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Minimal decoherence from inflation

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Abstract. We compute the rate with which super-Hubble cosmological fluctuations are decohered during inflation, by their gravitational interactions with unobserved shorter-wavelength scalar and tensor modes. We do so using Open Effective Field Theory methods, that remain under control at the late times of observational interest, contrary to perturbative calculations. Our result is minimal in the sense that it only incorporates the self-interactions predicted by General Relativity in single-clock models (additional interaction channels should only speed up decoherence). We find that decoherence is both suppressed by the first slow-roll parameter and by the energy density during inflation in Planckian units, but that it is enhanced by the volume comprised within the scale of interest, in Hubble units. This implies that, for the scales probed in the Cosmic Microwave Background, decoherence is effective as soon as inflation proceeds above $\sim 5 \times 10^9\,$GeV. Alternatively, if inflation proceeds at GUT scale decoherence is incomplete only for the scales crossing out the Hubble radius in the last ~ 13 e-folds of inflation. We also compute how short-wavelength scalar modes decohere primordial tensor perturbations, finding a faster rate unsuppressed by slow-roll parameters. Identifying the parametric dependence of decoherence, and the rate at which it proceeds, helps suggest ways to look for quantum effects.

Contents

1 Introduction

One of the most remarkable consequences of cosmology's recent conversion into a precision observational science has been the detection of large-scale correlations in the way that matter and radiation are distributed across the observable universe [1, 2]. These correlations point to a pattern of nearly scale-invariant primordial fluctuations inherited from the much-earlier universe about which otherwise little is known.

Even more remarkably these primordial fluctuations share the spectral properties expected of quantum fluctuations, if these were stretched across the sky in the remote past by some sort of accelerated universal expansion [3–8]. A natural question in any such a picture is how initial quantum fluctuations become the classical fluctuations that are known to describe the later observations so well [9–17]. Part of this question concerns precisely what is meant by 'classical' in this context (*e.g.* tree-level vs loop-level; WKB-squeezed vs generic quantum states; decoherence and so on).

In this paper our interest is the evolution of a quantum field's reduced density matrix, $\langle \varphi_1 \,|\, \varrho \,|\, \varphi_2 \rangle$, in the field basis, where $\varrho = \mathrm{Tr}_{\rm env}\, \rho$ is obtained by tracing out certain 'unobserved' degrees of freedom (in the language of open systems what is called the 'environment'). Classicalization will mean decoherence, in the sense that ϱ evolves from a pure to a mixed state; in particular one whose off-diagonal elements rapidly fall to zero in the field basis. Once ϱ becomes diagonal in this way it is indistinguishable from a non-quantum statistical ensemble of classical field configurations with probability distribution $P[\varphi] = \langle \varphi \,|\, \varrho \,|\, \varphi \rangle$.

Although it has long been recognized that existing cosmological measurements are largely insensitive to any off-diagonal components $\langle \varphi_1 \,|\, \varrho \,|\, \varphi_2 \rangle$ [11, 12], proposals are now being made to circumvent this to seek observational evidence for a quantum origin for primordial fluctuations [18–21]. The efficiency of any primordial decoherence is likely relevant to such searches, and in particular rapid decoherence can make it unlikely that quantum effects survive to the present day to be seen [22].

An obstacle must be surmounted and a choice must be made in order to describe such primordial quantum-to-classical transitions. Decoherence requires an environment: not all degrees of freedom should be measured. So the choice is to identify the environmental modes whose removal decoheres the fluctuations we can see. Since super-Hubble modes play a special role by freezing in early-universe effects, smaller wavelength sub-Hubble modes are natural candidates for a decohering environment. We therefore study how rapidly super-Hubble metric fluctuations are decohered by shorter-wavelength metric perturbations, using only the mutual gravitational self-interactions predicted by General Relativity.

The obstacle to be surmounted is more technical: the time interval between fluctuation generation and detection can be extremely large, and effects that were initially small can have enough time to become large during the long wait in between. This can cause a breakdown of perturbative methods; what are often called 'secular growth' effects in cosmology [23–25]. Similar late-time breakdown of perturbative methods are ubiquitous elsewhere in physics because no matter how small a perturbing interaction $\mathcal{H}_{\rm int} \ll \mathcal{H}_0$ is, it is always true that $e^{-i(\mathcal{H}_0 + \mathcal{H}_{\rm int})t}$ is not well-approximated by $e^{-i\mathcal{H}_0 t}(1 - i\mathcal{H}_{\rm int} t + \cdots)$ at sufficiently late times. The good news is: because these problems are ubiquitous, tools for circumventing them and making reliable late-time predictions are also very well-developed [26]. All that is required is to adapt these tools to cosmology [27].

In this paper we follow up on earlier work adapting to gravity well-developed tools from the quantum theory of open systems,[1] whose use to explore late-time evolution we call Open Effective Field Theory (Open EFT) [17, 27, 34–41]. These tools show how the evolution equation for the reduced density matrix in the interaction picture can be brought into an approximate (schematic) form

$$\partial_t \varrho = -i\left[\overline{\mathcal{H}}_{\rm int}\,,\varrho\right] + \mathscr{L}_2\left(\varrho\right) + \mathcal{O}\left(\mathcal{H}^3_{\rm int}\right) \text{ terms}\,, \tag{1.1}$$

where $\mathcal{H}_{\rm int}$ denotes the terms in the interaction Hamiltonian that couple the environment to the measured degrees of freedom and $\overline{\mathcal{H}}_{\rm int}$ denotes its average over the environment. $\mathscr{L}_2$ contains all terms second order in $\mathcal{H}_{\rm int}$ and in many circumstances has a Lindblad form [42, 43], which is derived below in some detail. Lindblad equations can have solutions that re-sum late-time behaviour even if the evolution equation itself is only computed perturbatively. Any terms not written explicitly are at least cubic in $\mathcal{H}_{\rm int}$.

We here apply these tools to compute the decoherence of super-Hubble scalar fluctuations of the metric in the simplest single-clock near-de Sitter (inflationary) cosmologies that are the best-explored explanations of primordial fluctuations [44]. An important observation is that the linear term in $\mathcal{H}_{\rm int}$ never contributes to decoherence because it simply represents Liouville evolution (which can never take pure states to mixed states). All decoherence effects necessarily first arise at second order in $\mathcal{H}_{\rm int}$, and this plays an important role when identifying the dominant interactions.

We work within the standard joint slow-roll and semi-classical expansions that track powers of small slow-roll parameters, ε_i, and powers of the small loop-counting parameter $GH^2 = H^2/(8\pi M_p^2)$ where G is Newton's constant, H is the inflationary Hubble scale and M_p is the reduced Planck mass.[2] For super-Hubble modes of co-moving momentum $\boldsymbol{k}$ these are supplemented by an additional expansion in powers of $k/(aH)$ where $k = |\boldsymbol{k}|$.

General relativity predicts that in such an expansion fluctuations of the metric self-interact. Of these only the cubic interactions – whose detailed form is worked out for near-de Sitter geometries in ref. [47] (summarized for convenience in Appendix A) – contribute at leading order when considering how interactions with shorter wavelength modes decohere the long-wavelength super-Hubble modes relevant for primordial fluctuations.

There is a simple reason why only cubic interactions dominate. As argued above, decoherence first arises at second order in $\mathcal{H}_{\rm int}$ and consequently does so at order $1/M_p^2$. Although quartic interactions can also contribute to fluctuation evolution at order $1/M_p^2$ they do so in (1.1) only through terms linear in $\mathcal{H}_{\rm int}$, and so cannot cause decoherence. Quick inspection of the interactions listed in ref. [47] (and Appendix A) shows that all but two of these are additionally suppressed for super-Hubble modes, either by additional factors of slow-roll parameters or by additional powers of $k/(aH)$.

For super-Hubble scalar metric fluctuations the two relevant interactions involve either $v\,\partial^i v\,\partial_i v$ or $v\,\partial^k v^{ij}\,\partial_k v_{ij}$ (where v denotes the Mukhanov-Sasaki scalar perturbation and v_{ij} is the tensor perturbation) and both contribute with equal strength. These respectively describe decoherence generated by short-wavelength scalar and tensor fluctuations. In passing we also compute how the interaction $v^{ij}\,\partial_i v\,\partial_j v$ allows short-wavelength scalar modes to decohere super-Hubble tensor modes, finding them to be less suppressed by slow-roll parameters. We do not

[1] refs. [28–31] explore the use of Open EFT techniques for the much simpler case where late-time predictions are only sought for an Unruh-DeWitt qubit detector [32, 33], rather than for the entire ϕ field, for which very explicit calculations can be performed.

[2] We follow the power-counting estimates of refs. [45, 46] for these two expansion parameters and use fundamental units throughout (for which $\hbar = c = 1$).

compute the similar-sized contribution of short-wavelength tensor modes towards the decoherence of long-wavelength tensors.

To determine the effect of the dominant cubic interactions we compute the evolution equation for the reduced density matrix describing the quantum state of the subset of modes visible to late-time observers like ourselves. We show why this evolution is very quickly well-approximated by a Lindblad equation describing Markovian evolution for super-Hubble modes during inflation. We then integrate this equation to identify the late-time evolution of ϱ where perturbation theory naively breaks down. We use this to compute a mode's decoherence over time and show that it is already very rapid despite the feeble gravitational strength of the interaction. Inclusion of other interactions is likely only to speed up the decoherence process.

As has been remarked elsewhere [17, 48] the squeezing of modes during inflation [49] explains in a simple way why the density matrix diagonalizes in a basis of field eigenstates; making these the system's natural 'pointer' basis. It is the surviving diagonal elements $P[\varphi] = \langle \varphi \,|\, \varrho \,|\, \varphi \rangle$ at which stochastic [50–54] and de Sitter EFT [55–57] methods ultimately aim.

Our result for the amplitude of decoherence is given by eq. (4.17) and arises suppressed by the gravitational loop-counting parameter $(H/M_{\rm p})^2$ and by the first slow-roll parameter [58–60], $\varepsilon_1 = -\dot{H}/H^2$, leading to an amplitude controlled by[3]

$$\frac{\varepsilon_1 H^2}{8\pi M_{\rm p}^2} \sim \varepsilon_1^2 \, \mathcal{P}_\zeta \lesssim 10^{-4} \times 10^{-10} \tag{1.2}$$

where $\mathcal{P}_\zeta(k) \simeq H^2/(8\pi^2 \varepsilon_1 M_{\rm p}^2) \sim 10^{-10}$ is the observed size of scalar perturbations and $\varepsilon_1 \lesssim 10^{-2}$ is bounded above by the non-observance of primordial tensor perturbations [61]. But this small amplitude is abundantly compensated by an exponential growth since (4.17) grows during inflation proportional to $(aH/k)^3 \propto e^{3Ht}$.

For $\rho_{\rm inf}^{1/4} \gtrsim 5 \times 10^9$GeV, or equivalently a tensor to scalar ratio $r \gtrsim 6.5 \times 10^{-28}$, this predicts classicalization of CMB scales is long completed before inflation ends. Alternatively, if $r \sim 10^{-3}$ (and so the discovery of primordial tensor fluctuations is within reach) then decoherence becomes important for modes that spend more than around ~ 13 e-folds outside the Hubble scale during inflation. All of these estimates assume no additional decoherence occurs (or disappears) after inflation ends, or occurs during inflation due to other interactions with short-wavelength modes, or due to interactions with other environmental degrees of freedom.

Our presentation is structured as follows. sec. 2 starts by setting up the Open EFT relevant to the scalar fluctuations of the metric using only the standard building blocks of single-clock inflation. We focus on the implications of the dominant cubic self-interactions, taken from amongst the cubic interaction terms outlined in ref. [47]. We focus initially on interactions involving only scalar modes (returning to include tensors in sec. 3) setting up the system and environmental degrees of freedom in terms of the super-Hubble and sub-Hubble modes of the Mukhanov-Sasaki field v.

sec. 3 derives the relevant late-time evolution equation for super-Hubble modes, showing that it has the form of a Lindblad equation for each mode $\boldsymbol{k}$ of the field. This is done by first passing through the intermediate step of deriving a Nakajima-Zwanzig master equation and then carefully identifying the regime in which it becomes approximately Markovian. The environmental correlation functions appearing in this Lindblad evolution are evaluated explicitly and it is shown how the ultraviolet (UV) divergences encountered when doing so can be renormalized. We also provide here a preliminary discussion of the issues of the gauge-dependence of our formalism.

[3]The same arguments imply tensor modes decohere with an amplitude $H^2/M_{\rm p}^2$ (*i.e.* unsuppressed by ε_1).

sec. 4 then applies these results to compute some implications for observable modes from the removal of their shorter wavelength counterparts. Two observables computed are (*i*) very small corrections that are predicted for the power spectrum and (*ii*) the late-time decoherence that is implied for super-Hubble modes by the tracing out of these unobserved short-wavelength degrees of freedom.

We conclude in sec. 5 with a brief discussion of the open ends that our calculation does not resolve and possible next steps. Included in this discussion is a calculation of how short-wavelength scalar modes decohere super-Hubble tensor modes during inflation. This suffices to confirm the dependence on small parameters predicted by power-counting arguments but leaves open the contribution of short-wavelength tensor modes to the decoherence of primordial tensor fluctuations.

2 Open system of super-Hubble metric modes

This section sets up the open-system framework for describing the self-interactions of metric fluctuations in a near-de Sitter geometry. The system of interest is as found in many of the simplest single-clock inflationary models, with the metric $g_{\mu\nu}$ coupled to a real scalar field φ through

$$S = \int \mathrm{d}^4x \, \sqrt{-g} \left[\frac{M_\mathrm{p}^2}{2} R - \frac{1}{2} g^{\mu\nu} \, \partial_\mu \varphi \, \partial_\nu \varphi - V(\varphi) \right] \tag{2.1}$$

where $M_\mathrm{p}^{-2} = 8\pi G$, R is the Ricci scalar and $V(\varphi)$ is the potential energy of the inflaton φ. Our focus is on the late-time evolution of fluctuations about a homogeneous near de Sitter geometry given by $\varphi = \phi(t)$ and the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\, \mathrm{d}\boldsymbol{x}^2 = a^2(\eta) \left(-\mathrm{d}\eta^2 + \mathrm{d}\boldsymbol{x}^2 \right) \tag{2.2}$$

with scale factor

$$a \simeq e^{Ht} \simeq -\frac{1}{H\eta} \,. \tag{2.3}$$

Here H is the usual Hubble scale, $H = \dot{a}/a$, computed[4] using cosmic time t (to which conformal time η is related by $a\, \mathrm{d}\eta = \mathrm{d}t$).

2.1 Curvature perturbation and self-interactions

To this end we expand the scalar field about its homogeneous background, $\varphi(t, \boldsymbol{x}) = \phi(t) + \delta\varphi(t, \boldsymbol{x})$, and employ the ADM decomposition to describe small metric fluctuations about metric by foliating the space-time into a family of space-like hyper-surfaces,

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + h_{ij} \big(\mathrm{d}x^i + N^i \mathrm{d}t \big) \big(\mathrm{d}x^j + N^j \mathrm{d}t \big) \,. \tag{2.4}$$

After picking a gauge to fix time and spatial reparametrizations, standard arguments reveal that the scalar fluctuations described by the action (2.1) end up being described by a single physical scalar degree of freedom plus tensor fluctuations. More specifically, we follow ref. [47] and write the metric fluctuation to second order as

$$h_{ij} = a^2 e^{2\zeta} \hat{h}_{ij} \quad \text{with} \quad \hat{h}_{ij} = \delta_{ij} + \gamma_{ij} + \frac{1}{2} \delta^{kl} \gamma_{ik} \gamma_{lj} + \cdots \,, \tag{2.5}$$

where $\det \hat{h}_{ij} = 1$ and $\delta^{ij} \partial_i \gamma_{jk} = \delta^{ij} \gamma_{ij} = 0$. Two convenient gauge choices are then obtained by either setting $\delta\varphi = 0$ (co-moving gauge) or setting $\zeta = 0$ (spatially-flat gauge). Dependence

[4] We denote derivatives with respect to t with overdots and derivatives with respect to η using primes.

on the slow-roll parameters is easier to follow when $\delta\varphi$ is the scalar variable (since its expected amplitude does not involve the slow-roll parameters) but super-Hubble time-evolution is clearer using the variable ζ (because ζ as defined in (2.5) becomes time-independent to all orders in fluctuations in the limit $k \to 0$). This is why hereafter we work in the co-moving gauge. We here temporarily drop the tensor fluctuation,[5] γ_{ij}, and focus exclusively on scalar perturbations, since these are the ones that have arguably been observed through cosmological measurements, but circle back to reconsider tensor fluctuations in sec. 3.4.

The leading (quadratic) part of the action governing fluctuations comes from expanding (2.1) in powers of ζ and has the form (see for example refs. [47, 62, 63] – with some details also given in Appendix A)

$$ {}^{(2)}S = \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}\, \frac{\dot{\phi}^2}{2H^2}\left[a^3\dot{\zeta}^2 - a(\partial\zeta)^2\right], \tag{2.6} $$

where we recall that $\phi(t)$ denotes the background value of the inflaton and $(\partial\zeta)^2 = \delta^{ij}\partial_i\zeta\,\partial_j\zeta$. The kinetic term can be made canonical by re-expressing in terms of the Mukhanov-Sasaki variable [3, 62], given by

$$ v(\eta,\boldsymbol{x}) = aM_{\mathrm{p}}\sqrt{2\varepsilon_1}\,\zeta(\eta,\boldsymbol{x}), \tag{2.7} $$

where

$$ \varepsilon_1 = -\frac{\dot{H}}{H^2} \tag{2.8} $$

is the first slow-roll parameter – related to the field velocity through $\dot{\phi}^2 = 2H^2M_{\mathrm{p}}^2\varepsilon_1$ – which is small if the background geometry is near de Sitter. In terms of v the quadratic part of the action given in eq. (2.6) takes the canonical form [63]

$$ {}^{(2)}S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3\boldsymbol{x}\left[(v')^2 - \delta^{ij}\partial_i v\,\partial_j v + \frac{\left(a\sqrt{\varepsilon_1}\right)''}{a\sqrt{\varepsilon_1}}\,v^2\right]. \tag{2.9} $$

Interactions arise at cubic and higher orders in the fluctuations, with $S_{\mathrm{int}} = {}^{(3)}S + {}^{(4)}S + \cdots$ where ${}^{(n)}S$ involves n powers of the fluctuation fields. Amongst the self-interactions involving just ζ obtained in this way is

$$ {}^{(3)}S \supset \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}\, \frac{\dot{\phi}^4 a}{4H^4M_{\mathrm{p}}^2}\,(\partial\zeta)^2\,\zeta \tag{2.10} $$

where the symbol "$\supset$" emphasizes that there are other cubic interactions in ${}^{(3)}S$ that are not explicitly written (for a full list of the cubic scalar interactions see Eq. (3.9) of ref. [47], or eq. (A.15) in appendix A).

As argued in the introduction, quartic and higher interactions beyond those cubic in v need not be considered when computing decoherence of super-Hubble modes because they can contribute only sub-dominantly in $1/M_{\mathrm{p}}$. The variable v is convenient when counting factors of $1/M_{\mathrm{p}}$ in this way because its lowest-order correlations functions are independent of M_{p}. In terms of v the cubic interaction (2.10) becomes

$$ {}^{(3)}S \supset \int \mathrm{d}\eta\, \mathrm{d}^3\boldsymbol{x}\, \frac{\sqrt{\varepsilon_1}}{2\sqrt{2}\,M_{\mathrm{p}}\,a}\left(\delta^{ij}\partial_i v\partial_j v\right)v\,, \tag{2.11} $$

revealing it also to be order $\sqrt{\varepsilon_1}$ in slow-roll. What is important for our later purposes is that all of the other cubic interactions listed in ref. [47] are either higher-order in slow-roll parameters

[5] The variable v_{ij} used above is equivalent to γ_{ij}, just normalized differently – see (A.17).

or trade two spatial derivatives for two time derivatives.[6] The freezing of ζ on super-Hubble scales implies $\dot\zeta/\zeta \propto k^2/(aH)^2$ for $k \ll aH$ and so implies time derivatives contribute only sub-dominantly in powers of $k/(aH)$ for super-Hubble modes.

The momentum conjugate to v is $p = \delta S/\delta v' = v'$, and the Hamiltonian is

$$\mathcal{H}(\eta) = \mathcal{H}_0(\eta) + \mathcal{H}_{\rm int}(\eta) \tag{2.12}$$

with free part

$$\mathcal{H}_0 := \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,\left[p^2 + \delta^{ij}\partial_i v\,\partial_j v - \frac{\left(a\sqrt{\varepsilon_1}\right)''}{a\sqrt{\varepsilon_1}}\,v^2\right]. \tag{2.13}$$

The interaction corresponding to (2.11) is

$$\mathcal{H}_{\rm int} \supset -\frac{\sqrt{\varepsilon_1}}{2\sqrt{2}M_{\rm p}a}\int \mathrm{d}^3\boldsymbol{x}\;\delta^{ij}v\,\partial_i v\,\partial_j v + \cdots. \tag{2.14}$$

This form of the interaction Hamiltonian is obtained in the co-moving gauge. Although gauge-independent observables are difficult to construct at cubic and higher order in cosmological perturbation theory, in ref. [47] it was checked that a calculation performed in the spatially-flat gauge gives the same result for the bispectrum. This supports the idea that physical results obtained from our calculations using this interaction Hamiltonian will be gauge independent.

Our goal is to describe dynamics perturbatively in $\mathcal{H}_{\rm int}$ and so it is useful first to diagonalize $\mathcal{H}_0$. This is achieved in momentum space,

$$v(\eta,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\;v_{\boldsymbol{k}}(\eta)\,e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2.15}$$

and writing $v_{\boldsymbol{k}}(\eta)$ in terms of mode functions $u_{\boldsymbol{k}}(\eta)$

$$v_{\boldsymbol{k}}(\eta) = u_{\boldsymbol{k}}(\eta)c_{\boldsymbol{k}} + u^*_{-\boldsymbol{k}}(\eta)c^\dagger_{-\boldsymbol{k}}. \tag{2.16}$$

shows that hermiticity in real space $v(\eta,\boldsymbol{x}) = v^\dagger(\eta,\boldsymbol{x})$ implies $v_{-\boldsymbol{k}}(\eta) = v^\dagger_{\boldsymbol{k}}(\eta)$ in momentum space, and these are both equivalent to having particles and antiparticles not being independent of one another. A similar expression holds for the conjugate momentum field $p_{\boldsymbol{k}}$. Equal-time commutation relations for the operators $v_{\boldsymbol{k}}(\eta)$ and $p_{\boldsymbol{k}}(\eta)$

$$\left[v_{\boldsymbol{k}}(\eta), p_{\boldsymbol{q}}(\eta)\right] = i\delta(\boldsymbol{k}+\boldsymbol{q}), \tag{2.17}$$

are equivalent to $[c_{\boldsymbol{k}}, c^\dagger_{\boldsymbol{q}}] = \delta(\boldsymbol{k}-\boldsymbol{q})$, provided the $u_{\boldsymbol{k}}(\eta)$ are normalized by $u_{\boldsymbol{k}}u^{*\prime}_{-\boldsymbol{k}} - u^*_{\boldsymbol{k}}u'_{-\boldsymbol{k}} = i$.

Plugging this decomposition into (2.13) gives, at the classical level,

$$\mathcal{H}(\eta) := \frac{1}{2}\int \mathrm{d}^3\boldsymbol{k}\;\left[p_{\boldsymbol{k}}(\eta)p^*_{\boldsymbol{k}}(\eta) + \omega^2(\boldsymbol{k},\eta)v_{\boldsymbol{k}}(\eta)v^*_{\boldsymbol{k}}(\eta)\right]. \tag{2.18}$$

On quantization noncommuting operators are replaced by their symmetrized product – such as $p_{\boldsymbol{k}}p^*_{\boldsymbol{k}} \to \frac{1}{2}\{p_{\boldsymbol{k}}\,, p^\dagger_{\boldsymbol{k}}\}$ – so that hermiticity is preserved. The time-dependent frequency is

$$\omega^2(\boldsymbol{k},\eta) := k^2 - \frac{(a\sqrt{\varepsilon_1})''}{a\sqrt{\varepsilon_1}}. \tag{2.19}$$

[6]The neglect of time derivatives relative to spatial derivatives acting on environmental modes is only justified when the environmental modes are also super-Hubble and need not be a good approximation for modes with $k/a \sim H$. Our calculations shed some light on the validity of this approximation, though the issue remains a partially open problem. See sec. 4 for further discussion.

In the limit $\varepsilon_1 \to 0$ this frequency function $\omega(\boldsymbol{k}, \eta)$ takes the well-known de Sitter form

$$\omega^2(\boldsymbol{k}, \eta) \simeq k^2 - \frac{2}{\eta^2}\,, \tag{2.20}$$

and describes adiabatic evolution in the regime $k^2\eta^2 \gg 1$.

In what follows we quantize using a field basis rather than the particle Fock space built using the creation and annihilation operators $c_{\boldsymbol{k}}$ and $c^\dagger_{\mathbf{k}}$. For the free system this is the analog of treating harmonic oscillators using states $\langle x|\Psi\rangle$ and density matrices $\langle x|\rho|y\rangle$ described in the position basis (rather than using occupation-number representations $\langle n|\Psi\rangle$ and $\langle n|\rho|m\rangle$ built from the raising and lowering operators c and $c^\dagger$). We briefly pause here to clarify an issue that arises due to the reality condition $v_{-\boldsymbol{k}} = v^\dagger_{\boldsymbol{k}}$.

Normally the position eigenvalue x for an oscillator is real and so having a complex field $v_{\boldsymbol{k}}$ for each $\boldsymbol{k}$ sounds like it contains too many coordinates to describe a single oscillator for each $\boldsymbol{k}$. For a complex field (for which particles and antiparticles are not identified) this is correct: complex coordinates correspond to two sets of real position coordinates and these correspond to the two types of oscillator – one each for particle and antiparticle – that exist for each $\boldsymbol{k}$.

For real fields the condition $v_{-\boldsymbol{k}} = v^\dagger_{\boldsymbol{k}}$ cuts the number of oscillators in half and so leaves a single oscillator for each $\boldsymbol{k}$. There are two equivalent ways to frame the field representation in this case. We can either restrict ourselves to only half of the total available momentum labels and keep the complex variables $v_{\boldsymbol{k}}$ arbitrary, or we can keep all momentum labels and use the reality condition to have effectively only a single real field for each $\boldsymbol{k}$.

To see how these are related in detail we follow ref. [37] and write the real and imaginary parts of $v_{\boldsymbol{k}}$

$$v_{\boldsymbol{k}}(\eta) =: \frac{v^{(\mathrm{R})}_{\boldsymbol{k}}(\eta) + i\, v^{(\mathrm{I})}_{\boldsymbol{k}}(\eta)}{\sqrt{2}}, \tag{2.21}$$

for which $v_{-\boldsymbol{k}} = v^\dagger_{\boldsymbol{k}}$ implies $v^{\mathrm{R}}_{\boldsymbol{k}} = v^{\mathrm{R}}_{-\boldsymbol{k}}$ while $v^{\mathrm{I}}_{\boldsymbol{k}} = -v^{\mathrm{I}}_{-\boldsymbol{k}}$. $v^{\mathrm{R}}_{\boldsymbol{k}}$ and $v^{\mathrm{I}}_{\boldsymbol{k}}$ evolve separately under linear evolution and this evolution is identical provided the Hamiltonian is invariant under reflections in $\boldsymbol{k}$, which is true in particular if the physics involved is parity invariant or if it is invariant under arbitrary rotations. We may therefore treat the system as if it involves a single real field, $\tilde{v}_{\boldsymbol{k}} = \tilde{v}^\dagger_{\boldsymbol{k}}$ for *all* $\boldsymbol{k}$ and then identify $\sqrt{2}\; v^{\mathrm{R}}_{\boldsymbol{k}} = \tilde{v}_{\boldsymbol{k}} + \tilde{v}_{-\boldsymbol{k}}$ and $\sqrt{2}\,i\; v^{\mathrm{I}}_{\boldsymbol{k}} = \tilde{v}_{\boldsymbol{k}} - \tilde{v}_{-\boldsymbol{k}}$ respectively as its even and odd parts under reflection. The evolution equation for $\tilde{v}$, v^{R} and v^{I} are all identical in the applications below. In what follows our interest is in the matrix elements of the density matrix $\rho_{\boldsymbol{k}}$ for the single oscillator that arises for each $\boldsymbol{k}$, and for fixed $\boldsymbol{k}$ we compute their evolution in an eigenbasis of the real field $\tilde{v}_{\boldsymbol{k}}$, since this simplifies the notation by allowing us to drop the superscripts 'R' and 'I' on the fields. A similar story applies also to the momentum which we denote $\tilde{p}_{\boldsymbol{k}}$.

2.2 The system and the environment

We next divide the Hilbert space of states for this system into the 'system' (*i.e.* degrees of freedom we choose to follow because they appear in observations made at late times) and an 'environment' (consisting of all modes that are not observed), see also Fig. 1. Having made this split we trace out over the environment modes and follow only state evolution within the observed sector.

Present-day measurements only sample primordial fluctuations whose co-moving momenta have magnitudes $k = |\boldsymbol{k}|$ that lie within a finite range

$$k_{\mathrm{IR}} < k < k_{\mathrm{UV}} \tag{2.22}$$

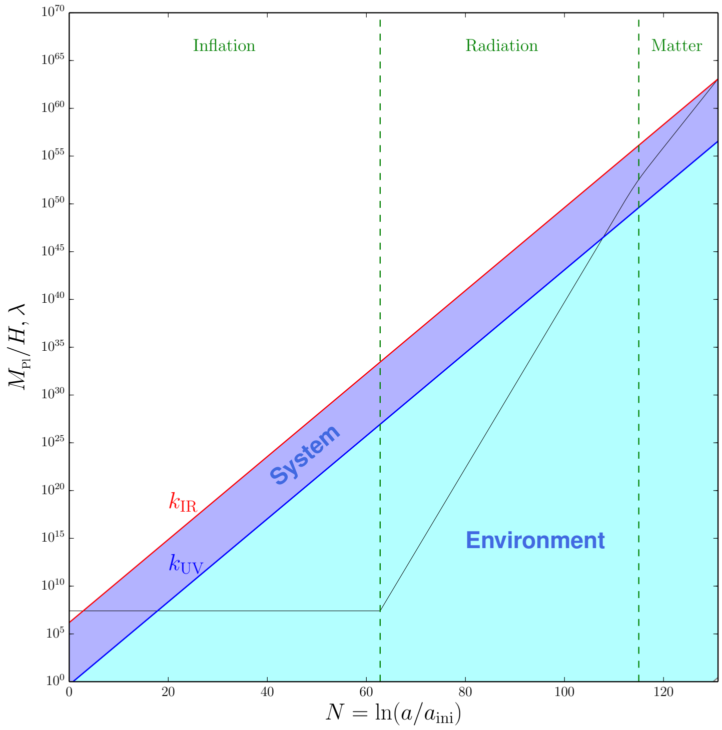

Figure 1: We sketch out the domain of the system and environment modes. The black line denotes the Hubble radius and the coloured lines stand for the mode wavelengths. The system is comprised of co-moving scales between $k_{\rm IR}$ and $k_{\rm UV}$, both of which are outside the Hubble radius at the end of inflation. The environment is made of all scales such that $k > k_{\rm UV}$.

where $k_{\rm IR}/a_0 \sim 0.05\, a_0{\rm Mpc}^{-1}$ and $k_{\rm UV}/a_0$ (with $k_{\rm UV} \sim 2500\, k_{\rm IR}$) are the smallest and largest currently observable physical momenta (such as through CMB or large-scale structure observations) and a_0 is the present-day scale factor. fig. 1 shows this schematically (not to scale).

In what follows we define the observed system (system A) to be those modes whose co-moving momenta satisfy[7] $k < k_{\rm UV}$ while the environment (system B) satisfies $k > k_{\rm UV}$, and so write the position-space field as

$$v(\eta, \boldsymbol{x}) = v_A(\eta, \boldsymbol{x}) \otimes \mathcal{I}_B \; + \mathcal{I}_A \otimes v_B(\eta, \boldsymbol{x}) \tag{2.23}$$

with v_A denoting the system and v_B representing the environment:

$$v_A(\eta, \boldsymbol{x}) := \int \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\, \Theta\big(k_{\rm UV} - k\big) v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \int_{k<k_{\rm UV}} \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\, v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \tag{2.24}$$

$$v_B(\eta, \boldsymbol{x}) := \int \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\, \Theta\big(k - k_{\rm UV}\big) v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \int_{k>k_{\rm UV}} \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\, v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{2.25}$$

[7] In practice we define our system to include both observable modes and those with $k < k_{\rm IR}$ whose wavelengths are too long to have been observed. Such modes are expected to be absorbable into the definition of the background geometry, and their inclusion should not affect our later discussion since they do not contribute to the observables (*e.g.* decoherence) on which we ultimately focus. (See ref. [64] for an open-system treatment of IR modes.)

where $\Theta(x)$ is the Heaviside step function and $\mathcal{I}_A$ and $\mathcal{I}_B$ representing the appropriate unit operators. The free Hamiltonian similarly becomes

$$\mathcal{H}_0(\eta) = \mathcal{H}_A(\eta) \otimes \mathcal{I}_B + \mathcal{I}_A \otimes \mathcal{H}_B(\eta) \tag{2.26}$$

where $\mathcal{H}_A$ and $\mathcal{H}_B$ are both given by (2.18) but with the momentum range respectively restricted to the intervals $k < k_{\rm UV}$ and $k > k_{\rm UV}$.

Inserting the decomposition (2.23) into the cubic interaction Hamiltonian $\mathcal{H}_{\rm int}$ defined in eq. (2.14) gives several contributions, of the schematic form v_A^3, $v_A^2 v_B$, $v_A v_B^2$ and v_B^3. Of these, only the cross terms ($v_A^2 v_B$ and $v_A v_B^2$) couple the system to the environment (and so contribute, say, to decoherence), and of these momentum conservation suppresses the $v_A^2 v_B$ interactions because it is impossible to sum two small momenta to get a large one. The largest interactions also come when derivatives act only on large-momentum (environment) fields. For these reasons we focus primarily on the contribution that has the form $v_A \delta^{ij}(\partial_i v_B)(\partial_j v_B)$ and so take

$$\mathcal{H}_{\rm int}(\eta) \supset -\frac{\sqrt{\varepsilon_1}}{2\sqrt{2}\,M_{\rm p} a(\eta)} \int {\rm d}^3\boldsymbol{x}\; v_A(\eta,\boldsymbol{x}) \otimes \delta^{ij}\partial_i v_B(\eta,\boldsymbol{x}) \partial_j v_B(\eta,\boldsymbol{x}) \; . \tag{2.27}$$

We therefore seek an open-system description of evolution using the Hamiltonian

$$\mathcal{H}(\eta) = \mathcal{H}_A(\eta) \otimes \mathcal{I}_B + \mathcal{I}_A \otimes \mathcal{H}_B(\eta) + \mathcal{H}_{\rm int}(\eta) \tag{2.28}$$

with Hamiltonians given explicitly by eqs. (2.18) and (2.27). We circle back to include mixed tensor-scalar cubic interactions in sec. 3.4.

2.3 State evolution

We seek to predict the evolution of the system's state $\rho(\eta)$ given the above choice of Hamiltonian, at first working perturbatively. We do so within both Schrödinger picture and interaction picture, since each can be more convenient for some kinds of questions.

The Schrödinger-picture density matrix for the full system-plus-environment, $\rho_{\rm S}(\eta)$, evolves through the standard Liouville equation,

$$\frac{\partial \rho_{\rm S}}{\partial \eta} = -i\Big[\mathcal{H}_{\rm S}(\eta), \rho_{\rm S}(\eta)\Big] \; . \tag{2.29}$$

The interaction picture density matrix is defined relative to this by

$$\rho(\eta) = U_0^\dagger(\eta,\eta_{\rm in}) \rho_{\rm S}(\eta)\, U_0(\eta,\eta_{\rm in}) \; , \tag{2.30}$$

where

$$U_0(\eta_1,\eta_2) := \mathcal{T} \exp\left(-i \int_{\eta_2}^{\eta_1} {\rm d}\eta\; \mathcal{H}_{0{\rm S}}(\eta) \right) \; , \tag{2.31}$$

and so satisfies

$$\frac{\partial \rho}{\partial \eta} = -i\Big[\mathcal{H}_{\rm int}(\eta), \rho(\eta)\Big] \, . \tag{2.32}$$

In particular, both pictures agree at the initial time $\rho(\eta_{\rm in}) = \rho_{\rm S}(\eta_{\rm in})$.

Field operators $v_{\rm S}(\boldsymbol{x})$ are time-independent in Schrödinger picture, but interaction-picture fields

$$v(\eta,\boldsymbol{x}) \;:=\; U_0^\dagger(\eta,\eta_{\rm in}) v_{\rm S}(\boldsymbol{x})\, U_0(\eta,\eta_{\rm in}) \tag{2.33}$$

evolve only via the free part of the Hamiltonian with initial condition $v(\eta_{\rm in},\boldsymbol{x}) = v_{\rm S}(\boldsymbol{x})$. Because the free evolution factorizes between system and environment

$$U_0(\eta_1,\eta_2) := U_{0A}(\eta_1,\eta_2)\otimes U_{0B}(\eta_1,\eta_2) \tag{2.34}$$

where

$$U_{0A}(\eta_1,\eta_2) := \mathcal{T}\exp\left(-i\int_{\eta_2}^{\eta_1}{\rm d}\eta\;\mathcal{H}_{AS}(\eta)\right) \quad\text{and}\quad U_{0B}(\eta_1,\eta_2) := \mathcal{T}\exp\left(-i\int_{\eta_2}^{\eta_1}{\rm d}\eta\;\mathcal{H}_{BS}(\eta)\right)\,, \tag{2.35}$$

the system and environment parts of the fields

$$v_{\rm S}(\boldsymbol{x}) = v_{AS}(\boldsymbol{x})\otimes\mathcal{I}_B + \mathcal{I}_A\otimes v_{BS}(\boldsymbol{x}) \tag{2.36}$$

evolve independently under free evolution:

$$v_A(\eta,\boldsymbol{x}) = U_{0A}^\dagger(\eta,\eta_{\rm in})v_{AS}(\boldsymbol{x})U_{0A}(\eta,\eta_{\rm in}) \quad\text{and}\quad v_B(\eta,\boldsymbol{x}) = U_{0B}^\dagger(\eta,\eta_{\rm in})v_{BS}(\boldsymbol{x})U_{0B}(\eta,\eta_{\rm in})\,. \tag{2.37}$$

We evolve the state assuming that the system and environment are uncorrelated at $\eta=\eta_{\rm in}$ and all modes are prepared in the Bunch-Davies vacuum [65]:

$$\rho(\eta_{\rm in}) = \rho_{\rm S}(\eta_{\rm in}) = |0\rangle\langle 0| = |0_A\rangle\langle 0_A|\otimes|0_B\rangle\langle 0_B|\,. \tag{2.38}$$

Here

$$|0_A\rangle := \bigotimes_{k<k_{\rm UV}}|0_{\boldsymbol{k}}\rangle \quad\text{and}\quad |0_B\rangle := \bigotimes_{k>k_{\rm UV}}|0_{\boldsymbol{k}}\rangle \quad\text{with}\quad c_{\boldsymbol{k}}(\eta_{\rm in})|0_{\boldsymbol{k}}\rangle = 0 \quad\text{for all } \boldsymbol{k}\,, \tag{2.39}$$

with the mode functions $u_{\boldsymbol{k}}(\eta)$ appearing in eq. (2.16) given (for massless states) by

$$u_{\boldsymbol{k}}(\eta) = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1-\frac{i}{k\eta}\right)\,. \tag{2.40}$$

The time evolution for observations restricted to be only in sector A is completely determined by the evolution of the reduced density matrix, ϱ, obtained by tracing out all the environment degrees of freedom of sector B from the full density matrix.

$$\varrho(\eta) := \operatorname*{Tr}_B\Big[\rho(\eta)\Big]\,. \tag{2.41}$$

In the absence of interactions the free evolution of the density matrix factorizes in momentum space, with separate momenta remaining uncorrelated

$$\varrho(\eta) = \bigotimes_{k<k_{\rm UV}}\varrho_{\boldsymbol{k}}(\eta)\,, \tag{2.42}$$

at all times. The time-dependence of each factor describes the squeezing of super-Hubble modes due to their non-adiabatic evolution in the presence of the time-dependent Hamiltonian. It is the deviations from this that are of most interest in what follows.

3 Evolution equations

This section contains the core derivation on which our results ultimately depend: we derive here how the interaction (2.27) alters the late-time evolution of the reduced density matrix for the observed modes. Because this interaction is linear in the long-wavelength field our evolution equation remains quadratic in this field even at second order, and so evolution still proceeds separately for each super-Hubble mode $\boldsymbol{k}$, greatly simplifying the analysis.

We do so because this proves to be the dominant interaction through which short-wavelength scalar modes decohere long-wavelength scalar fluctuations. Simplified evolution emerges for super-Hubble modes at late times and its domain of validity is studied in some detail because within it predictions can be made at unusually late times. Finally, these arguments are repeated for gravitational interactions coupling scalar and tensor metric modes to determine ones through which these modes dominantly decohere one another.

3.1 Nakajima-Zwanzig equation

As an intermediate step we first derive the Nakajima-Zwanzig equation for the reduced density matrix ϱ. This equation explicitly eliminates the unseen environmental degrees of freedom to rewrite the Liouville equation purely in terms of the observed degrees of freedom.

Although very general, this equation computes $\partial_t \varrho$ as a convolution of the earlier values of ϱ throughout its past history, and so is not so in itself. We show how this equation simplifies when specializing to super-Hubble modes at very late times (compared with the Hubble time), because it then becomes Markovian. The resulting Lindblad-type evolution equation for the reduced density matrix ϱ lends itself to making reliable late-time predictions that would otherwise lie beyond the reach of perturbative methods.

3.1.1 General derivation

To simplify later formulae it is useful to write the interaction-picture interaction Hamiltonian (2.27) as

$$\mathcal{H}_{\rm int}(\eta) = G(\eta) \int \mathrm{d}^3\boldsymbol{x}\; v_{\scriptscriptstyle A}(\eta, \boldsymbol{x}) \otimes B(\eta, \boldsymbol{x})\,, \tag{3.1}$$

where B denotes the relevant environmental field combination

$$B(\eta, \boldsymbol{x}) := \delta^{ij} \partial_i v_{\scriptscriptstyle B}(\eta, \boldsymbol{x}) \partial_j v_{\scriptscriptstyle B}(\eta, \boldsymbol{x}), \tag{3.2}$$

and the coupling strength is

$$G(\eta) := -\frac{\sqrt{\varepsilon_1}}{2\sqrt{2}\, M_{\rm p}\, a(\eta)}\,. \tag{3.3}$$

To derive the Nakajima-Zwanzig equation (see ref. [28] for a similar derivation in a simpler setting) we define the projection super-operator $\mathcal{P}$ to act on an arbitrary operator $\mathcal{O}$ in the Hilbert space by

$$\mathcal{P}\{\mathcal{O}\} = \underset{B}{\mathrm{Tr}}\big[\mathcal{O}\big] \otimes |0_{\scriptscriptstyle B}\rangle\langle 0_{\scriptscriptstyle B}|\ \ , \tag{3.4}$$

where $|0_{\scriptscriptstyle B}\rangle\langle 0_{\scriptscriptstyle B}|$ is the Bunch Davies vacuum for the environment sector as given in eq. (2.39). This satisfies $\mathcal{P}^2 = \mathcal{P}$ as does its complement $\mathcal{Q} = 1 - \mathcal{P}$, which is also a projection super-operator. In particular, $\mathcal{P}$ maps the full density matrix $\rho(\eta)$ onto the reduced density matrix $\varrho(\eta)$ as follows:

$$\mathcal{P}\{\rho(\eta)\} = \varrho(\eta) \otimes |0_{\scriptscriptstyle B}\rangle\langle 0_{\scriptscriptstyle B}|\ , \tag{3.5}$$

where ϱ is the reduced density matrix for the system defined by (2.41).

The Nakajima-Zwanzig equation for $\varrho(\eta)$ is derived by applying $\mathcal{P}$ to the interaction picture Liouville equation, written in terms of a Liouville super-operator:

$$\partial_\eta \rho(\eta) = \mathscr{L}_\eta\{\rho(\eta)\} \quad \text{with} \quad \mathscr{L}_\eta\{\rho(\eta)\} := -i\Big[\mathcal{H}_{\rm int}(\eta), \rho(\eta)\Big]\,, \tag{3.6}$$

with the goal of expressing it purely in terms of ϱ. Using $\mathcal{P} + \mathcal{Q} = 1$ this leads to

$$\mathcal{P}\,\{\partial_\eta \rho(\eta)\} = \mathcal{P}\mathscr{L}_\eta\mathcal{P}\{\rho(\eta)\} + \mathcal{P}\mathscr{L}_\eta\mathcal{Q}\{\rho(\eta)\} \tag{3.7}$$
$$\mathcal{Q}\{\partial_\eta \rho(\eta)\} = \mathcal{Q}\mathscr{L}_\eta\mathcal{P}\{\rho(\eta)\} + \mathcal{Q}\mathscr{L}_\eta\mathcal{Q}\{\rho(\eta)\}\,. \tag{3.8}$$

These can be regarded as evolution equations for $\mathcal{P}\{\rho\}$ and $\mathcal{Q}\{\rho\}$ provided $\mathcal{P}\{\partial_\eta\rho\} = \partial_\eta\mathcal{P}\{\rho\}$, which is true because the Bunch-Davies vacuum used in the definition (3.5) is time-independent in the interaction picture.

