qquad \left. + (p_z + qA_z)\,\frac{\partial A_z}{\partial x}\right] \\
& \frac{ds}{d\tau} = \frac{\partial H}{\partial p_s} = -\frac{1}{m_0}\,\frac{1}{1 + hx}\left(\frac{p_s}{1 + hx} + qA_s\right) \\
& \frac{dp_s}{d\tau} = -\frac{\partial H}{\partial s} = \frac{q}{m_0}\left[(p_x + qA_x)\,\frac{\partial A_x}{\partial s} + \left(\frac{p_s}{1 + hx} + qA_s\right)\left(\frac{\partial A_s}{\partial s} - \frac{hx'}{q(1+hx)^2}\right)\right. \\
& \qquad\qquad \left. + (p_z + qA_z)\,\frac{\partial A_z}{\partial s}\right] \\
& \frac{dz}{d\tau} = \frac{\partial H}{\partial p_z} = -\frac{1}{m_0}\,(p_z + qA_z) \\
& \frac{dp_z}{d\tau} = -\frac{\partial H}{\partial z} = \frac{q}{m_0}\left[(p_x + qA_x)\,\frac{\partial A_x}{\partial z} + \left(\frac{p_s}{1 + hx} + qA_s\right)\frac{\partial A_s}{\partial z}\right. \\
& \qquad\qquad \left. + (p_z + qA_z)\,\frac{\partial A_z}{\partial z}\right] .
\end{aligned}
\right. \tag{73}
$$

From the first equation one has

$$
\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\,\frac{d}{dt} = \gamma\,\frac{ds}{dt}\frac{d}{ds} = \gamma\,\dot{s}\,\frac{d}{ds} . \tag{74}
$$

Using this in (73) one obtains

$$\left\{\begin{aligned}
x' &= \frac{dx}{ds} = -\frac{1}{p}\frac{v}{\dot{s}}(p_x + qA_x) \\
z' &= \frac{dz}{ds} = -\frac{1}{p}\frac{v}{\dot{s}}(p_z + qA_z) \\
\dot{s} &= -\frac{v}{p}\frac{1}{1+hx}\left(\frac{p_s}{1+hx} + qA_s\right) \\
p'_x &= \frac{dp_x}{ds} = -q\left[x'\frac{\partial A_x}{\partial x} + hA_s + (1+hx)\frac{\partial A_s}{\partial x} + z'\frac{\partial A_z}{\partial x}\right] \\
&\qquad - p\frac{\dot{s}}{v}(1+hx) \\
p'_z &= \frac{dp_z}{ds} = -q\left[x'\frac{\partial A_x}{\partial z} + (1+hx)\frac{\partial A_s}{\partial z} + z'\frac{\partial A_z}{\partial z}\right] \\
p'_s &= \frac{dp_s}{ds} = -q\left[x'\frac{\partial A_x}{\partial s} + (1+hx)\frac{\partial A_s}{\partial s} + h'xA_s + z'\frac{\partial A_z}{\partial s}\right] \\
&\qquad - p\frac{\dot{s}}{v}(1+hx)h'x\ .
\end{aligned}\right. \tag{75}$$

If one solves for p_x, p_s, and p_z in the first three equations of (75), takes the derivative with respect to s for p_x, p_s, p_z, puts this equal to the last three equations using Eq. (48), and remembers that the vector potential is time independent

$$\begin{aligned}
A'_x &= x'\frac{\partial A_x}{\partial x} + \frac{\partial A_x}{\partial s} + z'\frac{\partial A_x}{\partial z} \\
A'_s &= x'\frac{\partial A_s}{\partial x} + \frac{\partial A_s}{\partial s} + z'\frac{\partial A_s}{\partial z} \\
A'_z &= x'\frac{\partial A_z}{\partial x} + \frac{\partial A_z}{\partial s} + z'\frac{\partial A_z}{\partial z}\ ,
\end{aligned} \tag{76}$$

