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The unpolarized and polarized single-mass three-loop heavy flavor operator matrix elements $A_{gg,Q}$ and $\Delta A_{gg,Q}$

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Abstract: We calculate the gluonic massive operator matrix elements in the unpolarized and polarized cases, $A_{gg,Q}(x,\mu^2)$ and $\Delta A_{gg,Q}(x,\mu^2)$, at three-loop order for a single mass. These quantities contribute to the matching of the gluon distribution in the variable flavor number scheme. The polarized operator matrix element is calculated in the Larin scheme. These operator matrix elements contain finite binomial and inverse binomial sums in Mellin N-space and iterated integrals over square root-valued alphabets in momentum fraction x-space. We derive the necessary analytic relations for the analytic continuation of these quantities from the even or odd Mellin moments into the complex plane, present analytic expressions in momentum fraction x-space and derive numerical results. The present results complete the gluon transition matrix elements both of the single- and double-mass variable flavor number scheme to three-loop order.

Keywords: Deep Inelastic Scattering or Small-x Physics, Higher-Order Perturbative Calculations, Parton Distributions, Quark Masses

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1 Introduction

The heavy flavor corrections to deeply inelastic scattering exhibit different scaling violations if compared to the light flavor contributions. At the present experimental accuracy of the deep-inelastic data, precision determinations of the strong coupling constant $\alpha_s(M_Z^2)$ [1–4] and the determination of the parton distribution functions [5, 6] require their calculation to next-to-next-to leading order (NNLO). If the virtuality, Q^2, is significantly larger than the heavy quark mass squared, m_Q^2, i.e. $Q^2/m_Q^2 \gtrsim 10$, one may calculate the heavy flavor corrections analytically [7]. Here m_Q denotes the heavy quark mass. This corresponds to Q^2 values above about 25 GeV^2 in the case of charm. In refs. [8–10] it has been shown to NNLO in the single- and double-mass cases, how to express the massive coefficient functions for the structure functions in terms of massive operator matrix elements (OMEs) A_{ij} and the massless coefficient functions [11, 12] at large scales, as well as the corresponding relations for the parton distribution functions in the variable flavor number scheme (VFNS).[1] From NNLO onward, five of the OMEs [25–27] are needed to express the deep-inelastic structure functions [28–30]. The additional OMEs, $A_{gg,Q}$ and $A_{gq,Q}$, at NNLO contribute only to the relations in the VFNS. The three-loop corrections to $A_{gq,Q}$ have been computed in ref. [31].

[1]The two-loop corrections can be found in refs. [7, 13–24].

In this paper we calculate the massive OMEs $A_{gg,Q}$ and $\Delta A_{gg,Q}$ to three-loop order. In refs. [32, 33] the $O(T_F^2 N_F)$ and $O(T_F^2)$ contributions to the operator matrix element have been calculated in the unpolarized case already. All logarithmic contributions have been derived in refs. [10, 34]. A series of moments and scalar integrals in the double-mass case were obtained in refs. [9, 35, 36]. The $O(T_F)$ contribution to the three-loop anomalous dimensions $(\Delta)\gamma_{gg,Q}^{(2)}$, resulting from the single pole term of the unrenormalized OME, has been calculated in refs. [37–39]. The double-mass three-loop contributions were computed in refs. [40, 41]. Here both charm and bottom quark lines are contained in the corresponding Feynman diagrams. Contributions of this kind emerge first at two-loop order due to reducible diagrams [23, 42].

We calculate the massive OME using the techniques described in ref. [43]. These include the method of (generalized) hypergeometric functions [44–53], the Mellin-Barnes method [54, 55], the method of ordinary differential equations [56–64], and the (multivartiate) Almkvist-Zeilberger algorithm [65–67]. All these methods finally map the problem to multiply nested sums, which are solved using the packages `Sigma` [68, 69], `EvaluateMultisums` and `SumProduction` [70–73], based on difference-ring theory [74–86], as well as the packages `OreSys` [87–89], and `MultiIntegrate` [67, 90].[2] For an efficient treatment of the different sum- and function algebras, such as the harmonic sums [93, 94] and harmonic polylogarithms [95], generalized harmonic sums and polylogarithms [96, 97], cyclotomic harmonic sums and polylogarithms [98], iterated integrals induced by quadratic forms [99], and finite binomial sums and square root-valued iterated integrals [100], we use the package `HarmonicSums` [93–95, 97–112]. We mention, in particular, that our solution methods do not require to refer to special bases of master integrals.

One obtains a general expression for the constant part of the unrenormalized massive OMEs $(\Delta)\hat{\hat{A}}_{gg,Q}$, at even (odd) integer values of the Mellin variable N, according to the light-cone expansion in the unpolarized and polarized cases, cf. [113]. The representation is given in terms of synchronized sum-product representations. In the present case it turned out that all but two diagrams could be calculated in this way and for the latter cases the problem had to be solved in x-space, deriving the N-space representation afterwards.

We consider the case of a single heavy quark. With the present results, both the unpolarized and polarized gluon distributions can be mapped in the single-mass VFNS [24] and the double-mass VFNS [9] to three-loop order.

The paper is organized as follows. In section 2 we present the basic formalism and describe the new functional aspects in the present case, both in Mellin N and x-space. In section 3 the results in Mellin N-space are presented for $a_{gg,Q}^{(3)}$ and $\Delta a_{gg,Q}^{(3)}$. The characteristic aspects of the x-space results are discussed in section 4. Because the full x-space results are rather lengthy, we describe its principal structure and present it in full form in a computer-readable file in the supplementary material attached to this paper. Numerical results are discussed in section 5 deriving a fast and precise numerical representation, and section 6 contains the conclusions. A series of technical aspects are presented in the appendices A–D.

[2] For surveys on the computation methods and the related function and number spaces, see refs. [91, 92].

2 Basic Formalism and the computation method

The unrenormalized massive OMEs $\hat{\hat{A}}_{gg,Q}^{(k)}$ from one- to three-loop order for a single heavy quark and N_F massless quarks, cf. ref. [8], are given by

$$\hat{\hat{A}}_{gg,Q}^{(1)} = \left(\frac{\hat{m}^2}{\mu^2}\right)^{\varepsilon/2}\left[\frac{\hat{\gamma}_{gg}^{(0)}}{\varepsilon} + a_{gg,Q}^{(1)} + \varepsilon\overline{a}_{gg,Q}^{(1)} + \varepsilon^2\overline{\overline{a}}_{gg,Q}^{(1)}\right] + O\left(\varepsilon^3\right), \tag{2.1}$$

$$\hat{\hat{A}}_{gg,Q}^{(2)} = \left(\frac{\hat{m}^2}{\mu^2}\right)^{\varepsilon}\left[\frac{1}{\varepsilon^2}c_{gg,Q,(2)}^{(-2)} + \frac{1}{\varepsilon}c_{gg,Q,(2)}^{(-1)} + c_{gg,Q,(2)}^{(0)} + \varepsilon c_{gg,Q,(2)}^{(1)}\right] + O\left(\varepsilon^2\right), \tag{2.2}$$

$$\hat{\hat{A}}_{gg,Q}^{(3)} = \left(\frac{\hat{m}^2}{\mu^2}\right)^{3\varepsilon/2}\left[\frac{1}{\varepsilon^3}c_{gg,Q,(3)}^{(-3)} + \frac{1}{\varepsilon^2}c_{gg,Q,(3)}^{(-2)} + \frac{1}{\varepsilon}c_{gg,Q,(3)}^{(-1)} + a_{gg,Q}^{(3)}\right] + O\left(\varepsilon\right). \tag{2.3}$$