(3.8) can then be employed to eliminate $\mathcal{Q}\{\rho(\eta)\}$, using the formal solution

$$\mathcal{Q}\{\rho(\eta)\} = \mathcal{G}(\eta,\eta_{\rm in})\mathcal{Q}\{\rho(\eta_{\rm in})\} + \int_{\eta_{\rm in}}^{\eta} {\rm d}\tau\; \mathcal{G}(\eta,\tau)\mathcal{Q}\mathscr{L}_\eta\mathcal{P}\{\rho(\tau)\} \tag{3.9}$$

where

$$\mathcal{G}(\eta,\tau) := 1 + \sum_{n=1}^{\infty} \int_\tau^\eta {\rm d}\tau_1 \;\cdots \int_\tau^{\tau_{n-1}} {\rm d}\tau_n\; \mathcal{Q}\mathscr{L}_{\tau_1}\cdots\mathcal{Q}\mathscr{L}_{\tau_n}\,. \tag{3.10}$$

Inserting this into eq. (3.7) yields

$$\mathcal{P}\{\partial_\eta\rho(\eta)\} = \mathcal{P}\mathscr{L}_\eta\mathcal{P}\{\rho(\eta)\} + \mathcal{P}\mathscr{L}_\eta\mathcal{G}(\eta,0)\mathcal{Q}\{\rho(\eta_{\rm in})\} + \int_{\eta_{\rm in}}^{\eta} {\rm d}s\; \mathcal{K}(\eta,s)\{\rho(s)\} \tag{3.11}$$

with kernel

$$\mathcal{K}(\eta,s) = \mathcal{P}\mathscr{L}_\eta\mathcal{G}(\eta,s)\mathcal{Q}\mathscr{L}_s\mathcal{P}\,. \tag{3.12}$$

For the uncorrelated initial state used here the second term in eq. (3.11) vanishes, since

$$\mathcal{Q}\Big\{|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B|\Big\} = 0\,. \tag{3.13}$$

This gives an evolution equation that involves only $\mathcal{P}\{\rho\}$, which can be expanded to any desired order in $\mathcal{H}_{\rm int}$. Stopping at second order – for which we may use $\mathcal{G}(\eta,s) \simeq 1$ – we find

$$\partial_\eta\mathcal{P}\{\rho(\eta)\} = \mathcal{P}\{\partial_\eta\rho(\eta)\} \simeq \mathcal{P}\mathscr{L}_\eta\mathcal{P}\{\rho(\eta)\} + \int_{\eta_{\rm in}}^{\eta} {\rm d}s\; \mathcal{P}\mathscr{L}_\eta\mathcal{Q}\mathscr{L}_s\mathcal{P}\{\rho(s)\}\,, \tag{3.14}$$

which, using the definitions of $\mathcal{P}$, $\mathcal{Q}$ and $\mathscr{L}_\eta$, becomes the following equation for ϱ:

$$\begin{aligned}\frac{\partial\varrho}{\partial\eta} \simeq -i\,\underset{B}{\rm Tr}\Big\{\Big[\mathcal{H}_{\rm int}(\eta), \varrho(\eta)\otimes|0_B\rangle\langle 0_B|\Big]\Big\} - \int_{\eta_{\rm in}}^{\eta} {\rm d}\eta'\; \underset{B}{\rm Tr}\Big\{\Big[\mathcal{H}_{\rm int}(\eta)\,,\, \Big[\mathcal{H}_{\rm int}(\eta')\,,\, \varrho(\eta')\otimes|0_B\rangle\langle 0_B|\Big] \\ - \underset{B}{\rm Tr}\Big\{\Big[\mathcal{H}_{\rm int}(\eta'), \varrho(\eta')\otimes|0_B\rangle\langle 0_B|\Big]\Big\}\otimes|0_B\rangle\langle 0_B|\Big]\Big\}\,.\end{aligned} \tag{3.15}$$

Specializing to the interaction (3.1) the evolution equation for ϱ finally simplifies to

$$\frac{\partial\varrho}{\partial\eta} \simeq -iG(\eta)\mathscr{B}(\eta)\int {\rm d}^3\boldsymbol{x}\;\; [v_A(\eta,\boldsymbol{x}), \varrho(\eta)] - \int {\rm d}^3\boldsymbol{x}\int {\rm d}^3\boldsymbol{x}'\int_{\eta_{\rm in}}^{\eta} {\rm d}\eta'\; G(\eta)G(\eta')$$

$$\times \left\{ [v_A(\eta, \boldsymbol{x}), v_A(\eta', \boldsymbol{x}')\varrho(\eta')]\, C_B(\eta, \eta'; \boldsymbol{x} - \boldsymbol{x}') + [\varrho(\eta')v_A(\eta', \boldsymbol{x}'), v_A(\eta, \boldsymbol{x})]\, C_B^*(\eta, \eta'; \boldsymbol{x} - \boldsymbol{x}') \right\} \tag{3.16}$$

whose right-hand side neglects $\mathcal{O}(G^3)$ terms. The required expectation values of the environment operator B are

$$\mathscr{B}(\eta) := \langle 0_B | B(\eta, \boldsymbol{x}) | 0_B \rangle = \int_{k > k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} |\boldsymbol{k}|^2 |u_{\boldsymbol{k}}(\eta)|^2 \tag{3.17}$$

and

$$C_B(\eta, \eta'; \boldsymbol{x} - \boldsymbol{x}') = \langle 0_B |\, [B(\eta, \boldsymbol{x}) - \mathscr{B}(\eta)]\, [B(\eta', \boldsymbol{x}') - \mathscr{B}(\eta')]\, | 0_B \rangle. \tag{3.18}$$

The second equality in (3.17) uses the translation-invariance of the Bunch-Davies state as well as the specific for the operator given in (3.2), and a similar expression for C_B is given in eq. (B.9). The integral in (3.17) diverges and part of the later discussion shows how such divergences are handled.

3.1.2 Nakajima-Zwanzig equation for each mode

The second-order Nakajima-Zwanzig equation (3.16) simplifies considerably when the interaction is linear in v_A – as it is in (3.1) – because evolution does not mix modes, similar to free evolution. To make this explicit define the momentum-space correlation function $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ using

$$C_B(\eta, \eta'; \boldsymbol{y}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \mathscr{C}_{\boldsymbol{k}}(\eta, \eta') e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\,. \tag{3.19}$$

$\mathscr{C}_{\boldsymbol{k}}$ is computed explicitly in Appendix B.2 for $k < k_{\rm UV}$, which shows in particular that $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ only depends on the modulus $k = |\boldsymbol{k}|$ – and so in particular $\mathscr{C}_{-\boldsymbol{k}}(\eta, \eta') = \mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$.

Factorizing the density matrix as in eq. (2.42) shows that $\varrho_{\boldsymbol{k}}$ for each mode $\boldsymbol{k}$ evolves independently. eq. (3.16) implies that (for $\boldsymbol{k} \neq 0$) each factor separately satisfies

$$\frac{\mathcal{V}}{(2\pi)^3} \frac{\partial \varrho_{\boldsymbol{k}}}{\partial \eta} = -(2\pi)^{3/2} \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta'\, G(\eta)\, G(\eta') \Big\{ [\tilde{v}_{\boldsymbol{k}}(\eta), \tilde{v}_{\boldsymbol{k}}(\eta')\varrho_{\boldsymbol{k}}(\eta')]\, \mathscr{C}_{\boldsymbol{k}}(\eta, \eta') + [\varrho_{\boldsymbol{k}}(\eta')\tilde{v}_{\boldsymbol{k}}(\eta'), \tilde{v}_{\boldsymbol{k}}(\eta)]\, \mathscr{C}^*_{\boldsymbol{k}}(\eta, \eta') \Big\}, \tag{3.20}$$

where $\tilde{v}_{\boldsymbol{k}}$ is the proxy for $v^{\rm R}_{\boldsymbol{k}}$ and $v^{\rm I}_{\boldsymbol{k}}$ defined below (2.21) and the factor $\mathcal{V}$ denotes the volume of space and arises when keeping track of the normalization of momentum modes in the continuum limit (see appendix D). Its presence ensures the final expressions remains finite as $\mathcal{V} \to \infty$ when $\boldsymbol{k}$ is taken to be continuum normalized. A similar expression also holds for $\boldsymbol{k} = 0$ but also includes a contribution from $\mathscr{B} = \langle B \rangle_{\rm env}$.

3.1.3 Environmental correlations

Later sections explore implications of the Nakajima-Zwanzig equation (3.20) and in particular what it predicts for very late times. The result depends in detail on the environmental correlator $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$, which is computed explicitly in appendix B, see eq. (B.62). It is plotted as a function of $k\eta'$ in fig. 2. For later use this section summarizes several useful limits of the result found there.

In the coincidence limit $\eta' \to \eta$, the correlator is singular and can be expanded as

$$\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') \simeq \frac{k^5}{32(2\pi)^{7/2}} \left[\frac{1}{(-k\eta)^4} \left(-32\kappa - 8 - \frac{64}{3\kappa} + \frac{4}{\kappa^2} \right) + \frac{1}{(-k\eta)^2} \left(-\frac{64}{3}\kappa^3 - 16\kappa^2 + 32\kappa \right. \right.$$

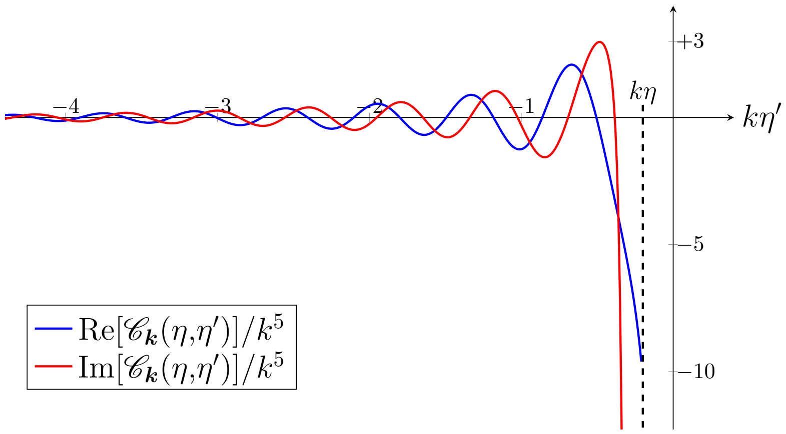

Figure 2: $\mathrm{Re}[\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')]$ and $\mathrm{Im}[\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')]$ as a function of $k\eta'$ for $k\eta = -0.2$ and $k_{\rm UV}/k = 5$. Note the singularity at $\eta' \simeq \eta$.

$$
\begin{aligned}
&+\frac{16}{3}+\frac{64}{15\kappa}\bigg)-\frac{32}{5}\kappa^5-8\kappa^4+\frac{32}{9}\kappa^3+4\kappa^2-\frac{64}{15}\kappa-\frac{44}{45}\bigg]+\frac{\pi}{32(2\pi)^{7/2}}\bigg[\delta''''(\eta-\eta')\\
&+\frac{4(\eta-\eta')}{\eta\eta'}\delta'''(\eta-\eta')+\frac{4\big[\eta^2+(\eta')^2+\big(\frac{5}{6}k^2\eta\eta'-4\big)\eta\eta'\big]}{\eta^2(\eta')^2}\delta''(\eta-\eta')\\
&+\frac{4(\eta-\eta')\big(3k^2\eta\eta'-4\big)}{\eta^2(\eta')^2}\delta'(\eta-\eta')+\frac{\frac{43}{15}k^4\eta^2(\eta')^2+\frac{4}{3}k^2\big[9\eta^2-32\eta\eta'+9(\eta')^2\big]+16}{\eta^2(\eta')^2}\\
&\times\delta(\eta-\eta')\bigg]+\frac{i}{32(2\pi)^{7/2}}\bigg[\frac{24}{(\eta'-\eta)^5}+\frac{20(k^2\eta^2-6)}{3\eta^2(\eta'-\eta)^3}+\frac{40}{\eta^3(\eta'-\eta)^2}\\
&+\frac{k^2(43k^2\eta^2-460)}{15\eta^2(\eta'-\eta)}-\frac{40}{\eta^5}+\frac{92k^2}{3\eta^3}\bigg].
\end{aligned}
\tag{3.21}
$$

where we define

$$
\kappa := \frac{k_{\rm UV}}{k} > 1\,. \tag{3.22}
$$

The above expression assumes $\eta > \eta'$ and the result for $\eta < \eta'$ is found using $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \mathscr{C}^*_{\boldsymbol{k}}(\eta',\eta)$. Notice that the imaginary part is completely contained in the last lines of this expression, and is totally independent of the parameter $k_{\rm UV}$.

Alternatively, for $\eta' \to -\infty$ (for fixed η and parametrically making $|k\eta'| \gg 1$) the correlator instead has the form

$$
\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') \simeq \frac{k^5}{32(2\pi)^{7/2}}\Bigg\{-\frac{4\left[i+\kappa(-k\eta)\right]^2\left(2\kappa^2-1\right)^2}{\kappa^2(-k\eta)^2(-k\eta')^2}\,e^{-2ik_{\rm UV}(\eta-\eta')} \tag{3.23}
$$

$$
+\frac{16\kappa\,(\kappa+1)\left[i+\kappa(-k\eta)\right]\left[i+(-k\eta)(\kappa+1)\right]}{(-k\eta)^2(-k\eta')^2}\,e^{-i(1+2\kappa)k(\eta-\eta')}\Bigg\}. \tag{3.24}
$$

which falls off for large η' proportional to $(\eta')^{-2}$.

3.2 Markovian approximation

Now comes the main approximation. The main observation is that the Nakajima-Zwanzig equation (3.20) for each mode simplifies if the contribution to the kernel $G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$ is so sharply peaked about $\eta' \simeq \eta$ that $\varrho_{\boldsymbol{k}}(\eta)$ does not vary significantly in the integration region where the kernel has appreciable support. When this is true the evolution becomes Markovian in the sense that $\partial_\eta \varrho_{\boldsymbol{k}}$ depends only on $\varrho_{\boldsymbol{k}}$ evaluated at the same time (rather than on its entire earlier history).

3.2.1 A false start

Before proceeding we first pause to describe a common method often used in the literature, which in this case does not reveal the correct Markovian limit. In this method one expands the density matrix in a Taylor series,

$$\varrho_{\boldsymbol{k}}(\eta') \simeq \varrho_{\boldsymbol{k}}(\eta) + (\eta' - \eta)\partial_\eta \varrho_{\boldsymbol{k}}(\eta) + \mathcal{O}\big[(\eta' - \eta)^2\big] \tag{3.25}$$

with the result truncated at leading order to obtain the Markovian regime. Such an expansion seems very likely to be a good approximation because eq. (2.32) shows that in the interaction picture all contributions to $\partial_\eta \varrho_{\boldsymbol{k}}$ are suppressed by the perturbative coupling (in our case $H^2/M_{\rm p}^2$).

Interestingly, we find that this derivation leads in the present instance to an unphysical Markovian limit, whose evolution equation can violate the fundamental positivity conditions that density matrices must satisfy. This signals a failure of the approximations used. Since this procedure is frequently used in the literature, we here describe in more detail the way in which it fails in the current setting.

Inserting the leading term of eq. (3.25) into eq. (3.20) leads to the evolution equation[8]

$$\frac{\mathcal{V}}{(2\pi)^3}\frac{\partial \varrho_{\boldsymbol{k}}}{\partial \eta} = -i\Big[\mathcal{H}_{\mathrm{eff}\,\boldsymbol{k}}(\eta), \varrho_{\boldsymbol{k}}\Big] + \sum_{n,m=1}^{2} h_{\boldsymbol{k},nm}\left[O_{\boldsymbol{k},n}\varrho_{\boldsymbol{k}}(\eta)O^\dagger_{\boldsymbol{k},m} - \frac{1}{2}\left\{O^\dagger_{\boldsymbol{k},m}O_{\boldsymbol{k},n}, \varrho_{\boldsymbol{k}}(\eta)\right\}\right], \tag{3.26}$$

where we define

$$O_{\boldsymbol{k},1} := k^2\,\tilde{v}_{\boldsymbol{k}}\,, \qquad O_{\boldsymbol{k},2} := k\,\tilde{p}_{\boldsymbol{k}}\,, \tag{3.27}$$

and

$$\mathcal{H}_{\mathrm{eff}\,\boldsymbol{k}}(\eta) = \mathrm{Im}\left[\mathfrak{A}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]\,(\tilde{v}_{\boldsymbol{k}})^2 + \mathrm{Im}\left[\mathfrak{B}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]\tilde{v}_{\boldsymbol{k}}\tilde{p}_{\boldsymbol{k}}\,. \tag{3.28}$$

The matrix of coefficients is

$$h_{\boldsymbol{k},nm} = \frac{1}{k^4}\begin{pmatrix} 2\mathrm{Re}\left[\mathfrak{A}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right] & k\,\mathfrak{B}^*_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \\ k\,\mathfrak{B}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) & 0 \end{pmatrix}, \tag{3.29}$$

where

$$\mathfrak{A}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := i(2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta}\mathrm{d}\eta'\,G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')\Big[-u^{*\prime}_{\boldsymbol{k}}(\eta)u_{\boldsymbol{k}}(\eta') + u'_{\boldsymbol{k}}(\eta)u^*_{\boldsymbol{k}}(\eta')\Big], \tag{3.30}$$

$$\mathfrak{B}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := i(2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta}\mathrm{d}\eta'\,G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')\Big[u^*_{\boldsymbol{k}}(\eta)u_{\boldsymbol{k}}(\eta') - u_{\boldsymbol{k}}(\eta)u^*_{\boldsymbol{k}}(\eta')\Big]. \tag{3.31}$$

Now comes the main point: evolution using eq. (3.26) only keeps the eigenvalues of ϱ real and between 0 and 1 (as required for probabilities) if the eigenvalues

$$\lambda^{\pm}_{\boldsymbol{k}} = k^{-4}\mathrm{Re}\left[\mathfrak{A}_{\boldsymbol{k}}(\eta)\right] \pm k^{-4}\sqrt{\mathrm{Re}\left[\mathfrak{A}_{\boldsymbol{k}}(\eta)\right]^2 + k^2\,\mathrm{Re}\left[\mathfrak{B}_{\boldsymbol{k}}(\eta)\right]^2 + k^2\,\mathrm{Im}\left[\mathfrak{B}_{\boldsymbol{k}}(\eta)\right]^2} \tag{3.32}$$

[8] In order to arrive at this equation using these steps, one must expand $v_{\boldsymbol{k}}(\eta')$ in terms of ladder operators which are then inverted with a Bogoliubov transformation to give rise to the operators in eq. (3.27).

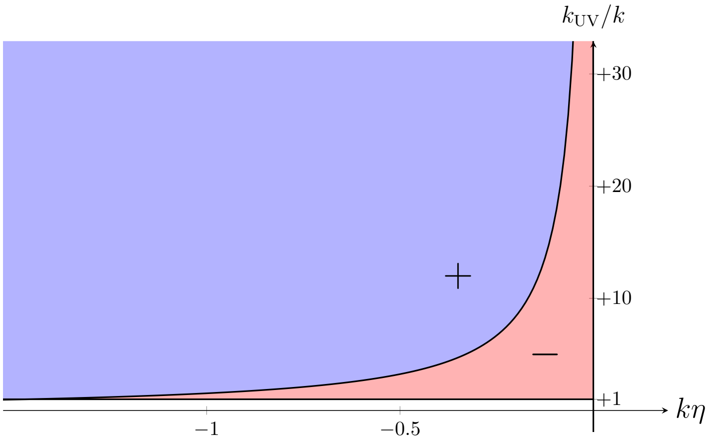

Figure 3: A plot of the sign of Re $\mathfrak{A}_{\boldsymbol{k}}$ as a function of $k\eta$ and $\kappa = k_{\rm UV}/k$ (for the case $k\eta_{\rm in} \to -\infty$), with blue representing positive and red being negative. The boundary between the two signs follows roughly the curve $k_{\rm UV}\eta \simeq \sqrt{3}$.

of $h_{\boldsymbol{k},nm}$ are strictly non-negative. But in the super-Hubble limit $-k\eta \ll 1$ with $-k\eta_{\rm in}$ fixed and $\kappa = k_{\rm UV}/k \gg 1$ one finds

$$\mathrm{Re}[\mathfrak{A}_{\boldsymbol{k}}] \simeq -\frac{3\epsilon_1 H^2}{256\pi^2 M_{\rm p}^2}\frac{k^3}{k_{\rm UV}(-k\eta)^3} + \cdots \tag{3.33}$$

is negative and this implies $\lambda^-_{\boldsymbol{k}}$ is also negative. The sign of $\mathrm{Re}(\mathfrak{A}_{\boldsymbol{k}})$ is evaluated numerically and shown in fig. 3 as a function of $k_{\rm UV}/k$ and $k\eta$ (for the case of $k\eta_{\rm in} \to -\infty$); showing that the leading small $-k\eta$ limit is negative whenever $-k_{\rm UV}\eta \lessapprox \sqrt{3}$ (and so $k_{\rm UV}$ is super-Hubble) but is positive otherwise.

3.2.2 A Markovian regime

Another strategy is to jointly expand all terms that multiply the sharply peaked kernel $G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$ in the integrand of eq. (3.20),

$$\Big[\tilde{v}_{\boldsymbol{k}}(\eta), \tilde{v}_{\boldsymbol{k}}(\eta')\varrho_{\boldsymbol{k}}(\eta')\Big] \simeq \Big[\tilde{v}_{\boldsymbol{k}}(\eta), \tilde{v}_{\boldsymbol{k}}(\eta)\varrho_{\boldsymbol{k}}(\eta)\Big] + (\eta'-\eta)\Big[\tilde{v}_{\boldsymbol{k}}(\eta), [\tilde{v}_{\boldsymbol{k}}(\eta)\partial_\eta\varrho_{\boldsymbol{k}}(\eta) + \tilde{p}_{\boldsymbol{k}}(\eta)\varrho_{\boldsymbol{k}}(\eta)]\Big] + \cdots, \tag{3.34}$$

and

$$\Big[\varrho_{\boldsymbol{k}}(\eta')\tilde{v}_{\boldsymbol{k}}(\eta'), \tilde{v}_{\boldsymbol{k}}(\eta)\Big] \simeq \Big[\varrho_{\boldsymbol{k}}(\eta)\tilde{v}_{\boldsymbol{k}}(\eta), \tilde{v}_{\boldsymbol{k}}(\eta)\Big] + (\eta'-\eta)\Big[[\partial_\eta\varrho_{\boldsymbol{k}}(\eta)\tilde{v}_{\boldsymbol{k}}(\eta) + \varrho_{\boldsymbol{k}}(\eta)\tilde{p}_{\boldsymbol{k}}(\eta)], \tilde{v}_{\boldsymbol{k}}(\eta)\Big] + \cdots, \tag{3.35}$$

and seek the regime where the first term dominates the integral.

When this is a good approximation eq. (3.20) becomes the following interaction-picture Lindblad equation,

$$\frac{\mathcal{V}}{(2\pi)^3}\frac{\partial\varrho_{\boldsymbol{k}}}{\partial\eta} \simeq -\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\,\Big[\tilde{v}_{\boldsymbol{k}}(\eta), \big[\tilde{v}_{\boldsymbol{k}}(\eta), \varrho_{\boldsymbol{k}}(\eta)\big]\Big] - i\,\mathrm{Im}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\,\Big[[\tilde{v}_{\boldsymbol{k}}(\eta)]^2, \varrho_{\boldsymbol{k}}(\eta)\Big]\,, \tag{3.36}$$

where we define the integrated environmental coefficient

$$\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := (2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta} {\rm d}\eta'\, G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') \qquad \text{(scalar environment)}\,. \tag{3.37}$$

eq. (3.36) describes Markovian evolution and, for the same reasons as described above for the false start, the evolution (3.36) is only consistent with a probability interpretation for $\varrho_{\boldsymbol{k}}$ if $\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}] > 0$, so this must be true in any valid approximation. We verify that it *is* true for several explicit limits below.

Experience with open systems suggests that the solutions to eq. (3.37) can sometimes be trusted well into the future in a way that those of eq. (3.20) cannot. A necessary condition for this extended domain of validity is that the right-hand side not make explicit reference to the initial time $\eta_{\rm in}$, since it is only then that one can expect the evolution equation to have a broader domain of validity than its perturbative derivation (for the reasons outlined in detail in ref. [26]). For this to be useful in the present instance would require the function $\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})$ to be approximately independent of $\eta_{\rm in}$.

We therefore evaluate eq. (3.37) in some detail in appendix C, encountering along the way ultraviolet divergences that we renormalize after first regularizing using a variant of dimensional regularization. The general expression for $\mathfrak{F}_{\boldsymbol{k}}$ we obtain is somewhat unwieldy for general values of its arguments η, $\eta_{\rm in}$ and $k_{\rm UV} > k$ and so we quote here only several useful limiting forms. In particular we find that $\mathrm{Re}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]$ is UV finite and becomes universal in the late-time super-Hubble limit $-k\eta \ll 1$, with

$$\mathrm{Re}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right] \simeq \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2}\left\{\frac{20\pi}{(-k\eta)^2} + \frac{g\left(\kappa, k\eta_{\rm in}\right)}{(-k\eta)} + \mathcal{O}\left[(-k\eta)^0\right]\right\}\,. \tag{3.38}$$

This is universal in the sense that all dependence on the parameters $k_{\rm UV}$ and $\eta_{\rm in}$ first appear at subdominant order in $k\eta$; within the known function $g(\kappa, k\eta_{\rm in})$. We show below that $\mathrm{Re}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]$ is the quantity relevant to decoherence and its universal form for late times opens up the possibility of also trusting its solutions at very late times.

In the slightly more restrictive regime $-k\eta \ll -k\eta_{\rm in} \ll 1$ we similarly have – see formula (C.59) – $\mathfrak{F}_{\boldsymbol{k}} = \mathfrak{F}_{\boldsymbol{k}}^{(\rm div)} + \mathfrak{F}_{\boldsymbol{k}}^{(\rm fin)}$ where the UV-divergent part is

$$\mathfrak{F}_{\boldsymbol{k}}^{(\rm div)}(\eta,\eta_{\rm in}) = \frac{i\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2}\left[\frac{40}{(-k\eta)^2} - \frac{92}{3} + \frac{43}{15}(-k\eta)^2\right]\left\{\frac{1}{\epsilon} + \log\left[\frac{k_{\rm UV}}{\mu}\left(2+\frac{1}{\kappa}\right)\right]\right\} \tag{3.39}$$

and so only contributes to $\mathrm{Im}\,\mathfrak{F}_{\boldsymbol{k}}$. The remaining finite part is

$$\begin{aligned}\mathfrak{F}_{\boldsymbol{k}}^{(\rm fin)}(\eta,\eta_{\rm in}) \simeq \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2}\Bigg(&\frac{20\pi}{(-k\eta)^2} + \frac{4i}{(-k\eta)^2}\left\{7 - 10\log\left[e^{\gamma}(2\kappa+1)(-k\eta)\right]\right\} \\ &+ \frac{1}{(-k\eta)}\left[\frac{4}{3}\left(24\kappa + 6 + \frac{16}{\kappa} - \frac{3}{\kappa^2}\right)\log\left(\frac{\eta}{\eta_{\rm in}}\right) + \frac{40i}{z_{\rm in}}\right] - \frac{46\pi}{3} - \frac{128i}{3} \\ &+ \frac{92i}{3}\log\left[e^{\gamma}(2\kappa+1)(-k\eta)\right] + \mathcal{O}\left(-k\eta, -k\eta_{\rm in}\right)\Bigg),\end{aligned} \tag{3.40}$$

where γ is Euler-Mascheroni constant. The divergence is visible in the limit that the regularization parameter $0 < |\epsilon| \ll 1$ tends to zero, and $\mu > 0$ is the usual associated arbitrary mass scale. Although the formula (3.40) above neglects terms $\mathcal{O}(-k\eta)$, we explicitly write out the divergence proportional to $(-k\eta)^2$ in formula (3.39) for completeness, and for later use when asking whether and how the divergences appearing in eq. (3.39) can be renormalized by appropriate choice of counter-term.

The goal is to solve eq. (3.36) and extract the physical observables from it, such as the decoherence rate and the (very small) corrections to the power spectrum. Before pursuing this we must tie up two loose ends: understand how to renormalize the divergences appearing in eq. (3.40) (so that we can understand why they do not affect the physical predictions) and verify the validity of the underlying Markovian approximation.

3.2.3 Domain of validity of the Markovian approximation

As the example of sec. 3.2.1 shows, truncating a Taylor expansion inside the integral need not always be a good approximation. It should be a good approximation however if the time-scale T over which the Taylor expanded quantity varies is much longer than the width τ of the correlation function's peak, since then subsequent terms should be suppressed by powers of τ/T. We here show that the leading corrections to eq. (3.36) are suppressed by powers of $k\eta$ in the late-time super-Hubble limit (for which $k\eta \to 0$).

To see why, insert the subdominant term of eq. (3.35) into eq. (3.20), yielding

$$\begin{aligned}\frac{\mathcal{V}}{(2\pi)^3}\frac{\partial \varrho_{\boldsymbol{k}}}{\partial \eta} &= -\,\mathrm{Re}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]\,[\tilde{v}_{\boldsymbol{k}}(\eta),[\tilde{v}_{\boldsymbol{k}}(\eta),\varrho_{\boldsymbol{k}}(\eta)]] - i\,\mathrm{Im}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]\,\left[\left(\tilde{v}_{\boldsymbol{k}}(\eta)\right)^2,\,\varrho_{\boldsymbol{k}}(\eta)\right] \\ &- \mathrm{Re}\left[\mathfrak{M}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]\,[\tilde{v}_{\boldsymbol{k}}(\eta),[\tilde{p}_{\boldsymbol{k}}(\eta),\varrho_{\boldsymbol{k}}(\eta)]] - i\,\mathrm{Im}\left[\mathfrak{M}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right]\,[\tilde{v}_{\boldsymbol{k}}(\eta),\{\tilde{p}_{\boldsymbol{k}}(\eta),\varrho_{\boldsymbol{k}}(\eta)\}]\,, \end{aligned}\tag{3.41}$$

where the first line is the same as eq. (3.36) and $\mathfrak{F}_{\boldsymbol{k}}$ is as defined in eq. (3.37) while

$$\mathfrak{M}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \;:=\; (2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta}\mathrm{d}\eta'\,G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')\,(\eta'-\eta)\,. \tag{3.42}$$

We seek the regime where the terms involving $\mathfrak{F}_{\boldsymbol{k}}$ dominate those proportional to $\mathfrak{M}_{\boldsymbol{k}}$.

The function $\mathfrak{M}_{\boldsymbol{k}}$ is computed in Appendix C.4, where we find — *c.f.* formula (C.67) — that in the super-Hubble limit $0 < -k\eta \ll -k\eta_{\rm in} \ll 1$

$$\begin{aligned}\mathfrak{M}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \simeq \frac{\varepsilon_1 H^2 k}{1024\pi^2 M_{\rm p}^2}&\Bigg[-\frac{40i\log\left(\eta/\eta_{\rm in}\right)+\mathcal{O}(-k\eta_{\rm in})}{-k\eta}+\frac{40i}{(-k\eta_{\rm in})} \\ &+4\left(8\kappa+2+\frac{16}{3\kappa}-\frac{1}{\kappa^2}\right)\log\left(\frac{\eta}{e\eta_{\rm in}}\right)+\mathcal{O}(-k\eta)\Bigg]+\mathfrak{M}_{\boldsymbol{k}}^{(\rm div)}(\eta,\eta_{\rm in})\,,\end{aligned}\tag{3.43}$$

where we again encounter a $1/\epsilon$ divergence in the imaginary part of the form

$$\mathfrak{M}_{\boldsymbol{k}}^{(\rm div)}(\eta,\eta_{\rm in}) \simeq -\frac{5i\varepsilon_1 H^2 k}{768\pi^2 M_{\rm p}^2}(-k\eta)\left\{\frac{1}{\epsilon}+\log\left[\frac{2k_{\rm UV}+k}{\mu}\right]\right\}. \tag{3.44}$$

This can be absorbed by a counter-term in the same way as can the divergences in $\mathfrak{F}_{\boldsymbol{k}}$ (see below). By contrast, the real part $\mathrm{Re}[\mathfrak{M}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]$ is finite and $\mathcal{O}\left[(-k\eta)^0\right]$ in the super-Hubble limit. What is important is that this is subdominant to $\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})] \propto (k\eta)^{-2}$ in this regime; putting late times and super-Hubble scales squarely within the domain of validity of the Markovian methods.

3.2.4 Renormalization of the Lindblad equation

Next consider the issue of renormalization. How renormalization works is easier to see if we convert eq. (3.36) to Schrödinger picture, since this reintroduces the free Hamiltonian (whose parameters are presumably the ones that get renormalized) into the evolution.

Repeating the above steps leads to the Schrödinger picture Lindblad equation,

$$\frac{\mathcal{V}}{(2\pi)^3}\frac{\partial \varrho_{\mathrm{s}\boldsymbol{k}}}{\partial \eta} \simeq -\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}})]\ \left[\tilde{v}_{\mathrm{s}\boldsymbol{k}},\left[\tilde{v}_{\mathrm{s}\boldsymbol{k}},\varrho_{\mathrm{s}\boldsymbol{k}}(\eta)\right]\right] \tag{3.45}$$

$$- i\left[\ \mathcal{H}_{\mathrm{s}\boldsymbol{k}}(\eta) + \mathrm{Im}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}})]\tilde{v}^2_{\mathrm{s}\boldsymbol{k}},\ \varrho_{\mathrm{s}\boldsymbol{k}}(\eta)\ \right], \tag{3.46}$$

where the Hamiltonian in momentum space is $\mathcal{H}_{\mathrm{s}\boldsymbol{k}}(\eta) = \mathcal{H}^{(0)}_{\mathrm{s}\boldsymbol{k}}(\eta) + \delta\mathcal{H}_{\mathrm{s}\boldsymbol{k}}(\eta)$ where

$$\mathcal{H}^{(0)}_{\mathrm{s}\boldsymbol{k}}(\eta) = \frac{1}{2}\left[\tilde{p}^2_{\mathrm{s}\boldsymbol{k}} + \left(k^2 - \frac{2}{\eta^2}\right)\tilde{v}^2_{\mathrm{s}\boldsymbol{k}}\right], \tag{3.47}$$

is the free part coming from the action of eq. (2.1) and $\delta\mathcal{H}_{\mathrm{s}\boldsymbol{k}}(\eta)$ contains any order $1/M_{\mathrm{p}}^2$ terms coming from local corrections to this action. The counterterms that cancel the divergence in $\mathrm{Im}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}})\right]$ must come from the quadratic term $\tilde{v}_{\boldsymbol{k}}$ in the expansion of $\delta\mathcal{H}_{\mathrm{s}\boldsymbol{k}}(\eta)$. What is important in eq. (3.45) is that this is possible because these counter-terms only appear together with the imaginary part of $\mathcal{F}_{\boldsymbol{k}}$ that contains the $1/\epsilon$ divergences in eq. (3.40).

But what are the counter-terms into which divergences might go? This ultimately is determined by the parameters appearing in the Lagrangian, and for a renormalizable theory (like QED) this would simply be our starting Lagrangian (2.1). However General Relativity is famously *not* renormalizable in the same way, and so for it divergences must be handled within an effective field theory treatment in which eq. (2.1) is regarded as the leading part of a low-energy derivative expansion. In this case standard power-counting arguments [66] determine what kinds of terms must be added to it at any perturbative order to capture divergences. For the gravitational systems of interest here this means that counter-terms arise either as renormalizations of Newton's constant (or M_{p}) or as parameters appearing in four-derivative interactions, like curvature-squared or mixed scalar-derivative/curvature terms.

Although we cannot here definitely prove that divergences all get renormalized into these parameters, we can provide several consistency checks. We cannot be definitive because it was only when we specialize to decoherence that we are free to ignore all other interactions beyond our specific cubic interaction, even at order $1/M_{\mathrm{p}}^2$. But these other interactions can contribute UV divergences and it is only the complete set of divergences at a fixed order in the small couplings – powers of H^2/M_{p}^2 and ε_1 in the present instance [45, 46] – that are guaranteed to cancel. What we can do is check that the divergences we encounter have the dependence on k and η that is required if they are to be absorbable into the expected Einstein-Hilbert or curvature-squared counter-terms. Along the way we show that the decoherence calculation is UV finite, and so in particular is independent of the values of these renormalized parameters.

To see how this works[9] notice that formula (3.39) shows that the divergences encountered in eq. (3.45) all have the form

$$\mathrm{Im}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}})\right]\ \tilde{v}^2_{\mathrm{s}\boldsymbol{k}} \propto \frac{\varepsilon_1 H^2}{M_{\mathrm{p}}^2}\left(\frac{c_1}{\eta^2} + c_2\,k^2 + c_3\,k^4\eta^2\right)\tilde{v}^2_{\mathrm{s}\boldsymbol{k}} \tag{3.48}$$

for some real constants $c_{1,2,3}$ that all contain a $1/\epsilon$ divergence in dimensional regularization. Keeping in mind that $a(\eta) = -(H\eta)^{-1}$, these have the k- and η-dependence appropriate to the renormalization of terms in the Lagrangian of the form

$$\frac{c_1}{\eta^2}\tilde{v}^2_{\mathrm{s}\boldsymbol{k}} \subset \left[\frac{\mathrm{d}}{\mathrm{d}t}(av)\right]^2, \quad c_2\,k^2\tilde{v}^2_{\mathrm{s}\boldsymbol{k}} \subset c_2(\partial v)^2 \quad \text{and} \quad c_3\,k^4\eta^2\tilde{v}^2_{\mathrm{s}\boldsymbol{k}} \subset c_3\,(\partial^2 v)^2\,. \tag{3.49}$$

[9]Notice that these types of consistency checks are easiest to do when using dimensional regularization since this preserves the underlying gauge symmetries of the gravitational action.

These are among the kinds of operators that arise in the fluctuation expansion of the Einstein-Hilbert Lagrangian $\sqrt{-g}\,R$, or of a curvature-squared term like $\sqrt{-g}\,R^2$. (For more detail on this see Appendix A).

The same EFT reasoning also explain why terms proportional to η^2 – such as in eq. (3.39) – are not problems even at early times where η can be big (a limit that would also require choosing $\eta_{\rm in}$ to be big as well). These naively seem to be in danger of interfering with the physical arguments that select the adiabatic Bunch-Davies initial conditions at early times. To see why they are not a worry, consider for example a term of the form

$$\frac{\varepsilon_1 H^2}{M_{\rm p}^2}(k\eta)^2 \sim \frac{\varepsilon_1 \ell_{\rm p}^2}{\lambda_{\rm phys}^2} \tag{3.50}$$

where $\ell_{\rm p} = 1/M_{\rm p}$ is the Planck length and $\lambda_{\rm phys} \sim a/k = (-Hk\eta)^{-1}$. Such terms can only be important for physical wavelengths $\lambda_{\rm phys} \lesssim \sqrt{\varepsilon_1}\,\ell_{\rm p}$, and so lie well outside the domain of validity of any EFT of gravity.

3.3 Gaussian transport

Because the right-hand side of eq. (3.45) is quadratic in the fields it follows that an initially gaussian state – such as the Bunch-Davies vacuum – remains gaussian under evolution. When this is true eq. (3.45) can be converted into a direct late-time evolution equation for field correlations (rather than for the density matrix), as we now show. This alternative formulation is possible because for gaussian systems $\varrho_{\rm s\boldsymbol{k}}$ is completely characterized by the one- and two-point functions of fields and canonical momenta:

$$\langle \tilde v_{\rm s\boldsymbol{k}} \tilde v_{\rm s\boldsymbol{k}'}\rangle = P_{vv}(k)\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,, \quad \langle \tilde p_{\rm s\boldsymbol{k}} \tilde p_{\rm s\boldsymbol{k}'}\rangle = P_{pp}(k)\,\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{3.51}$$

$$\langle \tilde v_{\rm s\boldsymbol{k}} \tilde p_{\rm s\boldsymbol{k}'}\rangle = \left[P_{vp}(k) + \frac{i}{2}\right]\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{3.52}$$

Directly differentiating the definitions – *e.g.* $\langle \tilde v_{\boldsymbol{k}} \tilde v_{\boldsymbol{k}}\rangle = {\rm Tr}[\tilde v_{\rm s\boldsymbol{k}}^2\,\varrho_{\rm s\boldsymbol{k}}]$ – and evaluating the time derivative of $\varrho_{\rm s\boldsymbol{k}}$ using eq. (3.45), together with commutation relations like eq. (2.17), leads to the following transport equations for the power spectra

$$P_{vv}'(k,\eta) = 2\,P_{vp}(k,\eta)\,, \tag{3.53}$$

$$P_{vp}'(k,\eta) = P_{pp}(k,\eta) - \big\{\omega^2(k,\eta) + {\rm Im}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\big\}\,P_{vv}(k,\eta)\,, \tag{3.54}$$

$$P_{pp}'(k,\eta) = -2\,\big\{\omega^2(k,\eta) + {\rm Im}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\big\}\,P_{vp}(k,\eta) + 2\,{\rm Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\,. \tag{3.55}$$

Like the Lindblad equation these equations can be integrated to late times when the function $\mathfrak{F}_{\boldsymbol{k}}$ is independent of the initial time $\eta_{\rm in}$.

The effect of the cubic interaction in this evolution is twofold. First, it ensures that the frequency $\omega^2(k)$ only appears in the combination

$$\widetilde\omega^2(k,\eta) := \omega^2(k,\eta) + {\rm Im}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\,, \tag{3.56}$$

and so effectively alters the k-dependence of the dispersion relation by shifting $\omega^2(k) \to \widetilde\omega^2(k)$. This shift would also be expected from the form of eq. (3.45) since there ${\rm Im}(\mathfrak{F}_{\boldsymbol{k}})$ appears as an additive contribution proportional to $\tilde v_{\rm s\boldsymbol{k}}^2$ in the Hamiltonian.[10] As described above, it is because UV divergences only appear in ${\rm Im}(\mathfrak{F}_{\boldsymbol{k}})$ that they can be renormlized into parameters appearing

[10] In an unfortunate nomenclature corrections like these to the lowest-order dispersion relation have come to be referred to in the literature as 'Lamb shift' terms.

in $\omega^2(k)$. Such terms cannot drive decoherence because they contribute to evolution as would a correction to the Hamiltonian appearing in the Liouville equation, and so cannot evolve pure states into mixed states.

The cubic interaction's other effect is to add a source term $\mathrm{Re}(\mathfrak{F}_{\boldsymbol{k}})$ into the evolution equation for P_{pp}. sec. 4 below shows that this term is the one responsible for decoherence and because $\mathrm{Re}(\mathfrak{F}_{\boldsymbol{k}})$ is UV finite so must be the decoherence rate. Showing this involves solving these equations explicitly and this is facilitated by eliminating two of the variables to obtain a single third-order differential equation for P_{vv}:

$$P_{vv}'''(k,\eta) + 4\widetilde{\omega}^2(k,\eta)\,P_{vv}'(k,\eta) + 4\,\widetilde{\omega}(k,\eta)\,\widetilde{\omega}'(k,\eta)\,P_{vv} = 4\,\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]\,. \tag{3.57}$$

Once this is solved the remaining correlators P_{vp} and P_{pp} are found from eqs. (3.53) and (3.54).

3.4 Contribution from the tensor environment

Before finding solutions we conclude this section by computing the rate with which an environment of short-wavelength tensor modes decoheres the long-wavelength scalars, showing that it gives twice the rate found above from a scalar environment. The full rate for decohering visible scalar fluctuations is the sum of the contributions from smaller scalar and tensor modes.

We first show that a tensor environment contributes to decoherence with the same leading parametric dependence on ε_1, $H/M_{\rm p}$ and $-k\eta = k/(aH)$ as does a scalar environment. The same arguments as given above again imply that decoherence first arises at order $(H/M_{\rm p})^2$ and does so only through cubic interactions. Furthermore, short-wavelength tensor modes can only decohere super-Hubble scalar fluctuations through the tensor-tensor-scalar interactions listed in ref. [47] [see also eq. (A.20) and eq. (E.1)], and of these only the interaction

$$^{(3)}S \supset \frac{M_{\rm p}^2}{8}\int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}\,a\,\varepsilon_1\,\zeta\,\partial_l\gamma_{ij}\partial_l\gamma_{ij}\,, \tag{3.58}$$

contributes at leading order in slow-roll parameters and in powers of $k/(aH)$.

Appendix E shows in detail how scalar evolution is modified by the presence of the tensor environment, again leading to an evolution equation of the Nakajima-Zwanzig form (3.20), but with the correlator $\mathscr{C}_{\boldsymbol{k}}$ replaced by $\mathscr{C}_{\boldsymbol{k}} + \mathscr{T}_{\boldsymbol{k}}$, with $\mathscr{T}_{\boldsymbol{k}}$ defined by

$$C_{\rm T}(\eta,\eta';\boldsymbol{y}) := \langle 0_B|\,[B_{\rm T}(\eta,\boldsymbol{x}) - \mathscr{B}_{\rm T}(\eta)]\,[B_{\rm T}(\eta',\boldsymbol{x}') - \mathscr{B}_{\rm T}(\eta')]\,|0_B\rangle = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\,\mathscr{T}_{\boldsymbol{k}}(\eta,\eta')e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\,, \tag{3.59}$$

where $\mathscr{B}_{\rm T}(\eta) := \langle 0_B|B_{\rm T}(\eta,\boldsymbol{x})|0_B\rangle$ and

$$B_{\rm T}(\boldsymbol{x}) := \delta^{ij}\delta^{kl}\delta^{rs}\partial_i v_{kr}(\eta,\boldsymbol{x})\partial_j v_{ls}(\eta,\boldsymbol{x}) \tag{3.60}$$

is the new (tensor) environmental operator implied by eq. (3.58).

Appendix E also evaluates the leading behaviour of $\mathscr{T}_{\boldsymbol{k}}$ for small $(-k\eta)$, which turns out to be twice the contribution from $\mathscr{C}_{\boldsymbol{k}}$ alone. The combined tensor and scalar contributions are therefore three times larger than the scalar result alone, with the combination of scalar and tensor environments leading to a late-time limit $-k\eta \ll 1$ limit that is three times larger than in eq. (3.38) [see eq. (E.71)]:

$$\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})] \simeq \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2}\left[\frac{60\pi}{(-k\eta)^2} + \mathcal{O}\left(\frac{1}{-k\eta}\right)\right] \qquad \text{(scalar+tensor environment)}\,. \tag{3.61}$$

Again all dependence on $\eta_{\rm in}$ and $k_{\rm UV}$ appear only at subdominant order in $(-k\eta)$. We comment here that the $k_{\rm UV}$-independence of the result follows because the most important scales for decoherence are the ones that are closest to the scale k being decohered (both much larger than the Hubble length). Because the important scales are not at the cutoff, the decoherence is largely insensitive to the value used for $k_{\rm UV}$. This also helps to further underline why the $v_{\scriptscriptstyle A} \otimes (\partial v_{\scriptscriptstyle B})^2$ interaction is the dominant contribution in going from (2.14) to (2.27) — all other neglected interactions there contribute most near the value of $k_{\rm UV}$ and so are unimportant for decoherence (for the same reason the value of $k_{\rm UV}$ itself is not).

4 Late-time solutions

We now solve the Lindblad equation (3.45) to extract some of its physical implications.

4.1 Solution to the Lindblad equation

Because the right-hand side of the Lindblad equation is quadratic in the system field $v(\eta, \boldsymbol{x})$, the solutions for the reduced density matrix in the field eigenstate basis remain Gaussian. In Schrödinger picture this means

$$\langle \tilde{v}_{\mathrm{s}\boldsymbol{k},1}|\, \varrho_{\mathrm{s}\boldsymbol{k}}\, |\tilde{v}_{\mathrm{s}\boldsymbol{k},2}\rangle = \sqrt{\frac{\mathrm{Re}(a_k) - c_k}{\pi}}\; \exp\left(-\frac{a_k}{2}\, \tilde{v}^2_{\mathrm{s}\boldsymbol{k},1} - \frac{a_k^*}{2}\, \tilde{v}^2_{\mathrm{s}\boldsymbol{k},2} + c_k\, \tilde{v}_{\mathrm{s}\boldsymbol{k},1}\tilde{v}_{\mathrm{s}\boldsymbol{k},2}\right)\,, \tag{4.1}$$

for time-dependent functions $a_k(\eta)$ and $c_k(\eta)$. This state is properly normalised inasmuch as $\mathrm{Tr}(\varrho_{\mathrm{s}\boldsymbol{k}}) = 1$ and the requirement $\varrho^\dagger_{\mathrm{s}\boldsymbol{k}} = \varrho_{\mathrm{s}\boldsymbol{k}}$ implies c_k is real.