one finds

$$\begin{cases} x'' + x' \dfrac{\ddot{s}}{\dot{s}} - h(1 + hx) \\ \qquad = \dfrac{v}{\dot{s}} \dfrac{q}{p} \left[hA_s + (1 + hx) \dfrac{\partial A_s}{\partial x} - \dfrac{\partial A_x}{\partial s} + z'\left(\dfrac{\partial A_z}{\partial x} - \dfrac{\partial A_x}{\partial z}\right)\right] \\ \dfrac{\ddot{s}}{\dot{s}^2} (1 + hx) + 2hx' + h'x \\ \qquad = \dfrac{v}{\dot{s}} \dfrac{q}{p} \left[x' \left(\dfrac{1}{1 + hx} \dfrac{\partial A_x}{\partial s} - \dfrac{h}{1 + hx} A_s - \dfrac{\partial A_x}{\partial s}\right) + z'\left(\dfrac{1}{1 + hx} \dfrac{\partial A_z}{\partial s} - \dfrac{\partial A_s}{\partial z}\right)\right] \\ z'' + z' \dfrac{\ddot{s}}{\dot{s}^2} = \dfrac{v}{\dot{s}} \dfrac{q}{p} \left[x'\left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right) + (1 + hx) \dfrac{\partial A_s}{\partial z} - \dfrac{\partial A_z}{\partial s}\right] . \end{cases} \tag{77}$$

From (54) follows

$$\begin{aligned} x'' + x' \frac{\ddot{s}}{\dot{s}} - h(1 + hx) &= \frac{v}{\dot{s}} \frac{q}{p} [(1 + hx)B_z - z'B_s] \\ \frac{\ddot{s}}{\dot{s}^2} (1 + hx) + 2hx' + h'x &= \frac{v}{\dot{s}} \frac{q}{p} [z'B_x - x'B_z] \\ z'' + z' \frac{\ddot{s}}{\dot{s}^2} &= \frac{v}{\dot{s}} \frac{q}{p} [x'B_s - (1 + hx)B_x] . \end{aligned} \tag{78}$$

The second equation of (78) gives

$$\frac{\ddot{s}}{\dot{s}^2} = - \frac{2hx' + h'x}{1 + hx} + \frac{v}{\dot{s}} \frac{q}{p} [z'B_x - x'B_z] \tag{79}$$

and from the metric tensor $g_{\mu\nu}$

$$d\bar{r} = dx\hat{x} + (1 + hx)\, ds\, \hat{s} + dz\, \hat{z} \tag{80}$$

gives

$$v = \frac{d\bar{r}}{dt} = \sqrt{\dot{x}^2 + (1 + hx)\dot{s}^2 + \dot{z}^2} = \dot{s}\sqrt{x'^2 + (1 + hx)^2 + z'^2} \tag{81}$$

so that

$$\frac{v}{\dot{s}} = \sqrt{x'^2 + (1 + hx)^2 + z'^2} \; . \tag{82}$$

As expected, Eqs. (78) agree with Eqs. (23) obtained from the tensor equation.

The development of canonical transformations, Poisson brackets, and the Hamilton-Jacobi theory can also be applied to the covariant Hamiltonian [8].

If we introduce an eight-dimensional phase space (p^μ, x^μ), the canonical transformations may be defined by the equation

$$p_\mu u^\mu - H = p'_\mu u'^\mu - H' + \frac{dF}{d\tau} \tag{83}$$

where u^μ and u'^μ are four-velocities. In the particular case, for instance, where

$$F \equiv F(x^\mu, x'^\mu, \tau) \; ,$$

we have

$$H' = H + \frac{\partial F}{\partial \tau}$$

$$p_\mu = \frac{\partial F}{\partial x^\mu} \tag{84}$$

$$p'_\mu = \frac{\partial F}{\partial x'^\mu} \; .$$

6. CONCLUSIONS

We have shown that the Hamiltonian formalism, in the case when the curvature is a function of s, leads to the same second-order equations as those derived by K. Brown.

We have also shown that the exact equations of motion in the curvilinear system obtained by the Hamiltonian formalism are the same as those derived from a tensor equation.

Finally we have presented a Hamiltonian formalism where one does not have to expand the Hamiltonian. It is not a big problem to get equations of motion to second order by the normal formalism. The other formalism may, however, be preferable for higher-order calculations. It is also easier to extend from a given order, since we just have to find the new higher-order terms in the vector potential.

Of course, there still remain the problems of finding expansions for the vector potential, and of applying some sort of perturbation theory to find solutions of the non-linear equations, which have not been treated here.

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