Here and in the following $\varepsilon = D-4$ denotes the dimensional parameter, $c_{gg,Q,(k)}^{(-l)}$ are the expansion coefficients of the unrenormalized OME $\hat{\hat{A}}_{gg,Q}$, $\hat{m}$ denotes the unrenormalized heavy quark mass, μ is both the factorization and renormalization scale, ζ_i is Riemann's ζ-function at integer values $i \geq 2$, β_k and $\beta_{k,Q}$ are expansion coefficients of the β-function in Quantum Chromodynamics (QCD) in different schemes, $\gamma_{ij}^{(k)}$ are anomalous dimensions [11, 12, 27, 37, 39, 114–121] and $m_k^{(l)}$ are the expansion coefficients of the heavy mass, while $a_{ij}^{(k)}$, $\bar{a}_{ij}^{(k)}$, and $\bar{\bar{a}}_{ij}^{(k)}$ are the expansion coefficients in ε to $O(\varepsilon^0)$, $O(\varepsilon)$, and $O(\varepsilon^2)$ of the different OMEs. All explanations have been given in ref. [8], to which we refer. Analogous expressions hold for $\Delta A_{gg,Q}^{(k)}$. Here the anomalous dimensions have to be replaced by the polarized ones, and the coefficients $a_{ij}^{(k)}(\bar{a}_{ij}^{(k)}, \bar{\bar{a}}_{ij}^{(k)})$ by $\Delta a_{ij}^{(k)}(\Delta\bar{a}_{ij}^{(k)}, \Delta\bar{\bar{a}}_{ij}^{(k)})$.

The renormalization of $\hat{\hat{A}}_{gg,Q}^{(3)}$ includes mass and coupling renormalization, as well as the renormalization of the local operator, and the subtraction of the collinear singularities, see [8]. It is essential to work in a MOM-scheme for charge renormalization first and then transform to the $\overline{\text{MS}}$ scheme. The renormalized OME from first to third order are then obtained by

$$A_{gg,Q}^{(1),\overline{\mathsf{MS}}} = -\beta_{0,Q}\ln\left(\frac{m^2}{\mu^2}\right), \tag{2.4}$$

$$\begin{aligned} A_{gg,Q}^{(2),\overline{\mathsf{MS}}} =& \frac{1}{8}\left\{2\beta_{0,Q}\left(\gamma_{gg}^{(0)}+2\beta_0\right)+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}+8\beta_{0,Q}^2\right\}\ln^2\left(\frac{m^2}{\mu^2}\right)+\frac{\hat{\gamma}_{gg}^{(1)}}{2}\ln\left(\frac{m^2}{\mu^2}\right) \\ &-\frac{\zeta_2}{8}\left[2\beta_{0,Q}\left(\gamma_{gg}^{(0)}+2\beta_0\right)+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\right]+a_{gg,Q}^{(2)}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} A_{gg,Q}^{(3),\overline{\mathsf{MS}}} =& \frac{1}{48}\Biggl\{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\left(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-6\beta_0-4n_f\beta_{0,Q}-10\beta_{0,Q}\right)-4\Biggl(\gamma_{gg}^{(0)}\left[2\beta_0+7\beta_{0,Q}\right] \\ &+4\beta_0^2+14\beta_{0,Q}\beta_0+12\beta_{0,Q}^2\Biggr)\beta_{0,Q}\Biggr\}\ln^3\left(\frac{m^2}{\mu^2}\right)+\frac{1}{8}\Biggl\{\hat{\gamma}_{qg}^{(0)}\left(\gamma_{gq}^{(1)}+(1-n_f)\hat{\gamma}_{gq}^{(1)}\right) \\ &+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(1)}+4\gamma_{gg}^{(1)}\beta_{0,Q}-4\hat{\gamma}_{gg}^{(1)}\left[\beta_0+2\beta_{0,Q}\right]+2\gamma_{gg}^{(0)}\beta_{1,Q}+4\left[\beta_1+\beta_{1,Q}\right]\beta_{0,Q}\Biggr\}\ln^2\left(\frac{m^2}{\mu^2}\right)+ \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{16}\Biggl\{8\hat{\gamma}_{gg}^{(2)}-8n_f a_{gq,Q}^{(2)}\hat{\gamma}_{qg}^{(0)}-16a_{gg,Q}^{(2)}\left(2\beta_0+3\beta_{0,Q}\right)+8\gamma_{gq}^{(0)}a_{Qg}^{(2)}+8\gamma_{gg}^{(0)}\beta_{1,Q}^{(1)}\\
&+\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_2\left(\gamma_{gg}^{(0)}-\gamma_{qq}^{(0)}+6\beta_0+4n_f\beta_{0,Q}+6\beta_{0,Q}\right)\\
&+4\beta_{0,Q}\zeta_2\left(\gamma_{gg}^{(0)}+2\beta_0\right)\left(2\beta_0+3\beta_{0,Q}\right)\Biggr\}\ln\left(\frac{m^2}{\mu^2}\right)+2\left(2\beta_0+3\beta_{0,Q}\right)\overline{a}_{gg,Q}^{(2)}\\
&+n_f\hat{\gamma}_{qg}^{(0)}\overline{a}_{gq,Q}^{(2)}-\gamma_{gq}^{(0)}\overline{a}_{Qg}^{(2)}-\beta_{1,Q}^{(2)}\gamma_{gg}^{(0)}+\frac{\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)}\zeta_3}{48}\Biggl(\gamma_{qq}^{(0)}-\gamma_{gg}^{(0)}-2\left[2n_f+1\right]\beta_{0,Q}\\
&-6\beta_0\Biggr)+\frac{\beta_{0,Q}\zeta_3}{12}\left(\left[\beta_{0,Q}-2\beta_0\right]\gamma_{gg}^{(0)}+2\left[\beta_0+6\beta_{0,Q}\right]\beta_{0,Q}-4\beta_0^2\right)\\
&-\frac{\hat{\gamma}_{qg}^{(0)}\zeta_2}{16}\left(\gamma_{gq}^{(1)}+\hat{\gamma}_{gq}^{(1)}\right)+\frac{\beta_{0,Q}\zeta_2}{8}\left(\hat{\gamma}_{gg}^{(1)}-2\gamma_{gg}^{(1)}-2\beta_1-2\beta_{1,Q}\right)+\frac{\delta m_1^{(-1)}}{4}\Biggl(8a_{gg,Q}^{(2)}\\
&+24\delta m_1^{(0)}\beta_{0,Q}+8\delta m_1^{(1)}\beta_{0,Q}+\zeta_2\beta_{0,Q}\beta_0+9\zeta_2\beta_{0,Q}^2\Biggr)+\delta m_1^{(0)}\left(\beta_{0,Q}\delta m_1^{(0)}+\hat{\gamma}_{gg}^{(1)}\right)\\
&+\delta m_1^{(1)}\left(\hat{\gamma}_{qg}^{(0)}\gamma_{gq}^{(0)}+2\beta_{0,Q}\gamma_{gg}^{(0)}+4\beta_{0,Q}\beta_0+8\beta_{0,Q}^2\right)-2\delta m_2^{(0)}\beta_{0,Q}+a_{gg,Q}^{(3)}\,.
\end{aligned}
\tag{2.6}
$$