The Lindblad equation determines the functions $a_k(\eta)$ and $c_k(\eta)$, and because the state is Gaussian these are completely determined by the two-point functions $P_{vv}(k)$, $P_{vp}(k)$ and $P_{pp}(k)$, through the formulae

$$P_{vv}(k) = \frac{1}{2\,[\mathrm{Re}(a_k) - c_k]}\,, \quad P_{vp}(k) = -\frac{\mathrm{Im}(a_k)}{2\,[\mathrm{Re}(a_k) - c_k]}\,, \quad P_{pp}(k) = \frac{|a_k|^2 - c_k^2}{2\,[\mathrm{Re}(a_k) - c_k]}\,, \tag{4.2}$$

which invert to give

$$\mathrm{Re}(a_k) = \frac{1}{P_{vv}(k)}\left[P_{vv}(k)P_{pp}(k) - P^2_{vp}(k) + \frac{1}{4}\right]\,, \quad \mathrm{Im}(a_k) = -\frac{P_{vp}(k)}{P_{vv}(k)} \tag{4.3}$$

$$c_k = \frac{1}{P_{vv}(k)}\left[P_{vv}(k)P_{pp}(k) - P^2_{vp}(k) - \frac{1}{4}\right]\,. \tag{4.4}$$

A measure of the state's decoherence is given by its 'purity', defined by

$$\mathfrak{p}_{\boldsymbol{k}}(\eta) := \mathrm{Tr}\left[\varrho^2_{\mathrm{s}\boldsymbol{k}}(\eta)\right]\,. \tag{4.5}$$

This quantity satisfies $0 \le \mathfrak{p}_{\boldsymbol{k}} \le 1$ and $\mathfrak{p}_{\boldsymbol{k}} = 1$ if and only if $\varrho_{\boldsymbol{k}}$ is a pure state. Decoherence is said to be effective when $\mathfrak{p}_{\boldsymbol{k}} \ll 1$. For a Gaussian state the purity (4.1) evaluates to [22, 67–69]

$$\mathfrak{p}_{\boldsymbol{k}} = \sqrt{\frac{\mathrm{Re}(a_k) - c_k}{\mathrm{Re}(a_k) + c_k}} = \frac{1}{2\sqrt{P_{vv}(k)P_{pp}(k) - P^2_{vp}(k)}}\,, \tag{4.6}$$

where the second equality uses eqs. (4.3)-(4.4).

The state is pure if and only if $c_k = 0$, or equivalently $P_{vv}(k)P_{pp}(k) - P_{vp}^2(k) = 1/4$. By contrast, the state is strongly decohered when $\mathfrak{p}_{\boldsymbol{k}} \ll 1$ and so $P_{vv}(k)P_{pp}(k) - P_{vp}^2 \gg 1/4$. This corresponds to the case $c_k \simeq \mathrm{Re}(a_k)$ and so eq. (4.1) becomes

$$\left|\langle\tilde{v}_{\mathrm{s}\boldsymbol{k},1}|\,\varrho_{\mathrm{s}\boldsymbol{k}}\,|\tilde{v}_{\mathrm{s}\boldsymbol{k},2}\rangle\right| \propto \exp\left[-\frac{\mathrm{Re}(a_k)}{2}(\tilde{v}_{\mathrm{s}\boldsymbol{k},1} - \tilde{v}_{\mathrm{s}\boldsymbol{k},2})^2\right], \tag{4.7}$$

showing in particular that decoherence occurs in the field basis – *i.e.* $|\langle\tilde{v}_1|\varrho|\tilde{v}_2\rangle| \to \delta(\tilde{v}_1 - \tilde{v}_2)$ – if $\Re\mathrm{e}\,(a_k)$ also grows in this limit.

For later use we notice that eqs. (3.53)-(3.55) imply the following evolution equation for the combination of correlators that controls the purity:

$$\frac{\partial}{\partial\eta}\left[P_{vv}(k,\eta)P_{pp}(k,\eta) - P_{vp}^2(k,\eta)\right] = 2\,\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}})]\,P_{vv}(k)\,. \tag{4.8}$$

Among other things this confirms that decoherence is driven purely by the UV-finite quantity $\mathrm{Re}(\mathfrak{F}_{\boldsymbol{k}})$, as foreshadowed in earlier sections. Its late-time behaviour is reliable in the regime $-k\eta \ll 1$ because in this regime eq. (3.61) shows $\mathrm{Re}(\mathfrak{F}_{\boldsymbol{k}})$ is approximately independent of η_{in} and k_{UV}.

4.2 Solution to the transport equations

It remains to solve the Lindblad equation to determine the functions $a_k(\eta)$ and $c_k(\eta)$. We exploit the Gaussianity to do so directly using the equivalent transport equations (3.53) through (3.55) or their equivalent (3.57).

eq. (3.57) can be integrated when there is no source term on its right-hand side, with solution given by $P_{vv} = |\tilde{u}_{\boldsymbol{k}}|^2$, where $\tilde{u}_{\boldsymbol{k}}$ solves the Mukhanov-Sasaki equation $\tilde{u}_{\boldsymbol{k}}'' + \widetilde{\omega}^2(k)\tilde{u}_{\boldsymbol{k}} = 0$ obtained using the modified dispersion relation $\omega(k) \to \widetilde{\omega}(k)$. This solution builds in the initial condition that it approaches the Bunch-Davies vacuum $\tilde{u}_{\boldsymbol{k}} \to u_{\boldsymbol{k}}$ in the limit of vanishing cubic coupling.

Nonzero source term can then be included using the Green's function formalism, leading to

$$P_{vv}(k,\eta) = |\tilde{u}_{\boldsymbol{k}}(\eta)|^2 + 8\int_{\eta_{\mathrm{in}}}^{\eta}\mathrm{d}\eta'\,\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta',\eta_{\mathrm{in}})]\,\mathrm{Im}^2\left[\tilde{u}_{\boldsymbol{k}}(\eta')\tilde{u}_{\boldsymbol{k}}^*(\eta)\right]. \tag{4.9}$$

eqs. (3.54) and (3.55) then give the two other power spectra,

$$P_{vp}(k,\eta) = \mathrm{Re}\left[\tilde{u}_{\boldsymbol{k}}'(\eta)\tilde{u}_{\boldsymbol{k}}^*(\eta)\right] + 8\int_{\eta_{\mathrm{in}}}^{\eta}\mathrm{d}\eta'\,\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta',\eta_{\mathrm{in}})]\,\mathrm{Im}\left[\tilde{u}_{\boldsymbol{k}}(\eta')\tilde{u}_{\boldsymbol{k}}^*(\eta)\right]\,\mathrm{Im}\left[\tilde{u}_{\boldsymbol{k}}(\eta')\tilde{u}_{\boldsymbol{k}}^{*\prime}(\eta)\right] \tag{4.10}$$

$$P_{pp}(k,\eta) = |\tilde{u}_{\boldsymbol{k}}'(\eta)|^2 + 8\int_{\eta_{\mathrm{in}}}^{\eta}\mathrm{d}\eta'\,\mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta',\eta_{\mathrm{in}})]\,\mathrm{Im}^2\left[\tilde{u}_{\boldsymbol{k}}(\eta')\tilde{u}_{\boldsymbol{k}}^{*\prime}(\eta)\right]. \tag{4.11}$$

These in principle solve the Lindblad equation entirely once eqs. (4.9)-(4.10) are used in eqs. (4.3)-(4.4) to evaluate the density matrix (4.1). We do not write the corresponding expression here explicitly because our purposes are already well served by eqs. (4.9)-(4.10).

There are two distinct regimes in which these solutions can be used. In straight-up perturbation theory they can be used directly provided one works only to lowest order in the semiclassical expansion. This in turn requires replacing $\tilde{u}_{\boldsymbol{k}}$ with $u_{\boldsymbol{k}}$ inside the integrals given that $\mathfrak{F}_{\boldsymbol{k}}$ is already order $(H/M_{\mathrm{p}})^2$. An extended domain of validity could apply in circumstances where $\mathrm{Re}(\mathfrak{F}_{\boldsymbol{k}})$ and $\mathrm{Im}(\mathfrak{F}_{\boldsymbol{k}})$ are independent of η_{in}, but this must be checked on a case-by-case basis.

4.3 Quantifying decoherence

We may now compute the time-dependence of the state's purity by directly integrating eq. (4.8). Assuming that the state is pure at $\eta = \eta_{\rm in}$ we have

$$\mathfrak{p}_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{1 + \Xi_{\boldsymbol{k}}(\eta)}} \tag{4.12}$$

with

$$\Xi_{\boldsymbol{k}}(\eta) = 8 \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta' \, \mathrm{Re}[\mathfrak{F}_{\boldsymbol{k}}(\eta', \eta_{\rm in})] P_{vv}(k, \eta')\,, \tag{4.13}$$

where P_{vv} is given in eq. (4.9). We next evaluate this expression and assess its domain of validity.

To that end we first remark that P_{vv} never deviates much from its counterpart in the free theory, namely $P_{vv}^{\rm free} = |u_{\boldsymbol{k}}|^2$ where $u_{\boldsymbol{k}}$ is the Bunch-Davies mode function (2.40). Although eq. (3.40) shows that $\mathrm{Im}(\mathfrak{F}_{\boldsymbol{k}}) \propto (H/M_{\rm p})^2 \eta^{-2} \propto a^2 H^4/M_{\rm p}^2$ (up to logarithms) grows strongly on super-Hubble scales, it does not grow faster than $\omega^2 \propto \eta^{-2} \propto a^2 H^2$. Up to logarithms both quantities grow at the same rate and so $\mathrm{Im}(\mathfrak{F}_{\boldsymbol{k}})$ gives a correction to the effective mass of super-Hubble fluctuations (and so to the tilt of the power spectrum) of relative size $\mathrm{Im}(\mathfrak{F}_{\boldsymbol{k}})/\omega^2 \propto (H/M_{\rm p})^2$ (and therefore remains negligible even at late times).

The integral in eq. (4.9) can thus be evaluated by letting $\tilde{u}_{\boldsymbol{k}} \simeq u_{\boldsymbol{k}}$. Using the approximate form (3.61) (which includes the contribution of small-scale tensors) for Re $\mathfrak{F}_{\mathbf{k}}$ on super-Hubble scales, together with the super-Hubble limit

$$\mathrm{Im}\,[u_{\boldsymbol{k}}(\eta') u^*_{\boldsymbol{k}}(\eta)] \simeq \frac{(\eta^3 - \eta'^3)}{6\eta\eta'} + \mathcal{O}(k^2\eta^3), \tag{4.14}$$

which follows from eq. (2.40), leads to the power-spectrum correction

$$\Delta P_{vv} \simeq \frac{5\varepsilon_1 H^2}{192\pi M_{\rm p}^2} \left[\frac{1}{6} \left(\frac{\eta^4}{\eta_{\rm in}^3} - \frac{\eta_{\rm in}^3}{\eta^2} \right) - \eta \log\left(\frac{\eta}{\eta_{\rm in}} \right) \right]. \tag{4.15}$$

Here, $\eta_{\rm in}$ corresponds to the time at which the integration starts, which in this expression is assumed to be already super-Hubble, so that the Markovian approximation holds (see sec. 3.2.3). In practice, suppose $k = \sigma a_{\rm in} H_{\rm in}$, where $\sigma < 1$ denotes the ratio between the Hubble radius and the mode wavelength at the initial time. The above thus implies a very small, time-independent and scale-invariant fractional correction to the power spectrum

$$\frac{\Delta P_{vv}}{P_{vv}} \simeq \frac{5\varepsilon_1 H^2 \sigma^3}{576\pi M_{\rm p}^2}. \tag{4.16}$$

In summary, as expected gravitational mode coupling leaves only a tenuous imprint on the power spectrum: an extremely small correction to the its tilt that is suppressed by $(H/M_{\rm p})^2$, and a correction to its amplitude that is suppressed by $\varepsilon_1 (H/M_{\rm p})^2$ (see also [40]).

Returning now to the decoherence, we therefore approximate P_{vv} in eq. (4.13) by its Bunch-Davies counterpart, $P_{vv} = |u_{\boldsymbol{k}}|^2$. Recall that eq. (4.13) is derived by integrating the Markovian evolution equations and so strictly speaking its validity requires $-k\eta' \ll 1$ throughout the entire integration region. But within this region the small $k\eta'$ form of $\mathfrak{F}_{\boldsymbol{k}}(\eta', \eta_{\rm in})$ shows that the integrand strongly peaks[11] in the regime $-1 \ll k\eta' \leq k\eta$, with the dominant contribution

[11] As a technical aside, one might worry that because Re $[\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})]$ is singular in the coincident limit, integrating through the singular region away from small $k\eta$ might give a competing contribution. Appendix C.3 shows that these contributions are in fact subdominant in $k\eta$.

coming at late times driven by the universal leading behaviour shown in eq. (3.61). Using this and the super-Hubble form $P_{vv} \simeq a^2H^2/(2k^3) = 1/(2k^3\eta^2)$ in eq. (4.13) leads to

$$\Xi_{\boldsymbol{k}}(\eta) \simeq \frac{5\varepsilon_1}{64\pi}\left(\frac{H}{M_{\rm p}}\right)^2 \frac{1}{(-k\eta)^3} = \frac{5\varepsilon_1}{64\pi}\left(\frac{H}{M_{\rm p}}\right)^2 \left(\frac{aH}{k}\right)^3 , \tag{4.17}$$

showing how $\Xi_{\boldsymbol{k}}(\eta)$ grows strongly at late times. This type of growth proportional to a^3 is as expected fairly generally when decoherence is driven by a short-distance environment [14, 17], and is also seen in the strictly perturbative calculation of [70].

Eq. 4.17 leads to several interpretational questions. First, the strong a^3 growth implies that $\Xi_{\boldsymbol{k}}(\eta)$ can easily become order unity over 60 e-foldings of inflation despite the presence of the extremely small factors $\varepsilon_1 H^2/M_{\rm p}^2$. Can eqs. (4.12) and (4.17) still be trusted once $\Xi_{\boldsymbol{k}}$ is no longer small? We argue that they can because in the late-time regime where this growth dominates the evolution is controlled by eq. (3.36) – or eq. (3.45) – with a coefficient function $\mathfrak{F}_{\boldsymbol{k}}$ that is independent of $\eta_{\rm in}$. This is the regime for which the arguments of [28, 29, 35] (see also [26]) allow the domain of validity of the Lindblad evolution equation to be broader than the naive perturbative domain on which the Lindblad evolution itself is derived.[12] This is why we use the full form (4.12) rather than expanding $\mathfrak{p}_{\boldsymbol{k}}$ out to linear order in $\Xi_{\boldsymbol{k}}$.

Another conceptual question concerns the restriction $-k\eta_{\rm in} \ll 1$ that is required to keep the entire integration regime of eq. (4.13) within the Markovian regime. Ideally we'd instead like to take the limit $\eta_{\rm in} \to -\infty$, pushing the initial epoch when environment and system are uncorrelated to the distant past where the uncorrelated Bunch-Davies modes for the environment are deeply sub-Hubble and effectively behave as if they are in flat space.

Physically one expects that no decoherence arises for early times because at these times both the system modes of interest and their shorter-wavelength brethren in the environment are all sub-Hubble and so behave largely as they would in flat space. The vacuum then doesn't decohere for the same reason that short-wavelength vacuum modes don't spontaneously decohere long-wavelength quantum systems around us all the time in flat space. In principle this can be demonstrated explicitly in perturbation theory without the need for any late-time resummation (such as by going back to the full Nakajima-Zwanzig evolution equation (3.20) to compute how things evolve once $k\eta' \ll -1$ in the presence of the full correlation function $\mathscr{C}_{\boldsymbol{k}} + \mathscr{T}_{\boldsymbol{k}}$).[13]

One therefore expects decoherence to begin only once modes become of order Hubble size, and because decoherence rates are usually suppressed relative to the enivronment's underlying correlation scale (the Hubble scale in the de Sitter example) by factors of the system-environment coupling (in our case gravitational in size: $H^2/M_{\rm p}^2$) it is plausible that most of the decoherence happens only once modes are deep in the super-Hubble regime. In this paper we compute for technical reasons the late-time decoherence (with $k\eta_{\rm in}$ also chosen to be super-Hubble) but the fact that our leading results are universal (independent of $k_{\rm UV}$ and $\eta_{\rm in}$) strongly suggests that the decoherence rate we find also captures the dominant super-Hubble evolution experienced by modes prepared in the Bunch Davies vacuum at $\eta_{\rm in} \to -\infty$.

A final dangling question involves the relative importance of cubic interactions involving environmental fields differentiated with respect to time rather than space. As argued earlier, these are subdominant when the environmental modes are super-Hubble, so the validity of neglecting them depends on whether Hubble-sized or sub-Hubble modes in the environment contribute significantly to the decoherence of super-Hubble system states. At face value our calculation

[12]This is much the same way that the evolution equation $\mathrm{d}n/\mathrm{d}t = -\Gamma n$ reliably predicts the exponential decay law when Γt is greater than unity, despite the decay rate Γ itself usually being computed only perturbatively.

[13]It is important when doing so to keep in mind that all times are evaluated along a time contour that has a small negative imaginary part (as is also done here), since this is what projects the initial state onto the vacuum along the lines described in [47].

does not rule this out because it shows that the decoherence of super-Hubble sytem modes at time η is mostly driven by environmental correlations between times η and $\eta' \to \eta$. Since this receives contributions from very short wavelength fluctuations it is conceivable that sub-Hubble modes are significant, and so we are continuing to examine whether additional environmental operators can contribute significanly to the decoherence rate.

4.3.1 Numerical estimates

Let us re-express the small factor $\varepsilon_1 H^2/M_\mathrm{p}^2$ appearing in the decoherence parameter (4.13) in terms of common observables, in order to better understand its size.

In the single-field slow-roll inflationary models of interest here, the amplitude of the primordial scalar power spectrum is given by

$$\mathcal{P}_\zeta \simeq \frac{H^2}{8\pi^2 \varepsilon_1 M_\mathrm{p}^2} \simeq 2.2 \times 10^{-9} \tag{4.18}$$

where the numerical size is what is required for this to explain the observed primordial power spectrum [61]. Furthermore, in these models the first slow-roll parameter is directly related to the tensor-to-scalar ratio r by $r = 16\varepsilon_1$. It follows that

$$\Xi_{\boldsymbol{k}}(N) \simeq 1.7 \times 10^{-17} \left(\frac{\mathcal{P}_\zeta}{2.2 \times 10^{-9}}\right) \left(\frac{r}{10^{-3}}\right)^2 e^{3(N_\mathrm{end}-N_*)-3(N_\mathrm{end}-N)}\,, \tag{4.19}$$

where we trade time for the number of inflationary e-folds N, with N_end denoting the number of e-folds at the end of inflation while $N_* = N_*(k)$ is the number of e-folds evaluated at the horizon exit during inflation for mode k. The same expression can alternatively be written in terms of the energy scale at which inflation proceeds, $\rho_\mathrm{inf}^{1/4}$, leading to

$$\Xi_{\boldsymbol{k}}(N) = 1.6 \times 10^{5} \left(\frac{\mathcal{P}_\zeta}{2.2 \times 10^{-9}}\right)^{-1} \left(\frac{\rho_\mathrm{inf}^{1/4}}{M_\mathrm{p}}\right)^8 e^{3(N_\mathrm{end}-N_*)-3(N_\mathrm{end}-N)}\,. \tag{4.20}$$

For scales probed in CMB experiments one typically has $N_\mathrm{end} - N_* \simeq 50$ and if so eq. (4.19) or (4.20) show the purity for CMB scales at the end of inflation is given by

$$\mathfrak{p}_\mathrm{CMB}(N_\mathrm{end}) \simeq 6.5 \times 10^{-25} \left(\frac{10^{-3}}{r}\right) \simeq 2.1 \times 10^{-35} \left(\frac{M_\mathrm{p}}{\rho_\mathrm{inf}^{1/4}}\right)^4 . \tag{4.21}$$

Gravitational decoherence is efficient if inflation proceeds at energy scales larger than $\rho_\mathrm{inf}^{1/4} > 5.2 \times 10^9$ GeV or (equivalently) $r > 6.5 \times 10^{-28}$. Alternatively, a future detection of the tensor-to-scalar ratio r at a level above $r \sim 10^{-3}$ as targeted by future CMB polarization experiments [71] (which puts the energy scale of inflation above 10^{15}GeV) implies decoherence proceeds quickly after Hubble crossing during inflation.

More generally, the purity at the end of inflation, $\mathfrak{p}_\mathrm{end} := \mathfrak{p}_{\boldsymbol{k}}(N_\mathrm{end})$, depends on two parameters: the mode k of interest – characterized by $N_\mathrm{end} - N_*(k)$ – and the energy scale of inflation, $\rho_\mathrm{inf}^{1/4}$. This dependence is shown as a colour scale in Fig. 4, which reveals two regions – one for which $\mathfrak{p}_\mathrm{end} \ll 1$ (red 'decohered' region) and one for which $\mathfrak{p}_\mathrm{end} \simeq 1$ (black 'coherent' region), separated by the relatively abrupt transition represented by the thin white region. This transition corresponds to where $\Xi_{\boldsymbol{k}}$ in eq. (4.20) is order unity; *i.e.* by the straight line $N_\mathrm{end} - N_* \simeq 110 - 6.14 \ln \left[\rho_\mathrm{inf}^{1/4}/(1\mathrm{GeV})\right]$.

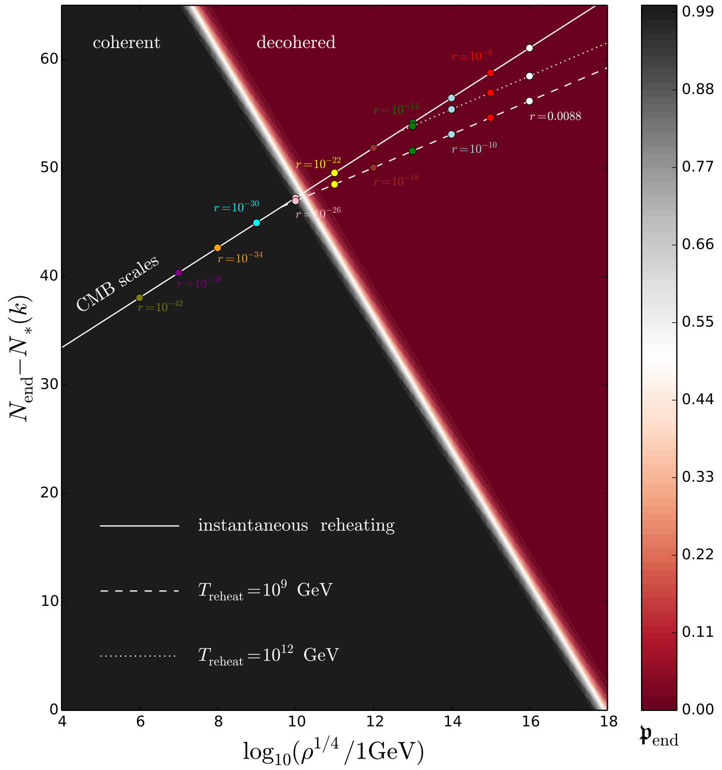

Figure 4: Purity as a function of the scale (namely, number of e-folds before the end of inflation) and the energy scale of inflation. Also plotted are predictions for how these two quantities are related for the value of k that is just re-entering the Hubble scale today, as a function of assumptions made about the post-inflationary reheat epoch, with the solid, dashed and dotted lines respectively representing instantaneous reheating at the end of inflation, $T_{\rm reh} = 10^{12}$GeV and $T_{\rm reh} = 10^9$GeV (with $g_* \simeq 1000$ and $w = 0$ during reheating for the latter two).

For comparison, for any fixed k a relationship is also predicted between $N_{\rm end} - N_*(k)$ and $\rho_{\rm inf}^{1/4}$ by following the mode's post-inflationary evolution through the reheating epoch up to the present day. For k corresponding to a physical wavelength that today equals the Hubble radius, $k/a_{\rm now} = H_{\rm now} = h\,(100 \text{ km/sec/Mpc})$ the prediction is

$$N_* - N_{\rm end} = \ln\left[(\Omega_{\gamma 0})^{-1/4}\,\frac{\rho_{\rm cri}^{1/4}}{1{\rm GeV}}\left(\frac{\pi^2 g_*}{30}\right)^{\frac{-1+3w}{12(1+w)}}\right] - \frac{2}{3}\left(\frac{1+3w}{1+w}\right)\ln\left(\frac{\rho_{\rm inf}^{1/4}}{1{\rm GeV}}\right) - \frac{1-3w}{3(1+w)}\ln\left(\frac{T_{\rm reheat}}{1{\rm GeV}}\right), \qquad (4.22)$$

where $\Omega_{\gamma 0} \simeq 2.471 \times 10^{-5}/h^2$ is the fraction of radiation today, $\rho_{\rm cri} \simeq 8.099\, h^2 \times 10^{-47}{\rm GeV}^{-4}$ is the critical energy density today, g_* the number of degrees of freedom after the end of inflation, $T_{\rm reheat}$ the reheat temperature and $w = p_{\rm rh}/\rho_{\rm rh}$ the equation of state parameter during reheating. The curves expressing eq. (4.22) are also shown in Fig. 4 for several choices of reheating properties.

If CMB polarization experiments detect a cosmic gravitational wave background in the foreseeable future then r cannot be too much smaller than $r \simeq 10^{-3}$ and the above expressions show that only modes that leave the Hubble radius less than $\simeq 12.9$ e-folds before the end of inflation do not have time to decohere. Whether the very small scales associated with such modes can be probed observationally is of course an open question.

5 Discussion

The main conceptual result of this paper is to apply open-system techniques to the evolution equation for the quantum state of observed metric modes during single-field inflation, including effects generated by their gravitational interaction with shorter wavelength modes. We identify those gravitational interactions that dominate the decoherence of long-wavelength modes, and show that their contribution to the evolution of longer-wavelength modes is given by equations like (3.20).

5.1 Decoherence of scalar modes

We compute the relevant environmental correlation functions – *i.e.* the function $\mathscr{C}_{\boldsymbol{k}}(\eta)$ appearing in eq. (3.20) and its tensor counterpart $\mathscr{T}_{\boldsymbol{k}}(\eta)$ once short-wavelength tensors are included in the environment – and show that their peaked form implies the evolution becomes Markovian for super-Hubble modes during inflation, leading to the approximate evolution equations (3.36) or (3.45) (in interaction and Schrödinger pictures, respectively) whose coefficient function $\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$ we also compute explicitly. In particular $\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$ turns out to be universal (depend only on $k\eta$ to leading approximation in the super-Hubble, late-time regime and grows strongly with dominant support at late times when $k\eta$ is small). All dependence on other parameters (like $\eta_{\rm in}$ and $k_{\rm UV}$) arising only at subdominant order in $k\eta$. We also derive an equivalent set of evolution equations – eqs. (3.53) through (3.55) – for correlation functions in this Markovian regime.

Because $\mathfrak{F}_{\boldsymbol{k}}$ peaks at small $k\eta$ it is super-Hubble modes in the environment that dominate the decoherence process. This in turn simplifies the selection of the dominant interactions because it implies that interactions can be neglected if they involve time derivatives (as opposed to spatial derivatives) acting on environmental fields.

Following refs. [26–29, 35] we argue that the universality of $\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$ allows the solutions to eqs. (3.36) and (3.45) to be trusted to later times than usual for perturbative reasoning, and so allows the late-time resummation of the secular growth found within the perturbative predictions for decoherence. We compute the late-time behaviour implied by the leading Markovian evolution and use it to evaluate the evolution of the mode purity.

The main practical results that emerge from this reasoning are given in eqs. (4.17) and (4.16) that respectively describe the rapid decoherence of observable primordial scalar metric fluctuations and the (very small) change to their power spectrum arising from their interactions with the environment of shorter-wavelength metric modes. Both scalar and tensor metric fluctuations in the environment act to decohere super-Hubble scalar modes and do so with the same dependence on inflationary parameters and the same efficiency per mode. The two spin states then ensure that tensor modes give twice the decoherence of scalars.

Because super-Hubble modes during inflation decohere at a rate proportional to $(aH/k)^3 \propto e^{3Ht}$, they grow exponentially quickly in cosmic time. This growth is initially small because it is suppressed by a small coupling prefactor $\propto \varepsilon_1 G H^2$, and so starts off at most 10^{-14} in single-field models. But it can become order unity during the 40-60 e-foldings of inflation that follow horizon exit and so can easily allow decoherence to be complete well before inflation ends. We believe our predictions continue to be valid even when decoherence is not small because the Lindblad evolution resums this late time growth. The associated modification to the power spectrum for scalar and tensor fluctuations remains very small (being loop-suppressed) at all times.

5.2 Decoherence of tensor modes by a scalar environment

We note in passing that the slow-roll suppression found above is not generic for all fluctuations. In particular super-Hubble tensor modes decohere *more quickly* than scalar modes because their dominant interactions are unsuppressed by slow-roll parameters. Since the factors of $H/M_{\rm p}$ and (aH/k) arise on very general grounds (leading order in the semiclassical expansion and the environment having shorter wavelength than the decohering modes) one is led to expect the decoherence of tensor modes to be of order $(H/M_{\rm p})^2(aH/k)^3$, and so be larger than our scalar result by a factor of $1/\varepsilon_1$.

This expectation can be tested using a relatively minor extension of the calculations described above that give the decoherence rate of tensor modes due to an environment of shorter-wavelength scalar metric fluctuations. We here provide a partial calculation of this; computing only the decoherence caused by interactions with environmental short-wavelength scalar metric fluctuations. A full calculation of the total rate with which tensor modes are decohered by their couplings to the short-wavelength tensor environment is in preparation.

5.2.1 Lindblad evolution

Repeating the same arguments as above shows that the leading cubic tensor-scalar-scalar interactions from the list in ref. [47] – *c.f.* eq. (A.19) – is

$$ {}^{(3)}S \supset \int {\rm d}t\, {\rm d}^3\boldsymbol{x}\, M_{\rm p}^2 \varepsilon_1 a \gamma^{ij} (\partial_i\zeta)(\partial_j\zeta)\,, \tag{5.1} $$

which contributes to the interaction hamiltonian the slow-roll unsuppressed interaction

$$ \mathcal{H}_{\rm int}(\eta) = -\frac{1}{M_{\rm p} a} \int {\rm d}^3\boldsymbol{x}\, v_{ij}(\eta,\boldsymbol{x}) \otimes B^{ij}(\eta,\boldsymbol{x})\,. \tag{5.2} $$

Here v_{ij} is the canonical field related to γ_{ij} by eq. (A.17) and the environmental operator

$$ B^{ij}(\boldsymbol{x}) := \delta^{ik}\delta^{jl}\partial_k v(\eta,\boldsymbol{x})\partial_l v(\eta,\boldsymbol{x})\,, \tag{5.3} $$

is related to the operator B of eq. (3.2) by $\delta_{ij}B^{ij} = B$. Crucially, the absence of slow-roll suppression in eq. (5.2) gives it an effective coupling

$$ \widetilde{G}(\eta) = -\frac{1}{M_{\rm p} a(\eta)} = 2\sqrt{\frac{2}{\varepsilon_1}}\, G(\eta)\,, \tag{5.4} $$

where $G(\eta)$ is the effective coupling defined in eq. (3.3).

Rotation invariance implies $\mathscr{B}^{ij} = \langle 0_B | B^{ij} | 0_B\rangle \propto \delta^{ij}$ and so because $\delta^{ij}v_{ij} = 0$ the mean of B^{ij} can be ignored in what follows. Similarly, the required correlation function is

$$ C^{iajb}(\eta,\eta';\boldsymbol{x}-\boldsymbol{x}') := \langle 0_B | \left[B^{ia}(\eta,\boldsymbol{x}) - \mathscr{B}^{ia}(\eta)\right]\left[B^{jb}(\eta',\boldsymbol{x}') - \mathscr{B}^{jb}(\eta')\right] |0_B\rangle \tag{5.5} $$

$$ = \int \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\, \mathscr{C}^{iajb}_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\,, \tag{5.6} $$

with

$$ \mathscr{C}^{iajb}_{\boldsymbol{k}} = \frac{2}{(2\pi)^{9/2}} \int_{q,p>k_{\rm UV}} {\rm d}^3\boldsymbol{q}\, {\rm d}^3\boldsymbol{p}\, p^i q^a p^j q^b\, u_q(\eta)u_p(\eta)u_q^*(\eta')u_p^*(\eta')\, \delta(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\,, \tag{5.7} $$

and comparing this with eq. (B.12) shows that $\mathscr{C}_{\boldsymbol{k}} = \delta_{ia}\delta_{jb}\mathscr{C}^{iajb}_{\boldsymbol{k}}$ is the quantity computed in sec. 3.1.3 above. Appendix F shows how rotation invariance can also be used to reduce this to a set of scalar integrals, related to those appearing in $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$. Once contracted with the graviton polarization tensors the required correlator at leading order in $k\eta$ is again universal – *i.e.* independent of $\eta_{\rm in}$ and $k_{\rm UV}$ – and related to our earlier result by

$$\mathscr{C}_{\boldsymbol{k}}^{iajb}\epsilon^{P}_{ia}\epsilon^{P'}_{jb} \simeq \frac{2}{15}\,\mathscr{C}_{\boldsymbol{k}}\delta^{PP'} \tag{5.8}$$

up to subdominant order in $k\eta$.

Keeping in mind the coupling (5.4), repeating the same steps as in earlier sections imply the analog of the Nakajima-Zwanzig evolution equation (3.20) involves $\mathscr{C}_{\boldsymbol{k}}^{ijkl}$, and in the super-Hubble limit again takes the Lindblad form (3.36) but with eq. (3.37) now given by the leading universal form

$$\mathrm{Re}\left[\mathfrak{T}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right] \simeq \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2}\left\{\frac{20\pi}{(-k\eta)^2} + \mathcal{O}\left[(-k\eta)^{-1}\right]\right\}\left(\frac{16}{15\varepsilon_1}\right) \simeq \frac{H^2}{48\pi M_{\rm p}^2\eta^2}\,, \tag{5.9}$$

with $\mathfrak{T}_{\boldsymbol{k}}$ the Lindblad coefficient analogous to $\mathfrak{F}_{\boldsymbol{k}}$ for the scalar (defined in eq. (F.40)).

Comparing this with the result (3.38) for how scalars modes decohere other scalar modes reveals the expected lack of slow-roll suppression.

5.2.2 Decoherence

The contribution of scalars already shows that tensor modes decohere more quickly than do scalar modes because their self-interactions are less suppressed by slow-roll parameters. The purity of tensor modes is given by an expression like eq. (4.12) with $\Xi_{\boldsymbol{k}}$ replaced by

$$\Xi^{\rm T}_{\boldsymbol{k}}(\eta) = 8\int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta'\ \mathrm{Re}[\mathfrak{T}_{\boldsymbol{k}}(\eta',\eta_{\rm in})]|u_{\boldsymbol{k}}(\eta)|^2\,, \tag{5.10}$$

and the universal late-time contribution of the short-wavelength scalar environment to $\mathfrak{F}_{\boldsymbol{k}}$ given by eq. (5.9). This implies a leading late-time contribution to $\Xi^{\rm T}_{\boldsymbol{k}}$ of size

$$\Xi^{\rm T}_{\boldsymbol{k}}(\eta) \simeq \frac{1}{36\pi}\left(\frac{H}{M_{\rm p}}\right)^2\frac{1}{(-k\eta)^3} = \frac{1}{36\pi}\left(\frac{H}{M_{\rm p}}\right)^2\left(\frac{aH}{k}\right)^3\,, \tag{5.11}$$

which is to be compared with equation (4.17). In particular it is unsuppressed by slow-roll parameters, as claimed (see also [72, 73]). The full expression for tensor decoherence requires adding to this the contribution of the tensor environment.

5.3 Open questions

The calculation described herein is clearly only the first step in a potentially long journey. Many things remain to be pinned down, and there is much value in doing so particularly if primordial fluctuations turn out to have a quantum origin (as they so far seem to do). We next list some of the open issues for scalar-mode decoherence, and then turn to what our result might imply for proposed late-time searches for observational quantum signatures [18–21].

One open issue concerns post-inflationary evolution of the reduced density matrix. We have shown how super-Hubble evolution during inflation acts to diagonalize the density matrix in a field basis, but we also know that the large factor $(aH/k)^3$ shrinks after inflation until it is again order unity at horizon re-entry. When saying that decoherence is efficient in the later universe we take for granted that the reduced density matrix remains effectively diagonal during the subsequent post-inflationary evolution before horizon re-entry. This seems intuitively reasonable: entanglement can be fragile and one rarely expects initially classical mixed systems to spontaneously self-purify. But this should be possible to prove, and we have not yet done so.

A related question concerns the effects of other degrees of freedom with other (possibly stronger) interactions or other variations like possible changes to the speed of sound (as in the EFT of inflation [74]). We have no general statements as to how more complicated interactions might change our result (4.17) apart from the general observation that most interactions are stronger than gravity and so likely produce a larger effect, making the decoherence progress faster — it is in this sense that our calculation is minimal. One suspects that many of these issues can benefit from further exchange of ideas between cosmology and the physics of open quantum systems.

5.4 Loopholes

At face value the efficiency of gravitational decoherence we find makes observing quantum coherence in measurements of primordial fluctuations much harder, assuming that these are generated by quantum fluctuations in single-field inflationary models and that the scale of inflation is not extremely low. So if evidence for coherent fluctuations should appear tomorrow, what might this mean?

Calculations like ours are useful for this question because they identify how decoherence depends on a theory's parameters. Even within the domain of validity of our result the inflationary scale might turn out to be very small. Or the modes in question might be short enough that they do not spend as much time outside the Hubble scale (even for $\varepsilon_1 \sim 10^{-2}$ modes can spend about 10 e-foldings outside the Hubble scale during inflation before decoherence stops being negligible).

There are also a variety of assumptions on which our calculations rely, at least one of which would have to break down. Perhaps the model involves more than the single inflaton; if so then ζ need not be conserved and this conservation played an important role in suppressing interactions involving time derivatives. Perhaps the important environmental scales are not at short wavelengths compared with observable modes; if so then their interactions need not be local and so the general argument of ref. [17] need no longer imply the same dependence on a^3 as found here. Perhaps additional interactions can re-cohere initially decohered states. Inquiring minds need to know.

Let us also note that the precise amount of decoherence it takes to erase a particular quantum feature can vary. For instance, in ref. [22], it was shown that decohered states in a de Sitter universe still carry a large quantum discord if decoherence is slow enough, i.e. if $\mathfrak{p}_{\boldsymbol{k}} \propto a^{-p}$ with $p < 4$. Given that we have found $p = 3/2$ in the case of gravitational decoherence, this implies that, although decoherence is very effective, the erasure of quantum discord is not, which might still leave open the possibility to detect quantum signatures. There is also the possibility to look for quantum effects at small scales, that spend too few e-folds outside the Hubble radius during inflation to efficiently decohere. If inflation proceeds at GUT scale, such wavelengths cannot be probed in the CMB, but they might be accessible in smaller-scale structures.

In many ways finding observational evidence for quantum coherence amongst primordial fluctuations is the most attractive option; by our present lights it would be the most surprising and so likely teach us the most.

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A Higher-derivative operators in the action

This appendix fleshes out the details of the derivation of the standard EFT of scalar and tensor fluctuations for single-field inflation (defined in ref. [47]), as well as the expansion of higher-curvature squared operators when this is understood as an EFT of gravity. The purpose of this appendix is two-fold: first, it is to review the set of cubic interactions considered in this work and in particular to further identify which interactions are dominant in driving decoherence. Secondly, it is to provide evidence that the counter-term operators renormalizing divergences in sec. 3.2.4 are indeed amongst those in the EFT of gravity in single-field inflation.

A.1 Single-field inflation

We start with a brief review of the standard fluctuation formalism that is to be used. As described in sec. 2, single-field inflation consists of gravity and a single scalar field in the form of the action (2.1), repeated here for convenience,

$$S[g_{\mu\nu},\varphi] = \int \mathrm{d}^4x\, \sqrt{-g}\left[\frac{M_\mathrm{p}^2}{2}R - \frac{1}{2}g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)\right]. \tag{A.1}$$

Homogeneous classical solutions for the inflaton $\phi(t)$ and Hubble parameter $H = \dot{a}/a$ (recall that a dot means derivative with respect to cosmic time) therefore obey

$$3M_\mathrm{p}^2H^2 = \tfrac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad M_\mathrm{p}^2\dot{H} = -\tfrac{1}{2}\dot{\phi}^2 \quad \text{and} \quad \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,. \tag{A.2}$$

We perturb about a near-de Sitter spacetime, $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\boldsymbol{x}^2$, working in the Arnowitt-Deser-Misner (ADM) formalism as in ref. [47] using the perturbed metric

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + h_{ij}\left(N^i\mathrm{d}t + \mathrm{d}x^i\right)\left(N^j\mathrm{d}t + \mathrm{d}x^j\right)\,, \tag{A.3}$$

with N the lapse function and N^i the shift vector and the inverse of spatial metric h^{ij} defined by $h^{ij}h_{jk} = \delta^i{}_k$. In terms of these variables the action (A.1) becomes

$$S = \int \mathrm{d}^4x\, \sqrt{h}N\left[\frac{M_\mathrm{p}^2}{2}\left(\mathcal{R} + \frac{E_{ij}E^{ij} - E^2}{N^2}\right) - V(\varphi) + \frac{\left(\dot{\varphi} - N^i\partial_i\varphi\right)^2}{2N^2} - \frac{1}{2}h^{ij}\partial_i\varphi\,\partial_j\varphi\right] \tag{A.4}$$

with $\mathcal{R}$ the 3D Ricci scalar built from the spatial metric h_{ij} and $K_{ij} = E_{ij}/N$ is the extrinsic curvature of these spatial slices, where

$$E_{ij} := \frac{1}{2}\left(\dot{h}_{ij} - \nabla_iN_j - \nabla_jN_i\right) \quad \text{and} \quad E := h^{ij}E_{ij}\,. \tag{A.5}$$

Specializing to the gauge where the inflaton has no perturbation, $\delta\varphi = 0$ and so $\varphi = \phi(t)$, the vanishing spatial derivative $\partial_j\varphi = 0$ allows the action to be simplified to

$$S = \int \mathrm{d}^4x\, \sqrt{h}N\left[\frac{M_\mathrm{p}^2}{2}\left(\mathcal{R} + \frac{E_{ij}E^{ij} - E^2}{N^2}\right) - V(\phi) + \frac{\dot{\phi}^2}{2N^2}\right]. \tag{A.6}$$

The constraint equations obtained by varying N and N^i then become

$$\nabla_i \left(\frac{E^i{}_j - \delta^i{}_j E}{N} \right) = 0, \quad \frac{M_{\rm p}^2}{2} \left(\mathcal{R} + \frac{E_{ij} E^{ij} - E^2}{N^2} \right) - V(\phi) + \frac{\dot{\phi}^2}{2N^2} = 0\,. \tag{A.7}$$

These constraint equations are solved for N and N^i as functions of the physical variables ζ and γ_{ij}, defined by

$$h_{ij} = a^2 e^{2\zeta} \left(\delta_{ij} + \gamma_{ij} + \frac{1}{2} \gamma_{i\ell} \gamma_{\ell j} + \dots \right) , \tag{A.8}$$

with $\partial_i \gamma_{ij} = \gamma_{ii} = 0$. The goal is to express the action (A.6) as a function of these variables after eliminating N and N^i using the constraints. Since our focus is mainly on scalar fluctuations we drop γ_{ij} in what follows, simply quoting when needed the graviton-dependent terms found elsewhere [47]. For the metric (A.8) the following relations prove useful:

$$\sqrt{h} = a^3 e^{3\zeta}, \quad \mathcal{R} = a^{-2} e^{-2\zeta} \left[-4(\partial^2 \zeta) - 2(\partial \zeta)^2 \right] , \tag{A.9}$$

where "∂" here denotes spatial differentiation.

A.2 Quadratic scalar action

We first verify the standard quadratic action for ζ. At leading order in ζ the lapse and shift are

$$N \simeq 1 + \frac{\dot{\zeta}}{H}, \quad N_i \simeq -\frac{\partial_i \zeta}{a^2 H} + \partial_i \chi\,, \tag{A.10}$$

where the field χ is defined as a solution to the equation $\partial^2 \chi = \dot{\phi}^2 \dot{\zeta} / (2 H^2 M_{\rm p}^2)$, and so

$$\chi := \frac{\dot{\phi}^2}{2 H^2 M_{\rm p}^2} \partial^{-2} \dot{\zeta} = \varepsilon_1 \partial^{-2} \dot{\zeta}\ . \tag{A.11}$$

Using the background equations of motion (A.2) and integrating by parts gives the quadratic action

$${}^{(2)}S = \int \mathrm{d}t\ \mathrm{d}^3\boldsymbol{x}\ M_{\rm p}^2 \varepsilon_1 \left[a^3 \dot{\zeta}^2 - a (\partial \zeta)^2 \right] \tag{A.12}$$

as given as eq. (2.6) in the main text, with $\varepsilon_1 = -\dot{H}/H^2 = \dot{\phi}^2/(2H^2 M_{\rm p}^2)$. This becomes canonical when expressed in terms of the Mukhanov-Sasaki field

$$v = a M_{\rm p} \sqrt{2 \varepsilon_1} \zeta \tag{A.13}$$

and when specialized to near-de Sitter geometries. eq. (A.12) has the same form as two of the three divergent terms found in eq. (3.49) that require renormalizing in the Lindblad equation:

$${}^{(2)}S \simeq \int \mathrm{d}\eta\ \mathrm{d}^3\boldsymbol{x}\ \left[\frac{1}{2} (v') + \frac{1}{\eta^2} v^2 - \frac{1}{2} (\partial v)^2 \right]\ . \tag{A.14}$$

A.3 Cubic scalar interactions

We next record the cubic self-interactions contained in the cubic part of the expansion $S = {}^{(2)}S + {}^{(3)}S + \dots$ of the action in powers of the fluctuations found in ref. [47]. The part cubic in the scalar perturbation can be written

$${}^{(3)}S = \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x} \left\{ \varepsilon_1^2 M_{\rm p}^2 \left[a \left(\partial \zeta \right)^2 \zeta + a^3 \dot{\zeta}^2 \zeta \right] - 2 \varepsilon_1^2 M_{\rm p}^2 a^3\, \dot{\zeta} \left(\partial_i \partial^{-2} \dot{\zeta} \right) \left(\partial_i \zeta \right) - \frac{1}{2} \epsilon_1^3 M_{\rm p}^2 a^3 \dot{\zeta}^2 \zeta \right.$$

$$+ 2\varepsilon_1 M_{\rm p}^4 a^3 \dot\zeta\zeta^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\ddot\phi}{2\dot\phi H} + \frac{\dot\phi^2}{4H^2M_{\rm p}^2}\right) + \frac{1}{2}\varepsilon_1^3 M_{\rm p}^2 a^3 \left(\partial_i\partial_j\partial^{-2}\dot\zeta\right)\left(\partial_i\partial_j\partial^{-2}\dot\zeta\right)\zeta\Bigg\} . \tag{A.15}$$

This way of writing the cubic action is organized in increasing powers of the slow-roll parameter, with the first line containing the dominant terms and the rest being subdominant. Although the first line is naively $\mathcal{O}(\varepsilon_1^2)$ the quadratic action (A.12) shows that correlations of ζ are themselves enhanced by slow-roll parameters. For this reason slow-roll behaviour is easier to read when the action is expressed in terms of v, and shows that the leading term of eq. (A.15) is actually $\mathcal{O}(\sqrt{\varepsilon_1})$. Notice also that for super-Hubble modes each time derivative of ζ counts as two spatial derivatives, so only the very first term dominates when both slow-roll parameters and $k/(aH)$ are small.