Here the heavy quark mass is renormalized in the on-shell scheme (OMS) and the strong coupling constant in the $\overline{\mathsf{MS}}$ scheme. The transformation from the OMS to the $\overline{\mathsf{MS}}$ scheme for the quark mass is described e.g. in eq. (5.13) of ref. [122], see also refs. [123–133].

The Feynman diagrams are generated by using `QGRAF` [8, 134], the spinor and Lorentz-algebra is performed using `FORM` [135, 136], and the color algebra by using `Color` [137]. The operator insertions are resummed into linear propagators as has been discussed in ref. [138], e.g. by

$$
\sum_{k=0}^{\infty}(\Delta.p)^k t^k = \frac{1}{1-t\Delta.p}, \tag{2.7}
$$

where t denotes an auxiliary parameter. Similar relations hold for the more complicated operator insertions. Taking the kth moment of (2.7), i.e. the coefficient of t^k, one obtains the contribution due to $(\Delta.p)^k$. Here Δ denotes a light-like vector.

In total, 642 irreducible Feynman diagrams contribute to the OMEs. The reduction to master integrals using the integration-by-parts relations [139–145] has been performed using `Reduze 2` [146, 147].

Unlike the case in later computations starting in 2017, we did not use the method of arbitrary high moments [148], establishing the corresponding difference equations by the method of guessing [149–151] and solving them using the package `Sigma` [68, 69]. Instead we calculated the master integrals directly in Mellin N-space using the different methods mentioned in section 1, from which the individual Feynman diagrams were calculated. Their first few moments were compared to Mellin moments computed using `Matad` [152], cf. ref. [8] in the unpolarized case. In the case of two Feynman diagrams, which are related to each other by reversal of the internal fermion line and which give the same result, we had to use a different computation method. Here we applied the method of differential equations [64] to the master integrals working in the auxiliary variable t used for the operator resummation.

The transition to momentum fraction x-space by an analytic continuation is described in detail in ref. [153]. This method avoids going to Mellin N-space. The result can be Mellin transformed analytically at the end of the calculation, cf. section 4. Working in Mellin N-space implies that one decides for expressions at either even or odd integer values of N starting with a value N_0 implied by the crossing relations [154, 155]. If one would like to extract the small x behavior of the OME, one has to do this in x-space, since the corresponding poles are situated at $N = 1$ in the unpolarized case and at $N = 0$ in the polarized case, for which no N-space representation is available.

In Mellin N-space, the present OMEs exhibit nested finite binomial sums [100]. Transforming to x-space, these structures lead to G-functions [100] containing also square root-valued letters. Along with these very many new constants $\mathrm{G}(\{a_i\}, 1)$ occur in intermediary steps. They can be rationalized by procedures contained in the package `HarmonicSums`, leading to tabulated cyclotomic constants [98]. The latter turn out to further reduce to multiple zeta values [156] in all cases. We show a series of examples in appendix B. Also the iterated integrals $\mathrm{G}(\{a_i\}, x)$ containing root-valued letters can be rationalized, leading to cyclotomic harmonic polylogarithms [98]. The latter representation can in principle be used in the numerical representation, cf. section 5.

Let us now discuss the new structures appearing in the present OMEs, which are nested finite binomial sums in Mellin N-space. Their Mellin inversion can be expressed by iterative G-functions and usual harmonic polylogarithms over the alphabet $\mathfrak{A}$

$$\mathfrak{A} = \{f_k(x)\}|_{k=1..6} = \left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{\sqrt{1-x}}{x}, \sqrt{x(1-x)}, \frac{1}{\sqrt{1-x}}\right\}. \tag{2.8}$$

The G-functions are defined by [100]

$$\mathrm{G}(\{b, \vec{a}\}, x) = \int_0^x dy f_b(y) \mathrm{G}(\{\vec{a}\}, y), \quad \mathrm{G}(\{\emptyset\}, y) = 1, \quad f_b(x), f_{a_i}(x) \in \mathfrak{A}. \tag{2.9}$$

In N-space we consider the objects

$$\mathsf{BS}_0(N) = \frac{1}{2N - (2l+1)}, \quad l \in \mathbb{N}, \tag{2.10}$$

$$\mathsf{BS}_1(N) = 4^N \frac{(N!)^2}{(2N)!}, \tag{2.11}$$

$$\mathsf{BS}_2(N) = \frac{1}{4^N} \frac{(2N)!}{(N!)^2}, \tag{2.12}$$

$$\mathsf{BS}_3(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1} (2\tau_1)!}{(\tau_1!)^2 \tau_1}, \tag{2.13}$$

$$\mathsf{BS}_4(N) = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1} (\tau_1!)^2}{(2\tau_1)! \tau_1^2}, \tag{2.14}$$

$$\mathsf{BS}_5(N) = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1} (\tau_1!)^2}{(2\tau_1)! \tau_1^3}, \tag{2.15}$$

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$$\mathsf{BS}_6(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^2}}{(\tau_1!)^2\tau_1}, \tag{2.16}$$

$$\mathsf{BS}_7(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^3}}{(\tau_1!)^2\tau_1}, \tag{2.17}$$

$$\mathsf{BS}_8(N) = \sum_{\tau_1=1}^{N} \frac{\sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2}{(2\tau_2)!\tau_2^2}}{\tau_1}, \tag{2.18}$$

$$\mathsf{BS}_9(N) = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)! \sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2 \sum_{\tau_3=1}^{\tau_2} \frac{1}{\tau_3}}{(2\tau_2)!\tau_2^2}}{(\tau_1!)^2\tau_1}, \tag{2.19}$$

$$\mathsf{BS}_{10}(N) = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}}{\binom{2\tau_1}{\tau_1}} \frac{1}{\tau_1^2} S_1(\tau_1), \tag{2.20}$$

where $S_{\vec{a}} \equiv S_{\vec{a}}(N)$ denote the nested harmonic sums [93, 94].

The above finite binomial sums obey the following recursion relations

$$\mathsf{BS}_3(N) - \mathsf{BS}_3(N-1) = \frac{1}{N}\mathsf{BS}_2(N), \tag{2.21}$$

$$\mathsf{BS}_4(N) - \mathsf{BS}_4(N-1) = \frac{1}{N^2}\mathsf{BS}_1(N), \tag{2.22}$$

$$\mathsf{BS}_5(N) - \mathsf{BS}_5(N-1) = \frac{1}{N^3}\mathsf{BS}_1(N), \tag{2.23}$$

$$\mathsf{BS}_6(N) - \mathsf{BS}_6(N-1) = \frac{1}{N}\mathsf{BS}_2(N)\mathsf{BS}_4(N), \tag{2.24}$$

$$\mathsf{BS}_7(N) - \mathsf{BS}_7(N-1) = \frac{1}{N}\mathsf{BS}_2(N)\mathsf{BS}_5(N), \tag{2.25}$$

$$\mathsf{BS}_8(N) - \mathsf{BS}_8(N-1) = \frac{1}{N}\mathsf{BS}_4(N), \tag{2.26}$$

$$\mathsf{BS}_9(N) - \mathsf{BS}_9(N-1) = \frac{1}{N}\mathsf{BS}_2(N)\mathsf{BS}_{10}(N), \tag{2.27}$$

$$\mathsf{BS}_{10}(N) - \mathsf{BS}_{10}(N-1) = \frac{1}{N^2}\mathsf{BS}_1(N)S_1. \tag{2.28}$$

In appendix D we will also calculate representations of $a_{gg,Q}^{(3)}(N)$ and $\Delta a_{gg,Q}^{(3)}(N)$ in the asymptotic region $|N| \gg 1$ up to $O(1/N^{10})$. In the analyticity region this and the recurrences allow one to compute $(\Delta)a_{gg,Q}^{(3)}(N)$ for $N \in \mathbb{C}$, see ref. [111]. The asymptotic expansion of $(\Delta)a_{gg,Q}^{(3)}$ can be obtained from the asymptotic expansions of its building blocks. The asymptotic representations of the finite binomial sums $\mathsf{BS}_k(N),\ k = 0\dots10$ are also given in appendix D. The constants contributing to the above sums can be calculated using infinite binomial and inverse binomial sums [100, 157, 158]. In the calculation, first partly different binomial sums appear after the sum reduction performed by `Sigma`, which have the following relations

$$\overline{\mathsf{BS}}_4 = \sum_{\tau_1=1}^{N} \frac{4^{-\tau_1}(2\tau_1)!}{(\tau_1!)^2(1+\tau_1)^2} = 3 - \frac{(1+2N)(3+2N)}{(1+N)^2}\frac{1}{4^N}\binom{2N}{N} - 2\mathsf{BS}_3, \tag{2.29}$$

$$\overline{\mathsf{BS}}_6 = \sum_{\tau_1=1}^{N} \frac{4^{\tau_1}(\tau_1!)^2 \sum_{\tau_2=1}^{\tau_1} \frac{4^{-\tau_2}(2\tau_2)!}{(\tau_2!)^2(1+\tau_2)^2}}{(2\tau_1)!\tau_1^2} = -2S_2 - 2S_3 - \frac{N(4+3N)}{(1+N)^2} + 3\mathsf{BS}_4 - 2\mathsf{BS}_3\mathsf{BS}_4 + 2\mathsf{BS}_6, \tag{2.30}$$

$$\overline{\mathsf{BS}}_9 = \sum_{\tau_1=1}^{N} \frac{\sum_{\tau_2=1}^{\tau_1} \frac{4^{\tau_2}(\tau_2!)^2 \sum_{\tau_3=1}^{\tau_2} \frac{4^{-\tau_3}(2\tau_3)!}{(\tau_3!)^2(1+\tau_3)^2}}{(2\tau_2)!\tau_2^2}}{\tau_1} = 2\mathsf{BS}_7 + 3\mathsf{BS}_8 - 2\mathsf{BS}_3\mathsf{BS}_8 - 2\mathsf{BS}_9 + \frac{N(5+4N)}{(1+N)^2} + 2\mathsf{BS}_6 S_1 - (3+2S_2+2S_3)S_1 - S_2 - 2S_3 - 2S_4 + 2S_{2,1} + 2S_{3,1}, \tag{2.31}$$

$$\overline{\mathsf{BS}}_{11} = \sum_{i_1=3}^{N} \frac{2^{2i_1}((-2+i_1)!)^2}{(2i_1)!} = -\frac{44}{3} + 3\mathsf{BS}_4 + \frac{(-1+3N)}{N^2}2^{1+2N}\binom{2N}{N}^{-1}, \tag{2.32}$$

$$\overline{\mathsf{BS}}_{12} = \sum_{i_1=3}^{N} \frac{2^{2i_1}((-2+i_1)!)^2}{(2i_1)!i_1} = -\frac{67}{3} + 4\mathsf{BS}_4 + \mathsf{BS}_5 + \frac{(-1+4N)}{N^2}2^{1+2N}\binom{2N}{N}^{-1}, \tag{2.33}$$

$$\overline{\mathsf{BS}}_{13} = \sum_{i_1=3}^{N} \frac{2^{2i_1}((-2+i_1)!)^2 S_1(i_1)}{(2i_1)!} = -23 + 3\mathsf{BS}_5 - 3\mathsf{BS}_8 + \mathsf{BS}_4(-2+3S_1) + \frac{2(-1+4N)}{N^2}4^N\binom{2N}{N}^{-1} + \frac{2(-1+3N)}{N^2}4^N\binom{2N}{N}^{-1}S_1. \tag{2.34}$$

The above functions emerge in products in part and one has to calculate the corresponding Mellin convolutions, also with harmonic sums. Besides the above binomial sums, the following 34 harmonic sums up to weight $\mathsf{w} = 5$ contribute

$$\{S_1, S_{-1}, S_2, S_{-2}, S_3, S_{-3}, S_4, S_{-4}, S_{-5}, S_5, S_{2,1}, S_{-2,1}, S_{2,-1}, S_{-2,-1}, S_{-2,2}, S_{3,1}, S_{-3,1}, S_{4,1}, S_{-4,1}, S_{2,3}, S_{2,-3}, S_{-2,3}, S_{-2,-3}, S_{3,1,1}, S_{-3,1,1}, S_{2,1,1}, S_{-2,1,1}, S_{2,2,1}, S_{2,1,-2}, S_{-2,1,-2}, S_{-2,2,1}, S_{-2,1,1,1}, S_{2,1,1,1}\} \tag{2.35}$$

after algebraic reduction [110]. Here we suppressed the argument N of the harmonic sums for brevity.