A.4 Tensor fluctuations

Tensor fluctuations are also straightforward to include by expanding the action (A.1), leading to the quadratic piece

$$^{(2)}S \supset \int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}\,\frac{M_{\rm p}^2}{8}\left[a^3\dot\gamma_{ij}\dot\gamma_{ij} - a(\partial_k\gamma_{ij})(\partial_k\gamma_{ij})\right] , \tag{A.16}$$

and revealing the canonically normalized field to be

$$v_{ij} = \frac{1}{2}aM_{\rm p}\gamma_{ij} . \tag{A.17}$$

The momentum-space mode expansion for this field then becomes

$$v_{ij}(\eta,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\sum_{P=+,\times}\epsilon_{ij}^P(\boldsymbol{k})v_{\boldsymbol{k}}^P(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{A.18}$$

where $\epsilon_{ij}^P(\boldsymbol{k})$ is the polarization tensor with properties $k^i\epsilon_{ij}^P(\boldsymbol{k}) = 0$ and $\epsilon_{ij}^P\epsilon_{ij}^{P'} = \delta^{PP'}$.

The cubic interactions that involve both scalar and tensor fluctuations are obtained along the same lines as above, leading to the following expressions [47] for the tensor-scalar-scalar interaction:

$$^{(3)}S \supset \int \mathrm{d}t\ \mathrm{d}^3\boldsymbol{x}\,M_{\rm p}^2\Big[\varepsilon_1 a\gamma_{ij}(\partial_i\zeta)(\partial_j\zeta) + \frac{1}{4}a^3\partial^2\gamma_{ij}(\partial_i\chi)(\partial_j\chi) + \frac{1}{2}\varepsilon_1 a^3\dot\gamma_{ij}(\partial_i\zeta)(\partial_j\chi)$$
$$+ \frac{1}{2}Ha^5\dot\gamma_{ij}\dot\gamma_{ij}\chi\Big], \tag{A.19}$$

which ignores redundant terms such as total derivatives and those that can be removed with field re-definitions. The χ-dependent terms become non-local expressions once χ is eliminated using eq. (A.11). Again only the very first term dominates in the slow-roll limit, but this time arises unsuppressed by ε_1 as is most easily seen when expressed using v rather than ζ.

The tensor-tensor-scalar interaction is found in an identical way and is given by

$$^{(3)}S \supset \int \mathrm{d}t\,\mathrm{d}^3\boldsymbol{x}\ M_{\rm p}^2\left(\frac{\varepsilon_1}{8}\,a\,\zeta\,\partial_l\gamma_{ij}\partial_l\gamma_{ij} + \frac{\varepsilon_1}{8}a^3\zeta\dot\gamma_{ij}\dot\gamma_{ij} - \frac{1}{4}a^3\dot\gamma_{ij}\partial_l\gamma_{ij}\partial_l\chi\right) , \tag{A.20}$$

where χ is again given by eq. (A.11) and redundant terms are dropped. For super-Hubble modes only the first term dominates in the slow-roll approximation and once powers of $k/(aH)$ are neglected, showing that the leading result is in this case $\mathcal{O}(\sqrt{\varepsilon_1})$.

A.5 Curvature-squared counterterms

We finally show that the final divergent contribution depends on $\boldsymbol{k}$ and η in a way consistent with it being absorbed into a curvature-squared counter-term, which can have the general form

$$S \supset \int \mathrm{d}^4x \sqrt{-g}\left(c_0 R^2 + d_0 R_{\mu\nu}R^{\mu\nu}\right), \tag{A.21}$$

where c_0 and d_0 are two constants. There is no $R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$ term here because we work in 4D where this term can be regarded as part of a topological invariant.

In particular, the operator proportional to $k^4\eta^2\big(v_{\mathrm{s}\boldsymbol{k}}^{(\alpha)}\big)^2$ in eq. (3.49) is found within the R^2 term, which when expanded in powers of ζ contains the contribution

$$\int \mathrm{d}^4x \sqrt{-g}\, R^2 = \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}\, a^3(1+3\zeta)\left(1+\frac{\dot\zeta}{H}\right)\left\{\frac{M_\mathrm{p}^2 a^{-2}e^{-2\zeta}\left[-4(\partial^2\zeta)-2(\partial\zeta)^2\right]}{2}+\cdots\right\}^2, \tag{A.22}$$

which when expanded out involves the term

$$\begin{aligned}\int \mathrm{d}^4x \sqrt{-g}\, R^2 &\supset \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}\, a^3\,(1+3\zeta)\left(1+\frac{\dot\zeta}{H}\right) M_\mathrm{p}^4 a^{-4}e^{-4\zeta}4\left(\partial^2\zeta\right)^2 \\ &\supset \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}\, 4M_\mathrm{p}^4 a^{-1}(\partial^2\zeta)^2 = \int \mathrm{d}\eta\, \mathrm{d}^3\boldsymbol{x}\, \frac{2M_\mathrm{p}^2}{\varepsilon_1 a^2}(\partial^2 v)^2 .\end{aligned} \tag{A.23}$$

Once translated to Fourier space (and using $a \propto \eta^{-1}$) this has the same time- and momentum-dependence — $k^4\eta^2\big(v_{\mathrm{s}\boldsymbol{k}}^{(\alpha)}\big)^2$ — that appears in the last operator of eq. (3.49).

B Environmental Correlations

B.1 Correlations in real space

In this Appendix, we explicitly compute the correlation function $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$ defined in eq. (3.19) for each mode $\boldsymbol{k}$ in the Lindblad equation associated with the environment operator

$$B(\eta,\boldsymbol{x}) = \delta^{ij}\partial_i v_{B}(\eta,\boldsymbol{x})\partial_j v_{B}(\eta,\boldsymbol{x}) . \tag{B.1}$$

Using the canonical commutation relations (2.17) it is easy to see that the one-point function defined in eq. (3.17) has the integral representation

$$\begin{aligned}\mathscr{B}(\eta) \;:=\; \langle 0_B|B(\eta,\boldsymbol{x})|0_B\rangle &= -\iint_{k,q>k_{\mathrm{UV}}} \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\,(\boldsymbol{k}\cdot\boldsymbol{q})\,u_{\boldsymbol{k}}(\eta)\,u^*_{\boldsymbol{q}}(\eta)\,\delta(\boldsymbol{k}+\boldsymbol{q})\,e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x}} \qquad \text{(B.2)}\\ &= \int_{k>k_{\mathrm{UV}}}\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,k^2\,|u_{\boldsymbol{k}}(\eta)|^2 \;. \qquad \text{(B.3)}\end{aligned}$$

This function is independent of the position $\boldsymbol{x}$ and is also formally divergent, although we avoid regulating it since it does not enter into any physical predictions in this work.

From here we similarly compute the two-point function:

$$\begin{aligned}\langle 0_B|B(\eta,\boldsymbol{x})B(\eta',\boldsymbol{x}')|0_B\rangle = &\iint_{k,q>k_{\mathrm{UV}}}\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3}\iint_{p,\ell>k_{\mathrm{UV}}}\frac{\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\\ &\times\;(\boldsymbol{k}\cdot\boldsymbol{q})(\boldsymbol{p}\cdot\boldsymbol{\ell})\langle 0_B|v_{\boldsymbol{k}}(\eta)v_{\boldsymbol{q}}(\eta)v_{\boldsymbol{p}}(\eta')v_{\boldsymbol{\ell}}(\eta')|0_B\rangle\; e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x}+i(\boldsymbol{p}+\boldsymbol{\ell})\cdot\boldsymbol{x}'} .\end{aligned} \tag{B.4}$$

In the expectation value above there are many combinations of creation and annihilation operators which occur, but only the following two survive:

$$\langle 0_B | c_{\boldsymbol{k}} c^\dagger_{-\boldsymbol{q}} c_{\boldsymbol{p}} c^\dagger_{-\boldsymbol{\ell}} | 0_B \rangle = \delta(\boldsymbol{k}+\boldsymbol{q})\delta(\boldsymbol{p}+\boldsymbol{\ell}) \tag{B.5}$$

$$\langle 0_B | c_{\boldsymbol{k}} c_{\boldsymbol{q}} c^\dagger_{-\boldsymbol{p}} c^\dagger_{-\boldsymbol{\ell}} | 0_B \rangle = \delta(\boldsymbol{k}+\boldsymbol{\ell})\delta(\boldsymbol{p}+\boldsymbol{q}) + \delta(\boldsymbol{k}+\boldsymbol{p})\delta(\boldsymbol{\ell}+\boldsymbol{q}). \tag{B.6}$$

This gives

$$\begin{aligned} \langle 0_B | B(\eta, \boldsymbol{x}) B(\eta', \boldsymbol{x}') | 0_B \rangle = &\iint_{k,q>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \iint_{p,\ell>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3} (\boldsymbol{k}\cdot\boldsymbol{q})(\boldsymbol{p}\cdot\boldsymbol{\ell})\, e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x} + i(\boldsymbol{p}+\boldsymbol{\ell})\cdot\boldsymbol{x}'} \\ &\times \big\{ u_{\boldsymbol{k}}(\eta) u^*_{\boldsymbol{q}}(\eta) u_{\boldsymbol{p}}(\eta') u^*_{\boldsymbol{\ell}}(\eta') \delta(\boldsymbol{k}+\boldsymbol{q})\delta(\boldsymbol{p}+\boldsymbol{\ell}) + u_{\boldsymbol{k}}(\eta) u_{\boldsymbol{q}}(\eta) u^*_{\boldsymbol{p}}(\eta') u^*_{\boldsymbol{\ell}}(\eta') \\ &\times [\delta(\boldsymbol{k}+\boldsymbol{\ell})\delta(\boldsymbol{p}+\boldsymbol{q}) + \delta(\boldsymbol{k}+\boldsymbol{p})\delta(\boldsymbol{\ell}+\boldsymbol{q})] \big\}. \end{aligned} \tag{B.7}$$

Using eq. (B.2) the first pair of δ-functions can be seen to give rise to a pair of 1-point functions. In addition to this, the term involving the second pair of δ-functions is easily seen to be equal to the term involving the third pair after a re-labeling of momenta giving

$$\begin{aligned} \langle 0_B | B(\eta, \boldsymbol{x}) B(\eta', \boldsymbol{x}') | 0_B \rangle = \mathscr{B}(\eta)\mathscr{B}(\eta') + \ 2 \iint_{k,q>k_{\rm UV}} &\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} (\boldsymbol{k}\cdot\boldsymbol{q})(\boldsymbol{k}\cdot\boldsymbol{q})\, e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot(\boldsymbol{x}-\boldsymbol{x}')} \\ &\times u_{\boldsymbol{k}}(\eta) u_{\boldsymbol{q}}(\eta) u^*_{\boldsymbol{k}}(\eta') u^*_{\boldsymbol{q}}(\eta'), \end{aligned} \tag{B.8}$$

and so

$$\begin{aligned} C_B(\eta, \eta'; \boldsymbol{x}-\boldsymbol{x}') &= \langle 0_B | B(\eta, \boldsymbol{x}) B(\eta', \boldsymbol{x}') | 0_B \rangle - \mathscr{B}(\eta)\mathscr{B}(\eta') \\ &= \iint_{q,p>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}}{(2\pi)^6}\, 2(\boldsymbol{q}\cdot\boldsymbol{p})^2 u_{\boldsymbol{q}}(\eta) u_{\boldsymbol{p}}(\eta) u^*_{\boldsymbol{q}}(\eta') u^*_{\boldsymbol{p}}(\eta') e^{i(\boldsymbol{q}+\boldsymbol{p})\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \end{aligned} \tag{B.9}$$

B.2 Correlations for each mode $\boldsymbol{k}$

B.2.1 Integral expression

Finally we here compute eq. (3.19) defined in the main text, which is the Fourier transform of the fluctuating part of the environment correlation function, where

$$\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') = \int \frac{\mathrm{d}^3\boldsymbol{y}}{(2\pi)^{3/2}}\, C_B(\eta, \eta'; \boldsymbol{y}) e^{-i\boldsymbol{k}\cdot\boldsymbol{y}} \ . \tag{B.10}$$

We note in particular that we are interested in $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ for modes $\boldsymbol{k} \in \mathbb{R}^{3+}$ in the open system with $0 < k < k_{\rm UV}$ since the Nakajima-Zwanzig equation (3.20) is written in terms of these modes. We begin with the position space representation (B.9)

$$C_B(\eta, \eta'; \boldsymbol{y}) = \iint_{q,p>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}}{(2\pi)^6}\, 2(\boldsymbol{q}\cdot\boldsymbol{p})^2\, u_{\boldsymbol{q}}(\eta) u_{\boldsymbol{p}}(\eta) u^*_{\boldsymbol{q}}(\eta') u^*_{\boldsymbol{p}}(\eta')\, e^{i(\boldsymbol{q}+\boldsymbol{p})\cdot\boldsymbol{y}}. \tag{B.11}$$

We notice that the (double) momentum integration is restricted to be in the environment only. Next notice that $C_B(\eta, \eta'; \boldsymbol{y})$ is a function of $y = |\boldsymbol{y}|$ only — to see why, note that any rotation $\boldsymbol{y} \to \boldsymbol{y}' = \mathbf{R}\boldsymbol{y}$ can be undone by a rotation of coordinates in the integrals, such that $\boldsymbol{q}' = \mathbf{R}^{-1}\boldsymbol{q}$ and $\boldsymbol{p}' = \mathbf{R}^{-1}\boldsymbol{p}$ (each with Jacobian one) so that clearly $C_B(\eta, \eta'; \boldsymbol{y}) = C_B(\eta, \eta; \mathbf{R}\boldsymbol{y})$ for any rotation matrix $\mathbf{R}$. This then of course implies the Fourier transform $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ is a function of $k = |\boldsymbol{k}|$ only.

Inserting the above expression for $C_B(\eta, \eta'; \boldsymbol{y})$ into our definition for $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ we get

$$\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') = \frac{2}{(2\pi)^{9/2}} \iint_{q,p>k_{\rm UV}} \mathrm{d}^3\boldsymbol{q}\, \mathrm{d}^3\boldsymbol{p}\, (\boldsymbol{q}\cdot\boldsymbol{p})^2\, u_{\boldsymbol{q}}(\eta)\, u_{\boldsymbol{p}}(\eta)\, u^*_{\boldsymbol{p}}(\eta')\, u^*_{\boldsymbol{q}}(\eta')\, \delta(\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{k}). \quad \text{(B.12)}$$

From here we convert to spherical coordinates $\boldsymbol{q} = (q, \theta_q, \varphi_q)$ and $\boldsymbol{p} = (p, \theta_p, \varphi_p)$, where we have

$$\boldsymbol{q}\cdot\boldsymbol{p} = q\, p\, [\sin\theta_q \sin\theta_p \cos(\varphi_q - \varphi_p) + \cos\theta_q \cos\theta_p]\,, \quad \text{(B.13)}$$

and, in addition, we have the identity (for any arbitrary vector $\boldsymbol{\ell}$)

$$\delta(\boldsymbol{q}-\boldsymbol{\ell}) = \frac{\delta(q-\ell)\delta(\theta_q-\theta_\ell)\delta(\varphi_q-\varphi_\ell)}{q^2|\sin\theta_q|}\,. \quad \text{(B.14)}$$

In eq. (B.12) we must express the vector $\boldsymbol{\ell} := \boldsymbol{k}-\boldsymbol{p}$ in spherical coordinates — to this end, we exploit the rotational symmetry of the function $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ in $\boldsymbol{k}$ and pick $\boldsymbol{k} = (0,0,k)$ to be pointing along the 3-axis, so that in Cartesian coordinates $\boldsymbol{\ell} := \boldsymbol{k}-\boldsymbol{p} = (-p_x, -p_y, k-p_z)$, which implies that

$$\ell = \sqrt{p^2+k^2-2kp\cos\theta_p}\,, \quad \theta_\ell = \cos^{-1}\left(\frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}\right), \quad \varphi_\ell = \varphi_p + \pi\,. \quad \text{(B.15)}$$

With these identities, eq. (B.12) becomes

$$\begin{aligned}\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') = {}& \frac{2}{(2\pi)^{9/2}} \int_{k_{\rm UV}}^\infty \mathrm{d}q\, q^2 \int_0^\pi \mathrm{d}\theta_q \sin\theta_q \int_0^{2\pi} \mathrm{d}\varphi_q \int_{k_{\rm UV}}^\infty \mathrm{d}p\, p^2 \int_0^\pi \mathrm{d}\theta_p \sin\theta_p \int_0^{2\pi} \mathrm{d}\varphi_p \\ & \times (qp)^2 [\sin\theta_q\sin\theta_p\cos(\varphi_q-\varphi_p)+\cos\theta_q\cos\theta_p]^2\, u_{\boldsymbol{q}}(\eta)\, u_{\boldsymbol{p}}(\eta)\, u^*_{\boldsymbol{p}}(\eta')\, u^*_{\boldsymbol{q}}(\eta') \\ & \times \frac{1}{q^2\sin\theta_q}\delta\left[q-\sqrt{p^2+k^2-2kp\cos\theta_p}\right]\delta\left[\theta_q - \cos^{-1}\left(\frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}\right)\right] \\ & \times \delta\left[\varphi_q - (\varphi_p+\pi)\right], \end{aligned} \quad \text{(B.16)}$$

Integrating over φ_q and then φ_p yields

$$\begin{aligned}\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') = {}& \frac{2}{(2\pi)^{7/2}} \int_{k_{\rm UV}}^\infty \mathrm{d}q \int_0^\pi \mathrm{d}\theta_q \int_{k_{\rm UV}}^\infty \mathrm{d}p \int_0^\pi \mathrm{d}\theta_p\, q^2\, p^4 \sin\theta_p\, (-\sin\theta_q\sin\theta_p + \cos\theta_q\cos\theta_p)^2 \\ & \times u_{\boldsymbol{q}}(\eta)\, u_{\boldsymbol{p}}(\eta)\, u^*_{\boldsymbol{p}}(\eta')\, u^*_{\boldsymbol{q}}(\eta')\, \delta\left[q-\sqrt{p^2+k^2-2kp\cos\theta_p}\right] \\ & \times \delta\left[\theta_q - \cos^{-1}\left(\frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}\right)\right]. \end{aligned} \quad \text{(B.17)}$$

The next step consists in integrating over θ_q. Using the identities

$$\cos\left[\cos^{-1}\left(\frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}\right)\right] = \frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}, \quad \text{(B.18)}$$

$$\sin\left[\cos^{-1}\left(\frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}\right)\right] = \frac{p\sin\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}, \quad \text{(B.19)}$$

this leads to

$$\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{2}{(2\pi)^{7/2}} \int_{k_{\rm UV}}^{\infty} \mathrm{d}q \int_{k_{\rm UV}}^{\infty} \mathrm{d}p \int_0^{\pi} \mathrm{d}\theta_p \, q^2 \, p^4 \sin\theta_p \frac{\left[(k - p\cos\theta_p)\cos\theta_p - p\sin^2\theta_p\right]^2}{p^2 + k^2 - 2kp\cos\theta_p}$$

$$u_{\boldsymbol{q}}\,(\eta)\, u_{\boldsymbol{p}}\,(\eta)\, u^*_{\boldsymbol{p}}\,(\eta')\, u^*_{\boldsymbol{q}}\,(\eta')\, \delta\left(q - \sqrt{p^2 + k^2 - 2kp\cos\theta_p}\right). \tag{B.20}$$

Now switching coordinates to $\mu := \cos\ \theta_p$ turns the above into

$$\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{2}{(2\pi)^{7/2}} \int_{k_{\rm UV}}^{\infty} \mathrm{d}q \int_{k_{\rm UV}}^{\infty} \mathrm{d}p \int_{-1}^{1} \mathrm{d}\mu \, q^2 \, p^4 \frac{(p - k\mu)^2}{p^2 + k^2 - 2kp\mu} u_{\boldsymbol{q}}\,(\eta)\, u_{\boldsymbol{p}}\,(\eta)\, u^*_{\boldsymbol{p}}\,(\eta')\, u^*_{\boldsymbol{q}}\,(\eta')$$

$$\times\, \delta\left(q - \sqrt{p^2 + k^2 - 2kp\mu}\right) . \tag{B.21}$$

Next, we notice that the δ-function is actually easiest to integrate over the μ variable, and so to this end we take note of the rule $\delta\big[f(\mu)\big] = \delta(\mu - \mu_0)/|f'(\mu_0)|$ where $\mu_0 = (p^2 + k^2 - q^2)/(2pk)$ is the (only) zero of the function $f(\mu) := q - \sqrt{p^2 + k^2 - 2pk\mu}$. This implies

$$\delta\left(q - \sqrt{p^2 + k^2 - 2pk\mu}\right) = \frac{q}{kp}\, \delta\left(\mu - \frac{p^2 + k^2 - q^2}{2pk}\right) \tag{B.22}$$

giving

$$\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{2}{(2\pi)^{7/2}k} \int_{k_{\rm UV}}^{\infty} \mathrm{d}q \int_{k_{\rm UV}}^{\infty} \mathrm{d}p \, q^3 p^3 u_{\boldsymbol{q}}\,(\eta)\, u_{\boldsymbol{p}}\,(\eta)\, u^*_{\boldsymbol{p}}\,(\eta')\, u^*_{\boldsymbol{q}}\,(\eta')$$

$$\times \int_{-1}^{1} \mathrm{d}\mu \, \frac{(p - k\mu)^2}{p^2 + k^2 - 2kp\mu}\, \delta\left(\mu - \frac{p^2 + k^2 - q^2}{2pk}\right) . \tag{B.23}$$

The μ-integration can now be performed in the sense that

$$\int_{-1}^{1} \mathrm{d}\mu \, \frac{(p - k\mu)^2}{p^2 + k^2 - 2kp\mu}\, \delta\left(\mu - \frac{p^2 + k^2 - q^2}{2pk}\right) = \frac{(q^2 + p^2 - k^2)^2}{4q^2p^2} \tag{B.24}$$

but only in the region of momentum space where

$$-1 \ < \ \frac{p^2 + k^2 - q^2}{2pk} \ < \ 1 \,, \tag{B.25}$$

which affects the region of integration in the (p, q)-plane. Note that this means the region which the above inequality bounds is actually rectangular — to see why note that $-1 < \frac{p^2+k^2-q^2}{2pk}$ implies $(p + k)^2 > q^2$ while $\frac{p^2+k^2-q^2}{2pk} < 1$ implies $(p - k)^2 < q^2$, which means that this region corresponds to

$$|p - k| < q < p + k \,. \tag{B.26}$$

Note that whenever $k < k_{\rm UV}$ (as we use in the main text), the above simplifies to the region in which $p - k < q < p + k$ (since $p > k_{\rm UV} > k$). Note however that we must have $p > k_{\rm UV}$ and $q > k_{\rm UV}$ in addition to the earlier inequality being satisfied — this means that the actual region being integrated is the quadrilateral region U depicted in fig. 5, giving

$$\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{1}{2(2\pi)^{7/2}k} \iint_U \mathrm{d}p\, \mathrm{d}q\, p\, q\, (q^2 + p^2 - k^2)^2\, u_{\boldsymbol{q}}\,(\eta)\, u_{\boldsymbol{p}}\,(\eta)\, u^*_{\boldsymbol{p}}\,(\eta')\, u^*_{\boldsymbol{q}}\,(\eta')\,. \tag{B.27}$$

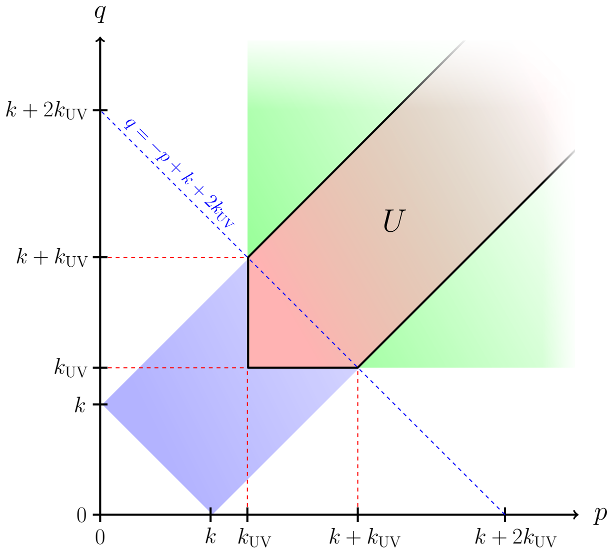

Figure 5: Defining the integration region in (p, q)-space for computing $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ in eq. (B.27) i.e. after the μ-integration is completed in eq. (B.23). The green region corresponds to $p, q > k_{\rm UV}$ and the blue region corresponds to $-1 < (p^2 + k^2 - q^2)/(2pk) < 1$ (assuming $k < k_{\rm UV}$) — the intersection of these regions (in red) is the resulting integration region U for computing $\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')$ in (p, q)-space in eq. (B.27).

In the current form it is complicated to integrate the above integrand — for this reason we transform integration variables to

$$p = \frac{P+Q}{2} \qquad \text{and} \qquad q = \frac{P-Q}{2} \tag{B.28}$$

with the Jacobian

$$\left|\frac{\partial(p, q)}{\partial(P, Q)}\right| = \frac{1}{2}\ . \tag{B.29}$$

The transformation $(p, q) \to (P, Q)$ rotates the region U by $\pi/4$ (and rescales it as well) giving rise to the set U' depicted in fig. 6 below. Then, using the explicit form of the mode functions, this leads to the following expression

$$\begin{aligned}\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') = \frac{1}{64(2\pi)^{7/2}k} \iint_{U'} \mathrm{d}P\, \mathrm{d}Q\, (P^2 + Q^2 - 2k^2)^2 \left[1 - \frac{2i}{(P-Q)\eta}\right] \left[1 + \frac{2i}{(P-Q)\eta'}\right] \\ \times \left[1 - \frac{2i}{(P+Q)\eta}\right] \left[1 + \frac{2i}{(P+Q)\eta'}\right]\, e^{-iP(\eta-\eta')}.\end{aligned} \tag{B.30}$$

By noting that the integrand in eq. (B.30) is symmetric under $Q \to -Q$, one can explicitly integrate over the region U' as

$$\begin{aligned}\mathscr{C}_{\boldsymbol{k}}(\eta, \eta') = \frac{1}{32(2\pi)^{7/2}k} \int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\, (P^2 + Q^2 - 2k^2)^2 \left[1 - \frac{2i}{(P-Q)\eta}\right] \left[1 + \frac{2i}{(P-Q)\eta'}\right] \\ \times \left[1 - \frac{2i}{(P+Q)\eta}\right] \left[1 + \frac{2i}{(P+Q)\eta'}\right]\, e^{-iP(\eta-\eta')}.\end{aligned} \tag{B.31}$$

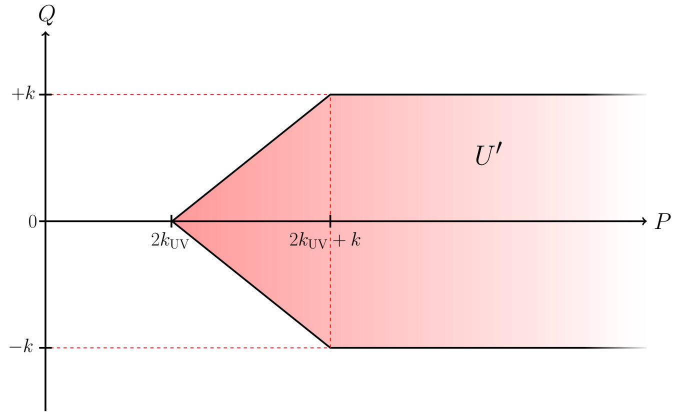

Figure 6: Defining the integration region U' in (P,Q)-space (after transforming $(p,q)\to(P,Q)$ in eq. (B.28)) for computing $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$ in eq. (B.30).

This representation of the correlator is most useful for computing the Lindblad coefficient $\mathfrak{F}_{\boldsymbol{k}}$ appearing in the Lindblad equation.

B.2.2 Integration of $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$

We begin with the double integral (B.31), and note that it is somewhat tricky to evaluate the P-integral because it is formally divergent in the ultraviolet. All this means is that we are to understand these integrals as distributions — to deal with this we organize the integrand in eq. (B.31) in terms of decreasing powers of P such that

$$\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{1}{32(2\pi)^{7/2}k}\int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\ [\mathcal{D}(P,Q)+\mathcal{F}(P,Q)]\ e^{-iP(\eta-\eta')}\ , \tag{B.32}$$

where we define the function $\mathcal{D}(P,Q)$ (giving rise to a formally divergent P-integral in the UV)

$$\begin{aligned}\mathcal{D}(P,Q) :=\ & P^4 + \frac{4i(\eta-\eta')}{\eta\eta'}P^3 + 2\left[Q^2-2k^2+\frac{8\eta\eta'-2\eta^2-2(\eta')^2}{\eta^2(\eta')^2}\right]P^2 \\ &+ \frac{16i(\eta-\eta')\left[1+\left(\frac{3}{4}Q^2-k^2\right)\eta\eta'\right]}{\eta^2(\eta')^2}P \\ &+ \frac{16\left\{1-4(k^2-Q^2)\eta\eta'+\left(k^2-\frac{3}{4}Q^2\right)\left[\eta^2+(\eta')^2\right]+\frac{(Q^2-2k^2)^2}{16}\eta^2(\eta')^2\right\}}{\eta^2(\eta')^2},\end{aligned} \tag{B.33}$$

as well as the function $\mathcal{F}(P,Q)$ (yielding a formally convergent P-integral in the UV)

$$\begin{aligned}\mathcal{F}(P,Q) := \frac{16(k^2-Q^2)}{\eta^2(\eta')^2(P^2-Q^2)^2}\Big(& i(\eta-\eta')\left[(k^2-Q^2)\eta\eta'-4\right]P^3 + \left\{\left[\eta^2+(\eta')^2\right](Q^2-k^2)\right. \\ &\left.-4(2Q^2-k^2)\eta\eta'-4\right\}P^2 + i(\eta-\eta')\left[Q^4\eta\eta'+k^2(4-Q^2\eta\eta')\right]P \\ &+\left\{4+Q^2[\eta^2+(\eta')^2]\right\}k^2 - Q^4\left[\eta^2-4\eta\eta'+(\eta')^2\right]\Big).\end{aligned} \tag{B.34}$$

To further simplify the calculation, we compute the above under the assumption that $\eta>\eta'$, which is the more useful case for the calculation in the main text (and furthermore, we can easily extract the opposing case $\eta < \eta'$ from the symmetry $\mathscr{C}^*_{\boldsymbol{k}}(\eta,\eta') = \mathscr{C}_{\boldsymbol{k}}(\eta',\eta)$ and so the calculation is performed without any loss of generality).

First we compute the part of the double integral (B.32) above involving $\mathcal{F}$ defined in eq. (B.34), namely the quantity

$$\frac{1}{k}\int_0^k \mathrm{d}Q \int_{Q+2k_{\mathrm{UV}}}^{\infty} \mathrm{d}P\,\mathcal{F}(P,Q)\,e^{-iP(\eta-\eta')} \tag{B.35}$$

which we note yields a convergent P-integral in the ultraviolet since for large P the associated integrand behaves as:

$$\mathcal{F}(P,Q) = \frac{16i(k^2-Q^2)(\eta-\eta')\left[(k^2-Q^2)\eta\eta'-4\right]}{\eta^2(\eta')^2}\frac{1}{P} + \mathcal{O}\left(P^{-2}\right). \tag{B.36}$$

It turns out that we can exactly write down the P-primitive of $\mathcal{F}(P,Q)\,e^{-iP(\eta-\eta')}$, where

$$\begin{aligned}\int \mathrm{d}P\,\mathcal{F}(P,Q)\,e^{-iP(\eta-\eta')} &= -\frac{32(k^2-Q^2)^2\left[P(1+Q^2\eta\eta')+iQ^2(\eta-\eta')\right]}{\eta^2(\eta')^2Q^2\left(P^2-Q^2\right)}\,e^{-iP(\eta-\eta')} \\ &- \frac{8(k^2-Q^2)(\eta-\eta')}{\eta^2(\eta')^2Q}\left\{\frac{k^2}{Q^2}\left[\frac{2}{\eta-\eta'}+2iQ-Q^2(\eta-\eta')+iQ^3\eta\eta'\right]\right. \\ &\left.+\frac{2}{\eta-\eta'}+2iQ+Q^2\frac{(\eta+\eta')^2}{\eta-\eta'}-iQ^3\eta\eta'\right\}\mathrm{Ei}\left[-i(P-Q)(\eta-\eta')\right]e^{-iQ(\eta-\eta')} \\ &+ \frac{8(k^2-Q^2)(\eta-\eta')}{\eta^2(\eta')^2Q}\left\{\frac{k^2}{Q^2}\left[\frac{2}{\eta-\eta'}-2iQ-Q^2(\eta-\eta')-iQ^3\eta\eta'\right]\right. \\ &\left.+\frac{2}{\eta-\eta'}-2iQ+Q^2\frac{(\eta+\eta')^2}{\eta-\eta'}+iQ^3\eta\eta'\right\}\mathrm{Ei}\left[-i(P+Q)(\eta-\eta')\right]e^{iQ(\eta-\eta')},\end{aligned} \tag{B.37}$$

where Ei is the exponential integral function defined for $z \in \mathbb{C}\backslash(-\infty,0]$ (i.e. there is a branch cut along the negative real axis) as

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty}\mathrm{d}t\,\frac{e^{-t}}{t}\,, \tag{B.38}$$

where the principal value of the integral is taken.

To evaluate the above primitive at the endpoint $P\to\infty$, we note the property

$$\mathrm{Ei}(-iy) \simeq -i\pi \qquad \text{for } y \gg 1\,, \tag{B.39}$$

which must be used since we are evaluating the correlator under the assumption that $\eta-\eta'>0$. Evaluating the P-integral for the required endpoints and simplifying yields

$$\begin{aligned}\int_{Q+2k_{\mathrm{UV}}}^{\infty}\mathrm{d}P\,\mathcal{F}(P,Q)\,e^{-iP(\eta-\eta')} &= -\frac{16\pi(k^2-Q^2)}{\eta^2(\eta')^2Q^3}\Big(Q(\eta-\eta')\left[2(k^2+Q^2)+(k^2-Q^2)Q^2\eta\eta'\right] \\ &\times\cos[Q(\eta-\eta')] + \left\{k^2\left[Q^2(\eta-\eta')^2-2\right]-Q^2\left[Q^2(\eta+\eta')^2+2\right]\right\}\sin[Q(\eta-\eta')]\Big) \\ &+\frac{32(k^2-Q^2)^2}{\eta^2(\eta')^2Q^2}\frac{[Q+2k_{\mathrm{UV}}(\eta)](1+Q^2\eta\eta')+iQ^2(\eta-\eta')}{[Q+2k_{\mathrm{UV}}(\eta)]^2-Q^2}e^{-i(Q+2k_{\mathrm{UV}})(\eta-\eta')}\end{aligned}$$

$$
\begin{aligned}
&+\frac{8(k^2-Q^2)(\eta-\eta')}{\eta^2(\eta')^2Q^3}\Bigg\{k^2\left[\frac{2}{\eta-\eta'}+2iQ-Q^2(\eta-\eta')+iQ^3\eta\eta'\right]\\
&+Q^2\left[\frac{2}{\eta-\eta'}+2iQ+Q^2\frac{(\eta+\eta')^2}{\eta-\eta'}-iQ^3\eta\eta'\right]\Bigg\}\mathrm{Ei}\left[-2ik_{\rm UV}(\eta-\eta')\right]e^{-iQ(\eta-\eta')}\\
&-\frac{8(k^2-Q^2)(\eta-\eta')}{\eta^2(\eta')^2Q^3}\Bigg\{k^2\left[\frac{2}{\eta-\eta'}-2iQ-Q^2(\eta-\eta')-iQ^3\eta\eta'\right]\\
&+Q^2\left[\frac{2}{\eta-\eta'}-2iQ+Q^2\frac{(\eta+\eta')^2}{\eta-\eta'}+iQ^3\eta\eta'\right]\Bigg\}\,\mathrm{Ei}\left[-2i(Q+k_{\rm UV})(\eta-\eta')\right]e^{iQ(\eta-\eta')}.
\end{aligned}
\tag{B.40}
$$

Next we need to integrate the above function with respect to Q, where we get exactly

$$
\begin{aligned}
&\int \mathrm{d}Q\int_{Q+2k_{\rm UV}}^{\infty}\mathrm{d}P\,\mathcal{F}(P,Q)\,e^{-iP(\eta-\eta')}=-\frac{16\pi k^4\sin[Q(\eta-\eta')]}{\eta^2(\eta')^2}\frac{1}{Q^2}\\
&+\frac{16k^4}{\eta^2(\eta')^2}\left\{\pi(\eta-\eta')\cos[Q(\eta-\eta')]-\frac{e^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{2k_{\rm UV}}\right\}\frac{1}{Q}+\frac{16\pi k^2(2-k^2\eta\eta')\sin[Q(\eta-\eta')]}{\eta^2(\eta')^2}\\
&+\frac{16ie^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{2k_{\rm UV}\,\eta^2(\eta-\eta')^5(\eta')^2}\Big\{k^4\eta(\eta-\eta')^4\eta'-4k^2(\eta-\eta')^2\left[\eta^2-3\eta\eta'+(\eta')^2\right]\\
&-8\left[\eta^2-5\eta\eta'+(\eta')^2\right]\Big\}+\frac{32k^2(\eta-\eta')}{\eta^2(\eta')^2}\Bigg(-\pi\cos[Q(\eta-\eta')]+\frac{e^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{2k^2k_{\rm UV}\,(\eta-\eta')^5}\\
&\times\left\{k^2(\eta-\eta')^2\left[\eta^2-4\eta\eta'+(\eta')^2\right]+4\left[\eta^2-5\eta\eta'+(\eta')^2\right]\right\}\Bigg)Q\\
&+\frac{16}{\eta^2(\eta')^2}\Bigg(\pi(2k^2\eta\eta'-1)\sin[Q(\eta-\eta')]-\frac{ie^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{k_{\rm UV}\,(\eta-\eta')^3}\big\{k^2\eta(\eta-\eta')^2\eta'\\
&-2\left[\eta^2-5\eta\eta'+(\eta')^2\right]\big\}\Bigg)Q^2+\frac{16(\eta-\eta')}{\eta^2(\eta')^2}\Bigg\{\pi\cos[Q(\eta-\eta')]-\frac{e^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{2k_{\rm UV}\,(\eta-\eta')^3}\big[\eta^2-6\eta\eta'\\
&+(\eta')^2\big]\Bigg\}Q^3+\frac{16}{\eta\eta'}\left\{-\pi\sin[Q(\eta-\eta')]+\frac{ie^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{2k_{\rm UV}\,(\eta-\eta')}\right\}Q^4\\
&+\frac{8(k^2-Q^2)^2}{Q^2\eta^2(\eta')^2}\Big\{(i-Q\eta)(i+Q\eta')\mathrm{Ei}\left[-2ik_{\rm UV}(\eta-\eta')\right]e^{-iQ(\eta-\eta')}-(i+Q\eta)(i-Q\eta')\\
&\times\mathrm{Ei}\left[-2i(Q+k_{\rm UV})(\eta-\eta')\right]e^{iQ(\eta-\eta')}\Big\}.
\end{aligned}
\tag{B.41}
$$

The first few terms are organized by increasing powers of Q, and the last term involves Ei functions. Finally, evaluating the above primitive at the endpoints (from $Q=0$ to $Q=k$) gives

$$
\begin{aligned}
&\int_0^k \mathrm{d}Q\int_{Q+2k_{\rm UV}}^{\infty}\mathrm{d}P\,\mathcal{F}(P,Q)\,e^{-iP(\eta-\eta')}=-\frac{192i}{k_{\rm UV}\;\eta^2(\eta-\eta')^5}\left[e^{-2ik_{\rm UV}(\eta-\eta')}-e^{-i(k+2k_{\rm UV})(\eta-\eta')}\right]\\
&\quad-\frac{192i}{k_{\rm UV}\;\eta^3(\eta-\eta')^4}\left[e^{-2ik_{\rm UV}(\eta-\eta')}-e^{-i(k+2k_{\rm UV})(\eta-\eta')}(1+ik\eta)\right]\\
&\quad-\frac{32i}{k_{\rm UV}\;\eta^4(\eta-\eta')^3}\left[e^{-2ik_{\rm UV}(\eta-\eta')}(4+k^2\eta^2)-2\,e^{-i(k+2k_{\rm UV})(\eta-\eta')}(1+ik\eta)(2+ik\eta)\right]\\
&\quad-\frac{32i}{k_{\rm UV}\;\eta^5(\eta-\eta')^2}\left[e^{-2ik_{\rm UV}(\eta-\eta')}(2+k^2\eta^2)-2\,e^{-i(k+2k_{\rm UV})(\eta-\eta')}(1+ik\eta)^2\right]
\end{aligned}
$$

$$
- \frac{8ik}{k_{\rm UV}\eta^5}\left[e^{-2ik_{\rm UV}(\eta-\eta')}k^3\eta^3 - 8i\,e^{-i(k+2k_{\rm UV})(\eta-\eta')}(1+ik\eta)\right]\left(\frac{1}{\eta-\eta'}+\frac{1}{\eta'}\right)
$$

$$
+ \frac{4i}{k_{\rm UV}^2\eta^5(\eta')^2}\left\{e^{-2ik_{\rm UV}(\eta-\eta')}\left[8k_{\rm UV}(2+k^2\eta^2) - ik^4\eta^3\right] - 16\,e^{-i(k+2k_{\rm UV})(\eta-\eta')}k_{\rm UV}(1+ik\eta)\right\} . \tag{B.42}
$$

Our next move consists in computing the part of the double integral (B.32) above involving $\mathcal{D}$ defined in eq. (B.33), namely

$$
\frac{1}{k}\int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\,\mathcal{D}(P,Q)e^{-iP(\eta-\eta')}. \tag{B.43}
$$

As mentioned previously, the above P-integral is formally divergent (the integrand scaling as $\propto P^4$ in the ultraviolet) however it is meaningful when understood as a distribution. Using the definition (B.33) we write the above integral over P as

$$
\int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\,\mathcal{D}(P,Q)\,e^{-iP(\eta-\eta')} = \alpha_4\big(\eta-\eta',Q+2k_{\rm UV}\big) + \frac{4i(\eta-\eta')}{\eta\eta'}\alpha_3\big(\eta-\eta',Q+2k_{\rm UV}\big)
$$

$$
+ 2\left[Q^2 - 2k^2 + \frac{8\eta\eta' - 2\eta^2 - 2(\eta')^2}{\eta^2(\eta')^2}\right]\alpha_2\big(\eta-\eta',Q+2k_{\rm UV}\big) + \frac{16i(\eta-\eta')\left[1+\left(\frac{3}{4}Q^2-k^2\right)\eta\eta'\right]}{\eta^2(\eta')^2}
$$

$$
\times\,\alpha_1\left(\eta-\eta',Q+2k_{\rm UV}\right) + \frac{16\left\{1-4(k^2-Q^2)\eta\eta' + \left(k^2-\frac{3}{4}Q^2\right)\left[\eta^2+(\eta')^2\right] + \frac{(Q^2-2k^2)^2}{16}\eta^2(\eta')^2\right\}}{\eta^2(\eta')^2}
$$

$$
\times\,\alpha_0\big(\eta-\eta',Q+2k_{\rm UV}\big), \tag{B.44}
$$

where we define the distributions for $m \in \{0,1,2,3,4\}$

$$
\alpha_m(x,y) := \int_y^{\infty} \mathrm{d}P\,P^m e^{-iPx}\,, \tag{B.45}
$$

which we must now compute. We first note the Fourier representation of the Heaviside step function

$$
\Theta(P) = \lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{e^{iPx}}{2\pi i(x-i\delta)} \tag{B.46}
$$

where we recall that $\Theta(x)=1$ for $x>0$ and $\Theta(x)=0$ for $x<0$. Inverting the above gives

$$
\int_0^{\infty}\mathrm{d}P\;e^{-iPx} = \frac{-i}{x-i\delta} = -\frac{i}{x} + \pi\delta(x) \tag{B.47}
$$

understood in the limit $\delta\to 0^+$ (in the last equality we have used $(x\pm i\delta)^{-1} = x^{-1}\mp i\pi\delta(x)$ — the so called "Sochocki-Plemelj" theorem). From this we easily find that

$$
\alpha_0(x,y) = \int_y^{\infty}\mathrm{d}P\,e^{-iPx} = \frac{-ie^{-ixy}}{x-i\delta}, \tag{B.48}
$$

which easily follows from a shift $P\to L$ in the integration variable $L = P-y$. By taking the limit $\delta\to 0^+$ in the above and using the property $f(x)\delta(x) = f(0)\delta(x)$ of δ-functions the above can be more simply written as

$$
\alpha_0(x,y) = \frac{-ie^{-ixy}}{x} + \pi\delta(x)\ . \tag{B.49}
$$

From here we can easily get the remaining set of required functions by noticing that $i\partial_x \alpha_m(x,y) = \alpha_{m+1}(x,y)$, giving:

$$\alpha_1(x,y) = \int_y^\infty \mathrm{d}P\, P e^{-iPx} = e^{-ixy}\left(-\frac{1}{x^2} - \frac{iy}{x}\right) + i\pi\delta'(x)\,, \tag{B.50}$$

$$\alpha_2(x,y) = \int_y^\infty \mathrm{d}P\, P^2 e^{-iPx} = e^{-ixy}\left(\frac{2i}{x^3} - \frac{2y}{x^2} - \frac{iy^2}{x}\right) - \pi\delta''(x)\,, \tag{B.51}$$

$$\alpha_3(x,y) = \int_y^\infty \mathrm{d}P\, P^3 e^{-iPx} = e^{-ixy}\left(\frac{6}{x^4} + \frac{6iy}{x^3} - \frac{3y^2}{x^2} - \frac{iy^3}{x}\right) - i\pi\delta'''(x)\,, \tag{B.52}$$

$$\alpha_4(x,y) = \int_y^\infty \mathrm{d}P\, P^4 e^{-iPx} = e^{-ixy}\left(-\frac{24i}{x^5} + \frac{24y}{x^4} + \frac{12iy^2}{x^3} - \frac{4y^3}{x^2} - \frac{iy^4}{x}\right) + \pi\delta''''(x)\,. \tag{B.53}$$