All logarithmic terms to the OMEs and the contributions to the constant term implied by renormalization were given in refs. [10, 34] before.

3 Results for $a_{gg,Q}^{(3)}(N)$ and $\Delta a_{gg,Q}^{(3)}(N)$

We now present the constant contributions of the unrenormalized massive OMEs $(\Delta)\hat{\hat{A}}_{gg,Q}^{(3)}$, denoted by $a_{gg,Q}^{(3)}$ and $\Delta a_{gg,Q}^{(3)}$. The expression for $a_{gg,Q}^{(3)}(N)$, valid for even moments $N \in \mathbb{N}, N \geq 2$, reads

$$
\begin{aligned}
a_{gg,Q}^{(3)}(N) = & \frac{1}{2}\Bigl(1+(-1)^N\Bigr) \\
& \times\Biggl\{C_A\Biggl[C_F T_F\Biggl(\frac{32 S_{-2,2} P_3}{(N-1) N^2 (N+1)^2 (N+2)} - \frac{64 S_{-2,1,1} P_{21}}{3 (N-1) N^2 (N+1)^2 (N+2)} \\
& - \frac{16 S_{-4} P_{35}}{3 (N-1) N^2 (N+1)^2 (N+2)} + \frac{4 S_4 P_{73}}{3 (N-2) (N-1) N^2 (N+1)^2 (N+2)} \\
& + \frac{32 [\mathsf{BS}_7 - \mathsf{BS}_9 + 7\zeta_3 \mathsf{BS}_3] P_{75}}{3 (N-1) N^2 (N+1)^2 (N+2)} - \frac{4 (2+N+N^2) S_1^3 P_{98}}{27 (N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + \frac{16 S_{3,1} P_{116}}{3 (N-2) (N-1) N^2 (N+1)^2 (N+2)} - \frac{16 S_{2,1,1} P_{122}}{3 (N-2) (N-1) N^2 (N+1)^2 (N+2)} \\
& + \frac{32 [S_{-1} S_2 - S_{2,-1} + S_{-2,-1}] P_{135}}{5 (N-3) (N-2) (N-1)^2 N^2 (N+1)^3 (N+2)} \\
& - \frac{16 S_{-2,1} P_{157}}{3 (N-3) (N-2) (N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + \frac{[\mathsf{BS}_8 - \mathsf{BS}_4 S_1 + 7\zeta_3] 2^{2-2N} \binom{2N}{N} P_{185}}{15 (N-3) (N-2) (N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + \frac{8 S_3 P_{189}}{135 (N-3) (N-2) (N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& - \frac{8 S_{2,1} P_{197}}{15 (N-3) (N-2) (N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + \frac{4 S_2 P_{207}}{45 (N-3) (N-2)^2 (N-1)^2 N^4 (N+1)^4 (N+2)^3} \\
& + \frac{P_{216}}{2430 (N-3) (N-2)^2 (N-1)^2 N^6 (N+1)^6 (N+2)^5} + \mathsf{B}_4 \Biggl(\frac{32 (4+N+N^2)(6+N+N^2)}{3 (N-1) N^2 (N+1)^2 (N+2)} \\
& - \frac{64}{3} S_1\Biggr) + \zeta_4 \Biggl(-\frac{48 (4+N+N^2)(6+N+N^2)}{(N-1) N^2 (N+1)^2 (N+2)} + 96 S_1\Biggr) + \Biggl(\frac{64 S_{-2,1} P_{22}}{3 (N-1) N^2 (N+1)^2 (N+2)} \\
& + \frac{32 S_3 P_{69}}{9 (N-2) (N-1) N^2 (N+1)^2 (N+2)} + \frac{32 S_{2,1} P_{77}}{3 (N-1) N^2 (N+1)^2 (N+2)} \\
& + \frac{4 S_2 P_{159}}{3 (N-2) (N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + \frac{8 P_{213}}{405 (N-3) (N-2) (N-1)^2 N^5 (N+1)^5 (N+2)^4}\Biggr) S_1 + \Biggl(-\frac{4 S_2 P_{41}}{3 (N-1) N^2 (N+1)^2 (N+2)}
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{4P_{178}}{27(N-2)(N-1)^2N^3(N+1)^4(N+2)^3}\Biggr)S_1^2-\frac{2(2+N+N^2)^2S_1^4}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{2(2+N+N^2)(-6+5N+5N^2)S_2^2}{(N-1)N^2(N+1)^2(N+2)}+\Biggl(-\frac{32S_1^2P_{23}}{3(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{32S_2P_{50}}{3(N-2)(N-1)N^2(N+1)^2(N+2)}-\frac{32S_{-1}P_{135}}{5(N-3)(N-2)(N-1)^2N^2(N+1)^3(N+2)}\\
&-\frac{16S_1P_{158}}{3(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\\
&-\frac{16P_{202}}{15(N-3)(N-2)^2(N-1)^2N^4(N+1)^4(N+2)^3}\Biggr)S_{-2}\\
&-\frac{32(2+N+N^2)(13+2N+2N^2)S_{-2}^2}{3(N-1)N^2(N+1)^2(N+2)}+\Biggl(-\frac{32S_1P_2}{3(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{8P_{162}}{3(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\Biggr)S_{-3}\\
&+\frac{32(-1-8N+N^2)(8+10N+N^2)S_{-3,1}}{3(N-1)N^2(N+1)^2(N+2)}+\Biggl(\frac{4S_1P_{167}}{3(N-1)^2N^3(N+1)^3(N+2)^2}\\
&+\frac{4P_{196}}{9(N-1)^2N^4(N+1)^4(N+2)^3}-\frac{4(2+N+N^2)^2S_1^2}{(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{12(2+N+N^2)^2S_2}{(N-1)N^2(N+1)^2(N+2)}-\frac{24(2+N+N^2)^2S_{-2}}{(N-1)N^2(N+1)^2(N+2)}\Biggr)\zeta_2\\
&+\Biggl(-\frac{8S_1P_{128}}{9(N-2)(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{P_{200}}{180(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\Biggr)\zeta_3\Biggr)\\
&+{\color{blue}T_F^2}\Biggl(-\frac{4S_1^2P_{72}}{135(N-1)N^2(N+1)^2(N+2)}+\frac{4S_2P_{111}}{135(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3]\,2^{2-2N}\binom{2N}{N}P_{140}}{45(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}\\
&+\frac{P_{195}}{3645(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)}+{\color{blue}N_F}\Biggl[-\frac{4S_1^2P_{70}}{27(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{4S_2P_{106}}{27(N-1)N^2(N+1)^2(N+2)}-\frac{8S_1P_{144}}{729(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{2P_{172}}{729(N-1)N^4(N+1)^4(N+2)}+\Biggl(\frac{4P_{93}}{27(N-1)N^2(N+1)^2(N+2)}-\frac{160}{27}S_1\Biggr)\zeta_2\\
&+\Biggl(-\frac{896(1+N+N^2)}{27(N-1)N(N+1)(N+2)}+\frac{448}{27}S_1\Biggr)\zeta_3\Biggr]-\frac{16(1-7N+4N^2+4N^3)[S_3-S_{2,1}]}{15(N-1)N(N+1)}\\
&-\frac{8P_{174}}{3645(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)}S_1+\Biggl(\frac{4P_{108}}{27(N-1)N^2(N+1)^2(N+2)}
\end{aligned}
$$