Inserting the distributions $\alpha_m(x,y)$ with $x = \eta - \eta'$ and $y = Q + 2k_{\mathrm{UV}}$ into eq. (B.44) yields the following expression

$$\begin{aligned}
&\int_{Q+2k_{\mathrm{UV}}}^\infty \mathrm{d}P\,\mathcal{D}(P,Q)e^{-iP(\eta-\eta')} = e^{-i(Q+2k_{\mathrm{UV}})(\eta-\eta')}\Bigg(\Bigg[-\frac{24i}{(\eta-\eta')^5} + \frac{24(Q+2k_{\mathrm{UV}})}{(\eta-\eta')^4} \\
&+ \frac{12i(Q+2k_{\mathrm{UV}})^2}{(\eta-\eta')^3} - \frac{4(Q+2k_{\mathrm{UV}})^3}{(\eta-\eta')^2} - \frac{i(Q+2k_{\mathrm{UV}})^4}{\eta-\eta'}\Bigg] + \frac{4i(\eta-\eta')}{\eta\eta'}\Bigg[\frac{6}{(\eta-\eta')^4} + \frac{6i(Q+2k_{\mathrm{UV}})}{(\eta-\eta')^3} \\
&- \frac{3(Q+2k_{\mathrm{UV}})^2}{(\eta-\eta')^2} - \frac{i(Q+2k_{\mathrm{UV}})^3}{\eta-\eta'}\Bigg] + 2\left[Q^2 - 2k^2 + \frac{8\eta\eta' - 2\eta^2 - 2(\eta')^2}{\eta^2(\eta')^2}\right]\Bigg[\frac{2i}{(\eta-\eta')^3} \\
&- \frac{2(Q+2k_{\mathrm{UV}})}{(\eta-\eta')^2} - \frac{i(Q+2k_{\mathrm{UV}})^2}{\eta-\eta'}\Bigg] - \frac{16i(\eta-\eta')\left[1+\left(\frac{3}{4}Q^2 - k^2\right)\eta\eta'\right]}{\eta^2(\eta')^2}\left[\frac{1}{(\eta-\eta')^2} + \frac{i(Q+2k_{\mathrm{UV}})}{\eta-\eta'}\right] \\
&+ \frac{16\left\{1 - 4(k^2-Q^2)\eta\eta' + \left(k^2 - \frac{3}{4}Q^2\right)\left[\eta^2 + (\eta')^2\right] + \frac{(Q^2-2k^2)^2}{16}\eta^2(\eta')^2\right\}}{\eta^2(\eta')^2}\left(\frac{-i}{\eta-\eta'}\right)\Bigg) \\
&+ \pi\Bigg(\delta''''(\eta-\eta') + \frac{4(\eta-\eta')}{\eta\eta'}\delta'''(\eta-\eta') - 2\left[Q^2 - 2k^2 + \frac{8\eta\eta' - 2\eta^2 - 2(\eta')^2}{\eta^2(\eta')^2}\right]\delta''(\eta-\eta') \\
&- \frac{16(\eta-\eta')\left[1+\left(\frac{3}{4}Q^2 - k^2\right)\eta\eta'\right]}{\eta^2(\eta')^2}\delta'(\eta-\eta') \\
&+ \frac{16\left\{1 - 4(k^2-Q^2)\eta\eta' + \left(k^2 - \frac{3}{4}Q^2\right)\left[\eta^2 + (\eta')^2\right] + \frac{(Q^2-2k^2)^2}{16}\eta^2(\eta')^2\right\}}{\eta^2(\eta')^2}\delta(\eta-\eta')\Bigg).
\end{aligned} \tag{B.54}$$

All that is left to do is to integrate over Q, which happens to be straightforward in this case with use of the integrals:

$$\int_0^k \mathrm{d}Q\, Q^n = \frac{k^{n+1}}{n+1}, \tag{B.55}$$

$$\int_0^k \mathrm{d}Q\, e^{-i(Q+2k_{\mathrm{UV}})(\eta-\eta')} = \frac{ie^{-2ik_{\mathrm{UV}}(\eta-\eta')}\left[-1 + e^{-ik(\eta-\eta')}\right]}{(\eta-\eta')}\,, \tag{B.56}$$

$$\int_0^k \mathrm{d}Q\, e^{-i(Q+2k_{\mathrm{UV}})(\eta-\eta')}Q = \frac{e^{-2ik_{\mathrm{UV}}(\eta-\eta')}\left\{-1 + \left[1 + ik(\eta-\eta')\right]e^{-ik(\eta-\eta')}\right\}}{(\eta-\eta')^2}\,, \tag{B.57}$$

$$\int_0^k \mathrm{d}Q\, e^{-i(Q+2k_{\mathrm{UV}})(\eta-\eta')}Q^2 = \frac{i\,e^{-2ik_{\mathrm{UV}}(\eta-\eta')}\left\{2 + \left[-2 - 2ik(\eta-\eta') + k^2(\eta-\eta')^2\right]e^{-ik(\eta-\eta')}\right\}}{(\eta-\eta')^3}\,, \tag{B.58}$$

$$\int_0^k \mathrm{d}Q\, e^{-i(Q+2k_{\rm UV})(\eta-\eta')}Q^3 = \frac{e^{-2ik_{\rm UV}(\eta-\eta')}}{(\eta-\eta')^4}\Big\{6+\big[-6-6ik(\eta-\eta')+3k^2(\eta-\eta')^2 + ik^3(\eta-\eta')^3\big]e^{-ik(\eta-\eta')}\Big\}\,, \tag{B.59}$$

$$\int_0^k \mathrm{d}Q\, \frac{e^{-i(Q+2k_{\rm UV})(\eta-\eta')}}{k}Q^4 = \frac{i\, e^{-2ik_{\rm UV}(\eta-\eta')}}{(\eta-\eta')^5}\Big\{-24+\big[24+24ik(\eta-\eta')-12k^2(\eta-\eta')^2 - 4ik^3(\eta-\eta')^3+k^4(\eta-\eta')^4\big]e^{-ik(\eta-\eta')}\Big\}\,. \tag{B.60}$$

The resulting expression contains many terms, but organizing the expression as a partial fraction expansion in terms of η' yields:

$$\begin{aligned}
&\int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\, \mathcal{D}(P,Q) e^{-iP(\eta-\eta')P} = -\frac{224}{(\eta-\eta')^6}\left[e^{-2ik_{\rm UV}(\eta-\eta')} - e^{-i(k+2k_{\rm UV})(\eta-\eta')}\right] \\
&- \frac{8i}{(\eta-\eta')^5}\left[32k_{\rm UV}e^{-2ik_{\rm UV}(\eta-\eta')} - (25k+32k_{\rm UV})e^{-i(k+2k_{\rm UV})(\eta-\eta')}\right] \\
&+ \frac{1}{\eta^2(\eta-\eta')^4}\Big\{32\left[10+(5k_{\rm UV}^2-k^2)\eta^2\right]e^{-2ik_{\rm UV}(\eta-\eta')} - 8\left[40+(7k^2+26kk_{\rm UV}+20k_{\rm UV}^2)\eta^2\right] \\
&\times e^{-i(k+2k_{\rm UV})(\eta-\eta')}\Big\} + \frac{1}{\eta^3(\eta-\eta')^3}\Big\{32\left[10+8ik_{\rm UV}\eta - ik_{\rm UV}(k^2-2k_{\rm UV}^2)\eta^3\right]e^{-2ik_{\rm UV}(\eta-\eta')} \\
&- 8\left[40+i(35k+32k_{\rm UV})\eta+2ik_{\rm UV}(3k^2+7kk_{\rm UV}+4k_{\rm UV}^2)\eta^3\right]e^{-i(k+2k_{\rm UV})(\eta-\eta')}\Big\} \\
&+ \frac{4}{\eta^4(\eta-\eta')^2}\left[56+64ik_{\rm UV}\eta+16(k^2-2k_{\rm UV}^2)\eta^2-(k^2-2k_{\rm UV}^2)^2\eta^4\right]e^{-2ik_{\rm UV}(\eta-\eta')} \\
&- \frac{8}{\eta^4(\eta-\eta')^2}\left[28+i(35k+32k_{\rm UV})\eta-(k+2k_{\rm UV})(7k+8k_{\rm UV})\eta^2-2k_{\rm UV}^2(k+k_{\rm UV})^2\eta^4\right] \\
&\times e^{-i(k+2k_{\rm UV})(\eta-\eta')} + \frac{32}{\eta^5}(2+ik_{\rm UV}\eta)[2+2ik_{\rm UV}\eta+(k^2-k_{\rm UV}^2)\eta^2]e^{-2ik_{\rm UV}(\eta-\eta')}\left(\frac{1}{\eta-\eta'}+\frac{1}{\eta'}\right) \\
&- \frac{8}{\eta^5}\left[16+4i(7k+6k_{\rm UV})\eta-(k+2k_{\rm UV})(7k+8k_{\rm UV})\eta^2-2ik_{\rm UV}(k+2k_{\rm UV})(k+k_{\rm UV})\eta^3\right] \\
&e^{-i(k+2k_{\rm UV})(\eta-\eta')}\left(\frac{1}{\eta-\eta'}+\frac{1}{\eta'}\right) - \frac{1}{\eta^4(\eta')^2}\Big\{16\left[6+4ik_{\rm UV}\eta+(k^2-k_{\rm UV}^2)\eta^2\right]e^{-2ik_{\rm UV}(\eta-\eta')} \\
&- 8\left[12+i(7k+8k_{\rm UV})\eta-2k_{\rm UV}(k+k_{\rm UV})\eta^2\right]e^{-i(k+2k_{\rm UV})(\eta-\eta')}\Big\} + \pi k\Big[\delta''''(\eta-\eta') \\
&+ \frac{4(\eta-\eta')}{\eta\eta'}\delta'''(\eta-\eta') + \frac{4\big[\eta^2+(\eta')^2+\big(\frac{5}{6}k^2\eta\eta'-4\big)\eta\eta'\big]}{\eta^2(\eta')^2}\delta''(\eta-\eta') \\
&+ \frac{4(\eta-\eta')\big(3k^2\eta\eta'-4\big)}{\eta^2(\eta')^2}\delta'(\eta-\eta') + \frac{\frac{43}{15}k^4\eta^2(\eta')^2+\frac{4}{3}k^2\left[9\eta^2-32\eta\eta'+9(\eta')^2\right]+16}{\eta^2(\eta')^2}\delta(\eta-\eta')\Big].
\end{aligned} \tag{B.61}$$

We are now in a position where the final result for $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')$ can be obtained. The correlator we seek is given by eq. (B.32). Using the derived results (B.42) and (B.61), and organizing in terms of a partial fraction expansion in η' the result is at last

$$
\begin{aligned}
\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{1}{32(2\pi)^{7/2}}\Bigg(&-\frac{224}{k(\eta-\eta')^6}\left[e^{-2ik_{\rm UV}(\eta-\eta')}-e^{-i(k+2k_{\rm UV})(\eta-\eta')}\right]\\
&-\frac{1}{k\,k_{\rm UV}\,\eta^2\,(\eta-\eta')^5}\Big\{64i(3+4k_{\rm UV}^2\eta^2)e^{-2ik_{\rm UV}(\eta-\eta')}-8i[24+k_{\rm UV}(25k+32k_{\rm UV})\eta^2]\\
&\times e^{-i(k+2k_{\rm UV})(\eta-\eta')}\Big\}+\frac{1}{k\,k_{\rm UV}\,\eta^3\,(\eta-\eta')^4}\Big\{32[-6i+10k_{\rm UV}\eta-k_{\rm UV}(k^2-5k_{\rm UV}^2)\eta^3]\\
&\times e^{-2ik_{\rm UV}(\eta-\eta')}+8[24i-8(3k+5k_{\rm UV})\eta-k_{\rm UV}(7k^2+26kk_{\rm UV}+20k_{\rm UV}^2)\eta^3]\\
&\times e^{-i(k+2k_{\rm UV})(\eta-\eta')}\Big\}+\frac{32}{k\,k_{\rm UV}\,\eta^4\,(\eta-\eta')^3}(-i+k_{\rm UV}\eta)[4+6ik_{\rm UV}\eta+(k^2-2k_{\rm UV}^2)\eta^2\\
&-ik_{\rm UV}(k^2-2k_{\rm UV}^2)\eta^3]e^{-2ik_{\rm UV}(\eta-\eta')}-\frac{8i}{k\,k_{\rm UV}\,\eta^4\,(\eta-\eta')^3}[-16-8i(3k+5k_{\rm UV})\eta\\
&+(8k^2+35kk_{\rm UV}+32k_{\rm UV}^2)\eta^2+2k_{\rm UV}^2(k+k_{\rm UV})(3k+4k_{\rm UV})\eta^4]e^{-i(k+2k_{\rm UV})(\eta-\eta')}\\
&-\frac{4}{k\,k_{\rm UV}\,\eta^5\,(\eta-\eta')^2}[16i-56k_{\rm UV}\eta+8i(k^2-8k_{\rm UV}^2)\eta^2-16k_{\rm UV}(k^2-2k_{\rm UV}^2)\eta^3\\
&+k_{\rm UV}(k^2-2k_{\rm UV}^2)^2\eta^5]e^{-2ik_{\rm UV}(\eta-\eta')}+\frac{8}{k\,k_{\rm UV}\,\eta^5\,(\eta-\eta')^2}[8i-4(4k+7k_{\rm UV})\eta\\
&-i(8k^2+35kk_{\rm UV}+32k_{\rm UV}^2)\eta^2+k_{\rm UV}(k+2k_{\rm UV})(7k+8k_{\rm UV})\eta^3+2k_{\rm UV}^3(k+k_{\rm UV})^2\eta^5]\\
&\times e^{-i(k+2k_{\rm UV})(\eta-\eta')}+\frac{8}{k\,k_{\rm UV}\,\eta^5}[16k_{\rm UV}+24ik_{\rm UV}^2\eta+8k_{\rm UV}(k^2-2k_{\rm UV}^2)\eta^2\\
&-i(k^2-2k_{\rm UV}^2)^2\eta^3]e^{-2ik_{\rm UV}(\eta-\eta')}\left(\frac{1}{\eta-\eta'}+\frac{1}{\eta'}\right)+\frac{8}{k\,k_{\rm UV}\,\eta^5}(k+2k_{\rm UV})[-8\\
&-4i(2k+3k_{\rm UV})\eta+k_{\rm UV}(7k+8k_{\rm UV})\eta^2+2ik_{\rm UV}^2(k+k_{\rm UV})\eta^3]e^{-i(k+2k_{\rm UV})(\eta-\eta')}\\
&\times\left(\frac{1}{\eta-\eta'}+\frac{1}{\eta'}\right)+\frac{4}{k\,k_{\rm UV}^2\,\eta^5\,(\eta')^2}[16ik_{\rm UV}-24k_{\rm UV}^2\eta+8ik_{\rm UV}(k^2-2k_{\rm UV}^2)\eta^2\\
&+(k^2-2k_{\rm UV}^2)^2\eta^3]e^{-2ik_{\rm UV}(\eta-\eta')}-\frac{8}{k\,k_{\rm UV}^2\,\eta^5\,(\eta')^2}k_{\rm UV}[8i-4(2k+3k_{\rm UV})\eta\\
&-ik_{\rm UV}(7k+8k_{\rm UV})\eta^2+2k_{\rm UV}^2(k+k_{\rm UV})\eta^3]e^{-i(k+2k_{\rm UV})(\eta-\eta')}+\pi\Bigg[\delta''''(\eta-\eta')\\
&+\frac{4(\eta-\eta')}{\eta\eta'}\delta'''(\eta-\eta')+\frac{4\big[\eta^2+(\eta')^2+\big(\frac{5}{6}k^2\eta\eta'-4\big)\eta\eta'\big]}{\eta^2(\eta')^2}\delta''(\eta-\eta')\\
&+\frac{4(\eta-\eta')\big(3k^2\eta\eta'-4\big)}{\eta^2(\eta')^2}\delta'(\eta-\eta')\\
&+\frac{\frac{43}{15}k^4\eta^2(\eta')^2+\frac{4}{3}k^2\left[9\eta^2-32\eta\eta'+9(\eta')^2\right]+16}{\eta^2(\eta')^2}\,\delta(\eta-\eta')\Bigg]\Bigg).
\end{aligned}
\tag{B.62}
$$

which assumes that $\eta > \eta'$ (the result for the opposing case of $\eta < \eta'$ can be extracted from the symmetry $\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \mathscr{C}^*_{\boldsymbol{k}}(\eta',\eta)$ easily). This is the result used in the main text. In sec. 3.1.3, we present the coincidence and the early time η' limits of the exact expression, see eqs. (3.21) and (3.23).

C Lindblad coefficients

In this appendix we explicitly compute the Lindblad coefficient $\mathfrak{F}_{\boldsymbol{k}}$ defined in eq. (3.37) used to derive the physical predictions of the Lindblad equation. We later also compute the validity coefficient $\mathfrak{M}_{\boldsymbol{k}}$ defined in eq. (3.42).

We begin with the integral $\mathfrak{F}_{\boldsymbol{k}}$ defined in eq. (3.37), repeated here,

$$\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) := (2\pi)^{3/2} \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta'\, G(\eta) G(\eta') \mathscr{C}_{\boldsymbol{k}}(\eta, \eta') \;, \tag{C.1}$$

where we here assume that $\eta_{\rm in}$ is fixed and arbitrary (assuming only that $\eta_{\rm in} < \eta < 0$). To evaluate this we use the representation (B.31) of $\mathcal{C}_{\boldsymbol{k}}(\eta, \eta')$, as well as $G(\eta)G(\eta') = \frac{\varepsilon_1 H^2}{8 M_{\rm p}^2} \eta\eta'$, see eq. (3.3), which expresses $\mathfrak{F}_{\boldsymbol{k}}$ as the triple integral

$$\begin{aligned}\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) = \frac{\varepsilon_1 H^2}{1024\pi^2 M_{\rm p}^2 k} \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta' \int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\, \eta\, \eta' \, (P^2 + Q^2 - 2k^2)^2 \left[1 - \frac{2i}{(P-Q)\eta}\right] \\ \times \left[1 + \frac{2i}{(P-Q)\eta'}\right] \left[1 - \frac{2i}{(P+Q)\eta}\right] \left[1 + \frac{2i}{(P+Q)\eta'}\right] e^{-iP(\eta - \eta')}.\end{aligned} \tag{C.2}$$

The utility of this representation is that it allows for a more standard/straightforward handling of the UV divergences that arise (which can be understood as UV divergences arising in the $P \to \infty$ limit). We proceed by computing the η' integrals first, where we note that

$$\begin{aligned}\int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta'\, \eta' \left[1 + \frac{2i}{(P-Q)\eta'}\right] \left[1 + \frac{2i}{(P+Q)\eta'}\right] e^{-iP(\eta-\eta')} = \frac{1}{P^2} - \frac{i\eta}{P} + \frac{4}{P^2 - Q^2} \\ - \frac{4e^{-iP\eta}\, \mathrm{Ei}(iP\eta)}{P^2 - Q^2} - e^{-iP(\eta - \eta_{\rm in})} \left(\frac{1}{P^2} - \frac{i\eta_{\rm in}}{P} + \frac{4}{P^2 - Q^2} \right) + \frac{4e^{-iP\eta}\, \mathrm{Ei}(iP\eta_{\rm in})}{P^2 - Q^2},\end{aligned} \tag{C.3}$$

where Ei is the exponential integral function defined around eq. (B.38).

To simplify the above expression we wish to express it in terms of dimensionless and positive variables, so to this end we define

$$z := -k\eta \;, \qquad z_{\rm in} := -k\eta_{\rm in} \tag{C.4}$$

and we also switch integration variables to

$$a := \frac{Q}{k} \qquad \text{and} \qquad b := \frac{P}{k} \;. \tag{C.5}$$

After using the result (C.3) as well as these variable definitions, we find that eq. (C.2) becomes

$$\begin{aligned}\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) = \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2} \int_0^1 \mathrm{d}a \int_{a+2\kappa}^{\infty} \mathrm{d}b\, z\, (a^2 + b^2 - 2)^2 \left[1 + \frac{2i}{(b-a)z}\right] \left[1 + \frac{2i}{(b+a)z}\right] \\ \times \left[-\frac{1}{b^2} - \frac{iz}{b} - \frac{4}{b^2 - a^2} + \frac{4e^{ibz} \mathrm{Ei}(-ibz)}{b^2 - a^2} + e^{-ib(z_{\rm in} - z)} \left(\frac{1}{b^2} + \frac{iz_{\rm in}}{b} + \frac{4}{b^2 - a^2} \right) \right. \\ \left. - \frac{4e^{ibz} \mathrm{Ei}(-ibz_{\rm in})}{b^2 - a^2} \right].\end{aligned} \tag{C.6}$$

To emphasize the terms which depend on $\eta_{\rm in}$, we split up the above function into two pieces,

$$\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) = \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2} \left[\mathcal{I}_{\boldsymbol{k}}(\eta) - \mathcal{G}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) \right] \tag{C.7}$$

where

$$\mathcal{I}_{\boldsymbol{k}}(\eta) := \int_0^1 \mathrm{d}a \int_{a+2\kappa}^{\infty} \mathrm{d}b\, z\,(a^2+b^2-2)^2 \left[1+\frac{2i}{(b-a)z}\right]\left[1+\frac{2i}{(b+a)z}\right] \times \left[-\frac{1}{b^2}-\frac{iz}{b}-\frac{4}{b^2-a^2}+\frac{4e^{ibz}\mathrm{Ei}(-ibz)}{b^2-a^2}\right] \tag{C.8}$$

$$\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) := \int_0^1 \mathrm{d}a \int_{a+2\kappa}^{\infty} \mathrm{d}b\, z\,(a^2+b^2-2)^2 \left[1+\frac{2i}{(b-a)z}\right]\left[1+\frac{2i}{(b+a)z}\right] \times \left[-e^{-ib(z_{\mathrm{in}}-z)}\left(\frac{1}{b^2}+\frac{iz_{\mathrm{in}}}{b}+\frac{4}{b^2-a^2}\right)+\frac{4e^{ibz}\mathrm{Ei}(-ibz_{\mathrm{in}})}{b^2-a^2}\right]. \tag{C.9}$$

We next evaluate each of these functions exactly, and also derive asymptotic series for each. We start by evaluating $\mathcal{G}_{\boldsymbol{k}}$ and then use this to get $\mathcal{I}_{\boldsymbol{k}}$ (which is straightforward to derive once we know what $\mathcal{G}_{\boldsymbol{k}}$ is).

C.1 The function $\mathcal{G}$

Here we exactly compute $\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}})$ defined in eq. (C.9). It turns out that it is easiest to do this by switching the order of integration relative to the formula (C.9), and so we compute it in two pieces

$$\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) = \mathcal{G}^{\triangle}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) + \mathcal{G}^{\square}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) \tag{C.10}$$

where $\mathcal{G}^{\triangle}_{\boldsymbol{k}}$ integrates over a triangular region

$$\mathcal{G}^{\triangle}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) := \int_{2\kappa}^{2\kappa+1} \mathrm{d}b \int_0^{b-2\kappa} \mathrm{d}a\; f(a,b,z,z_{\mathrm{in}}) \tag{C.11}$$

and where $\mathcal{G}^{\square}_{\boldsymbol{k}}$ integrates over a rectangular region

$$\mathcal{G}^{\square}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) := \int_{2\kappa+1}^{\infty} \mathrm{d}b \int_0^1 \mathrm{d}a\; f(a,b,z,z_{\mathrm{in}})\ , \tag{C.12}$$

using the shorthand

$$f(a,b,z,z_{\mathrm{in}}) := z\,(a^2+b^2-2)^2 \left[1+\frac{2i}{(b-a)z}\right]\left[1+\frac{2i}{(b+a)z}\right] \times \left[-e^{-ib(z_{\mathrm{in}}-z)}\left(\frac{1}{b^2}+\frac{iz_{\mathrm{in}}}{b}+\frac{4}{b^2-a^2}\right)+\frac{4e^{ibz}\,\mathrm{Ei}(-ibz_{\mathrm{in}})}{b^2-a^2}\right] \tag{C.13}$$

for the integrand given in the definition (C.9). For a visual representation of the domains of integration Δ and $\square$ see fig. 7.

In order to proceed we note that f has the following a-primitive (assuming $b>a$)

$$\int \mathrm{d}a\, f(a,b,z,z_{\mathrm{in}}) = e^{-ib(z_{\mathrm{in}}-z)}\Bigg\{-\frac{z(1+ibz_{\mathrm{in}})}{5b^2}a^5 + \left[\frac{4(i+bz)(i-bz_{\mathrm{in}})}{3b^2z} + \frac{2z(2+2ibz_{\mathrm{in}}+b^2-ib^3z_{\mathrm{in}})}{3b^2}\right]a^3 + \frac{z(4+4ibz_{\mathrm{in}}+12b^2-4ib^3z_{\mathrm{in}}-11b^4+ib^5z_{\mathrm{in}})}{b^2}a - \frac{8(i+bz)(6i-6bz_{\mathrm{in}}+ib^2+5b^3z_{\mathrm{in}})}{3bz} - \frac{4z(15+15ibz_{\mathrm{in}}+40b^2-20ib^3z_{\mathrm{in}}-43b^4+7ib^5z_{\mathrm{in}})}{15b} + \frac{16(b^2-1)^2(-1+ibz)}{b^2z}$$

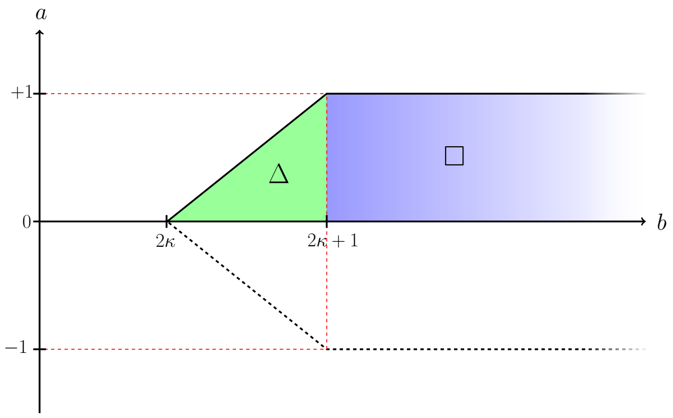

Figure 7: The triangular region Δ integrated over in equation (C.11) is shown in green and the rectangular region $\square$ integrated over in equation (C.12) is shown in blue.

$$
\begin{aligned}
&\times\left(\frac{1}{a+b}+\frac{1}{a-b}\right)+\left[-\frac{16(b^2-1)^2 z}{b}+\frac{16(b^2-1)(i+bz)(3i-bz_{\rm in}+ib^2+b^3z_{\rm in})}{b^3 z}\right]\\
&\times \coth^{-1}\left(\frac{b}{a}\right)\Bigg\} + \Bigg\{-\frac{z}{3}a^3+\left(-\frac{4}{z}+4z+4ib-3b^2z\right)a+\frac{2}{3}b\left(-\frac{6}{z}+6z+6ib-5b^2z\right)\\
&+\frac{4(b^2-1)^2(1-ibz)}{b^2 z}\left(\frac{1}{a+b}+\frac{1}{a-b}\right)+4(b^2-1)\left[\frac{2(b^2+1)(1-ibz)}{b^3z}+\frac{(b^2-1)z}{b}\right]\\
&\times\coth^{-1}\left(\frac{b}{a}\right)\Bigg\}\,4\,e^{ibz}\,\mathrm{Ei}(-ibz_{\rm in}).
\end{aligned}
\tag{C.14}
$$

Next we use this primitive to evaluate both $\mathcal{G}^{\Delta}_{\boldsymbol{k}}(\eta,\eta_{\rm in})$ and $\mathcal{G}^{\square}_{\boldsymbol{k}}(\eta,\eta_{\rm in})$.

Let us start with $\mathcal{G}^{\Delta}_{\boldsymbol{k}}$. Here we take the double integral $\mathcal{G}^{\Delta}_{\boldsymbol{k}}$ defined in eq. (C.11), and perform the a-integration by evaluating the a-primitive (C.14) at the end points $a=0$ to $a=b-2\kappa$:

$$
\begin{aligned}
\int_0^{b-2\kappa}\mathrm{d}a\, f(a,b,z,z_{\rm in}) = e^{-ib(z_{\rm in}-z)}\Bigg(&-\frac{28izz_{\rm in}}{15}b^4+4\left[-\frac{2i}{\kappa}-\frac{10z_{\rm in}}{3}+\left(\frac{43}{15}+2i\kappa z_{\rm in}\right)z\right]b^3\\
&+8\left\{\frac{3-5i\kappa z_{\rm in}}{3\kappa z}+\frac{2(i+6\kappa z_{\rm in})}{3}+\left[-3\kappa+\frac{2i(1-3\kappa^2)z_{\rm in}}{3}\right]z\right\}b^2\\
&+8\left\{\frac{2(-1+6i\kappa z_{\rm in})}{3z}+\frac{2i}{\kappa}+i\kappa+2(1-\kappa^2)z_{\rm in}+\frac{2\left[-2+i\kappa(4\kappa^2-3)z_{\rm in}\right]}{3}z\right\}b\\
&+4\Bigg\{-\frac{2\left[2+\kappa^2+2i\kappa(\kappa^2-1)z_{\rm in}\right]}{\kappa z}+2i(3\kappa^2-4)+\frac{8\kappa(\kappa^2-3)z_{\rm in}}{3}\\
&+\left[\frac{4\kappa(2\kappa^2+3)}{3}-i(2\kappa^2-1)^2z_{\rm in}\right]z\Bigg\}+\frac{4}{3}\Bigg\{\frac{2\left[3-6\kappa^2(\kappa^2-1)+4i\kappa^3(\kappa^2-3)z_{\rm in}\right]}{\kappa^2 z}\\
&-\frac{4i(2\kappa^4-6\kappa^2+3)}{\kappa}+\frac{2i\kappa(12\kappa^4-20\kappa^2+15)z_{\rm in}-15(2\kappa^2-1)^2}{5}z\Bigg\}\frac{1}{b}\\
&+\frac{8(\kappa^2-1)^2(-1+i\kappa z)}{\kappa^2 z(b-\kappa)}+\frac{8}{3}\left[\frac{4\kappa^4-12\kappa^2+6}{\kappa z}+\frac{\kappa(12\kappa^4-20\kappa^2+15)}{5}z\right]\frac{1}{b^2}
\end{aligned}
$$

$$
\begin{aligned}
&+16\Bigg[(z_{\rm in}-z)b^3+\frac{i(z_{\rm in}+z)}{z}b^2-\left(\frac{1}{z}-2z+2z_{\rm in}\right)b-\frac{2i(z_{\rm in}-z)}{z}-\frac{2+z^2-zz_{\rm in}}{bz}\\
&+\frac{i(z_{\rm in}-3z)}{zb^2}+\frac{3}{zb^3}\Bigg]\coth^{-1}\left(\frac{b}{b-2\kappa}\right)\Bigg)+\Bigg\{\frac{2(3i-5\kappa z)}{3\kappa}b^3+2\left(-\frac{1}{\kappa z}+i+4\kappa z\right)b^2\\
&+2\left[-\frac{1}{z}-\frac{i(5\kappa^2+2)}{\kappa}+2(1-\kappa^2)z\right]b+\frac{10\kappa^2+4}{\kappa z}-2i(\kappa^2-2)+\frac{8\kappa(\kappa^2-3)}{3}z\\
&+\frac{4i\kappa z-2}{\kappa^2 zb}+\frac{2(\kappa^2-1)^2(1-i\kappa z)}{\kappa^2 z(b-\kappa)}-\frac{4}{\kappa zb^2}+4\Bigg[zb^3-2ib^2+\left(\frac{2}{z}-2z\right)b+\frac{z}{b}\\
&+\frac{2i}{b^2}-\frac{2}{zb^3}\Bigg]\coth^{-1}\left(\frac{b}{b-2\kappa}\right)\Bigg\}\,4\,e^{+ibz}\,\mathrm{Ei}(-ibz_{\rm in})\,. \qquad \text{(C.15)}
\end{aligned}
$$

Next we find that we can explicitly write down the b-primitive (up to a constant) of the above:

$$
\begin{aligned}
&\int \mathrm{d}b\int_0^{b-2\kappa}\mathrm{d}a\,f(a,b,z,z_{\rm in})=e^{-ib(z_{\rm in}-z)}\Bigg(\frac{224z_{\rm in}^2}{5(z_{\rm in}-z)^5}+\frac{1}{(z_{\rm in}-z)^4}\Bigg[\frac{48i}{\kappa}-\frac{168z_{\rm in}}{5}\\
&+\frac{16i(14b-15\kappa)z_{\rm in}^2}{5}\Bigg]-\frac{1}{(z_{\rm in}-z)^3}\Bigg[\frac{8(90b-19\kappa)}{15\kappa}+\frac{8i(21b-20\kappa)z_{\rm in}}{5}\\
&+\frac{16(21b^2-45b\kappa+30\kappa^2-10)z_{\rm in}^2}{15}\Bigg]+\frac{64i}{\kappa z_{\rm in}z^3}+\frac{1}{(z_{\rm in}-z)^2}\Bigg\{-\frac{32i}{\kappa z_{\rm in}^2}\\
&+\frac{8i[-45b^2+19b\kappa+30(\kappa^2+1)]}{15\kappa}+\frac{4(63b^2-120b\kappa+60\kappa^2-20)z_{\rm in}}{15}\\
&-\frac{8i\left[14b^3-45b^2\kappa+20b\left(3\kappa^2-1\right)-40\kappa^3+30\kappa\right]z_{\rm in}^2}{15}\Bigg\}+\frac{32}{\kappa z^2}\left(\frac{i}{z_{\rm in}^2}+\frac{b-\kappa}{z_{\rm in}}\right)\\
&+\frac{1}{z_{\rm in}-z}\Bigg\{\frac{32(b-\kappa)}{\kappa z_{\rm in}^2}+\frac{8(b^2-2)b}{\kappa}-\frac{4(19b^2+20)}{15}-16\kappa(b-\kappa)\\
&+\frac{4ib(21b^2-60b\kappa+60\kappa^2-20)}{15}z_{\rm in}+\frac{4}{15}\Big[7b^4+60(b^2-1)\kappa^2-30(b^2-2)b\kappa-20b^2\\
&-80b\kappa^3+60\kappa^4+15\Big]z_{\rm in}^2\Bigg\}+\frac{8}{3z}\left[\frac{12(b-\kappa)}{\kappa z_{\rm in}^2}+\frac{(b-2\kappa)\left(5b^2-2b\kappa+2\kappa^2-6\right)}{b}\right]\\
&+\frac{1}{15}\Bigg\{4i\left(-43b^3+90b^2\kappa+40b-40\kappa^3-60\kappa\right)-4z_{\rm in}\Big[7b^4+60\left(b^2-1\right)\kappa^2\\
&-30\left(b^2-2\right)b\kappa-20b^2-80b\kappa^3+60\kappa^4+15\Big]-\frac{8\kappa\left(12\kappa^4-20\kappa^2+15\right)z}{b}\Bigg\}\Bigg)\\
&+\left[\frac{64i}{\kappa z^4}+\frac{32i}{\kappa z^2}-\frac{32}{z^3}+\frac{8i\left(4\kappa^4-12\kappa^2-3\right)}{3\kappa}-4\left(2\kappa^2-1\right)^2z+\frac{8i\kappa\left(12\kappa^4-20\kappa^2+15\right)}{15}z^2\right]\\
&\times\mathrm{Ei}\left[-ib(z_{\rm in}-z)\right]+\Bigg[\frac{64i}{\kappa z^4}-\frac{32}{z^3}+\frac{32i}{\kappa z^2}+\frac{8i\left(4\kappa^4-12\kappa^2-3\right)}{3\kappa}-4(2\kappa^2-1)^2z\\
&+\frac{8i\kappa\left(12\kappa^4-20\kappa^2+15\right)}{15}z^2\Bigg]\mathrm{Ei}\left[-ib(z_{\rm in}-z)\right]+8\Bigg\{\left(\frac{3}{\kappa z}+5i\right)\frac{b^3}{3}+\left(\frac{4i}{\kappa z^2}-\frac{3}{z}-4i\kappa\right)b^2\\
&+2\left[-\frac{4}{\kappa z^3}-\frac{2i}{z^2}+\frac{2\kappa^2-1}{\kappa z}+i(\kappa^2-1)\right]b-\frac{8i}{\kappa z^4}+\frac{4}{z^3}-\frac{4i}{\kappa z^2}-\frac{2\left(\kappa^2-1\right)}{z}-\frac{4i\kappa\left(\kappa^2-3\right)}{3}
\end{aligned}
$$

$$+ \frac{1}{\kappa z b}\Bigg\} e^{ibz}\mathrm{Ei}(-ibz_{\mathrm{in}}) + \frac{16\left(b^2-1\right)^2(1-ibz)}{b^2 z}\coth^{-1}\left(\frac{b}{b-2\kappa}\right)\Big[e^{ibz}\mathrm{Ei}(-ibz_{\mathrm{in}}) - e^{-ib(z_{\mathrm{in}}-z)}\Big]. \tag{C.16}$$

Finally we can evaluate this at the endpoints from $b = 2\kappa$ to $b = 2\kappa + 1$ and we are left with the answer

$$\begin{aligned}
\mathcal{G}^{\Delta}_{\boldsymbol{k}}(\eta,\eta_{\mathrm{in}}) = e^{-i(2\kappa+1)(z_{\mathrm{in}}-z)}\Bigg\{ & \frac{224 z_{\mathrm{in}}^2}{5(z_{\mathrm{in}}-z)^5} + \frac{1}{(z_{\mathrm{in}}-z)^4}\left[\frac{48i}{\kappa} - \frac{168 z_{\mathrm{in}}}{5} + \frac{16i(13\kappa+14)}{5} z_{\mathrm{in}}^2\right] \\
& + \frac{1}{(z_{\mathrm{in}}-z)^3}\left[-\frac{1288}{15} - \frac{48}{\kappa} - \frac{8i(22\kappa+21)z_{\mathrm{in}}}{5} - \frac{16(24\kappa^2+39\kappa+11)z_{\mathrm{in}}^2}{15}\right] + \frac{64i}{\kappa z_{\mathrm{in}} z^3} \\
& + \frac{1}{(z_{\mathrm{in}}-z)^2}\left[-\frac{32i}{\kappa z_{\mathrm{in}}^2} - \frac{8i(112\kappa^2+161\kappa+15)}{15\kappa} + \frac{4(72\kappa^2+132\kappa+43)z_{\mathrm{in}}}{15}\right. \\
& \left. - \frac{8i(12\kappa^3+48\kappa^2+29\kappa-6)z_{\mathrm{in}}^2}{15}\right] + \frac{32}{z^2}\left(\frac{i}{\kappa z_{\mathrm{in}}^2} + \frac{\kappa+1}{\kappa z_{\mathrm{in}}}\right) + \frac{1}{z_{\mathrm{in}}-z}\left[\frac{32(\kappa+1)}{\kappa z_{\mathrm{in}}^2}\right. \\
& + \frac{4(104\kappa^3+224\kappa^2+21\kappa-30)}{15\kappa} + \frac{4i\left(48\kappa^3+72\kappa^2+26\kappa+1\right)z_{\mathrm{in}}}{15} \\
& \left. + \frac{8(6\kappa^4+12\kappa^3+14\kappa^2+3\kappa+1)z_{\mathrm{in}}^2}{15}\right] + \frac{32(\kappa+1)}{\kappa z_{\mathrm{in}}^2 z} + \frac{8(18\kappa^2+18\kappa-1)}{3(2\kappa+1)z} \\
& - \frac{4i(24\kappa^3+156\kappa^2+148\kappa+3)}{15} - \frac{8\kappa(12\kappa^4-20\kappa^2+15)z}{15(2\kappa+1)} \\
& - \frac{8\left(6\kappa^4+12\kappa^3+14\kappa^2+3\kappa+1\right)z_{\mathrm{in}}}{15} + \frac{128i(\kappa+1)^2\kappa^2[i+(2\kappa+1)z]}{(2\kappa+1)^2 z}\log\left(\frac{\kappa+1}{\kappa}\right)\Bigg\} \\
& + e^{-2i\kappa(z_{\mathrm{in}}-z)}\Bigg\{-\frac{224 z_{\mathrm{in}}^2}{5(z_{\mathrm{in}}-z)^5} + \frac{1}{(z_{\mathrm{in}}-z)^4}\left(-\frac{48i}{\kappa} + \frac{168 z_{\mathrm{in}}}{5} - \frac{208 i\kappa z_{\mathrm{in}}^2}{5}\right) \\
& + \frac{1}{(z_{\mathrm{in}}-z)^3}\left[\frac{1288}{15} + \frac{176 i\kappa z_{\mathrm{in}}}{5} + \frac{32(12\kappa^2-5)z_{\mathrm{in}}^2}{15}\right] - \frac{64i}{\kappa z_{\mathrm{in}} z^3} + \frac{1}{(z_{\mathrm{in}}-z)^2}\left[\frac{32i}{\kappa z_{\mathrm{in}}^2}\right. \\
& \left. + \frac{16i(56\kappa^2-15)}{15\kappa} - \frac{16(18\kappa^2-5)z_{\mathrm{in}}}{15} + \frac{16i\kappa\left(6\kappa^2-5\right)z_{\mathrm{in}}^2}{15}\right] - \frac{32}{z^2}\left(\frac{i}{\kappa z_{\mathrm{in}}^2} + \frac{1}{z_{\mathrm{in}}}\right) \\
& - \frac{4}{z_{\mathrm{in}}-z}\left[\frac{8}{z_{\mathrm{in}}^2} + \frac{4(26\kappa^2-35)}{15} + \frac{8i\kappa(6\kappa^2-5)z_{\mathrm{in}}}{15} + \frac{(12\kappa^4-20\kappa^2+15)z_{\mathrm{in}}^2}{15}\right] \\
& - \frac{32}{z z_{\mathrm{in}}^2} + \frac{16i\kappa(6\kappa^2-5)}{15} + \frac{4\left(12\kappa^4-20\kappa^2+15\right)}{15}(z_{\mathrm{in}}+z)\Bigg\} + 8\left[-\frac{8i}{\kappa z^4} + \frac{4}{z^3} - \frac{4i}{\kappa z^2}\right. \\
& \left. - \frac{i(4\kappa^4-12\kappa^2-3)}{3\kappa} + \frac{(2\kappa^2-1)^2 z}{2} - \frac{i\kappa(12\kappa^4-20\kappa^2+15)z^2}{15}\right]\big\{\mathrm{Ei}\left[-2i\kappa(z_{\mathrm{in}}-z)\right] \\
& - \mathrm{Ei}\left[-i(2\kappa+1)(z_{\mathrm{in}}-z)\right]\big\} + 8\left[\frac{8i}{\kappa z^4} + \frac{12}{z^3} - \frac{4i(2\kappa^2-1)}{\kappa z^2} - \frac{4\kappa^4-4\kappa^2+1}{2\kappa^2 z}\right] \\
& \times e^{2i\kappa z}\mathrm{Ei}(-2i\kappa z_{\mathrm{in}}) + 8\Bigg\{-\frac{8i}{\kappa z^4} - \frac{4(3\kappa+2)}{\kappa z^3} + \frac{4i(2\kappa+3)}{z^2} + \frac{4\kappa^3+10\kappa^2+6\kappa-1}{(2\kappa+1)z} \\
& + \frac{i(18\kappa^2+18\kappa-1)}{3} + \frac{16(\kappa+1)^2\kappa^2[1-i(2\kappa+1)z]}{(2\kappa+1)^2 z}\log\left(\frac{\kappa+1}{\kappa}\right)\Bigg\} e^{i(2\kappa+1)z}
\end{aligned}$$

$$\times\,\mathrm{Ei}\left[-i(2\kappa+1)z_{\rm in}\right]\,. \tag{C.17}$$

We now turn to the calculation of $\mathcal{G}^{\square}_{\boldsymbol{k}}$. We follow similar steps for evaluating $\mathcal{G}^{\square}_{\boldsymbol{k}}$ defined in eq. (C.12). We first perform the a-integration by evaluating the a-primitive in eq. (C.14) at the endpoints from $a=0$ to $a=1$ giving:

$$\begin{aligned}\mathcal{G}^{\square}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = &\int_{2\kappa+1}^{+\infty}{\rm d}b\,\Bigg(e^{-ib(z_{\rm in}-z)}\Bigg\{\left(\frac{1}{z}-ib\right)\left[\frac{4(27b^2-13)}{3b^2}-\frac{4i(9b^2-11)z_{\rm in}}{3b}\right]\\ &+\left[11b^2-\frac{34}{3}-\frac{43}{15b^2}-\frac{i(15b^4-50b^2+43)z_{\rm in}}{15b}\right]z+\frac{16\left(b^2-1\right)}{b}\left[\frac{i(b^2-1)bz_{\rm in}-b^2-3}{b^2z}\right.\\ &\left.+\frac{i(b^2+3)}{b}+(b^2-1)(z_{\rm in}-z)\right]\coth^{-1}(b)\Bigg\}+4\Bigg\{\frac{4(2-3b^2)}{b}\left(\frac{1}{bz}-i\right)-\frac{9b^2-11}{3}z\\ &+\left[\frac{8(b^4-1)}{b^2}\left(\frac{1}{bz}-i\right)+\frac{4(b^2-1)^2}{b}z\right]\coth^{-1}(b)\Bigg\}\,e^{ibz}\,\mathrm{Ei}(-ibz_{\rm in})\Bigg).\end{aligned} \tag{C.18}$$

At this stage we must deal with a subtlety occurring in the above integration: the integral is formally divergent in the $b\to\infty$ limit (since $b\propto P$ this means these are UV divergences). To deal with this we separate out the diverging $b\to\infty$ behaviour from the rest of the integrand, so that

$$\mathcal{G}^{\square}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = \mathcal{G}^{\square,{\rm div}}_{\boldsymbol{k}}(\eta,\eta_{\rm in})+\mathcal{G}^{\square,{\rm reg}}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \tag{C.19}$$

where we define

$$\begin{aligned}\mathcal{G}^{\square,{\rm div}}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := &\int_{2\kappa+1}^{\infty}{\rm d}b\,\Bigg(4\pi e^{ibz}\left[-izb^2+4b+i\left(3z+\frac{4}{z}\right)\right]+e^{-ib(z_{\rm in}-z)}\Bigg\{-izz_{\rm in}b^3\\ &-(5z-4z_{\rm in})b^2+\frac{2ib}{3}\left[\frac{6z_{\rm in}}{z}-30+\left(\frac{6}{z_{\rm in}}+5z_{\rm in}\right)z\right]+2\Bigg(\frac{10}{z}-\frac{8}{z_{\rm in}}-6z_{\rm in}\\ &+\frac{23z_{\rm in}^2-6}{3z_{\rm in}^2}z\Bigg)+\frac{4i}{b}\left[-\left(\frac{4}{z_{\rm in}}+3z_{\rm in}\right)\frac{1}{z}-\frac{4}{z_{\rm in}^2}+\frac{41}{3}-\left(\frac{2}{z_{\rm in}^3}+\frac{3}{z_{\rm in}}+\frac{43z_{\rm in}}{60}\right)z\right]\Bigg\}\Bigg)\end{aligned} \tag{C.20}$$

$$\begin{aligned}\mathcal{G}^{\square,{\rm reg}}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := &\int_{2\kappa+1}^{\infty}{\rm d}b\,\Bigg(-4\pi e^{ibz}\left[-izb^2+4b+i\left(3z+\frac{4}{z}\right)\right]+e^{-ib(z_{\rm in}-z)}\Bigg\{-16(z_{\rm in}-z)b^2\\ &-\frac{4i(z+2z_{\rm in})^2b}{zz_{\rm in}}+4\left[\frac{4}{z}+\frac{4}{z_{\rm in}}+\frac{z}{z_{\rm in}^2}+\frac{20}{3}(z_{\rm in}-z)\right]+\frac{4i}{b}\Bigg(\frac{2z}{z_{\rm in}^3}+\frac{3z}{z_{\rm in}}+\frac{20z_{\rm in}}{3z}\\ &+\frac{4}{zz_{\rm in}}+\frac{4}{z_{\rm in}^2}-\frac{28}{3}\Bigg)-\frac{1}{b^2}\left(\frac{43z}{15}+\frac{52}{3z}\right)+\frac{16\left(b^2-1\right)}{b}\left[\frac{i(b^2-1)bz_{\rm in}-b^2-3}{b^2z}\right.\\ &\left.+\frac{i(b^2+3)}{b}+(b^2-1)(z_{\rm in}-z)\right]\coth^{-1}(b)\Bigg\}+4\Bigg\{\frac{4(2-3b^2)}{b}\left(\frac{1}{bz}-i\right)\\ &-\frac{9b^2-11}{3}z+\left[\frac{8(b^4-1)}{b^2}\left(\frac{1}{bz}-i\right)+\frac{4(b^2-1)^2}{b}z\right]\coth^{-1}(b)\Bigg\}e^{ibz}\mathrm{Ei}(-ibz_{\rm in})\Bigg)\,.\end{aligned} \tag{C.21}$$

One may check that summing $\mathcal{G}^{\square,{\rm div}}_{\boldsymbol{k}}+\mathcal{G}^{\square,{\rm reg}}_{\boldsymbol{k}}$ gives exactly eq. (C.18). The logic for this organization is that $\mathcal{G}^{\square,{\rm div}}_{\boldsymbol{k}}$ contains all the divergences, while $\mathcal{G}^{\square,{\rm reg}}_{\boldsymbol{k}}$ is a formally convergent integral (one may check that it falls off fast enough to converge, like $\propto b^{-1}e^{+ibz}+\ldots$, in the limit $b\to\infty$).

In order to make sense of the divergent integral $\mathcal{G}_{\boldsymbol{k}}^{\square,\mathrm{div}}$, we must understand it in the distributional sense — using the results (B.49) and (B.50) we have

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{-ib(z_{\rm in}-z)} b^3 = e^{-i(2\kappa+1)(z_{\rm in}-z)}\left[\frac{6}{(z_{\rm in}-z)^4}+\frac{6i(2\kappa+1)}{(z_{\rm in}-z)^3}-\frac{3(2\kappa+1)^2}{(z_{\rm in}-z)^2}-\frac{i(2\kappa+1)^3}{z_{\rm in}-z}\right] - i\pi\delta'''(z_{\rm in}-z)\,, \tag{C.22}$$

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{-ib(z_{\rm in}-z)} b^2 = e^{-i(2\kappa+1)(z_{\rm in}-z)}\left[\frac{2i}{(z_{\rm in}-z)^3}-\frac{2(2\kappa+1)}{(z_{\rm in}-z)^2}-\frac{i(2\kappa+1)^2}{z_{\rm in}-z}\right] - \pi\delta''(z_{\rm in}-z)\,, \tag{C.23}$$

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{-ib(z_{\rm in}-z)} b = e^{-i(2\kappa+1)(z_{\rm in}-z)}\left[-\frac{1}{(z_{\rm in}-z)^2}-\frac{i(2\kappa+1)}{z_{\rm in}-z}\right] + i\pi\delta'(z_{\rm in}-z)\,, \tag{C.24}$$

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{-ib(z_{\rm in}-z)} = \frac{-ie^{-i(2\kappa+1)(z_{\rm in}-z)}}{z_{\rm in}-z} + \pi\delta(z_{\rm in}-z), \tag{C.25}$$

and similarly

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{ibz} b^2 = e^{+i(2\kappa+1)z}\left[-\frac{2i}{z^3}-\frac{2(2\kappa+1)}{z^2}+\frac{i(2\kappa+1)^2}{z}\right] - \pi\delta''(z)\,, \tag{C.26}$$

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{+ibz} b = e^{+i(2\kappa+1)z}\left[-\frac{1}{z^2}+\frac{i(2\kappa+1)}{z}\right] - i\pi\delta'(z)\,, \tag{C.27}$$

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, e^{+ibz} = \frac{ie^{+i(2\kappa+1)z}}{z} + \pi\delta(z)\,. \tag{C.28}$$

A comment on the δ-functions is in order: we can ignore $\delta(z)=0$ and its derivatives, since we always assume that $z>0$. Since we will care about the coincident limit $z_{\rm in}\to z\to 0$ later on, we need to keep $\delta(z_{\rm in}-z)$ and its derivatives. Finally we also need to use

$$\int_{2\kappa+1}^{\infty} \mathrm{d}b\, \frac{e^{-ib(z_{\rm in}-z)}}{b} = -\left\{\mathrm{Ei}\left[-i(2\kappa+1)(z_{\rm in}-z)\right]+i\pi\right\}\ . \tag{C.29}$$

Putting the above pieces all together in eq. (C.20) leaves us with

$$\begin{aligned}
\mathcal{G}_{\boldsymbol{k}}^{\square,\mathrm{div}}(\eta,\eta_{\rm in}) = {}& 8\pi e^{i(2\kappa+1)z}\left[-\frac{5}{z^2}+\frac{3i(2\kappa+1)}{z}+2\kappa^2+2\kappa-1\right] + e^{-i(2\kappa+1)(z_{\rm in}-z)}\Bigg\{-\frac{6iz_{\rm in}^2}{(z_{\rm in}-z)^4}\\
&+\frac{2}{(z_{\rm in}-z)^3}\left[2i+3(2\kappa+1)z_{\rm in}\right]z_{\rm in}+\frac{1}{(z_{\rm in}-z)^2}\Bigg\{22i-4(2\kappa+1)z_{\rm in}\\
&+\frac{i[36\kappa(\kappa+1)-1]}{3}z_{\rm in}^2\Bigg\}-\frac{1}{z_{\rm in}-z}(2\kappa+1)\left[22+2i(2\kappa+1)z_{\rm in}+\frac{12\kappa^2+12\kappa-7}{3}z_{\rm in}^2\right]\\
&+\frac{4}{z}\left(-\frac{6i}{z_{\rm in}}+2\kappa+1\right)-\frac{4i}{z_{\rm in}^2}-\frac{4(2\kappa+1)}{z_{\rm in}}-\frac{i\left(60\kappa^2+60\kappa-31\right)}{3}\\
&+\frac{(2\kappa+1)\left(12\kappa^2+12\kappa-7\right)}{3}z_{\rm in}\Bigg\}+4i\left[\frac{1}{z}\left(\frac{4}{z_{\rm in}}+3z_{\rm in}\right)+\frac{4}{z_{\rm in}^2}-\frac{41}{3}\right.\\
&\left.+\left(\frac{2}{z_{\rm in}^3}+\frac{3}{z_{\rm in}}+\frac{43z_{\rm in}}{60}\right)z\right]\left\{\mathrm{Ei}\left[-i(2\kappa+1)(z_{\rm in}-z)\right]+i\pi\right\}+\pi\Big\{-zz_{\rm in}\delta'''(z_{\rm in}-z)\\
&+(5z-4z_{\rm in})\delta''(z_{\rm in}-z)-\frac{2}{3}\left(5zz_{\rm in}+\frac{6z}{z_{\rm in}}+\frac{6z_{\rm in}}{z}-30\right)\delta'(z_{\rm in}-z)
\end{aligned}$$

$$+2\left[\frac{10}{z}-\frac{8}{z_{\rm in}}-6z_{\rm in}+\left(\frac{23}{3}-\frac{2}{z_{\rm in}^2}\right)z\right]\delta(z_{\rm in}-z)\Bigg\}. \tag{C.30}$$

We recall that, in the above expression, we have ignored the Dirac functions $\delta(z)$ and its derivatives.

Next we compute $\mathcal{G}_{\boldsymbol{k}}^{\square,\rm reg}$. To this end note that the integrand in the definition (C.21) has an explicit b-primitive where

$$\begin{aligned}\mathcal{G}_{\boldsymbol{k}}^{\square,\rm reg}(\eta,\eta_{\rm in}) = \Bigg(&4\pi e^{ibz}\left(b^2+\frac{6ib}{z}-\frac{10}{z^2}-3\right)+e^{-ib(z_{\rm in}-z)}\Bigg\{-16ib^2+\left(\frac{16}{z}-\frac{4}{z_{\rm in}}\right)b\\&+\frac{4i}{3}\left[20-\frac{3(z+6z_{\rm in})}{zz_{\rm in}^2}\right]+\frac{43z^2-220}{15bz}+\frac{16\left(b^2-1\right)^2(-1+ibz)}{b^2z}\coth^{-1}(b)\Bigg\}\\&+\left[-\frac{40i}{z^2z_{\rm in}}+\frac{12i\left(z_{\rm in}^2-2\right)}{zz_{\rm in}^2}+\frac{43i}{15}z-\frac{4i(3z_{\rm in}^2+2)}{z_{\rm in}^3}\right](z_{\rm in}-z){\rm Ei}\left[-ib(z_{\rm in}-z)\right]\\&+4\left[\frac{10i}{z^2}+\frac{2(b^2-2)}{bz}+\frac{i(9b^2-11)}{3}+\frac{4(b^2-1)^2(1-ibz)}{b^2z}\coth^{-1}(b)\right]\\&\times e^{ibz}{\rm Ei}(-ibz_{\rm in})\Bigg)\Bigg|_{b=2\kappa+1}^{b\to\infty}. \end{aligned}\tag{C.31}$$

The integrand approaches $\pi(z-z_{\rm in})\left[\frac{40}{z^2z_{\rm in}}-\frac{12(z_{\rm in}^2-2)}{zz_{\rm in}^2}+\frac{4(3z_{\rm in}^2+2)}{z_{\rm in}^3}-\frac{43z}{15}\right]$ at the upper limit $b\to\infty$. We find then that the result is

$$\begin{aligned}\mathcal{G}_{\boldsymbol{k}}^{\square,\rm reg}(\eta,\eta_{\rm in}) =\ &\pi(z-z_{\rm in})\left[\frac{40}{z^2z_{\rm in}}-\frac{12(z_{\rm in}^2-2)}{zz_{\rm in}^2}+\frac{4(3z_{\rm in}^2+2)}{z_{\rm in}^3}-\frac{43z}{15}\right]-4\pi e^{i(2\kappa+1)z}\Bigg[-\frac{10}{z^2}\\&+\frac{6i(2\kappa+1)}{z}+4\kappa^2+4\kappa-2\Bigg]-e^{-i(2\kappa+1)(z_{\rm in}-z)}\Bigg\{-16i(2\kappa+1)^2+\frac{43z^2-220}{15(2\kappa+1)z}\\&+(2\kappa+1)\left(\frac{16}{z}-\frac{4}{z_{\rm in}}\right)+\frac{4i}{3}\left[20-\frac{3(z+6z_{\rm in})}{zz_{\rm in}^2}\right]+\frac{128\kappa^2(\kappa+1)^2[-1+i(2\kappa+1)z]}{(2\kappa+1)^2z}\\&\times\log\left(\frac{\kappa+1}{\kappa}\right)\Bigg\}-4\Bigg\{\frac{10i}{z^2}+\frac{2\left(4\kappa^2+4\kappa-1\right)}{(2\kappa+1)z}+\frac{i(36\kappa^2+36\kappa-2)}{3}\\&+\frac{32\kappa^2(\kappa+1)^2[1-i(2\kappa+1)z]}{(2\kappa+1)^2z}\log\left(\frac{\kappa+1}{\kappa}\right)\Bigg\}e^{i(2\kappa+1)z}{\rm Ei}\left[-i(2\kappa+1)z_{\rm in}\right]\\&-i\left[-\frac{40}{z^2z_{\rm in}}+\frac{12\left(z_{\rm in}^2-2\right)}{zz_{\rm in}^2}-\frac{4\left(3z_{\rm in}^2+2\right)}{z_{\rm in}^3}+\frac{43}{15}z\right](z_{\rm in}-z){\rm Ei}\left[-i(2\kappa+1)(z_{\rm in}-z)\right].\end{aligned}\tag{C.32}$$

We now have explicit expressions for $\mathcal{G}_{\boldsymbol{k}}^{\square,\rm div}$ and $\mathcal{G}_{\boldsymbol{k}}^{\square,\rm reg}$ and so we can finally write down an expression for $\mathcal{G}_{\boldsymbol{k}}$ using the sum

$$\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in})=\mathcal{G}_{\boldsymbol{k}}^{\triangle}(\eta,\eta_{\rm in})+\mathcal{G}_{\boldsymbol{k}}^{\square,\rm div}(\eta,\eta_{\rm in})+\mathcal{G}_{\boldsymbol{k}}^{\square,\rm reg}(\eta,\eta_{\rm in}). \tag{C.33}$$

Using the explicit formulae (C.17), (C.30) and (C.32) we arrive at last to the formula:

$$\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in})=e^{-i(2\kappa+1)(z_{\rm in}-z)}\Bigg\{\frac{224z_{\rm in}^2}{5(z_{\rm in}-z)^5}+\frac{1}{(z_{\rm in}-z)^4}\left[\frac{48i}{\kappa}-\frac{168z_{\rm in}}{5}+\frac{2i(104\kappa+97)}{5}z_{\rm in}^2\right]$$

$$
\begin{aligned}
&+\frac{64i}{\kappa z_{\rm in}z^3}+\frac{1}{(z_{\rm in}-z)^3}\left[-\frac{8(161\kappa+90)}{15\kappa}-\frac{4i(44\kappa+37)}{5}z_{\rm in}-\frac{2(192\kappa^2+222\kappa+43)}{15}z_{\rm in}^2\right]\\
&+\frac{32}{\kappa z^2}\left(\frac{i}{z_{\rm in}^2}+\frac{\kappa+1}{z_{\rm in}}\right)+\frac{1}{(z_{\rm in}-z)^2}\left[-\frac{32i}{\kappa z_{\rm in}^2}-\frac{2i(448\kappa^2+479\kappa+60)}{15\kappa}\right.\\
&\left.+\frac{8(36\kappa^2+51\kappa+14)}{15}z_{\rm in}-\frac{i(96\kappa^3+204\kappa^2+52\kappa-43)}{15}z_{\rm in}^2\right]\\
&+\frac{32(\kappa+1)}{\kappa z_{\rm in}^2 z}+\frac{1}{z_{\rm in}-z}\left[\frac{32(\kappa+1)}{\kappa z_{\rm in}^2}+\frac{2(208\kappa^2+118\kappa-\frac{60}{\kappa}-123)}{15}\right.\\
&\left.+\frac{2i(2\kappa+1)(48\kappa^2+18\kappa-13)}{15}z_{\rm in}+\frac{48\kappa^4-24\kappa^3-68\kappa^2+34\kappa+43}{15}z_{\rm in}^2\right]\\
&\left.-\frac{i}{15}\left(96\kappa^3-36\kappa^2-68\kappa+17\right)-\frac{1}{15}\left(48\kappa^4-24\kappa^3-68\kappa^2+34\kappa+43\right)(z_{\rm in}+z)\right\}\\
&+e^{-2i\kappa(z_{\rm in}-z)}\left\{-\frac{224z_{\rm in}^2}{5(z_{\rm in}-z)^5}+\frac{1}{(z_{\rm in}-z)^4}\left[-\frac{48i}{\kappa}+\frac{168}{5}z_{\rm in}-\frac{208i\kappa}{5}z_{\rm in}^2\right]-\frac{64i}{\kappa z_{\rm in}z^3}\right.\\
&+\frac{1}{(z_{\rm in}-z)^3}\left[\frac{1288}{15}+\frac{176i\kappa}{5}z_{\rm in}+\frac{32(12\kappa^2-5)}{15}z_{\rm in}^2\right]-\frac{32}{z^2}\left(\frac{i}{\kappa z_{\rm in}^2}+\frac{1}{z_{\rm in}}\right)\\
&+\frac{1}{(z_{\rm in}-z)^2}\left[\frac{32i}{\kappa z_{\rm in}^2}-\frac{16(18\kappa^2-5)}{15}z_{\rm in}+\frac{16i\left(56\kappa^2-15\right)}{15\kappa}+\frac{16i\kappa(6\kappa^2-5)}{15}z_{\rm in}^2\right]\\
&-\frac{4}{z_{\rm in}-z}\left[\frac{8}{z_{\rm in}^2}+\frac{4(26\kappa^2-35)}{15}+\frac{8i\kappa(6\kappa^2-5)}{15}z_{\rm in}+\frac{12\kappa^4-20\kappa^2+15}{15}z_{\rm in}^2\right]-\frac{32}{z_{\rm in}^2 z}\\
&\left.+\frac{16i(6\kappa^2-5)\kappa}{15}+\frac{4(12\kappa^4-20\kappa^2+15)}{15}(z+z_{\rm in})\right\}+\left(-\frac{40}{z^2}+\frac{92}{3}-\frac{43}{15}z^2\right)\\
&\times\left\{\pi-i{\rm Ei}\left[-i(2\kappa+1)(z_{\rm in}-z)\right]\right\}+4\left[-\frac{16i}{\kappa z^4}+\frac{8}{z^3}-\frac{8i}{\kappa z^2}-\frac{2i(4\kappa^4-12\kappa^2-3)}{3\kappa}\right.\\
&\left.+(2\kappa^2-1)^2 z-\frac{2i\kappa(12\kappa^4-20\kappa^2+15)}{15}z^2\right]\left\{{\rm Ei}\big[-2i\kappa(z_{\rm in}-z)\big]\right.\\
&\left.-{\rm Ei}\big[-i(2\kappa+1)(z_{\rm in}-z)\big]\right\}+4\left[\frac{16i}{\kappa z^4}+\frac{24}{z^3}-\frac{8i(2\kappa^2-1)}{\kappa z^2}+\frac{-4\kappa^4+4\kappa^2-1}{\kappa^2 z}\right]\\
&\times e^{2i\kappa z}{\rm Ei}(-2i\kappa z_{\rm in})+4\left[-\frac{16i}{\kappa z^4}-\frac{8(3\kappa+2)}{\kappa z^3}+\frac{2i(8\kappa+7)}{z^2}+\frac{4\kappa(\kappa+1)}{z}\right]\\
&\times e^{i(2\kappa+1)z}{\rm Ei}\left[-i(2\kappa+1)z_{\rm in}\right]+\pi\left\{-zz_{\rm in}\delta'''(z_{\rm in}-z)+(5z-4z_{\rm in})\delta''(z_{\rm in}-z)\right.\\
&-\frac{2}{3}\left(5zz_{\rm in}+\frac{6z}{z_{\rm in}}+\frac{6z_{\rm in}}{z}-30\right)\delta'(z_{\rm in}-z)+2\left[\frac{10}{z}-\frac{8}{z_{\rm in}}-6z_{\rm in}\right.\\
&\left.\left.+\left(\frac{23}{3}-\frac{2}{z_{\rm in}^2}\right)z\right]\delta(z_{\rm in}-z)\right\}\,. \qquad\qquad (C.34)
\end{aligned}
$$

Next we explore the various asymptotic limits of $\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in})$. First we note that in the limit $z_{\rm in}\gg 1$ (so that $\eta_{\rm in}\to-\infty$ in the distant past) we have

$$
\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\simeq 4\pi e^{2i\kappa z}\left[\frac{16}{\kappa z^4}-\frac{24i}{z^3}-\frac{8\left(2\kappa^2-1\right)}{\kappa z^2}+\frac{i\left(2\kappa^2-1\right)^2}{\kappa^2 z}\right]
$$

$$+ 8\pi e^{i(2\kappa+1)z}\left[-\frac{8}{\kappa z^4}+\frac{4i(3\kappa+2)}{\kappa z^3}+\frac{8\kappa+7}{z^2}-\frac{2i\kappa(\kappa+1)}{z}\right]+\mathcal{O}\left(\frac{1}{z_{\rm in}}\right), \qquad \text{(C.35)}$$

and note that if we furthermore take the super-Hubble limit of this then we get

$$\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \simeq -\frac{40\pi}{z^2}-\frac{4i\pi\left(24\kappa^3+6\kappa^2+16\kappa-3\right)}{3\kappa^2 z}+\frac{92\pi}{3}+\mathcal{O}(z)\ . \qquad \text{(C.36)}$$

Next let us fix arbitrary $z_{\rm in}$ and take the super-Hubble limit $z \ll 1$ so that

$$\begin{aligned}\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \simeq& -\frac{40\pi}{z^2}+\frac{16}{3z}\Bigg\{e^{-2i\kappa z_{\rm in}}\left[-\frac{2i(\kappa^2-3)}{\kappa z_{\rm in}}+\frac{4i}{\kappa z_{\rm in}^3}-\frac{5}{z_{\rm in}^2}\right]+e^{-i(2\kappa+1)z_{\rm in}}\left[-\frac{4i}{\kappa z_{\rm in}^3}+\frac{5\kappa+4}{\kappa z_{\rm in}^2}\right.\\ &\left.+\frac{i(4\kappa^2-5\kappa-8)}{2\kappa z_{\rm in}}\right]+\left(\kappa^2-9-\frac{3}{4\kappa^2}\right)\mathrm{Ei}(-2i\kappa z_{\rm in})+\left(-\kappa^2+6\kappa+\frac{4}{\kappa}+\frac{21}{2}\right)\\ &\times \mathrm{Ei}\left[-i(2\kappa+1)z_{\rm in}\right]\Bigg\}+\frac{92\pi}{3}+\mathcal{O}(z)\ . \end{aligned} \qquad \text{(C.37)}$$

Let us now take the above expression and take $z_{\rm in} \ll 1$ (assuming the hierarchy $z \ll z_{\rm in} \ll 1$)

$$\begin{aligned}\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \simeq& -\frac{40\pi}{z^2}+\frac{1}{z}\Bigg[-\frac{40i}{z_{\rm in}}+2\left(-\frac{1}{\kappa^2}+8\kappa+\frac{16}{3\kappa}+2\right)\left\{2\log\left[e^{\gamma}(2\kappa+1)z_{\rm in}\right]-i\pi\right\}\\ &+4\left(-\frac{4\kappa^2}{3}+\frac{1}{\kappa^2}+12\right)\log\left(\frac{2\kappa+1}{2\kappa}\right)-\frac{16(39\kappa+15+16/\kappa)}{9}+\mathcal{O}(z_{\rm in})\Bigg]\\ &+\frac{92\pi}{3}+\mathcal{O}(z) \end{aligned} \qquad \text{(C.38)}$$

where γ is Euler-Mascheroni constant.

Finally let us take the coincident limit for $z \to z_{\rm in}$ where we have

$$\begin{aligned}\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \simeq& -\frac{6iz_{\rm in}^2}{(z_{\rm in}-z)^4}+\frac{4iz_{\rm in}}{(z_{\rm in}-z)^3}+\frac{2i\left(33-5z_{\rm in}^2\right)}{3(z_{\rm in}-z)^2}+i\left(\frac{40}{z_{\rm in}^2}-\frac{92}{3}+\frac{43}{15}z_{\rm in}^2\right)\\ &\times\log\left[e^{\gamma}(2\kappa+1)(z_{\rm in}-z)\right]+\pi\Bigg\{-zz_{\rm in}\delta'''(z_{\rm in}-z)+(5z-4z_{\rm in})\delta''(z_{\rm in}-z)\\ &-\frac{2}{3}\left(5zz_{\rm in}+\frac{6z}{z_{\rm in}}+\frac{6z_{\rm in}}{z}-30\right)\delta'(z_{\rm in}-z)+2\Bigg[\frac{10}{z}-\frac{8}{z_{\rm in}}-6z_{\rm in}\\ &+\left(\frac{23}{3}-\frac{2}{z_{\rm in}^2}\right)z\Bigg]\delta(z_{\rm in}-z)\Bigg\}+\mathcal{O}\left[(z_{\rm in}-z)^0\right]\ . \end{aligned} \qquad \text{(C.39)}$$

Most important for us is the real part of the above, where we have

$$\begin{aligned}\mathrm{Re}\left[\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\right] \simeq \pi\Bigg\{& -zz_{\rm in}\delta'''(z_{\rm in}-z)+(5z-4z_{\rm in})\delta''(z_{\rm in}-z)-\frac{2}{3}\left(5zz_{\rm in}+\frac{6z}{z_{\rm in}}+\frac{6z_{\rm in}}{z}-30\right)\\ &\times\delta'(z_{\rm in}-z)+2\left[\frac{10}{z}-\frac{8}{z_{\rm in}}-6z_{\rm in}+\left(\frac{23}{3}-\frac{2}{z_{\rm in}^2}\right)z\right]\delta(z_{\rm in}-z)\Bigg\}+\mathcal{O}\left[(z_{\rm in}-z)^0\right]. \end{aligned} \qquad \text{(C.40)}$$

Notice that the singular part of the above is only coming from the δ-functions.

C.2 The function $\mathcal{I}$

Here we compute the integral (C.8), repeated here for convenience

$$\mathcal{I}_{\boldsymbol{k}}(\eta) := \int_0^1 \mathrm{d}a \int_{a+2\kappa}^{\infty} \mathrm{d}b\, z\,(a^2+b^2-2)^2 \left[1+\frac{2i}{(b-a)z}\right]\left[1+\frac{2i}{(b+a)z}\right] \times \left[-\frac{1}{b^2} - \frac{iz}{b} - \frac{4}{b^2-a^2} + \frac{4e^{ibz}\mathrm{Ei}(-ibz)}{b^2-a^2}\right]. \tag{C.41}$$

Luckily the computation of this is straightforward since we already know what the function $\mathcal{G}_{\boldsymbol{k}}(\eta, \eta_{\text{in}})$ is — the reason for this is that the integrand of $\mathcal{I}_{\boldsymbol{k}}(\eta)$ and $\mathcal{G}_{\boldsymbol{k}}(\eta, \eta_{\text{in}})$ are identical in the limit $\eta_{\text{in}} \to \eta$ (one must only be careful about the divergences in making this replacement).

As before, we split apart the integral $\mathcal{I}_{\boldsymbol{k}}(\eta)$ into a triangular and rectangular region,

$$\mathcal{I}_{\boldsymbol{k}}(\eta) = \mathcal{I}_{\boldsymbol{k}}^{\triangle}(\eta) + \mathcal{I}_{\boldsymbol{k}}^{\square}(\eta) \tag{C.42}$$

where we define

$$\mathcal{I}_{\boldsymbol{k}}^{\triangle}(\eta) := \int_{2\kappa}^{2\kappa+1} \mathrm{d}b \int_0^{b-2r} \mathrm{d}a\, f(a,b,z,z) \tag{C.43}$$

$$\mathcal{I}_{\boldsymbol{k}}^{\square}(\eta) := \int_{2\kappa+1}^{\infty} \mathrm{d}b \int_0^{1} \mathrm{d}a\, f(a,b,z,z)\,, \tag{C.44}$$

cf. eqs. (C.11) and (C.12) where we use

$$f(a,b,z,z) = z\,(a^2+b^2-2)^2 \left[1+\frac{2i}{(b-a)z}\right]\left[1+\frac{2i}{(b+a)z}\right] \times \left[-\frac{1}{b^2} - \frac{iz}{b} - \frac{4}{b^2-a^2} + \frac{4e^{ibz}\mathrm{Ei}(-ibz)}{b^2-a^2}\right], \tag{C.45}$$

where $f(a,b,z,z_{\text{in}})$ is the integrand of $\mathcal{G}_{\boldsymbol{k}}(\eta,\eta_{\text{in}})$, defined in eq. (C.13).

Note that

$$\mathcal{I}_{\boldsymbol{k}}^{\triangle}(\eta) = \lim_{z_{\text{in}}\to z} \left[\, \mathcal{G}_{\boldsymbol{k}}^{\triangle}(\eta,\eta_{\text{in}}) \,\right] \tag{C.46}$$

where we have the explicit formula for $\mathcal{G}_{\boldsymbol{k}}^{\triangle}(\eta,\eta_{\text{in}})$ given in eq. (C.17) and so we straightforwardly have an answer for $\mathcal{I}_{\boldsymbol{k}}^{\triangle}(\eta)$ (for brevity we avoid writing it down here).

For computing $\mathcal{I}_{\boldsymbol{k}}^{\square}(\eta)$ in eq. (C.44) we use the formula (C.18) (after a integration has been performed) and take the limit $z_{\text{in}} \to z$ giving

$$\mathcal{I}_{\boldsymbol{k}}^{\square}(\eta) = \mathcal{I}_{\boldsymbol{k}}^{\square,\text{div}}(\eta) + \mathcal{I}_{\boldsymbol{k}}^{\square,\text{reg}}(\eta)\;, \tag{C.47}$$

where

$$\mathcal{I}_{\boldsymbol{k}}^{\square,\text{div}}(\eta) := \int_{2\kappa+1}^{\infty} \mathrm{d}b \left\{ 4\pi e^{+ibz}\left[-izb^2+4b+i\left(3z+\frac{4}{z}\right)\right] + \left[-iz^2b^3 - zb^2 + \frac{2i(5z^2-18)}{3}b + \frac{10z}{3} + \frac{i}{b}\left(-\frac{40}{z^2}+\frac{92}{3}-\frac{43}{15}z^2\right)\right]\right\} \tag{C.48}$$

$$\mathcal{I}_{\boldsymbol{k}}^{\square,\text{reg}}(\eta) := \int_{2\kappa+1}^{\infty} \mathrm{d}b \left(-4\pi e^{+ibz}\left[-izb^2+4b+i\left(3z+\frac{4}{z}\right)\right] - 36ib + \frac{36}{z} + \frac{4i\,(z^2+30)}{3bz^2}\right.$$

$$+\frac{-43z^2-260}{15b^2z}+\frac{16\left(2ib^5z-b^4-2b^2-2ibz+3\right)}{b^3z}\coth^{-1}(b)+4\bigg\{-3b^2z+12ib$$
$$+\frac{11z^2-36}{3z}-\frac{8i}{b}+\frac{8}{b^2z}+\left[\frac{4(b^2-1)^2z}{b}+\frac{8}{b^2}(b^4-1)\left(\frac{1}{bz}-i\right)\right]\coth^{-1}(b)\bigg\}$$
$$\times e^{ibz}\mathrm{Ei}(-ibz)\Bigg) \tag{C.49}$$

cf. the definitions (C.20) and (C.21). Note in particular that

$$\mathcal{I}_{\boldsymbol{k}}^{\square,\mathrm{reg}}(\eta)=\lim_{z_{\mathrm{in}}\to z}\left[\mathcal{G}_{\boldsymbol{k}}^{\square,\mathrm{reg}}(\eta,\eta_{\mathrm{in}})\right], \tag{C.50}$$

where we can straightforwardly write down an expression for $\mathcal{I}_{\boldsymbol{k}}^{\square,\mathrm{reg}}(\eta)$ given we know what $\mathcal{G}_{\boldsymbol{k}}^{\square,\mathrm{reg}}(\eta,\eta_{\mathrm{in}})$ is in eq. (C.32) (although once again we avoid writing this down here for brevity).

One can almost make a similar statement for $\mathcal{I}_{\boldsymbol{k}}^{\square,1}(\eta)$, but not exactly because this integral is formally divergent — we need to use the distributions (C.26) to evaluate the first terms in eq. (C.48), as well as the following dimensional regularizations of various powers of b:

$$\int_{2\kappa+1}^{\infty}\mathrm{d}b\,b^j\left(\frac{b}{M}\right)^{\epsilon}=-\frac{(2\kappa+1)^{\epsilon+j+1}}{M^{\epsilon}(\epsilon+j+1)}\simeq-\frac{(2\kappa+1)^{j+1}}{j+1}+\mathcal{O}(\epsilon), \tag{C.51}$$
$$\int_{2\kappa+1}^{\infty}\mathrm{d}b\,\frac{1}{b}\left(\frac{b}{M}\right)^{\epsilon}=-\frac{(2\kappa+1)^{\epsilon}}{M^{\epsilon}\epsilon}\simeq-\frac{1}{\epsilon}-\log\left(\frac{2\kappa+1}{M}\right)+\mathcal{O}(\epsilon) \tag{C.52}$$

where the first relation applies for $j\in\{0,1,2,3\}$ and $M=\mu/k>0$ ($\epsilon\in\mathbb{C}$) for some mass scale $\mu>0$ and where we have expanded the above integrals for $0<|\epsilon|\ll 1$ (although the integrals are technically only convergent for $\mathrm{Re}(\epsilon)<-j-1$ and $\mathrm{Re}(\epsilon)<0$ respectively). With this, we find that

$$\mathcal{I}_{\boldsymbol{k}}^{\square,\mathrm{div}}(\eta)=i\left(\frac{40}{z^2}-\frac{92}{3}+\frac{43}{15}z^2\right)\left[\log\left(\frac{2\kappa+1}{M}\right)+\frac{1}{\epsilon}\right]+8\pi e^{i(2\kappa+1)z}\bigg[-\frac{5}{z^2}+\frac{3i(2\kappa+1)}{z}$$
$$+2\kappa^2+2\kappa-1\bigg]+6i(2\kappa+1)^2+\frac{(2\kappa+1)(4\kappa^2+4\kappa-9)}{3}z$$
$$+\frac{i(2\kappa+1)^2(12\kappa^2+12\kappa-17)}{12}z^2\,, \tag{C.53}$$

where we have ignored the Dirac functions. With the above, we can at last sum together all the pieces to get our result

$$\mathcal{I}_{\boldsymbol{k}}(\eta)=\mathcal{I}_{\boldsymbol{k}}^{\Delta}(\eta)+\mathcal{I}_{\boldsymbol{k}}^{\square,\mathrm{div}}(\eta)+\mathcal{I}_{\boldsymbol{k}}^{\square,\mathrm{reg}}(\eta)$$
$$=i\left(\frac{40}{z^2}-\frac{92}{3}+\frac{43}{15}z^2\right)\left[\frac{1}{\epsilon}+\log\left(\frac{2k_{\mathrm{UV}}+k}{\mu}\right)\right]+\frac{64}{\kappa z^3}-\frac{4i(\kappa+4)}{\kappa z^2}+\frac{2i}{9\kappa}\bigg(84\kappa^3+60\kappa^2$$
$$+79\kappa+27\bigg)+\frac{(32\kappa^3-40\kappa+1)z}{3}+\frac{i}{900}\left(-3611-3360\kappa+360\kappa^2+4320\kappa^3+720\kappa^4\right)$$
$$+\left[\frac{64i}{\kappa z^4}-\frac{32}{z^3}+\frac{32i}{\kappa z^2}+\frac{8i(4\kappa^4-12\kappa^2-3)}{3\kappa}-4(2\kappa^2-1)^2z+\frac{8i\kappa(12\kappa^4-20\kappa^2+15)}{15}z^2\right]$$
$$\times\log\left(\frac{2\kappa+1}{2\kappa}\right)+8e^{i(2\kappa+1)z}\mathrm{Ei}\left[-i(2\kappa+1)z\right]\bigg[-\frac{8i}{\kappa z^4}-\frac{4(3\kappa+2)}{\kappa z^3}+\frac{i(8\kappa+7)}{z^2}$$

$$+\frac{2\kappa(\kappa+1)}{z}\Bigg]+8e^{2i\kappa z}\mathrm{Ei}\left(-2i\kappa z\right)\left[\frac{8i}{\kappa z^4}+\frac{12}{z^3}-\frac{4i(2\kappa^2-1)}{\kappa z^2}-\frac{(2\kappa^2-1)^2}{2\kappa^2 z}\right] \tag{C.54}$$

Writing out the real and imaginary parts of the above function gives

$$\begin{aligned}\mathrm{Re}\left[\mathcal{I}_{\boldsymbol{k}}(\eta)\right] &= \frac{64}{\kappa z^3}+\frac{(32\kappa^3-40\kappa+1)z}{3}+\left[-\frac{32}{z^3}-4(2\kappa^2-1)^2 z\right]\log\left(\frac{2\kappa+1}{2\kappa}\right)\\ &+\mathrm{Re}\left\{8e^{i(2\kappa+1)z}\mathrm{Ei}\left[-i(2\kappa+1)z\right]\left[-\frac{8i}{\kappa z^4}-\frac{4(3\kappa+2)}{\kappa z^3}+\frac{i(8\kappa+7)}{z^2}+\frac{2\kappa(\kappa+1)}{z}\right]\right\}\\ &+\mathrm{Re}\left\{8e^{2i\kappa z}\mathrm{Ei}\big(-2i\kappa z\big)\left[\frac{8i}{\kappa z^4}+\frac{12}{z^3}-\frac{4i(2\kappa^2-1)}{\kappa z^2}-\frac{(2\kappa^2-1)^2}{2\kappa^2 z}\right]\right\}\end{aligned} \tag{C.55}$$

while the imaginary part is

$$\begin{aligned}\mathrm{Im}\left[\mathcal{I}_{\boldsymbol{k}}(\eta)\right] &= \left(\frac{40}{z^2}-\frac{92}{3}+\frac{43}{15}z^2\right)\left[\frac{1}{\epsilon}+\log\left(\frac{2k_{\rm UV}+k}{\mu}\right)\right]-\frac{4(\kappa+4)}{\kappa z^2}+\frac{2}{9\kappa}\Bigg(84\kappa^3+60\kappa^2\\ &+79\kappa+27\Bigg)+\frac{-3611-3360\kappa+360\kappa^2+4320\kappa^3+720\kappa^4}{900}z^2+\Bigg[\frac{64}{\kappa z^4}+\frac{32}{\kappa z^2}\\ &+\frac{8(4\kappa^4-12\kappa^2-3)}{3\kappa}+\frac{8\kappa(12\kappa^4-20\kappa^2+15)}{15}z^2\Bigg]\log\left(\frac{2\kappa+1}{2\kappa}\right)\\ &+\mathrm{Im}\left\{8e^{i(2\kappa+1)z}\mathrm{Ei}\left[-i(2\kappa+1)z\right]\left[-\frac{8i}{\kappa z^4}-\frac{4(3\kappa+2)}{\kappa z^3}+\frac{i(8\kappa+7)}{z^2}+\frac{2\kappa(\kappa+1)}{z}\right]\right\}\\ &+\mathrm{Im}\left\{8e^{2i\kappa z}\mathrm{Ei}\big(-2i\kappa z\big)\left[\frac{8i}{\kappa z^4}+\frac{12}{z^3}-\frac{4i(2\kappa^2-1)}{\kappa z^2}-\frac{(2\kappa^2-1)^2}{2\kappa^2 z}\right]\right\}.\end{aligned} \tag{C.56}$$

Next we explore the various asymptotic limits of $\mathcal{I}_{\boldsymbol{k}}(\eta)$. First we consider the super-Hubble limit $z \ll 1$ (which also assumes $\kappa z = -k_{\rm UV}\eta \ll 1$)

$$\begin{aligned}\mathcal{I}_{\boldsymbol{k}}(\eta) &\simeq i\left(\frac{40}{z^2}-\frac{92}{3}\right)\left[\frac{1}{\epsilon}+\log\left(\frac{2k_{\rm UV}+k}{\mu}\right)\right]+\frac{-20\pi+28i-40i\log[e^{\gamma}(2\kappa+1)z]}{z^2}\\ &+\frac{1}{z}\Bigg(\frac{4}{9}\Bigg\{-4\left(39\kappa+15+\frac{16}{\kappa}\right)+3\left(-4\kappa^2+36+\frac{3}{\kappa^2}\right)\log\left(\frac{2\kappa+1}{2\kappa}\right)+\Bigg(72\kappa+18\\ &+\frac{48}{\kappa}-\frac{9}{\kappa^2}\Bigg)\log\left[e^{\gamma}(2\kappa+1)z\right]\Bigg\}-2i\pi\left(8\kappa+2+\frac{16}{3\kappa}-\frac{1}{\kappa^2}\right)\Bigg)+\frac{1}{3}\Bigg\{46\pi\\ &+92i\log[e^{\gamma}(2\kappa+1)z]-128i\Bigg\}\quad+\mathcal{O}(z)\end{aligned} \tag{C.57}$$

Next we consider the early-time expansion $z \gg 1$ (which implicitly assumes that $z_{\rm in} \gg z \gg 1$), where

$$\begin{aligned}\mathcal{I}_{\boldsymbol{k}}(\eta) &\simeq i\left(-\frac{92}{3}+\frac{43}{15}z^2\right)\left[\frac{1}{\epsilon}+\log\left(\frac{2k_{\rm UV}+k}{\mu}\right)\right]+i\Bigg[\frac{720\kappa^4+4320\kappa^3+360\kappa^2-3360\kappa-3611}{900}\\ &+\frac{8\kappa\left(12\kappa^4-20\kappa^2+15\right)}{15}\log\left(\frac{2\kappa+1}{2\kappa}\right)\Bigg]z^2+\Bigg[\frac{32\kappa^3-40\kappa+1}{3}-4\left(1-2\kappa^2\right)^2\\ &\times\log\left(\frac{2\kappa+1}{2\kappa}\right)\Bigg]z+\frac{2i}{9\kappa}\left[84\kappa^3+60\kappa^2+79\kappa+27+12\left(4\kappa^4-12\kappa^2-3\right)\log\left(\frac{2\kappa+1}{2\kappa}\right)\right]\\ &+\mathcal{O}\left(\frac{1}{z}\right).\end{aligned} \tag{C.58}$$

C.3 Asymptotics of $\mathfrak{F}$

We can now take the above asymptotic expressions for $\mathcal{G}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$ and $\mathcal{I}_{\boldsymbol{k}}(\eta)$ and use formula (C.7) to obtain expressions for $\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$ in various limits used in the main text. Most importantly we consider the limit $z \ll z_{\rm in} \ll 1$ in which, using eqs. (C.37) and (C.57), we have

$$\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) = \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2}\left(\frac{40i}{z^2}\left[\frac{1}{\epsilon} + \log\left(\frac{2k_{\rm UV}+k}{\mu}\right)\right] - \frac{20\pi}{z^2} + \frac{i}{z^2}\left\{28 - 40\log\left[e^{\gamma}(2\kappa+1)z\right]\right\}\right.$$
$$\left. + \mathcal{O}\left(\frac{1}{z}\right) - \left[-\frac{40\pi}{z^2} + \mathcal{O}\left(\frac{1}{z}\right)\right]\right) \tag{C.59}$$

This expression (in fact an even more accurate version of this equation with higher order terms written explicitly) is given in eq. (3.40). eq. (3.38), which is also used in the main text, also corresponds to the limit $z \ll 1$ but for arbitrary $z_{\rm in}$.

We now turn to address the question raised in Footnote 11. This footnote asks whether the singularity of Re $[\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})]$ in the coincident limit might mean that integrating through the coincident limit might compete in the expression for $\Xi_{\boldsymbol{k}}$ even if $k\eta$ is not small. We show here that these corrections are in fact subdominant in $k\eta_{\rm in}$. In the coincidence limit, the integral in the denominator of eq. (4.13) can be written as

$$-\int_{z_0}^{z_{\rm in}} {\rm d}z' \,{\rm Re}\left[\mathcal{G}_{\boldsymbol{k}}\left(-\frac{z'}{k}, -\frac{z_{\rm in}}{k}\right)\right]\left[1 + \frac{1}{(z')^2}\right] \simeq -\pi\int_{z_0}^{z_{\rm in}} {\rm d}z' \left\{ - z' z_{\rm in}\delta'''(z_{\rm in} - z')\right.$$
$$+ (5z' - 4z_{\rm in})\delta''(z_{\rm in} - z') - \frac{2}{3}\left(5z' z_{\rm in} + \frac{6z'}{z_{\rm in}} + \frac{6z_{\rm in}}{z'} - 30\right)\delta'(z_{\rm in} - z') + 2\left[\frac{10}{z'} - \frac{8}{z_{\rm in}} - 6z_{\rm in}\right.$$
$$\left.\left. + \left(\frac{23}{3} - \frac{2}{z_{\rm in}^2}\right)z'\right]\delta(z_{\rm in} - z')\right\}\left[1 + \frac{1}{(z')^2}\right] \tag{C.60}$$

where we have used $\mathfrak{F}_{\boldsymbol{k}} \propto \mathcal{I}_{\boldsymbol{k}} - \mathcal{G}_{\boldsymbol{k}}$ and where $z \simeq z_0$ is close to the coincident limit so that $|z_0 - z_{\rm in}| \ll 1$ and where we keep the δ-functions because all other terms are regular in the coincident limit. The above integral can be calculated exactly by using the regularization $\delta_\epsilon(x) = \pi^{-1}\lim_{\epsilon\to 0}\epsilon/(x^2+\epsilon^2)$. Then, one obtains

$$-\int_{z_0}^{z_{\rm in}} {\rm d}z' \,{\rm Re}\left[\mathcal{G}_{\boldsymbol{k}}\left(-\frac{z'}{k}, -\frac{z_{\rm in}}{k}\right)\right]\left[1 + \frac{1}{(z')^2}\right] \simeq \frac{2\left(z_{\rm in}^2+1\right)}{\epsilon^3} - \frac{10z_{\rm in}^4 - 41z_{\rm in}^2 - 39}{3z_{\rm in}^2\epsilon}$$
$$- \frac{2\pi\left(5z_{\rm in}^2 - 24\right)}{3z_{\rm in}^3} + \mathcal{O}(\epsilon). \tag{C.61}$$

We see that the ϵ-dependence is power-law, and any $z_{\rm in}$-dependence is subdominant to the result quoted in the main text.