$$
-\frac{560}{27}S_1\Big)\zeta_2+\Big(-\frac{7P_{49}}{270(N-1)N(N+1)(N+2)}-\frac{1120}{27}S_1\Big)\zeta_3\Big)\Big]
$$
$$
+C_FT_F^2\Big[\frac{16S_1^2P_{82}}{27(N-1)N^3(N+1)^3(N+2)}-\frac{16S_2P_{82}}{9(N-1)N^3(N+1)^3(N+2)}
$$
$$
+\frac{[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3]\,2^{4-2N}\binom{2N}{N}P_{92}}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)}
$$
$$
-\frac{2P_{203}}{243(N-1)N^5(N+1)^5(N+2)(2N-3)(2N-1)}+N_F\Big[-\frac{16S_2P_{82}}{9(N-1)N^3(N+1)^3(N+2)}
$$
$$
+\frac{16S_1^2P_{107}}{27(N-1)N^3(N+1)^3(N+2)}-\frac{2P_{192}}{243(N-1)N^5(N+1)^5(N+2)}
$$
$$
-\Big(\frac{32P_{141}}{81(N-1)N^4(N+1)^4(N+2)}+\frac{16(2+N+N^2)^2S_2}{3(N-1)N^2(N+1)^2(N+2)}\Big)S_1
$$
$$
-\frac{112(2+N+N^2)^2S_1^3}{27(N-1)N^2(N+1)^2(N+2)}+\frac{160(2+N+N^2)^2S_3}{27(N-1)N^2(N+1)^2(N+2)}
$$
$$
-\Big(\frac{4P_{136}}{9(N-1)N^3(N+1)^3(N+2)}+\frac{16(2+N+N^2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Big)\zeta_2
$$
$$
-\frac{448(2+N+N^2)^2\zeta_3}{9(N-1)N^2(N+1)^2(N+2)}\Big]-\Big(\frac{32P_{165}}{81(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)}
$$
$$
+\frac{16(2+N+N^2)^2S_2}{3(N-1)N^2(N+1)^2(N+2)}\Big)S_1+\frac{16(2+N+N^2)^2S_1^3}{27(N-1)N^2(N+1)^2(N+2)}
$$
$$
-\frac{352(2+N+N^2)^2S_3}{27(N-1)N^2(N+1)^2(N+2)}+\frac{64(2+N+N^2)^2S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)}
$$
$$
+\Big(-\frac{8P_{139}}{9(N-1)N^3(N+1)^3(N+2)}+\frac{16(2+N+N^2)^2S_1}{3(N-1)N^2(N+1)^2(N+2)}\Big)\zeta_2
$$
$$
+\frac{P_{74}}{9(N-1)N^2(N+1)^2(N+2)}\zeta_3\Big]+C_F^2T_F\Big[\frac{32[\mathsf{B}_4+\mathsf{BS}_7-\mathsf{BS}_9+7\zeta_3\mathsf{BS}_3](2+N+N^2)}{N^2(N+1)^2}
$$
$$
-\frac{144(2+N+N^2)\zeta_4}{N^2(N+1)^2}+\frac{4(2+N+N^2)S_1^3P_{19}}{9(N-1)N^3(N+1)^3(N+2)}
$$
$$
-\frac{64S_{-1}S_2P_{125}}{15(N-3)(N-2)(N-1)^2N^2(N+1)^3}+\frac{64S_{2,-1}P_{125}}{15(N-3)(N-2)(N-1)^2N^2(N+1)^3}
$$
$$
-\frac{64S_{-2,-1}P_{125}}{15(N-3)(N-2)(N-1)^2N^2(N+1)^3}+\frac{32S_{-2,1}P_{149}}{3(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}
$$
$$
+\frac{16S_3P_{152}}{45(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}-\frac{16S_{2,1}P_{153}}{15(N-3)(N-2)(N-1)^2N^3(N+1)^3}
$$
$$
-\frac{[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3]\,2^{5-2N}\binom{2N}{N}P_{166}}{15(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}
$$