C.4 The validity coefficient

Here we compute the integral $\mathfrak{M}_{\boldsymbol{k}}$ defined in eq. (3.42)

$$\mathfrak{M}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) := (2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta} {\rm d}\eta' \, G(\eta)G(\eta')\mathscr{C}_{\boldsymbol{k}}(\eta, \eta')\,(\eta' - \eta)\ . \tag{C.62}$$

Noting the earlier definition (3.37) of $\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$, it turns out that we can decompose $\mathfrak{M}_{\boldsymbol{k}}$ as follows

$$\mathfrak{M}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) = \mathfrak{N}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) - \eta\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) \tag{C.63}$$

with $\mathfrak{N}_{\boldsymbol{k}}$ is defined by

$$\mathfrak{N}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := (2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta} {\rm d}\eta'\, G(\eta)\, G(\eta')\, \mathscr{C}_{\boldsymbol{k}}(\eta,\eta')\,\eta' \ . \tag{C.64}$$

Following similar steps for the computation as for $\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})$ as in Appendix C, we find that $\mathfrak{N}_{\boldsymbol{k}}(\eta,\eta_{\rm in})$ is given by the following expression

$$\begin{aligned}
\mathfrak{N}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = \frac{\varepsilon_1 H^2 k}{1024\pi^2 M_{\rm p}^2}\Bigg(&-32\kappa - 8 - \frac{64}{3\kappa} + \frac{4}{\kappa^2} - \frac{2z^2}{3}\bigg[32\kappa^3 - 40\kappa + 1 - 12\left(2\kappa^2-1\right)^2 \\
&\times \log\left(\frac{2\kappa+1}{2r}\right)\bigg] - iz\left(\frac{40}{z^2} - 24 + \frac{43z^2}{15}\right)\left[\frac{1}{\epsilon} + \log\left(\frac{2\kappa+1}{M}\right)\right] + \frac{i}{z}\bigg[2\left(\frac{1}{\kappa^3} + \frac{2}{2\kappa+1} + \frac{4}{\kappa}\right) \\
&- \frac{32}{\kappa}\log\left(\frac{2\kappa+1}{2\kappa}\right)\bigg] + \frac{iz}{\kappa}\left[8\left(4\kappa^4+1\right)\log\left(\frac{2\kappa+1}{2\kappa}\right) - \frac{2\left(36\kappa^4+24\kappa^3+6\kappa^2+9\kappa+4\right)}{2\kappa+1}\right] \\
&+ \frac{iz^3}{15}\left[-12\kappa^4 - 72\kappa^3 - 6\kappa^2 + 56\kappa + \frac{3611}{60} - 8\left(12\kappa^4-20\kappa^2+15\right)\kappa\log\left(\frac{2\kappa+1}{2r}\right)\right] \\
&- 8\left[\frac{4i}{\kappa z} - i\left(4\kappa^3+\frac{1}{\kappa}\right)z - \left(1-2\kappa^2\right)^2 z^2 + \frac{i\kappa(12\kappa^4-20\kappa^2+15)}{15}z^3\right]{\rm Ei}\left[-2i\kappa(z_{\rm in}-z)\right] \\
&- 8\bigg[-\frac{i}{z}\left(\frac{4}{\kappa}+5\right) + \frac{iz}{\kappa}\left(4\kappa^4+3\kappa+1\right) + \left(1-2\kappa^2\right)^2 z^2 - \frac{iz^3}{120}\big(96\kappa^5 - 160\kappa^3 + 120\kappa \\
&+ 43\big)\bigg]{\rm Ei}\left[i(2\kappa+1)(z-z_{\rm in})\right] - \pi\left(\frac{43z^3}{15} - 24z + \frac{40}{z}\right) \\
&+ e^{-2i\kappa(z_{\rm in}-z)}\bigg\{-\frac{224z_{\rm in}^3}{5(z_{\rm in}-z)^5} + \frac{16z_{\rm in}}{5(z_{\rm in}-z)^4}\left(-13i\kappa z_{\rm in}^2 - \frac{15i}{\kappa} + 7z_{\rm in}\right) + \frac{16}{15(z_{\rm in}-z)^3} \\
&\times\left[2\left(12\kappa^2-5\right)z_{\rm in}^3 + 27i\kappa z_{\rm in}^2 - \frac{15i}{\kappa} + 84z_{\rm in}\right] + \frac{16}{15(z_{\rm in}-z)^2}\bigg[i\kappa\left(6\kappa^2-5\right)z_{\rm in}^3 - 12\kappa^2 z_{\rm in}^2 \\
&+ \frac{3i\left(21\kappa^2-5\right)z_{\rm in}}{\kappa} + \frac{30i}{\kappa z_{\rm in}} + 27\bigg] + \frac{4}{15(z_{\rm in}-z)}\bigg[-12i\kappa\left(6\kappa^2-5\right)z_{\rm in}^2 - 24\left(3\kappa^2-5\right)z_{\rm in} \\
&+ \frac{12i\left(3\kappa^2-5\right)}{\kappa} - \left(12\kappa^4-20\kappa^2+15\right)z_{\rm in}^3 + \frac{120i}{\kappa z_{\rm in}^2} - \frac{120}{z_{\rm in}}\bigg] + \frac{2}{z}\bigg[-\frac{i\left(2\kappa^2-1\right)^2}{\kappa^3} + \frac{16i}{\kappa z_{\rm in}^2} \\
&- \frac{16}{z_{\rm in}}\bigg] + \frac{4}{15}\left[8i\kappa\left(6\kappa^2-5\right)z_{\rm in} + \left(12\kappa^4-20\kappa^2+15\right)z_{\rm in}^2 - \frac{36\kappa^4+20\kappa^2+15}{\kappa^2}\right] \\
&+ \frac{4z^2}{15}\left(12\kappa^4-20\kappa^2+15\right) + \frac{2z}{15}\left[2\left(12\kappa^4-20\kappa^2+15\right)z_{\rm in} + \frac{i\left(108\kappa^4-100\kappa^2+15\right)}{\kappa}\right]\bigg\} \\
&+ e^{-i(2\kappa+1)(z_{\rm in}-z)}\bigg\{\frac{224z_{\rm in}^3}{5(z_{\rm in}-z)^5} + \frac{2z_{\rm in}}{5(z_{\rm in}-z)^4}\left[i(104\kappa+97)z_{\rm in}^2 + \frac{120i}{\kappa} - 56z_{\rm in}\right] \\
&+ \frac{1}{15(z_{\rm in}-z)^3}\bigg[-2\left(192\kappa^2+222\kappa+43\right)z_{\rm in}^3 - 18i(24\kappa+17)z_{\rm in}^2 - \frac{48(28\kappa+15)z_{\rm in}}{\kappa} \\
&+ \frac{240i}{\kappa}\bigg] + \frac{1}{15(z_{\rm in}-z)^2}\bigg[-\frac{6i\left(168\kappa^2+169\kappa+20\right)z_{\rm in}}{\kappa} - i\left(96\kappa^3+204\kappa^2+52\kappa-43\right)z_{\rm in}^3 \\
&+ 6(2\kappa+1)(16\kappa+23)z_{\rm in}^2 - \frac{480i}{\kappa z_{\rm in}} - \frac{48(9\kappa+5)}{\kappa}\bigg] + \frac{1}{15(z_{\rm in}-z)}\bigg[3i\big(96\kappa^3+44\kappa^2 \\
&- 28\kappa - 3\big)z_{\rm in}^2 + \left(48\kappa^4 - 24\kappa^3 - 68\kappa^2 + 34\kappa + 43\right)z_{\rm in}^3 - \frac{480i}{\kappa z_{\rm in}^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{6(2\kappa+1)(3\kappa+4)(8\kappa-5)z_{\rm in}}{\kappa}+\frac{480(\kappa+1)}{\kappa z_{\rm in}}-\frac{6i(3\kappa+4)(8\kappa-5)}{\kappa}\Bigg]\\
&+\frac{16}{z}\left[-\frac{2i}{\kappa z_{\rm in}^2}+\frac{2(\kappa+1)}{\kappa z_{\rm in}}+\frac{i\kappa(\kappa+1)}{2\kappa+1}\right]+\frac{1}{15}\Big[-2i\left(96\kappa^3-36\kappa^2-68\kappa+17\right)z_{\rm in}\\
&+\frac{2\left(72\kappa^3-18\kappa^2+43\kappa+60\right)}{\kappa}-\left(48\kappa^4-24\kappa^3-68\kappa^2+34\kappa+43\right)z_{\rm in}^2\Big]-\frac{z}{15}\Big[\big(48\kappa^4\\
&-24\kappa^3-68\kappa^2+34\kappa+43\big)z_{\rm in}+\frac{i\left(432\kappa^4+24\kappa^3-412\kappa^2-34\kappa+77\right)}{2\kappa+1}\Big]\\
&-\frac{z^2}{15}\left(48\kappa^4-24\kappa^3-68\kappa^2+34\kappa+43\right)\Big\}+\pi\Big\{-zz_{\rm in}^2\delta'''(z_{\rm in}-z)\\
&+2z_{\rm in}(3z-2z_{\rm in})\delta''(z_{\rm in}-z)+\frac{2}{3}\left[-\frac{6z_{\rm in}^2}{z}-5z\left(z_{\rm in}^2+3\right)+36z_{\rm in}\right]\delta'(z_{\rm in}-z)\\
&+\frac{4}{3}\left(14zz_{\rm in}+\frac{18z_{\rm in}}{z}-9z_{\rm in}^2-30\right)\delta(z_{\rm in}-z)\Big\}\Bigg). \qquad \text{(C.65)}
\end{aligned}
$$

The limit $z \ll z_{\rm in} \ll 1$ of the above is

$$
\begin{aligned}
\mathfrak{N}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \simeq \frac{\varepsilon_1 H^2 k}{1024\pi^2 M_{\rm p}^2}\Bigg(&-\frac{20\pi}{z}+\frac{4i}{z}\left\{10\log\left[e^{\gamma}(2\kappa+1)z_{\rm in}\right]-7\right\}+\left(-32\kappa-8-\frac{64}{3\kappa}+\frac{4}{\kappa^2}\right)\\
&-i\frac{40}{z}\left[\frac{1}{\epsilon}+\log\left(\frac{2\kappa+1}{M}\right)\right]+\mathcal{O}(z)\Bigg). \qquad \text{(C.66)}
\end{aligned}
$$

With this formula, along with the earlier formula (C.59) in the same limit, we get

$$
\begin{aligned}
\mathfrak{M}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) &= \mathfrak{N}_{\boldsymbol{k}}(\eta,\eta_{\rm in})-\eta\,\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = \mathfrak{N}_{\boldsymbol{k}}(\eta,\eta_{\rm in})+\frac{z}{k}\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\\
&= \frac{\varepsilon_1 H^2 k}{1024\pi^2 M_{\rm p}^2}\Bigg\{-\frac{40i}{z}\log\left(\frac{z}{z_{\rm in}}\right)+\frac{40i}{z_{\rm in}}-\frac{20i}{3}z\left[\frac{1}{\epsilon}+\log\left(\frac{2\kappa+1}{M}\right)\right]\\
&+4\left(-\frac{1}{\kappa^2}+8\kappa+\frac{16}{3\kappa}+2\right)\log\left(\frac{z}{e z_{\rm in}}\right)+\mathcal{O}(z,z_{\rm in})\Bigg\}, \qquad \text{(C.67)}
\end{aligned}
$$

where (as before) $M = \mu/k$.

D Infrared volume factors

After splitting apart the Nakajima-Zwanzig equation into different (continuous) momenta $\boldsymbol{k} \in \mathbb{R}^{3+}$ in eq. (3.20) there appear volume factors $\mathcal{V}$ on the LHS so that the equations make sense dimensionally. This Appendix derives the necessity of these volume factors, in the simpler setting where there are no interactions at all where

$$
\frac{\mathcal{V}}{(2\pi)^3}\frac{\partial \varrho_{\rm s\boldsymbol{k}}}{\partial \eta} = -i\left[\mathcal{H}_{\rm s\boldsymbol{k}}(\eta), \varrho_{\rm s\boldsymbol{k}}(\eta)\right] \qquad \text{(D.1)}
$$

which is derived from the free Liouville equation. For clarity of notation, we omit the label $\alpha = \mathrm{R}, \mathrm{I}$ from eq. (2.21) in the main text in this Appendix. This is done by assuming an ansatz of the form (2.42) for the reduced density matrix, repeated here,

$$
\varrho(\eta) = \bigotimes_{q<k_{\rm UV}} \varrho_{\boldsymbol{q}}(\eta) \ , \qquad \text{(D.2)}
$$

which assumes the momentum label is continuous. We justify the presence of the volume factors in eq. (D.1) here by passing to the limit where discrete momenta are considered, so that the system is placed inside a box of volume $\mathcal{V}$. In this case the conversion between discrete and continuum normalization is given by

$$\sum_{\boldsymbol{k}} = \frac{\mathcal{V}}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}\,, \quad C_{\boldsymbol{k}}(\eta_{\rm in}) = \left[\frac{(2\pi)^3}{\mathcal{V}}\right]^{1/2} c_{\boldsymbol{k}}(\eta_{\rm in})\,, \quad V_{\boldsymbol{k}}(\eta) = \left[\frac{(2\pi)^3}{\mathcal{V}}\right]^{1/2} v_{\boldsymbol{k}}(\eta)\,, \tag{D.3}$$

where $C_{\boldsymbol{k}}(\eta_{\rm in})$ are the discretely normalized annihilation operators, and $V_{\boldsymbol{k}}(\eta)$ are the discretely normalized Fourier transforms of the Mukhanov-Sasaki field, and so on. With this, continuum momentum field expansions like

$$v(\eta,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} v_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \Big[u_{\boldsymbol{k}}(\eta)c_{\boldsymbol{k}}(\eta_{\rm in}) + u^*_{\boldsymbol{k}}(\eta)c^\dagger_{-\boldsymbol{k}}(\eta_{\rm in})\Big]\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{D.4}$$

become instead

$$v(\eta,\boldsymbol{x}) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\boldsymbol{k}} V_{\boldsymbol{k}}(\eta) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\boldsymbol{k}} \Big[u_{\boldsymbol{k}}(\eta)C_{\boldsymbol{k}}(\eta_{\rm in}) + u^*_{\boldsymbol{k}}(\eta)C^\dagger_{-\boldsymbol{k}}(\eta_{\rm in})\Big]\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{D.5}$$

The tensor product structure of the ansatz (D.2) is now the same but over a discrete label — this allows us to compute the time-derivative of $\varrho_{\rm S}$ as

$$\begin{aligned}\frac{\partial \varrho_{\rm S}}{\partial\eta} &= \frac{\partial}{\partial\eta}\left[\bigotimes_{k<k_{\rm UV}} \varrho_{{\rm S}\boldsymbol{k}}(\eta)\right] = \sum_{k<k_{\rm UV}} \frac{\partial \varrho_{{\rm AS}\boldsymbol{k}}}{\partial\eta} \bigotimes_{q<k_{\rm UV},\boldsymbol{q}\neq\boldsymbol{k}} \varrho_{{\rm S}\boldsymbol{q}}(\eta) \\ &= \frac{\mathcal{V}}{(2\pi)^3}\int_{k<k_{\rm UV}} \mathrm{d}^3\boldsymbol{k}\, \frac{\partial \varrho_{{\rm AS}\boldsymbol{k}}}{\partial\eta} \bigotimes_{q<k_{\rm UV},\boldsymbol{q}\neq\boldsymbol{k}} \varrho_{{\rm S}\boldsymbol{q}}(\eta)\,.\end{aligned} \tag{D.6}$$

On the other hand, the RHS of the free Liouville equation *does not* end up with a volume factor by the above logic. To see why, recall the form of the free Hamiltonian (2.18)

$$\begin{aligned}\mathcal{H}_{\rm AS}(\eta) &= \frac{1}{2}\int_{k<k_{\rm UV}} \mathrm{d}^3\boldsymbol{k}\, \Big[p_{{\rm S}\boldsymbol{k}}(\eta)p^\dagger_{{\rm S}\boldsymbol{k}}(\eta) + \omega^2(\boldsymbol{k},\eta)v_{{\rm S}\boldsymbol{k}}(\eta)v^\dagger_{{\rm S}\boldsymbol{k}}(\eta)\Big] = \frac{1}{2}\int_{k<k_{\rm UV}} \mathrm{d}^3\boldsymbol{k}\, \mathcal{H}_{{\rm S}\boldsymbol{k}}(\eta) \\ &= \frac{1}{2}\sum_{k<k_{\rm UV}} \Big[P_{{\rm S}\boldsymbol{k}}(\eta)P^\dagger_{{\rm S}\boldsymbol{k}}(\eta) + \omega^2(\boldsymbol{k},\eta)V_{{\rm S}\boldsymbol{k}}(\eta)V^\dagger_{{\rm S}\boldsymbol{k}}(\eta)\Big] = \sum_{k<k_{\rm UV}} \mathfrak{h}_{{\rm S}\boldsymbol{k}}(\eta)\end{aligned} \tag{D.7}$$

where we define the shorthand $\mathfrak{h}_{{\rm S}\boldsymbol{k}}(\eta) := \frac{1}{2}\big[P_{{\rm S}\boldsymbol{k}}(\eta)P^\dagger_{{\rm S}\boldsymbol{k}}(\eta) + \omega^2(\boldsymbol{k},\eta)V_{{\rm S}\boldsymbol{k}}(\eta)V^\dagger_{{\rm S}\boldsymbol{k}}(\eta)\big]$ in the last line, which is the discrete normalized version of the (free) Hamiltonian density $\mathcal{H}_{{\rm S}\boldsymbol{k}}(\eta)$ — note that there is no volume factor since this operator is built from two fields. The above implies that:

$$-i\left[\mathcal{H}_{\rm AS}(\eta), \varrho_{\rm AS}(\eta)\right] = -i\left[\sum_{k<k_{\rm UV}} \mathfrak{h}_{{\rm S}\boldsymbol{k}}(\eta), \bigotimes_{q<k_{\rm UV}} \varrho_{{\rm S}\boldsymbol{q}}(\eta)\right] \tag{D.8}$$

$$= -i\sum_{k<k_{\rm UV}} \left[\mathfrak{h}_{{\rm S}\boldsymbol{k}}(\eta), \varrho_{{\rm S}\boldsymbol{k}}(\eta)\right] \bigotimes_{q<k_{\rm UV},\boldsymbol{q}\neq\boldsymbol{k}} \varrho_{{\rm S}\boldsymbol{q}}(\eta) \tag{D.9}$$

$$= -i\int_{k<k_{\rm UV}} \mathrm{d}^3\boldsymbol{k}\, \left[\mathcal{H}_{{\rm S}\boldsymbol{k}}(\eta), \varrho_{{\rm S}\boldsymbol{k}}(\eta)\right] \bigotimes_{q<k_{\rm UV},\boldsymbol{q}\neq\boldsymbol{k}} \varrho_{{\rm S}\boldsymbol{q}}(\eta)\,. \tag{D.10}$$

Using the above in the free Liouville equation yields eq. (D.1) with the desired volume factors.

E Scalar decoherence from a tensor environment

E.1 Correlation function in real space

Here we consider the $\zeta\gamma\gamma$ interaction,

$$S_{\rm int} \;=\; \frac{M_{\rm p}^2}{8}\int {\rm d}t\,{\rm d}^3\boldsymbol{x}\, a\,\varepsilon_1\,\zeta\partial_\ell\gamma_{ij}\partial_\ell\gamma_{ij} \tag{E.1}$$

which, after using $a\,{\rm d}\eta = {\rm d}t$ corresponds to the interaction Hamiltonian

$$\mathcal{H}_{\rm int}(\eta) \;=\; -\frac{M_{\rm p}^2\varepsilon_1}{8}a^2\int {\rm d}^3\boldsymbol{x}\,\zeta(\eta,\boldsymbol{x})\otimes\partial_\ell\gamma_{ij}(\eta,\boldsymbol{x})\partial_\ell\gamma_{ij}(\eta,\boldsymbol{x})\;. \tag{E.2}$$

Our interest is in how the environment of short-wavelength tensors decohering long-wavelength scalar fluctuations.

In terms of the canonical fields $v = aM_{\rm p}\sqrt{2\varepsilon_1}\,\zeta$ and $v_{ij} = \frac{1}{2}aM_{\rm p}\gamma_{ij}$ – see eq. (A.17) and using $a^{-1} = -H\eta$ – we have

$$\begin{aligned}\mathcal{H}_{\rm int}(\eta) \;&=\; -\frac{\sqrt{\varepsilon_1}}{2\sqrt{2}\,a(\eta)M_{\rm p}}\int {\rm d}^3\boldsymbol{x}\,v(\eta,\boldsymbol{x})\otimes\partial_\ell v_{ij}(\eta,\boldsymbol{x})\partial_\ell v_{ij}(\eta,\boldsymbol{x})\\ &=\; G(\eta)\int {\rm d}^3\boldsymbol{x}\,v(\eta,\boldsymbol{x})\otimes B_{\rm T}(\eta,\boldsymbol{x})\end{aligned} \tag{E.3}$$

where $G(\eta)$ is the same as that defined in eq. (3.3) and

$$B_{\rm T}(\eta,\boldsymbol{x}) \;:=\; \partial_\ell v_{ij}(\eta,\boldsymbol{x})\partial_\ell v_{ij}(\eta,\boldsymbol{x})\;. \tag{E.4}$$

Next we concern ourselves with the mode expansion for the graviton, noting that the free part of the graviton action is

$$\begin{aligned}{}^{(2)}S[\gamma] &= \frac{M_{\rm p}^2}{8}\int {\rm d}\eta\,{\rm d}^3\boldsymbol{x}\,a^2\left(\gamma'_{ij}\gamma'_{ij} - \partial_\ell\gamma_{ij}\partial_\ell\gamma_{ij}\right) = \frac{1}{2}\int {\rm d}\eta\,{\rm d}^3\boldsymbol{x}\left(v'_{ij}v'_{ij} - \partial_\ell v_{ij}\partial_\ell v_{ij} + \frac{2}{\eta^2}v_{ij}v_{ij}\right)\\ &= \frac{1}{2}\sum_{P=+,\times}\int {\rm d}\eta\,{\rm d}^3\boldsymbol{k}\left[|(v^P_{\boldsymbol{k}})'|^2 - \left(k^2 - \frac{2}{\eta^2}\right)|v^P_{\boldsymbol{k}}|^2\right]\end{aligned} \tag{E.5}$$

which uses $\dot\gamma_{ij} = a^{-1}\gamma'_{ij}$ as well as $a\gamma'_{ij} = \frac{2}{M_{\rm p}}(v'_{ij} + v_{ij}/\eta)$, and the expansion in terms of graviton modes, already introduced in eq. (A.18), and repeated here for convenience

$$v_{ij}(\eta,\boldsymbol{x}) = \int\frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\sum_{P=+,\times}\epsilon^P_{ij}(\boldsymbol{k})v^P_{\boldsymbol{k}}(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{E.6}$$

where we recall that $\epsilon^P_{ij}(\boldsymbol{k})$ is the polarization tensor. Let us explain how they are defined concretely. We here follow the conventions of ref. [75], up to numerical factors. For a given momentum vector $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$ pointing along the direction (θ,φ) in polar coordinates we define the vectors

$$\boldsymbol{e}^x(\hat{\boldsymbol{k}}) = \begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix},\qquad \boldsymbol{e}^y(\hat{\boldsymbol{k}}) \;=\; \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}\;. \tag{E.7}$$

These vectors are perpendicular to $\boldsymbol{k}$ as well as to each other where

$$\boldsymbol{e}^L(\hat{\boldsymbol{k}})\cdot\boldsymbol{e}^{L'}(\hat{\boldsymbol{k}}) \;=\; \delta_{LL'},\qquad \boldsymbol{k}\cdot\boldsymbol{e}^L(\hat{\boldsymbol{k}}) = 0,\qquad L,L'\in(x,y)\;, \tag{E.8}$$

and under reflection $\boldsymbol{k} \to -\boldsymbol{k}$ these satisfy (this means taking $\theta \to \pi - \theta$ and $\varphi \to \varphi + \pi$)

$$\boldsymbol{e}^x(-\hat{\boldsymbol{k}}) = \boldsymbol{e}^x(\hat{\boldsymbol{k}}), \qquad \boldsymbol{e}^y(-\hat{\boldsymbol{k}}) = -\boldsymbol{e}^y(\hat{\boldsymbol{k}}) \ . \tag{E.9}$$

Note that when $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$ with $\varphi = \theta = 0$ then the above reduce to $\boldsymbol{e}^x(\hat{\boldsymbol{z}}) = (1,0,0)$ and $\boldsymbol{e}^y(\hat{\boldsymbol{z}}) = (0,1,0)$. Next construct $\boldsymbol{\epsilon}^+ := \frac{1}{\sqrt{2}}(\boldsymbol{e}^x \otimes \boldsymbol{e}^x - \boldsymbol{e}^y \otimes \boldsymbol{e}^y)$ and $\boldsymbol{\epsilon}^\times := \frac{1}{\sqrt{2}}(\boldsymbol{e}^x \otimes \boldsymbol{e}^y + \boldsymbol{e}^y \otimes \boldsymbol{e}^x)$, or more simply in terms of components as:

$$\epsilon^+_{ij}(\boldsymbol{k}) := \frac{1}{\sqrt{2}}\left[e^x_i(\hat{\boldsymbol{k}})e^x_j(\hat{\boldsymbol{k}}) - e^y_i(\hat{\boldsymbol{k}})e^y_j(\hat{\boldsymbol{k}})\right], \tag{E.10}$$

$$\epsilon^\times_{ij}(\boldsymbol{k}) := \frac{1}{\sqrt{2}}\left[e^x_i(\hat{\boldsymbol{k}})e^y_j(\hat{\boldsymbol{k}}) + e^y_i(\hat{\boldsymbol{k}})e^x_j(\hat{\boldsymbol{k}})\right]. \tag{E.11}$$

These are the standard linear polarizations, which for the familiar case of $\boldsymbol{k} \propto \hat{\boldsymbol{z}}$ simplify to

$$\boldsymbol{\epsilon}^+(\hat{\boldsymbol{z}}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \boldsymbol{\epsilon}^\times(\hat{\boldsymbol{z}}) = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \ . \tag{E.12}$$

Note that these are normalized as

$$\epsilon^{P}_{ij}(\boldsymbol{k})\epsilon^{P'}_{ij}(\boldsymbol{k}) = \delta_{PP'} \ . \tag{E.13}$$

Importantly the symmetry under $\boldsymbol{k} \to -\boldsymbol{k}$ means that $\epsilon^+_{ij}(-\boldsymbol{k}) = \epsilon^+_{ij}(\boldsymbol{k})$ as well as $\epsilon^\times_{ij}(-\boldsymbol{k}) = -\epsilon^\times_{ij}(\boldsymbol{k})$. This implies that

$$\epsilon^+_{ij}(-\boldsymbol{k})\epsilon^+_{ij}(\boldsymbol{k}) = 1, \qquad \epsilon^\times_{ij}(-\boldsymbol{k})\epsilon^\times_{ij}(\boldsymbol{k}) = -1 \ . \tag{E.14}$$

Furthermore we have the identify, see Eq. (2.21) of ref. [76]

$$\sum_P \epsilon^P_{ij}(\boldsymbol{k})\epsilon^P_{nm}(\boldsymbol{k}) = \frac{1}{2}\left[\perp_{in}(\boldsymbol{k})\perp_{jm}(\boldsymbol{k}) + \perp_{im}(\boldsymbol{k})\perp_{jn}(\boldsymbol{k}) - \perp_{ij}(\boldsymbol{k})\perp_{nm}(\boldsymbol{k})\right] \tag{E.15}$$

where $\perp_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$ is a (symmetric) projection tensor such that $k_i \perp_{ij} = 0$.

As for scalars, the reality of $v_{ij}(\eta, \boldsymbol{x})$ implies that $v^{+*}_{\boldsymbol{k}}(\eta) = v^{+*}_{-\boldsymbol{k}}(\eta)$ and $v^{\times *}_{\boldsymbol{k}}(\eta) = -v^{\times *}_{-\boldsymbol{k}}(\eta)$. This has the same mode expansion as the Mukhanov-Sasaki field with an extra polarization label summing over $P = +, \times$. We then expand

$$v^P_{\boldsymbol{k}}(\eta) \;=\; u_{\boldsymbol{k}}(\eta)\, c^P_{\boldsymbol{k}} \;+\; s_P\, u^*_{\boldsymbol{k}}(\eta)\, c^{P\dagger}_{-\boldsymbol{k}} \tag{E.16}$$

with $s_P = 1$ for $P = +$ and $s_P = -1$ for $P = \times$, where $u_{\boldsymbol{k}}$ are the Bunch-Davies mode functions (same as for the scalar) and the ladder operators satisfy $[c^P_{\boldsymbol{k}}, c^{P'\dagger}_{\boldsymbol{k}}] = \delta(\boldsymbol{k} - \boldsymbol{q})\delta_{PP'}$ and so on. The above then implies that the operator $B_{\rm T}(\eta, \boldsymbol{x})$ has the expansion

$$B_{\rm T}(\eta, \boldsymbol{x}) \;:=\; -\sum_{P,P'=+,\times}\int_{k,q>k_{\rm UV}} \frac{{\rm d}^3\boldsymbol{k}\,{\rm d}^3\boldsymbol{q}}{(2\pi)^3}\,(\boldsymbol{k}\cdot\boldsymbol{q})\,\epsilon^P_{ij}(\boldsymbol{k})\epsilon^{P'}_{ij}(\boldsymbol{q})v^P_{\boldsymbol{k}}(\eta)v^{P'}_{\boldsymbol{q}}(\eta)e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x}} \ , \tag{E.17}$$

which can be used to simplify the one-point function,

$$\mathscr{B}_{\rm T}(\eta) := \langle 0_B\,|B_{\rm T}(\eta, \boldsymbol{x})|\,0_B\rangle \tag{E.18}$$

$$= - \sum_{P,P'=+,\times} \int_{k,q>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \left(\boldsymbol{k}\cdot\boldsymbol{q}\right) \epsilon^{P}_{ij}(\boldsymbol{k})\epsilon^{P'}_{ij}(\boldsymbol{q}) s_{P'} u_{\boldsymbol{k}}(\eta)u^*_{\boldsymbol{q}}(\eta)\delta(\boldsymbol{k}+\boldsymbol{q})\delta_{PP'}e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x}} \tag{E.19}$$

$$= 2\int_{k>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} k^2 \left|u_{\boldsymbol{k}}(\eta)\right|^2 \ , \tag{E.20}$$

where we have used $\epsilon^{P}_{ij}(-\boldsymbol{k})s_P = \epsilon^{P}_{ij}(\boldsymbol{k})$ and then $\epsilon^{P}_{ij}(\boldsymbol{k})\epsilon^{P}_{ij}(\boldsymbol{k}) = 1$ for each $P = +, \times$. Using the one-point function $\mathscr{B}(\eta)$ defined in eq. (B.2), we find that

$$\mathscr{B}_{\rm T}(\eta) = 2\,\mathscr{B}(\eta)\ . \tag{E.21}$$

Let us now turn to the calculation of the two-point correlation function. Using the expectation values

$$\left\langle 0_B \left| c^{P}_{\boldsymbol{k}} c^{P'\dagger}_{-\boldsymbol{q}} c^{P''}_{\boldsymbol{p}} c^{P'''\dagger}_{-\boldsymbol{\ell}} \right| 0_B \right\rangle = \delta(\boldsymbol{k}+\boldsymbol{q})\delta(\boldsymbol{p}+\boldsymbol{\ell})\delta_{PP'}\delta_{P''P'''}, \tag{E.22}$$

$$\left\langle 0_B \left| c^{P}_{\boldsymbol{k}} c^{P'}_{\boldsymbol{q}} c^{P''\dagger}_{-\boldsymbol{p}} c^{P'''\dagger}_{-\boldsymbol{\ell}} \right| 0_B \right\rangle = \delta(\boldsymbol{k}+\boldsymbol{\ell})\delta(\boldsymbol{p}+\boldsymbol{q})\delta_{PP'''}\delta_{P'P''} + \delta(\boldsymbol{k}+\boldsymbol{p})\delta(\boldsymbol{\ell}+\boldsymbol{q})\delta_{PP''}\delta_{P'P'''} \tag{E.23}$$

this quantity simplifies to

$$\begin{aligned}\left\langle 0_B \left| B_{\rm T}(\eta,\boldsymbol{x})B_{\rm T}(\eta',\boldsymbol{x}')\right| 0_B \right\rangle = \mathscr{B}_{\rm T}(\eta)\mathscr{B}_{\rm T}(\eta') + 2\sum_{P,P',P'',P'''}\int_{k,q,p,\ell>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^6} \\ \times \epsilon^{P}_{ij}(\boldsymbol{k})\epsilon^{P'}_{ij}(\boldsymbol{q})\epsilon^{P''}_{nm}(\boldsymbol{p})\epsilon^{P'''}_{nm}(\boldsymbol{\ell})\left(\boldsymbol{k}\cdot\boldsymbol{q}\right)(\boldsymbol{p}\cdot\boldsymbol{\ell})e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{x}+i(\boldsymbol{p}+\boldsymbol{\ell})\cdot\boldsymbol{x}'} \\ \times u_{\boldsymbol{k}}(\eta)u_{\boldsymbol{q}}(\eta)u^*_{\boldsymbol{p}}(\eta')u^*_{\boldsymbol{\ell}}(\eta')s_{P''}s_{P'''}\delta(\boldsymbol{k}+\boldsymbol{\ell})\delta(\boldsymbol{p}+\boldsymbol{q})\delta_{PP'''}\delta_{P'P''}\ .\end{aligned} \tag{E.24}$$

Our interest is in the centred two-point function, defined by

$$\begin{aligned}C_{\rm T}(\eta,\eta';\boldsymbol{x}-\boldsymbol{x}') &:= \left\langle 0_B \left| \left[B_{\rm T}(\eta,\boldsymbol{x}) - \mathscr{B}_{\rm T}(\eta)\right]\left[B_{\rm T}(\eta',\boldsymbol{x}') - \mathscr{B}_{\rm T}(\eta')\right]\right| 0_B \right\rangle \\ &= \left\langle 0_B \left| B_{\rm T}(\eta,\boldsymbol{x})B_{\rm T}(\eta',\boldsymbol{x}')\right| 0_B \right\rangle - \mathscr{B}_{\rm T}(\eta)\mathscr{B}_{\rm T}(\eta')\ ,\end{aligned} \tag{E.25}$$

which after using eq. (E.24), integrating over the δ-functions and using $s_P\epsilon^{P}_{nm}(-\boldsymbol{k}) = \epsilon^{P}_{nm}(\boldsymbol{k})$ gives rise to

$$\begin{aligned}C_{\rm T}(\eta,\eta';\boldsymbol{y}) = \sum_{P,P'}\int_{k,q>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^6}\,\epsilon^{P}_{ij}(\boldsymbol{k})\epsilon^{P'}_{ij}(\boldsymbol{q})\epsilon^{P'}_{nm}(\boldsymbol{q})\epsilon^{P}_{nm}(\boldsymbol{k})\,2\left(\boldsymbol{k}\cdot\boldsymbol{q}\right)^2 \\ \times u_{\boldsymbol{k}}(\eta)u_{\boldsymbol{q}}(\eta)u^*_{\boldsymbol{q}}(\eta')u^*_{\boldsymbol{k}}(\eta')e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{y}}\ .\end{aligned} \tag{E.26}$$

Using the identity (E.15) involving the projection $\perp_{ij}(\boldsymbol{k}) := \delta_{ij} - k_ik_j/k^2$ the summations over polarizations can be simplified to

$$\begin{aligned}C_{\rm T}(\eta,\eta';\boldsymbol{y}) = \int_{k,q>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{q}}{(2\pi)^6}\,\frac{1}{2}\left[\perp_{in}(\boldsymbol{k})\perp_{jm}(\boldsymbol{k}) + \perp_{im}(\boldsymbol{k})\perp_{jn}(\boldsymbol{k}) - \perp_{ij}(\boldsymbol{k})\perp_{nm}(\boldsymbol{k})\right] \\ \times\frac{1}{2}\left[\perp_{in}(\boldsymbol{q})\perp_{jm}(\boldsymbol{q}) + \perp_{im}(\boldsymbol{q})\perp_{jn}(\boldsymbol{q}) - \perp_{ij}(\boldsymbol{q})\perp_{nm}(\boldsymbol{q})\right]2\left(\boldsymbol{k}\cdot\boldsymbol{q}\right)^2 \\ \times u_{\boldsymbol{k}}(\eta)u_{\boldsymbol{q}}(\eta)u^*_{\boldsymbol{q}}(\eta')u^*_{\boldsymbol{k}}(\eta')e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{y}}\ .\end{aligned} \tag{E.27}$$

There is a sum over the indices i, j, n, m, and so to this end we note that

$$\perp_{in}(\boldsymbol{k})\perp_{in}(\boldsymbol{q}) = 1 + \frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{k^2q^2}, \tag{E.28}$$

$$\perp_{in}(\boldsymbol{k})\perp_{in}(\boldsymbol{q})\perp_{jm}(\boldsymbol{k})\perp_{jm}(\boldsymbol{q}) = \left[1+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{k^2q^2}\right]^2 , \tag{E.29}$$

$$\perp_{in}(\boldsymbol{k})\perp_{im}(\boldsymbol{q})\perp_{jm}(\boldsymbol{k})\perp_{jn}(\boldsymbol{q}) = 1+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^4}{k^4q^4}, \tag{E.30}$$

which are the only two types of contractions that occur in the above (after re-labeling i, j, n, m in various ways). After some manipulation the above implies that

$$C_{\rm T}(\eta,\eta';\boldsymbol{y}) = \int_{k,q>k_{\rm UV}} \frac{{\rm d}^3\boldsymbol{k}\,{\rm d}^3\boldsymbol{q}}{(2\pi)^6}\,\frac{1}{4}\left[1+6\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2}{k^2q^2}+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^4}{k^4q^4}\right]2(\boldsymbol{k}\cdot\boldsymbol{q})^2 \\ \times u_{\boldsymbol{k}}(\eta)u_{\boldsymbol{q}}(\eta)u^*_{\boldsymbol{q}}(\eta')u^*_{\boldsymbol{k}}(\eta')\,e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{y}}\ . \tag{E.31}$$

E.2 Fourier Transform of $C_{\rm T}$

What appears in the Lindblad equation for this interaction is of course the Fourier transform of the above correlator which we define as

$$\mathscr{T}_{\boldsymbol{k}}(\eta,\eta') := \int\frac{{\rm d}^3\boldsymbol{y}}{(2\pi)^{3/2}}\;C_{\rm T}(\eta,\eta';\boldsymbol{y})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\ . \tag{E.32}$$

We again are restricted to modes with $0<k<k_{\rm UV}$ and using eq. (E.31) we find

$$\mathscr{T}_{\boldsymbol{k}}(\eta,\eta') = \frac{2}{(2\pi)^{9/2}}\int_{q,p>k_{\rm UV}}{\rm d}^3\boldsymbol{q}\,{\rm d}^3\boldsymbol{p}\,(\boldsymbol{q}\cdot\boldsymbol{p})^2\left[\frac{1}{4}+\frac{3}{2}\frac{(\boldsymbol{q}\cdot\boldsymbol{p})^2}{q^2p^2}+\frac{1}{4}\frac{(\boldsymbol{q}\cdot\boldsymbol{p})^4}{q^4p^4}\right] \\ \times u_{\boldsymbol{q}}(\eta)\,u_{\boldsymbol{p}}(\eta)\,u^*_{\boldsymbol{p}}(\eta')\,u^*_{\boldsymbol{q}}(\eta')\,\delta(\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{k})\,. \tag{E.33}$$

From here the integration over the angles goes over exactly in the same manner as in Appendix B.2, eventually giving rise to the expression

$$\mathscr{T}_{\boldsymbol{k}}(\eta,\eta') = \frac{1}{32(2\pi)^{7/2}k}\int_0^k{\rm d}Q\int_{Q+2k_{\rm UV}}^{\infty}{\rm d}P\left[\frac{1}{4}+\frac{3}{2}\frac{(P^2+Q^2-2k^2)^2}{(P^2-Q^2)^2}+\frac{1}{4}\frac{(P^2+Q^2-2k^2)^4}{(P^2-Q^2)^4}\right] \\ \times\left(P^2+Q^2-2k^2\right)^2\left[1-\frac{2i}{(P-Q)\eta}\right]\left[1+\frac{2i}{(P-Q)\eta'}\right]\left[1-\frac{2i}{(P+Q)\eta}\right] \\ \times\left[1+\frac{2i}{(P+Q)\eta'}\right]e^{-i(\eta-\eta')P}. \tag{E.34}$$

E.3 Super-Hubble limit of Lindblad coefficient

Next we must compute the super-Hubble limit of the Lindblad coefficient

$$\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) := (2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta}{\rm d}\eta'\;G(\eta)G(\eta')\left[\mathscr{C}_{\boldsymbol{k}}(\eta,\eta')+\mathscr{T}_{\boldsymbol{k}}(\eta,\eta')\right]\ . \tag{E.35}$$

We use the earlier representations (B.31) and (E.34) of $\mathcal{C}_{\boldsymbol{k}}$ and $\mathscr{T}_{\boldsymbol{k}}$ above, as well as $G(\eta)G(\eta') = \varepsilon_1H^2\eta\eta'/(8M_{\rm p}^2)$, which expresses $\mathfrak{F}_{\boldsymbol{k}}$ as the triple integral

$$\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = \frac{\varepsilon_1H^2}{1024\pi^2M_{\rm p}^2k}\int_{\eta_{\rm in}}^{\eta}{\rm d}\eta'\int_0^k{\rm d}Q\int_{Q+2k_{\rm UV}}^{\infty}{\rm d}P\,\eta\,\eta'\left[\frac{5}{4}+\frac{3}{2}\frac{(P^2+Q^2-2k^2)^2}{(P^2-Q^2)^2}\right. \\ \left.+\frac{1}{4}\frac{(P^2+Q^2-2k^2)^4}{(P^2-Q^2)^4}\right]\left(P^2+Q^2-2k^2\right)^2\left[1-\frac{2i}{(P-Q)\eta}\right]\left[1+\frac{2i}{(P-Q)\eta'}\right]$$

$$\times \left[1 - \frac{2i}{(P+Q)\eta}\right] \left[1 + \frac{2i}{(P+Q)\eta'}\right] e^{-iP(\eta-\eta')} \,. \tag{E.36}$$

We first evaluate the η'-integral using the formula (C.3),

$$\int_{\eta_{\rm in}}^{\eta} {\rm d}\eta'\, \eta' \left[1 + \frac{2i}{(P-Q)\eta'}\right] \left[1 + \frac{2i}{(P+Q)\eta'}\right] e^{-i(\eta-\eta')P}$$
$$= \left[e^{-iP(\eta-\eta')} \left(\frac{1}{P^2} - \frac{i\eta'}{P} + \frac{4}{P^2-Q^2} \right) - \frac{4e^{-iP\eta}{\rm Ei}(iP\eta')}{P^2-Q^2} \right] \Bigg|_{\eta'\to\eta_{\rm in}}^{\eta'\to\eta} \tag{E.37}$$
$$= \frac{1}{P^2} - \frac{i\eta}{P} + \frac{4}{P^2-Q^2} - \frac{4e^{-iP\eta}\left[{\rm Ei}(iP\eta) + i\pi\right]}{P^2-Q^2} - e^{-iP(\eta-\eta_{\rm in})}\left(\frac{1}{P^2} - \frac{i\eta_{\rm in}}{P} + \frac{4}{P^2-Q^2} \right)$$
$$+ \frac{4e^{-iP\eta}\left[{\rm Ei}(iP\eta_{\rm in}) + i\pi\right]}{P^2-Q^2} \,, \tag{E.38}$$

where we have added and subtracted an extra factor of $4i\pi e^{-iP\eta}/(P^2-Q^2)$ for convenience later on. With this we find that

$$\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) = \frac{\varepsilon_1 H^2}{1024\pi^2 M_{\rm p}^2} \left[h(\eta) + g(\eta, \eta_{\rm in})\right] \tag{E.39}$$

with the definitions

$$h(\eta) := \int_0^k {\rm d}Q \int_{Q+2k_{\rm UV}}^{\infty} {\rm d}P\, \eta \left[\frac{5}{4} + \frac{3}{2}\frac{(P^2+Q^2-2k^2)^2}{(P^2-Q^2)^2} + \frac{1}{4}\frac{(P^2+Q^2-2k^2)^4}{(P^2-Q^2)^4} \right]$$
$$\times \frac{(P^2+Q^2-2k^2)^2}{k} \left[1 - \frac{2i}{(P-Q)\eta}\right] \left[1 - \frac{2i}{(P+Q)\eta}\right] \left\{ \frac{1}{P^2} - \frac{i\eta}{P} + \frac{4}{P^2-Q^2} \right.$$
$$\left. - \frac{4e^{-iP\eta}\left[{\rm Ei}(iP\eta) + i\pi\right]}{P^2-Q^2} \right\} , \tag{E.40}$$

$$g(\eta, \eta_{\rm in}) := \int_0^k {\rm d}Q \int_{Q+2k_{\rm UV}}^{\infty} {\rm d}P\, \eta \left[\frac{5}{4} + \frac{3}{2}\frac{(P^2+Q^2-2k^2)^2}{(P^2-Q^2)^2} + \frac{1}{4}\frac{(P^2+Q^2-2k^2)^4}{(P^2-Q^2)^4} \right]$$
$$\times \frac{(P^2+Q^2-2k^2)^2}{k} \left[1 - \frac{2i}{(P-Q)\eta}\right] \left[1 - \frac{2i}{(P+Q)\eta}\right] \left\{ - e^{-iP(\eta-\eta_{\rm in})} \left(\frac{1}{P^2} \right.\right.$$
$$\left.\left. - \frac{i\eta_{\rm in}}{P} + \frac{4}{P^2-Q^2} \right) + \frac{4e^{-iP\eta}\left[{\rm Ei}(iP\eta_{\rm in}) + i\pi\right]}{P^2-Q^2} \right\} . \tag{E.41}$$

Our goal will be write down the $0 \ll -k\eta \ll -k\eta_{\rm in} \ll 1$ limit of $\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})$.