$$
\begin{aligned}
&+\frac{4S_2P_{177}}{15(N-3)(N-2)(N-1)^2N^4(N+1)^4(N+2)}\\
&+\frac{2P_{208}}{45(N-3)(N-2)(N-1)N^6(N+1)^6(N+2)}+\Bigg(\frac{4S_2P_{91}}{3(N-1)N^3(N+1)^3(N+2)}\\
&+\frac{8P_{186}}{15(N-3)(N-2)(N-1)N^5(N+1)^5(N+2)}+\frac{16(2+N+N^2)(-46+13N+13N^2)S_3}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{32(2+N+N^2)(-2+3N+3N^2)S_{2,1}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{256(2+N+N^2)S_{-2,1}}{(N-1)N^2(N+1)^2(N+2)}\Bigg)S_1\\
&+\Bigg(-\frac{4P_{133}}{3(N-1)N^4(N+1)^4(N+2)}+\frac{4(2+N+N^2)(-22+5N+5N^2)S_2}{3(N-1)N^2(N+1)^2(N+2)}\Bigg)S_1^2\\
&+\frac{2(2+N+N^2)^2S_1^4}{9(N-1)N^2(N+1)^2(N+2)}-\frac{2(2+N+N^2)(30+31N+31N^2)S_2^2}{3(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{4(2+N+N^2)(54+11N+11N^2)S_4}{3(N-1)N^2(N+1)^2(N+2)}+\Bigg(\frac{64S_{-1}P_{125}}{15(N-3)(N-2)(N-1)^2N^2(N+1)^3}\\
&-\frac{32S_1P_{149}}{3(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}-\frac{128(2+N+N^2)S_1^2}{(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{32P_{161}}{15(N-3)(N-2)(N-1)N^4(N+1)^4(N+2)}-\frac{256(2+N+N^2)S_2}{3(N-1)N^2(N+1)^2(N+2)}\Bigg)S_{-2}\\
&+\frac{64(2+N+N^2)S_{-2}^2}{(N-1)N^2(N+1)^2(N+2)}+\Bigg(-\frac{16P_{148}}{3(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}\\
&-\frac{128(2+N+N^2)S_1}{3(N-1)N^2(N+1)^2(N+2)}\Bigg)S_{-3}+\frac{128(2+N+N^2)S_{-4}}{3(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{128(-1+N+N^2)(2+N+N^2)S_{3,1}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{256(2+N+N^2)S_{-2,2}}{3(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{128(2+N+N^2)S_{-3,1}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{32(2+N+N^2)(-10+7N+7N^2)S_{2,1,1}}{3(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{256(2+N+N^2)S_{-2,1,1}}{(N-1)N^2(N+1)^2(N+2)}+\Bigg(\frac{4(2+N+N^2)S_1P_{20}}{(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{2(2+N+N^2)P_{105}}{(N-1)N^4(N+1)^4(N+2)}+\frac{4(2+N+N^2)^2S_1^2}{(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{12(2+N+N^2)^2S_2}{(N-1)N^2(N+1)^2(N+2)}\Bigg)\zeta_2+\Bigg(\frac{P_{180}}{15(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}\\
&-\frac{16(-118+N+N^2)(2+N+N^2)S_1}{3(N-1)N^2(N+1)^2(N+2)}\Bigg)\zeta_3\Bigg]+C_A^2T_F\Bigg[\frac{256S_{-2,2}P_7}{9(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{16S_{-2,3}P_{11}}{3N(N+1)}-\frac{16S_{-4,1}P_{12}}{3N(N+1)}+\frac{32S_{-5}P_{16}}{9N(N+1)}-\frac{32S_{2,1,-2}P_{28}}{9N(N+1)}+\frac{64S_{-2,-3}P_{29}}{9N(N+1)}-\frac{32S_{-2,1,-2}P_{30}}{9N(N+1)}\\
&+\frac{16S_{2,-3}P_{31}}{9N(N+1)}-\frac{16S_5P_{38}}{9N(N+1)}+\frac{16S_{2,3}P_{39}}{9N(N+1)}+\frac{32S_{-2,1,1}P_{52}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{16S_{-3,1}P_{64}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{8S_2^2P_{67}}{9(N-1)N^2(N+1)^2}-\frac{8[\mathsf{BS}_7-\mathsf{BS}_9+7\zeta_3\mathsf{BS}_3]P_{84}}{3(N-1)N^2(N+1)^2(N+2)}
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{16S_{2,1,1}P_{117}}{3(N-2)(N-1)N^2(N+1)^2(N+2)}+\frac{4S_4P_{120}}{9(N-2)(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{8S_{3,1}P_{123}}{9(N-2)(N-1)N^2(N+1)^2(N+2)}-\frac{16[S_{-1}S_2-S_{2,-1}+S_{-2,-1}]P_{137}}{15(N-3)(N-2)(N-1)^2N^2(N+1)^3(N+2)}\\
&-\frac{16S_{-2,1}P_{181}}{81(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}\\
&-\frac{[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3]\,2^{1-2N}\binom{2N}{N}P_{184}}{15(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\\
&-\frac{4S_3P_{198}}{405(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\\
&-\frac{2S_{2,1}P_{199}}{45(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\\
&+\frac{P_{215}}{14580(N-3)(N-2)^2(N-1)^2N^5(N+1)^5(N+2)^5}+\zeta_4\Bigg(\frac{96P_{13}}{(N-1)N^2(N+1)^2(N+2)}\\
&-96S_1\Bigg)+\mathrm{B}_4\Bigg(-\frac{8P_{25}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{32}{3}S_1\Bigg)+\Bigg(-\frac{32S_{-2,1}P_{56}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{8S_{2,1}P_{96}}{9(N-1)N^2(N+1)^2(N+2)}-\frac{16S_3P_{126}}{27(N-2)(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{4S_2P_{183}}{81(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}+\frac{16}{9}S_2^2+\frac{592}{9}S_4-\frac{832}{9}S_{3,1}-\frac{128}{9}S_{-2,2}\\
&+\frac{2P_{214}}{3645(N-3)(N-2)(N-1)^2N^5(N+1)^5(N+2)^4}-\frac{128}{9}S_{-3,1}+\frac{32}{3}S_{2,1,1}+\frac{256}{9}S_{-2,1,1}\Bigg)S_1\\
&+\Bigg(\frac{4S_2P_{54}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{P_{190}}{27(N-2)(N-1)^2N^4(N+1)^3(N+2)^3}+\frac{272}{9}S_3\\
&-\frac{32}{9}S_{2,1}-\frac{128}{9}S_{-2,1}\Bigg)S_1^2+\Bigg(-\frac{64(2N+1)P_6}{27(N-1)^2N^2(N+1)^2(N+2)^2}+\frac{64}{27}S_2\Bigg)S_1^3\\
&+\Bigg(-\frac{64S_{-2,1}P_5}{3N(N+1)}-\frac{16S_3P_{43}}{27N(N+1)}+\frac{P_{209}}{405(N-3)(N-2)^2(N-1)^2N^4(N+1)^4(N+2)^3}\\
&-\frac{80}{9}S_{2,1}\Bigg)S_2+\Bigg(\frac{64S_{2,1}P_{18}}{9N(N+1)}+\frac{16S_1^2P_{58}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{16S_2P_{114}}{9(N-2)(N-1)N^2(N+1)^2(N+2)}+\frac{16S_{-1}P_{137}}{15(N-3)(N-2)(N-1)^2N^2(N+1)^3(N+2)}\\
&+\frac{16S_1P_{182}}{81(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)}+\frac{128}{27}S_1^3-\frac{320}{27}S_3+\frac{64}{9}S_{-2,1}\\
&+\frac{16P_{204}}{405(N-3)(N-2)^2(N-1)^2N^4(N+1)^4(N+2)^3}\Bigg)S_{-2}+\Bigg(\frac{16P_{100}}{27(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{64}{9}S_1\Bigg)S_{-2}^2+\Bigg(\frac{32S_2P_{17}}{9N(N+1)}-\frac{16S_{-2}P_{32}}{9N(N+1)}-\frac{16S_1P_{62}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{64}{9}S_1^2\\
&+\frac{8P_{191}}{81(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\Bigg)S_{-3}+\Bigg(\frac{16P_{79}}{9(N-1)N^2(N+1)^2(N+2)}
\end{aligned}
$$