Let us now study the super-Hubble limit of h. As before, we make use of the variables $z = -k\eta$, $z_{\rm in} = -k\eta_{\rm in}$ and $\kappa := k_{\rm UV}/k$ and change the (positive) integration variables to $x = -Q\eta$ and $y = -P\eta$ which turns eq. (E.40) into

$$h(\eta) := \int_0^z {\rm d}x\, \frac{f(x,z)}{z^3} \tag{E.42}$$

with f defined by

$$f(x,z) := \int_{x+2\kappa z}^{\infty} {\rm d}y \left[\frac{5}{4} + \frac{3}{2}\frac{(x^2+y^2-2z^2)^2}{(x^2-y^2)^2} + \frac{1}{4}\frac{(x^2+y^2-2z^2)^4}{(x^2-y^2)^4} \right] (x^2+y^2-2z^2)^2$$

$$\times \left(1-\frac{2i}{x-y}\right)\left(1+\frac{2i}{x+y}\right)\left\{-\frac{1}{y^2}-\frac{i}{y}+\frac{4}{x^2-y^2}-\frac{4e^{iy}\left[\mathrm{Ei}(-iy)+i\pi\right]}{x^2-y^2}\right\}. \tag{E.43}$$

We next notice the Taylor series about $z=0$,

$$\int_0^z \mathrm{d}x\, f(x,z) \simeq\; f(0,0)z \;+\; \mathcal{O}(z^{-1})\,, \tag{E.44}$$

for $0<z\ll 1$, which implies that f contributes to h in eq. (E.42) at order z^{-2} for small z, such that

$$h(\eta) \simeq \frac{f(0,0)}{z^2} + \mathcal{O}(z^{-1}) \tag{E.45}$$

with coefficient through eq. (E.43) given as

$$f(0,0) := 3\int_0^\infty \mathrm{d}y\; y^4\left(1+\frac{2i}{y}\right)^2\left\{-\frac{5+iy}{y^2}+\frac{4e^{iy}}{y^2}\left[\mathrm{Ei}(-iy)+i\pi\right]\right\}\;. \tag{E.46}$$

In order to compare to the earlier section, let us focus on the real part of the above where

$$\begin{aligned}\mathrm{Re}\left[f(0,0)\right] = 3\int_0^\infty \mathrm{d}y\bigg(&20-y^2+4(y^2-4)\,\mathrm{Re}\left\{e^{iy}\left[\mathrm{Ei}(-iy)+i\pi\right]\right\}\\ &-16y\,\mathrm{Im}\left\{e^{iy}\left[\mathrm{Ei}(-iy)+i\pi\right]\right\}\bigg)\,.\end{aligned} \tag{E.47}$$

We next use (for $y>0$)

$$e^{iy}\left[\mathrm{Ei}(-iy)+i\pi\right] = \mathrm{Ci}(y)\cos y + \left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\sin(y) + i\left\{\mathrm{Ci}(y)\sin y - \left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\cos y\right\}\,, \tag{E.48}$$

where the functions $\mathrm{Ci}(y)$ and $\mathrm{Si}(y)$ are defined by

$$\mathrm{Ci}(y) \;=\; -\int_y^\infty \mathrm{d}t\;\frac{\cos t}{t}, \qquad \mathrm{Si}(y) \;=\; \int_0^y \mathrm{d}t\;\frac{\sin t}{t}\,, \tag{E.49}$$

to write

$$\begin{aligned}\mathrm{Re}\left[f(0,0)\right] = 3\int_0^\infty \mathrm{d}y\,\bigg(&-y^2+20+4(y^2-4)\left\{\mathrm{Ci}(y)\cos y+\left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\sin y\right\}\\ &-16y\left\{\mathrm{Ci}(y)\sin y-\left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\cos y\right\}\bigg)\;.\end{aligned} \tag{E.50}$$

For $y\gg 1$ the integrand in this expression behaves as

$$\begin{aligned}&-y^2+20+4(y^2-4)\left\{\mathrm{Ci}(y)\cos y+\left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\sin y\right\}-16y\left\{\mathrm{Ci}(y)\sin y-\left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\cos y\right\}\\ &\simeq -y^2+\frac{72}{y^2}+\mathcal{O}\left(y^{-4}\right)\end{aligned} \tag{E.51}$$

and so the integral diverges in the UV. A similar divergence also arose when summing over scalar fluctuations, and corresponded to a distributional singularity in the correlation function near $\eta=\eta'$, see the discussion surrounding eq. (C.22). The distributional singularities do not contribute to the integration over η' performed here, and for the present purposes it is convenient to have an alternative approach that provides a short-cut to the small-z behaviour without fully evaluating the position-space correlation function for all values of its arguments. We adopt here a regularization of this divergence that properly reproduces the small-z behaviour found in the more complete treatment of Appendix C.

To this end we regulate the divergence by first isolating the UV-divergent part $\propto y^2$ of the function, leading to

$$\mathrm{Re}\,[f(0,0)] = -3\int_0^\infty \mathrm{d}y\, y^2 + 3\int_0^\infty \mathrm{d}y \left(20 - 4(y^2-4)\left\{\mathrm{Ci}(y)\cos y + \left[\mathrm{Si}(y) - \frac{\pi}{2}\right]\sin y\right\} - 16y\left\{\mathrm{Ci}(y)\sin(y) - \left[\mathrm{Si}(y) - \frac{\pi}{2}\right]\cos y\right\}\right). \tag{E.52}$$

We regulate the divergent integral $\int_0^\infty \mathrm{d}y\; y^2$ in the spirit of dimensional regularization by writing

$$\int_0^\infty \mathrm{d}y\, y^2 \to \lim_{q\to 0^+}\lim_{n\to 0}\int_0^\infty \mathrm{d}y\, y^n\left(\frac{y^4}{y^2+q^2}\right) = \lim_{q\to 0^+}\lim_{n\to 0}\left[\frac{\pi q^{n+3}}{2\cos(n\pi/2)}\right] = 0, \tag{E.53}$$

where the initial integral only converges when $\mathrm{Re}(n) < -5$ and we introduce a parameter $q > 0$ to regulate the associated divergence in the IR (whose presence ultimately cancels the UV divergence). This leaves the convergent integral

$$\begin{aligned}\mathrm{Re}\,[f^{\mathrm{reg}}(0,0)] &= 3\int_0^\infty \mathrm{d}y\left(20 + 4(y^2-4)\left\{\mathrm{Ci}(y)\cos y + \left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\sin y\right\}\right.\\ &\quad\left. - 16y\left\{\mathrm{Ci}(y)\sin y - \left[\mathrm{Si}(y)-\frac{\pi}{2}\right]\cos y\right\}\right) &\text{(E.54)}\\ &= \left\{-4y + 4\mathrm{Ci}(y)\left[6y\cos y + (y^2-10)\sin y\right] - 4\left[\left(y^2-10\right)\cos y - 6y\sin y\right]\right.\\ &\quad\left.\times\left[\mathrm{Si}(y) - \frac{\pi}{2}\right]\right\}\Bigg|_{y\to 0}^{y\to\infty} = 60\pi &\text{(E.55)}\end{aligned}$$

which is the result used in eq. (E.45).

Let us finally compute the super-Hubble behaviour of g defined in eq. (E.41). We again use the variables z and z_{in} and we here argue that

$$g(\eta,\eta_{\mathrm{in}}) \;\sim\; \mathcal{O}(z^{-1}) \tag{E.56}$$

in the $z \ll 1$ limit and so is subdominant to h. To see why it is easier to use the integration $a := Q/k$ and $b := P/k$ which turns eq. (E.41) into

$$\begin{aligned}g(\eta,\eta_{\mathrm{in}}) := \int_0^1 \mathrm{d}a\int_{a+2\kappa}^\infty \mathrm{d}b\left[\frac{5}{4} + \frac{3}{2}\frac{(a^2+b^2-2)^2}{(a^2-b^2)^2} + \frac{1}{4}\frac{(a^2+b^2-2)^4}{(a^2-b^2)^4}\right](a^2+b^2-2)^2 z\\ \times\left[1 - \frac{2i}{(a-b)z}\right]\left[1+\frac{2i}{(a+b)z}\right]\left\{e^{ib(z-z_{\mathrm{in}})}\left(\frac{1}{b^2} + \frac{iz_{\mathrm{in}}}{b} - \frac{4}{a^2-b^2}\right)\right.\\ \left. + \frac{4e^{izb}\left[\mathrm{Ei}(-iz_{\mathrm{in}}b) + i\pi\right]}{a^2-b^2}\right\}.\end{aligned} \tag{E.57}$$

We then write the above as

$$g(z,z_{\mathrm{in}}) := \int_0^1 \mathrm{d}a\,\mathscr{G}(a,z,z_{\mathrm{in}}) \tag{E.58}$$

where we have

$$\mathscr{G}(a,z,z_{\rm in}) := \int_{a+2\kappa}^{\infty} {\rm d}b\,\left[\frac{\chi_{-1}(a,b,z_{\rm in})}{z} + \chi_0(a,b,z_{\rm in}) + \chi_{+1}(a,b,z_{\rm in})z\right] e^{ibz} \tag{E.59}$$

with the definitions:

$$\begin{aligned}\chi_{-1}(a,b,z_{\rm in}) &:= \frac{4}{a^2-b^2}\left[\frac{5}{4}+\frac{3}{2}\frac{(a^2+b^2-2)^2}{(a^2-b^2)^2}+\frac{1}{4}\frac{(a^2+b^2-2)^4}{(a^2-b^2)^4}\right](a^2+b^2-2)^2\\ &\quad\times\left\{e^{-ibz_{\rm in}}\left(\frac{1}{b^2}+\frac{iz_{\rm in}}{b}-\frac{4}{a^2-b^2}\right)+\frac{4\,[{\rm Ei}(-iz_{\rm in}b)+i\pi]}{a^2-b^2}\right\}\end{aligned} \tag{E.60}$$

$$\begin{aligned}\chi_{0}(a,b,z_{\rm in}) &:= -\frac{4ib}{a^2-b^2}\left[\frac{5}{4}+\frac{3}{2}\frac{(a^2+b^2-2)^2}{(a^2-b^2)^2}+\frac{1}{4}\frac{(a^2+b^2-2)^4}{(a^2-b^2)^4}\right](a^2+b^2-2)^2\\ &\quad\times\left\{e^{-ibz_{\rm in}}\left(\frac{1}{b^2}+\frac{iz_{\rm in}}{b}-\frac{4}{a^2-b^2}\right)+\frac{4\,[{\rm Ei}(-iz_{\rm in}b)+i\pi]}{a^2-b^2}\right\}\end{aligned} \tag{E.61}$$

$$\begin{aligned}\chi_{+1}(a,b,z_{\rm in}) &:= \left[\frac{5}{4}+\frac{3}{2}\frac{(a^2+b^2-2)^2}{(a^2-b^2)^2}+\frac{1}{4}\frac{(a^2+b^2-2)^4}{(a^2-b^2)^4}\right](a^2+b^2-2)^2\\ &\quad\times\left\{e^{-ibz_{\rm in}}\left(\frac{1}{b^2}+\frac{iz_{\rm in}}{b}-\frac{4}{a^2-b^2}\right)+\frac{4\,[{\rm Ei}(-iz_{\rm in}b)+i\pi]}{a^2-b^2}\right\}\end{aligned} \tag{E.62}$$

Since eq. (E.59) reveals that $\mathscr{G}$ is a Fourier transform, we note that its $0 < z \ll 1$ limit (for fixed $z_{\rm in}$ and κ) is governed by the $b \gg 1$ behaviour of the functions χ_j, in which limit we have:

$$\chi_{-1}(a,b,z_{\rm in}) \simeq e^{-ibz_{\rm in}}\left\{-12ib - 60z_{\rm in} + \mathcal{O}\left(b^{-1}\right)\right\}\,, \tag{E.63}$$

$$\chi_{0}(a,b,z_{\rm in}) \simeq e^{-ibz_{\rm in}}\left\{-12z_{\rm in}b^2 + 60ib + \left[\left(80-68a^2\right)z_{\rm in} + \frac{48}{z_{\rm in}}\right] + \mathcal{O}\left(b^{-1}\right)\right\}\,, \tag{E.64}$$

$$\chi_{+1}(a,b,z_{\rm in}) \simeq e^{-ibz_{\rm in}}\left\{3iz_{\rm in}b^3 + 15b^2 + \frac{2i\left[\left(7a^2-10\right)z_{\rm in}^2-6\right]}{z_{\rm in}}b + \frac{12}{z_{\rm in}^2} + 82a^2 - 100 + \mathcal{O}\left(b^{-1}\right)\right\}\,. \tag{E.65}$$

To derive the required asymptotics of each of the above Fourier transforms, we write

$$\int_{a+2\kappa}^{\infty} {\rm d}b\,\chi_{-1}(a,b,z_{\rm in})e^{ibz} = \int_{a+2\kappa}^{\infty} {\rm d}b\,e^{-ibz_{\rm in}}\left(-12ib-60z_{\rm in}\right)e^{ibz} + \Xi_{-1}(z,a,z_{\rm in}) \tag{E.66}$$

with the definition

$$\Xi_{-1}(z,a,z_{\rm in}) := \int_{a+2\kappa}^{\infty} {\rm d}b\,\left[\chi_{-1}(a,b,z_{\rm in}) - e^{-ibz_{\rm in}}\left(-12ib-60z_{\rm in}\right)\right]e^{ibz}\,. \tag{E.67}$$

First note that $\Xi_{-1}(z,a,z_{\rm in}) \sim \mathcal{O}(z^0)$ in the $z \ll 1$ limit, which follows from the fact that its integrand converges to a z-independent constant when $z \to 0^+$. What remains then is to evaluate the distribution

$$\begin{aligned}\int_{a+2\kappa}^{\infty} {\rm d}b\,e^{-ibz_{\rm in}}\left(-12ib-60z_{\rm in}\right)e^{ibz} &= e^{-i(z_{\rm in}-z)(a+2\kappa)}\left[\frac{12i}{(z_{\rm in}-z)^2} - \frac{12\left(a+2\kappa-5i\right)}{z_{\rm in}-z}\right]\\ &\quad + 12\pi\delta'(z_{\rm in}-z) - 60\pi z_{\rm in}\delta(z_{\rm in}-z)\,,\end{aligned} \tag{E.68}$$

which follows from eqs. (B.49) and (B.50). Fixing a and $z_{\rm in}$ in the above, and taking the $0 < z \ll 1$ limits then implies that (also assuming that $\kappa z \ll 1$)

$$\int_{a+2\kappa}^{\infty} {\rm d}b\,e^{-ibz_{\rm in}}\left(-12ib-60z_{\rm in}\right)e^{+ibz} \simeq e^{-iz_{\rm in}(a+2\kappa)}\left[\frac{12i}{z_{\rm in}^2} - \frac{12\left(a+2\kappa-5i\right)}{z_{\rm in}}\right] + \mathcal{O}\left(z\right)\,, \tag{E.69}$$

which implies that

$$\int_{a+2\kappa}^{\infty} \mathrm{d}b\, \chi_{-1}(a,b,z_{\rm in}) e^{ibz} \sim \mathcal{O}\left(z^0\right) \, . \tag{E.70}$$

Similar computations show that in the super-Hubble limit $0 < z \ll 1$ with $\kappa z \ll 1$, $\int_{a+2\kappa}^{\infty} \mathrm{d}b \;\; \chi_{0,+1}(a,b,z_{\rm in})e^{ibz} \sim \mathcal{O}(z^0)$, which when combined with eq. (E.59) show that $\mathscr{G}(a,z,z_{\rm in}) \sim \mathcal{O}(z^{-1})$ in the same limit. Integrating from $a=0$ to $a=1$ in eq. (E.58) leaves the same z-dependence and so we conclude that $g(z,z_{\rm in}) \sim \mathcal{O}(z^{-1})$.

Combining the above dependences for h and g in eq. (E.39) implies

$$\mathfrak{F}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \;\simeq\; \frac{\varepsilon_1 H^2 k^2}{1024\pi^2 M_{\rm p}^2} \frac{60\pi}{(-k\eta)^2} + \mathcal{O}\left[(-k\eta)^{-1}\right] \tag{E.71}$$

in the super-Hubble limit, as claimed in the main text, see eq. (3.61).

F Tensor decoherence from a scalar environment

F.1 General considerations

Here we consider the interaction Hamiltonian (5.2)

$$\mathcal{H}_{\rm int}(\eta) = \tilde{G}(\eta) \int \mathrm{d}^3\boldsymbol{x}\; v_{ij}(\eta,\boldsymbol{x}) \otimes B^{ij}(\eta,\boldsymbol{x})\,, \tag{F.1}$$

with $\tilde{G} = -(2M_{\rm p}a)^{-1}$ defined in eq. (5.4), v_{ij} the canonical tensor mode for the system modes (related to γ_{ij} by eq. (A.17)) and the environmental operator $B^{ij}(\boldsymbol{x}) = \delta^{ik}\delta^{jl}\partial_k v(\eta,\boldsymbol{x})\partial_l v(\eta,\boldsymbol{x})$ defined in eq. (5.3).

Using an analogous setup as that used for the scalar, the Nakajima-Zwanzig equation for the reduced density matrix (now for the graviton) at second-order in $\tilde{G}$ is given by

$$\begin{aligned}
\frac{\partial \varrho}{\partial \eta} \simeq & -i\tilde{G}(\eta)\mathscr{B}^{ia}(\eta) \int \mathrm{d}^3\boldsymbol{x}\, \Big[v_{ia}(\eta,\boldsymbol{x}), \varrho(\eta)\Big] \\
& - \int \mathrm{d}^3\boldsymbol{x} \int \mathrm{d}^3\boldsymbol{x}' \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta'\; \tilde{G}(\eta)\tilde{G}(\eta') \bigg\{ \Big[v_{ia}(\eta,\boldsymbol{x}), v_{jb}(\eta',\boldsymbol{x}')\varrho(\eta')\Big] C^{iajb}(\eta,\eta';\boldsymbol{x}-\boldsymbol{x}') \\
& + \Big[\varrho(\eta')v_{jb}(\eta',\boldsymbol{x}'), v_{ia}(\eta,\boldsymbol{x})\Big] C^{iajb*}(\eta,\eta';\boldsymbol{x}-\boldsymbol{x}') \bigg\}
\end{aligned} \tag{F.2}$$

cf. eq. (3.16), with the definitions

$$\mathscr{B}^{ia}(\eta) := \langle 0_B | B^{ia}(\eta,\boldsymbol{x}) | 0_B \rangle \tag{F.3}$$

and

$$C^{iajb}(\eta,\eta';\boldsymbol{x}-\boldsymbol{x}') = \langle 0_B \left| \left[B^{ia}(\eta,\boldsymbol{x}) - \mathscr{B}^{ia}(\eta) \right] \left[B^{jb}(\eta',\boldsymbol{x}') - \mathscr{B}^{jb}(\eta') \right] \right| 0_B \rangle \, . \tag{F.4}$$

Using the mode expansion of the scalar as usual one finds that

$$\mathscr{B}^{ia}(\eta) = \int_{k>k_{\rm UV}} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} k^i k^a |u_{\boldsymbol{k}}(\eta)|^2 \;=\; \frac{1}{3}\delta^{ia}\mathscr{B}(\eta) \tag{F.5}$$

with $\mathscr{B}$ given in eq. (3.17)[14] and

$$C^{iajb}(\eta,\eta';\boldsymbol{x}-\boldsymbol{x}') = 2\int_{q,p>k_{\rm UV}} \frac{{\rm d}^3\boldsymbol{q}\,{\rm d}^3\boldsymbol{p}}{(2\pi)^6}\; p^i q^a p^j q^b\, u_q(\eta)u_p(\eta)u_q^*(\eta')u_p^*(\eta')\, e^{i(\boldsymbol{q}+\boldsymbol{p})\cdot(\boldsymbol{x}-\boldsymbol{x}')}. \quad \text{(F.6)}$$

Using the mode expansion for v_{ij} given in eq. (A.18), repeated here for convenience

$$v_{ij}(\eta,\boldsymbol{x}) = \int \frac{{\rm d}^3\boldsymbol{k}}{(2\pi)^{3/2}} \sum_{P=+,\times} \epsilon^P_{ij}(\boldsymbol{k})v^P_{\boldsymbol{k}}(\eta)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\ , \quad \text{(F.7)}$$

one can re-write the above equation as

$$\begin{aligned}\frac{\partial\varrho}{\partial\eta} =& -(2\pi)^{3/2}\sum_{P,P'}\int_{k<k_{\rm UV}}{\rm d}^3\boldsymbol{k}\int_{\eta_{\rm in}}^{\eta}{\rm d}\eta'\ \tilde{G}(\eta)\tilde{G}(\eta')\epsilon^P_{ia}(\boldsymbol{k})\epsilon^{P'}_{jb}(-\boldsymbol{k})\\ &\times\left\{\left[v^P_{\boldsymbol{k}}(\eta), v^{P'}_{-\boldsymbol{k}}(\eta')\varrho(\eta')\right]\mathscr{C}^{iajb}_{-\boldsymbol{k}}(\eta,\eta') + \left[\varrho(\eta')v^{P'}_{-\boldsymbol{k}}(\eta'), v^P_{\boldsymbol{k}}(\eta)\right]\mathscr{C}^{iajb*}_{-\boldsymbol{k}}(\eta,\eta')\right\}, \quad \text{(F.8)}\end{aligned}$$

where we define the Fourier transform $\mathscr{C}^{iajb}_{\boldsymbol{k}}$ of the correlator $C^{iajb}(\eta,\eta';\boldsymbol{y})$ in the variable $\boldsymbol{y}$ where

$$\begin{aligned}\mathscr{C}^{iajb}_{\boldsymbol{k}}(\eta,\eta') =& \int\frac{{\rm d}^3\boldsymbol{y}}{(2\pi)^{3/2}}\ C^{iajb}(\eta,\eta';\boldsymbol{y})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\\ =&\frac{2}{(2\pi)^{9/2}}\int_{q,p>k_{\rm UV}}{\rm d}^3\boldsymbol{q}\,{\rm d}^3\boldsymbol{p}\; p^i q^a p^j q^b\, u_q(\eta)u_p(\eta)u^*_q(\eta')u^*_p(\eta')\,\delta\left(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k}\right)\ . \quad \text{(F.9)}\end{aligned}$$

From here we note that eq. (F.9) implies $\mathscr{C}^{iajb}_{-\boldsymbol{k}}(\eta,\eta') = \mathscr{C}^{iajb}_{\boldsymbol{k}}(\eta,\eta')$, and we also use the symmetry $\epsilon^{P'}_{jb}(-\boldsymbol{k})v^{P'}_{-\boldsymbol{k}}(\eta') = \epsilon^{P'}_{jb}(\boldsymbol{k})v^{P'*}_{\boldsymbol{k}}(\eta')$ for each polarization P' giving

$$\begin{aligned}\frac{\partial\varrho}{\partial\eta} =& -(2\pi)^{3/2}\sum_{P,P'}\int_{k<k_{\rm UV}}{\rm d}^3\boldsymbol{k}\int_{\eta_{\rm in}}^{\eta}{\rm d}\eta'\ \tilde{G}(\eta)\tilde{G}(\eta')\,\epsilon^P_{ia}(\boldsymbol{k})\epsilon^{P'}_{jb}(\boldsymbol{k})\\ &\times\left\{\left[v^P_{\boldsymbol{k}}(\eta), v^{P'\dagger}_{\boldsymbol{k}}(\eta')\varrho(\eta')\right]\mathscr{C}^{iajb}_{\boldsymbol{k}}(\eta,\eta') + \left[\varrho(\eta')v^{P'\dagger}_{\boldsymbol{k}}(\eta'), v^P_{\boldsymbol{k}}(\eta)\right]\mathscr{C}^{iajb*}_{\boldsymbol{k}}(\eta,\eta')\right\}. \quad \text{(F.10)}\end{aligned}$$

We next split the density matrix into a product over modes as before, with mode labels $\boldsymbol{k}$ and polarizations $P=+,\times$

$$\rho\ =\ \bigotimes_{\boldsymbol{k},P}\rho^P_{\boldsymbol{k}} \quad \text{(F.11)}$$

and repeating the arguments used in the scalar case leads to the separate evolution equation for each label:

$$\frac{\mathcal{V}}{(2\pi)^{3/2}}\frac{\partial\varrho^P_{\boldsymbol{k}}}{\partial\eta} = -(2\pi)^{3/2}\int_{\eta_{\rm in}}^{\eta}{\rm d}\eta'\tilde{G}(\eta)\tilde{G}(\eta')\left\{\mathscr{S}_{\boldsymbol{k}}(\eta,\eta')\left[\tilde{v}^P_{\boldsymbol{k}}(\eta),\tilde{v}^P_{\boldsymbol{k}}(\eta')\varrho^P_{\boldsymbol{k}}(\eta')\right] + {\rm h.c.}\right\}. \quad \text{(F.12)}$$

with $\tilde{v}$ a proxy for the real and imaginary parts of the field (as in the text below (2.21)) and where we define

$$\epsilon^P_{ia}(\boldsymbol{k})\epsilon^{P'}_{jb}(\boldsymbol{k})\ \mathscr{C}^{iajb}_{\boldsymbol{k}}(\eta,\eta') = \delta^{PP'}\mathscr{S}_{\boldsymbol{k}}(\eta,\eta')\,. \quad \text{(F.13)}$$

[14] Notice that since $\mathscr{B}^{ia}\propto\delta^{ia}$ then the contraction in the first term of eq. (F.2) vanishes since we assume the graviton is traceless, where $\delta^{ia}\gamma_{ia}=0$

F.2 Computing the correlation function

We first note that eq. (F.9) implies $\mathscr{C}_{\boldsymbol{k}}^{iajb}(\eta,\eta')$ has the symmetries $\mathscr{C}_{\boldsymbol{k}}^{iajb} = \mathscr{C}_{\boldsymbol{k}}^{jaib} = \mathscr{C}_{\boldsymbol{k}}^{ibja} = \mathscr{C}_{\boldsymbol{k}}^{aibj}$. The most general form possible consistent with these symmetries is

$$\begin{aligned}\mathscr{C}_{\boldsymbol{k}}^{iajb} = \frac{A}{2}\left(\delta^{ia}\delta^{jb}+\delta^{ib}\delta^{ja}\right) + B\,\delta^{ij}\delta^{ab} + \frac{C}{4}\left(\delta^{ia}k^jk^b+\delta^{ja}k^ik^b+\delta^{ib}k^jk^a+\delta^{jb}k^ik^a\right)\\ +\frac{D}{2}\left(\delta^{ij}k^ak^b+\delta^{ab}k^ik^j\right) + E\,k^ik^jk^ak^b\,,\end{aligned} \tag{F.14}$$

where the coefficients A through E are functions of η, η' and k^2. Contracting the above form with two polarization tensors as in eq. (F.13) yields

$$\epsilon^{P}_{ia}(\boldsymbol{k})\epsilon^{P'}_{jb}(\boldsymbol{k})\,\mathscr{C}_{\boldsymbol{k}}^{iajb}(\eta,\eta') = \delta^{PP'}\left(\frac{A}{2}+B\right)\,, \tag{F.15}$$

which uses $\epsilon^{P}_{ij}(\boldsymbol{k}) = \epsilon^{P}_{ji}(\boldsymbol{k})$, $\epsilon^{P}_{ij}(\boldsymbol{k})\epsilon^{P'}_{ij}(\boldsymbol{k}) = \delta^{PP'}$ and $\epsilon^{P}_{ii}(\boldsymbol{k}) = k^i\epsilon_{ij}(\boldsymbol{k}) = 0$, showing that

$$\mathscr{S}_{\boldsymbol{k}}(\eta,\eta') = \frac{A}{2}+B \tag{F.16}$$

is the function appearing in eq. (F.13). The required coefficients A and B can be computed by inverting the following five expressions for rotation-invariant integrals:

$$\delta_{ij}\delta_{ab}\mathscr{C}_{\boldsymbol{k}}^{iajb} = 3A+9B+k^2C+3k^2D+k^4E\,, \tag{F.17}$$

$$\delta_{ia}\delta_{jb}\mathscr{C}_{\boldsymbol{k}}^{iajb} = \mathscr{C}_{\boldsymbol{k}} = 6A+3B+2k^2C+k^2D+k^4E\,, \tag{F.18}$$

$$k_ik_j\delta_{ab}\mathscr{C}_{\boldsymbol{k}}^{iajb} = k^2A+3k^2B+k^4C+2k^4D+k^6E\,, \tag{F.19}$$

$$\delta_{ia}k_jk_b\mathscr{C}_{\boldsymbol{k}}^{iajb} = 2k^2A+k^2B+\frac{3}{2}k^4C+k^4D+k^6E\,, \tag{F.20}$$

and

$$k_ik_ak_jk_b\mathscr{C}_{\boldsymbol{k}}^{iajb} = k^4A+k^4B+k^6C+k^6D+k^8E\,. \tag{F.21}$$

Inverting these formulas gives the coefficients A through E in terms of integrals, and using the result in eq. (F.16) yields

$$\mathscr{S}_{\boldsymbol{k}}(\eta,\eta') = \frac{\delta_{ij}\delta_{ab}\mathscr{C}_{\boldsymbol{k}}^{iajb}}{4} - \frac{k_ik_j\delta_{ab}\mathscr{C}_{\boldsymbol{k}}^{iajb}}{2k^2} + \frac{k_ik_ak_jk_b\mathscr{C}_{\boldsymbol{k}}^{iajb}}{4k^4} \tag{F.22}$$

$$\begin{aligned}=\frac{2}{(2\pi)^{9/2}}\int_{q,p>k_{\rm UV}} {\rm d}^3\boldsymbol{q}\,{\rm d}^3\boldsymbol{p}\,u_q(\eta)u_p(\eta)u_q^*(\eta')u_p^*(\eta')\left[\frac{q^2p^2}{4} - \frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2q^2}{2k^2}\right.\\ \left.+\frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2(\boldsymbol{k}\cdot\boldsymbol{p})^2}{4k^4}\right]\delta(\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{k})\end{aligned} \tag{F.23}$$

$$\begin{aligned}=\frac{2}{(2\pi)^{9/2}}\int_{q,p>k_{\rm UV}} {\rm d}q\,{\rm d}p\,p^4q^4u_q(\eta)u_p(\eta)u_q^*(\eta')u_p^*(\eta')\int_0^{4\pi}{\rm d}^2\Omega_q\,{\rm d}^2\Omega_p\left[\frac{1}{4}\right.\\ \left.-\frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2}{2k^2p^2} + \frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2(\boldsymbol{k}\cdot\boldsymbol{p})^2}{4k^4p^2q^2}\right]\delta(\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{k})\,,\end{aligned} \tag{F.24}$$

which writes the integrals in polar coordinates. The three terms mainly differ in their angular integrations. Rotation invariance allows us to choose $\boldsymbol{k}$ (as in Appendix B) to point along the z axis: $\boldsymbol{k} = (0,0,k)$. Using this choice, the angular integrals become

$$\mathcal{J}(p,q,k) := \int_0^{4\pi}{\rm d}^2\Omega_q\,{\rm d}^2\Omega_p\left[\frac{1}{4} - \frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2}{2k^2p^2} + \frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2(\boldsymbol{k}\cdot\boldsymbol{p})^2}{4k^4p^2q^2}\right]\delta(\boldsymbol{q}+\boldsymbol{p}-\boldsymbol{k}) \tag{F.25}$$

$$
\begin{aligned}
=&\frac{1}{q^2}\int_{-1}^{1}\mathrm{d}\cos\theta_q\int_0^{2\pi}\mathrm{d}\varphi_q\int_{-1}^{1}\mathrm{d}\cos\theta_p\int_0^{2\pi}\mathrm{d}\varphi_p\left[\frac{1}{4}-\frac{\cos^2\theta_p}{2}+\frac{\cos^2\theta_q\cos^2\theta_p}{4}\right]\\
&\times\delta\left(q-\sqrt{p^2+k^2-2kp\cos\theta_p}\right)\delta\left(\cos\theta_q-\frac{k-p\cos\theta_p}{\sqrt{p^2+k^2-2kp\cos\theta_p}}\right)\\
&\times\delta\left[\varphi_q-(\varphi_p+\pi)\right].
\end{aligned}\tag{F.26}
$$

Integrating over φ_q, φ_p and then θ_q, and writing $\mu=\cos\theta_p$ yields

$$
\mathcal{J}(p,q,k)=\frac{2\pi}{pqk}\int_{-1}^{1}\mathrm{d}\mu\left[\frac{1}{4}-\frac{\mu^2}{2}+\frac{(k-p\mu)^2\mu^2}{4(p^2+k^2-2kp\mu)}\right]\delta\left(\mu-\frac{p^2+k^2-q^2}{2pk}\right),\tag{F.27}
$$

which uses the identity $\delta\big(q-\sqrt{p^2+k^2-2kp\mu}\,\big)=\frac{q}{kp}\,\delta\left(\mu-\frac{p^2+k^2-q^2}{2pk}\right)$. The result vanishes unless $-2pk\le p^2+k^2-q^2\le 2pk$, which requires the p and q integrals to run over the region U depicted in fig. 5. For p,q in this region the final angular integral gives

$$
\mathcal{J}(p,q,k)=\frac{\pi}{2pqk}\left\{1-\frac{(p^2+k^2-q^2)^2}{2p^2k^2}+\frac{[k^4-(p^2-q^2)^2]^2}{16p^2q^2k^4}\right\},\tag{F.28}
$$

which when used in eq. (F.22) together with the Bunch-Davies mode functions (2.40) gives

$$
\begin{aligned}
\mathscr{S}_{\boldsymbol{k}}(\eta,\eta')=&\frac{1}{8(2\pi)^{7/2}k}\iint_U\mathrm{d}q\mathrm{d}p\left\{q^2p^2-\frac{(k^2+p^2-q^2)^2q^2}{2k^2}+\frac{\left[k^4-(p^2-q^2)^2\right]^2}{16k^4}\right\}\\
&\times\left(1-\frac{i}{q\eta}\right)\left(1+\frac{i}{q\eta'}\right)\left(1-\frac{i}{p\eta}\right)\left(1+\frac{i}{p\eta'}\right)e^{-i(q+p)(\eta-\eta')}.
\end{aligned}\tag{F.29}
$$

The integral over the region U is performed using the coordinate change $p=\frac{1}{2}(P+Q)$ and $q=\frac{1}{2}(P-Q)$, leading to

$$
\begin{aligned}
\mathscr{S}_{\boldsymbol{k}}(\eta,\eta')=&\frac{1}{32(2\pi)^{7/2}k}\int_{-k}^{k}\mathrm{d}Q\int_{|Q|+2k_{\mathrm{UV}}}^{\infty}\mathrm{d}P\left[\frac{(P^2-Q^2)^2}{8}-\frac{(k^2+PQ)^2(P-Q)^2}{4k^2}\right.\\
&\left.+\frac{(k^4-P^2Q^2)^2}{8k^4}\right]\left[1-\frac{2i}{(P-Q)\eta}\right]\left[1+\frac{2i}{(P-Q)\eta'}\right]\\
&\times\left[1-\frac{2i}{(P+Q)\eta}\right]\left[1+\frac{2i}{(P+Q)\eta'}\right]e^{-i(\eta-\eta')P}.
\end{aligned}\tag{F.30}
$$

Note that this integrand is *not* symmetric under $Q\to -Q$. By splitting up the Q-integral to be over $[-k,0]$ and $[0,k]$ and then taking $Q\to -Q$ in the first piece the above simplifies to

$$
\begin{aligned}
\mathscr{S}_{\boldsymbol{k}}(\eta,\eta')=&\frac{1}{128(2\pi)^{7/2}k^5}\int_0^k\mathrm{d}Q\,(k^2-Q^2)^2\int_{Q+2k_{\mathrm{UV}}}^{\infty}\mathrm{d}P\,(k^2-P^2)^2\left[1-\frac{2i}{(P-Q)\eta}\right]\\
&\times\left[1+\frac{2i}{(P-Q)\eta'}\right]\left[1-\frac{2i}{(P+Q)\eta}\right]\left[1+\frac{2i}{(P+Q)\eta'}\right]e^{-i(\eta-\eta')P}.
\end{aligned}\tag{F.31}
$$

Our interest is in the leading behaviour as for small $-k\eta$ and so we focus on the $k\to 0$ (and so also $Q\to 0$) limit of eq. (F.31), which is

$$
\mathscr{S}_{\boldsymbol{k}}(\eta,\eta')\simeq\frac{1}{128(2\pi)^{7/2}k^5}\int_0^k\mathrm{d}Q\,(k^2-Q^2)^2\int_{2k_{\mathrm{UV}}}^{\infty}\mathrm{d}P\,P^4\left(1-\frac{2i}{P\eta}\right)^2\left(1+\frac{2i}{P\eta'}\right)^2e^{-i(\eta-\eta')P}\tag{F.32}
$$

$$\simeq \frac{1}{240(2\pi)^{7/2}} \int_{2k_{\rm UV}}^{\infty} \mathrm{d}P\, P^4 \left(1-\frac{2i}{P\eta}\right)^2 \left(1+\frac{2i}{P\eta'}\right)^2 e^{-i(\eta-\eta')P}\,, \tag{F.33}$$

where the last expression is valid in the limit $k \to 0$. This is to be compared with the same limit for the scalar contribution (B.31), which is

$$\mathscr{C}_{\boldsymbol{k}}(\eta,\eta') = \frac{1}{32(2\pi)^{7/2}} \int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\, \frac{(P^2+Q^2-2k^2)^2}{k} \left[1-\frac{2i}{(P-Q)\eta}\right]\left[1+\frac{2i}{(P-Q)\eta'}\right]$$
$$\times \left[1-\frac{2i}{(P+Q)\eta}\right]\left[1+\frac{2i}{(P+Q)\eta'}\right] e^{-i(\eta-\eta')P} \tag{F.34}$$
$$\simeq \frac{1}{32(2\pi)^{7/2}} \int_{2k_{\rm UV}}^{\infty} \mathrm{d}P\, P^4 \left(1-\frac{2i}{P\eta}\right)^2 \left(1+\frac{2i}{P\eta'}\right)^2 e^{-i(\eta-\eta')P}, \tag{F.35}$$

where, again, the last expression is valid for $k \to 0$. Therefore, we see that for small k

$$\mathscr{S}_{\boldsymbol{k}}(\eta,\eta') \simeq \frac{2}{15}\, \mathscr{C}_{\boldsymbol{k}}(\eta,\eta')\,. \tag{F.36}$$

This can also be derived directly by taking the $\boldsymbol{k} \to 0$ limit of eq. (F.25), which becomes

$$\mathcal{J}(p,q,k\to 0) := \frac{1}{q^2}\delta(p-q) \int_0^{4\pi} \mathrm{d}^2\Omega_q\, \mathrm{d}^2\Omega_p \left[\frac{1}{4} - \frac{(\boldsymbol{k}\cdot\boldsymbol{p})^2}{2k^2p^2} + \frac{(\boldsymbol{k}\cdot\boldsymbol{q})^2(\boldsymbol{k}\cdot\boldsymbol{p})^2}{4k^4p^2q^2}\right] \delta\left(\frac{\boldsymbol{q}}{q}+\frac{\boldsymbol{p}}{p}\right) \tag{F.37}$$
$$= \frac{2\pi}{q^2}\,\delta(p-q) \int_{-1}^{1} \mathrm{d}\mu \left(\frac{1}{4} - \frac{\mu^2}{2} + \frac{\mu^4}{4}\right) = \frac{4}{15}\left(\frac{2\pi}{q^2}\right)\delta(p-q)\,, \tag{F.38}$$

which is $\frac{2}{15}$ times the result obtained when the square bracket is 1.

F.3 Lindblad Coefficient

Beginning with the Nakajima-Zwanzig equation (F.39), and applying the Markovian approximation in the same spirit as earlier gives

$$\frac{\mathcal{V}}{(2\pi)^{3/2}} \frac{\partial \varrho^P_{\boldsymbol{k}}}{\partial \eta} = -\mathfrak{T}_{\boldsymbol{k}}(\eta,\eta_{\rm in})\,[\tilde{v}^P_{\boldsymbol{k}}(\eta), \tilde{v}^P_{\boldsymbol{k}}(\eta')\varrho^P_{\boldsymbol{k}}(\eta')] - \mathfrak{T}^*_{\boldsymbol{k}}(\eta,\eta_{\rm in})\,[\varrho^P_{\boldsymbol{k}}(\eta')\tilde{v}^P_{\boldsymbol{k}}(\eta'), \tilde{v}^P_{\boldsymbol{k}}(\eta)]\,, \tag{F.39}$$

where we define

$$\mathfrak{T}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) \;=\; (2\pi)^{3/2} \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta'\, \tilde{G}(\eta)\, \tilde{G}(\eta')\, \mathscr{S}_{\boldsymbol{k}}(\eta,\eta') \tag{F.40}$$

and we note that again ${\rm Re}\,[\mathfrak{T}_{\boldsymbol{k}}(\eta,\eta_{\rm in})]$ drives the decoherence. Using $\tilde{G}(\eta) = H\eta/M_{\rm p}$ as well as eq. (F.31) means that eq. (F.40) can be written as the triple integral

$$\mathfrak{T}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = \frac{H^2}{512\pi^2 M_{\rm p}^2 k^5} \int_{\eta_{\rm in}}^{\eta} \mathrm{d}\eta' \int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\, \eta\eta'\,(k^2-P^2)^2(k^2-Q^2)^2 \left[1-\frac{2i}{(P-Q)\eta}\right]$$
$$\times \left[1+\frac{2i}{(P-Q)\eta'}\right]\left[1-\frac{2i}{(P+Q)\eta}\right]\left[1+\frac{2i}{(P+Q)\eta'}\right] e^{-i(\eta-\eta')P}\,. \tag{F.41}$$

We first evaluate the η'-integral using the formula (C.3), giving

$$\mathfrak{T}_{\boldsymbol{k}}(\eta,\eta_{\rm in}) = \frac{H^2}{512\pi^2 M_{\rm p}^2 k^5} \int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P\, \eta\,(k^2-P^2)^2(k^2-Q^2)^2 \left[1-\frac{2i}{(P-Q)\eta}\right]$$

$$
\times \left[1 - \frac{2i}{(P+Q)\eta}\right] \left\{ \frac{1}{P^2} - \frac{i\eta}{P} + \frac{4}{P^2 - Q^2} - \frac{4e^{-iP\eta}\big[\mathrm{Ei}(iP\eta) + i\pi\big]}{P^2 - Q^2} \right.
$$

$$
\left. - e^{-iP(\eta - \eta_{\rm in})} \left(\frac{1}{P^2} - \frac{i\eta_{\rm in}}{P} + \frac{4}{P^2 - Q^2} \right) + \frac{4e^{-iP\eta}\big[\mathrm{Ei}(iP\eta_{\rm in}) + i\pi\big]}{P^2 - Q^2} \right\}. \tag{F.42}
$$

The terms involving $\eta_{\rm in}$ can be handled in precisely the same way as in the discussion around eq. (E.56), showing that they contribute only subdominantly for small $(-k\eta)$, and so for our purposes can be neglected. This leaves

$$
\mathfrak{T}_{\boldsymbol{k}}(\eta, \eta_{\rm in}) \simeq \frac{H^2}{512\pi^2 M_{\rm p}^2 k^5} \int_0^k \mathrm{d}Q \int_{Q+2k_{\rm UV}}^{\infty} \mathrm{d}P \; \eta \, (k^2 - P^2)^2 (k^2 - Q^2)^2 \left[1 - \frac{2i}{(P-Q)\eta}\right]
$$

$$
\times \left[1 - \frac{2i}{(P+Q)\eta}\right] \left\{ \frac{1}{P^2} - \frac{i\eta}{P} + \frac{4}{P^2 - Q^2} - \frac{4e^{-iP\eta}\big[\mathrm{Ei}(iP\eta) + i\pi\big]}{P^2 - Q^2} \right\}, \tag{F.43}
$$

which can be rewritten using the change of variables already used before, reproduced here for convenience, $z = -k\eta$, $\kappa = k_{\rm UV}/k$, $x = -Q\eta$ and $y = -P\eta$ as

$$
\mathfrak{T}_{\boldsymbol{k}} \simeq \frac{H^2 k^2}{512\pi^2 M_{\rm p}^2 z^3} \int_0^z \mathrm{d}x \, \left(\frac{x^2}{z^2} - 1 \right)^2 h(x, z) \tag{F.44}
$$

with

$$
h(x, z) := \int_{x+2\kappa z}^{\infty} \mathrm{d}y \; (y^2 - z^2)^2 \left(1 - \frac{2i}{x - y} \right) \left(1 + \frac{2i}{x + y} \right) \left\{ -\frac{1}{y^2} - \frac{i}{y} + \frac{4 - 4e^{iy}\left[\mathrm{Ei}(-iy) + i\pi\right]}{x^2 - y^2} \right\}. \tag{F.45}
$$

This integral diverges (as did the previous examples), since the integrand approaches $-y^2$ for large y. We handle this divergence the same way as before by subtracting the divergent large-y form, leaving a convergent result (see the discussion around eq. (E.53)).

Taking the $z \to 0$ limit (which also implies $x \to 0$) then gives the leading contribution

$$
\mathfrak{T}_{\boldsymbol{k}} \simeq \frac{H^2 k^2}{512\pi^2 M_{\rm p}^2 z^3} h^{\rm reg}(0, 0) \int_0^z \mathrm{d}x \, \left(\frac{x^2}{z^2} - 1 \right)^2 = \frac{H^2 k^2}{512\pi^2 M_{\rm p}^2 z^2} \left[\frac{8 h^{\rm reg}(0, 0)}{15} \right] \tag{F.46}
$$

where (compare to eqs. (E.46) and (E.54))

$$
h^{\rm reg}(0, 0) = \int_0^{\infty} \mathrm{d}y \left(y^4 \left(1 + \frac{2i}{y} \right)^2 \left\{ -\frac{1}{y^2} - \frac{i}{y} - \frac{4 - 4e^{iy}\left[\mathrm{Ei}(-iy) + i\pi\right]}{y^2} \right\} + y^2 \right) \tag{F.47}
$$

$$
= \frac{f^{\rm reg}(0, 0)}{3} = 20\pi \, . \tag{F.48}
$$

This leads to the following $\eta_{\rm in}$-independent leading behaviour

$$
\mathrm{Re}\left[\mathfrak{T}_{\boldsymbol{k}}(\eta, \eta_{\rm in})\right] \simeq \frac{H^2 k^2}{48\pi M_{\rm p}^2 z^2} = \frac{H^2 k^2}{1024\pi^2 M_{\rm p}^2} \left[\frac{64\pi}{3z^2} + \mathcal{O}(z^{-1}) \right] \tag{F.49}
$$

and so comparing with eq. (3.38) we see that

$$
\mathrm{Re}\left[\mathfrak{T}_{\boldsymbol{k}}(\eta, \eta_{\rm in})\right] \simeq \frac{16}{15\,\varepsilon_1} \mathrm{Re}\left[\mathfrak{F}_{\boldsymbol{k}}(\eta, \eta_{\rm in})\right] \tag{F.50}
$$

in the super-Hubble limit, as claimed in the main text. The factor $16/(15\varepsilon_1)$ has two sources: the factor of $\frac{2}{15}$ seen in eq. (F.36) together with the factor of $8/\varepsilon_1$ coming from the change in effective coupling noted in eq. (5.9).

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