$$+\frac{64}{9}S_1\Big)S_{-4}-96S_{4,1}+16S_{2,2,1}+\frac{416}{3}S_{3,1,1}+\frac{128}{9}S_{-2,2,1}+\frac{128}{9}S_{-3,1,1}-\frac{160}{9}S_{2,1,1,1}$$
$$-\frac{256}{9}S_{-2,1,1,1}+\Big(-\frac{4P_{176}}{27(N-1)^2N^3(N+1)^3(N+2)^3}+\Big[-\frac{16P_{130}}{27(N-1)^2N^2(N+1)^2(N+2)^2}$$
$$+\frac{32}{3}S_2\Big]S_1-\frac{64(1+N+N^2)S_2}{3(N-1)N(N+1)(N+2)}+\frac{16}{3}S_3+\Big(-\frac{64(1+N+N^2)}{3(N-1)N(N+1)(N+2)}$$
$$+\frac{32}{3}S_1\Big)S_{-2}+\frac{16}{3}S_{-3}-\frac{32}{3}S_{-2,1}\Big)\zeta_2+\Big(-\frac{32S_2P_5}{3N(N+1)}-\frac{32S_{-2}P_{11}}{3N(N+1)}-\frac{80}{3}S_1^2$$
$$+\frac{4S_1P_{129}}{27(N-2)(N-1)N^2(N+1)^2(N+2)}$$
$$+\frac{P_{201}}{1080(N-3)(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\Big)\zeta_3\Big]+\frac{64}{27}T_F^3\zeta_3\Big\}, \tag{3.1}$$

with

$$\mathsf{B}_4=-4\zeta_2\ln^2(2)+\frac{2}{3}\ln^4(2)-\frac{13}{2}\zeta_4+16\mathrm{Li}_4\left(\frac{1}{2}\right). \tag{3.2}$$

The polynomials P_i are listed in appendix A. Eq. (3.1) possesses a removable pole at $N=2$. By a series expansion one obtains eq. (8.67) of ref. [8]. Furthermore, also the even moments for $N=4$ to $N=10$ agree with the result in ref. [8]. There are removable poles also at $N=3, N=1/2$ and $N=3/2$, see also ref. [33]. To see their cancellation, one has to expand the Mellin inversion to x-space around $x=0$. In the present case one finds the most singular terms $\propto \ln(x)/x$ and $\propto 1/x$, i.e. it is proven that rightmost pole is situated at $N=1$, as expected in the unpolarized gluonic case. Note that in general one may not expect to cancel the above poles by just using the N-space representation.

It is also instructive to see how accurate the asymptotic expansion of $a_{gg,Q}^{(3)}(N)$ represents higher moments. We compare the expressions for the N_F-independent part only, since the N_F-dependent part solely consists of harmonic sums. One may represent $a_{gg,Q}^{(3)}(N)$ by

$$a_{gg,Q}^{(3)}(N)=a_{gg,Q,\delta}^{(3)}+a_{gg,Q,\mathrm{pl}}^{(3)}(N)+\tilde{a}_{gg,Q}^{(3)}(N). \tag{3.3}$$

Here $a_{gg,Q,\delta}^{(3)}$ denotes the N-independent part, cf. (4.6), $a_{gg,Q,\mathrm{pl}}^{(3)}(N)$ the part $\propto L$ with

$$L=\ln(N)+\gamma_E, \tag{3.4}$$

where γ_E denotes the Euler-Mascheroni constant. The explicit expressions for the asymptotic expansion of $(\Delta)\tilde{a}_{gg,Q}^{(3)}(N)$ for the N_F-independent part are given in appendix D. The asymptotic representation for $\tilde{a}_{gg,Q}^{(3)}(N)$ for positive even integer values converges very quickly, as shown in table 1. Already for $N=4$ a reasonable approximation is obtained by expanding to $O(1/N^{10})$.

We turn now to $\Delta a_{gg,Q}(N)$ which is given in the Larin scheme [159] by

$$\Delta a_{gg,Q}^{(3)}(N)=\frac{1}{2}\left(1-(-1)^N\right)$$
$$\times\Big\{C_A\Big[C_FT_F\Big(\frac{32S_{-2,2}P_8}{(N-1)N^2(N+1)^2(N+2)}+\frac{32S_{-3,1}P_{10}}{3(N-1)N^2(N+1)^2(N+2)}$$

N	complete expression	rel. asymp. accuracy
4	–430.337594532836914	−2.10169D–3
6	–261.554324759832203	−3.13097D–5
8	–193.698203673549029	−1.29985D–6
10	–156.572494406521072	−9.09255D–8
12	–132.845604857074259	−7.66447D–9
22	–79.9576964391278831	3.81320D–11
42	–48.0359673792636099	1.41090D–13
102	–24.2090141858051135	3.14898D–17

Table 1. Numerical comparison of $\tilde{a}_{gg,Q}^{(3)}(N)$ in QCD with its asymptotic representation for $N_F = 0$ retaining 10 terms of the asymptotic expansion.

$$
\begin{aligned}
&+\frac{32\left[\mathsf{BS}_7-\mathsf{BS}_9+7\zeta_3\mathsf{BS}_3\right]P_{14}}{3N^2(N+1)^2}-\frac{64S_{-2,1,1}P_{27}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{4S_1^3P_{34}}{27N^3(N+1)^3}\\
&-\frac{16S_{-4}P_{36}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{4S_4P_{47}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{16S_{2,1,1}P_{66}}{3(N-1)N^2(N+1)^2}\\
&+\frac{32\left[S_{-1}S_2-S_{2,-1}+S_{-2,-1}\right]P_{81}}{(N-2)(N-1)N^3(N+1)^2(N+2)}+\frac{16S_{3,1}P_{83}}{3(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{16S_{-2,1}P_{115}}{3(N-2)(N-1)N^3(N+1)^3(N+2)}+\frac{\left[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3\right]2^{2-2N}\binom{2N}{N}P_{132}}{3(N-2)(N-1)N^3(N+1)^3}\\
&+\frac{8S_3P_{150}}{27(N-2)(N-1)N^3(N+1)^3(N+2)}-\frac{8S_{2,1}P_{160}}{3(N-2)(N-1)N^3(N+1)^3(N+2)}\\
&+\frac{4S_2P_{188}}{9(N-2)(N-1)^2N^4(N+1)^4(N+2)}+\frac{P_{211}}{486(N-2)(N-1)^2N^6(N+1)^6(N+2)^2}\\
&+\left(\frac{32(N-3)(N+4)}{3N^2(N+1)^2}-\frac{64S_1}{3}\right)\mathsf{B}_4+\left(-\frac{48(N-3)(N+4)}{N^2(N+1)^2}+96S_1\right)\zeta_4\\
&+\Biggl(\frac{64S_{-2,1}P_{26}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{32S_3P_{37}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{32S_{2,1}P_{76}}{3(N-1)N^2(N+1)^2(N+2)}+\frac{4S_2P_{121}}{3(N-1)N^3(N+1)^3(N+2)}\\
&+\frac{8P_{205}}{81(N-2)(N-1)^2N^5(N+1)^5(N+2)^2}\Biggr)S_1+\Biggl(-\frac{4S_2P_{40}}{3(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{4P_{127}}{27(N-1)N^3(N+1)^4(N+2)}\Biggr)S_1^2-\frac{2(N-1)(N+2)S_1^4}{9N^2(N+1)^2}+\frac{2(6+5N+5N^2)S_2^2}{N^2(N+1)^2}\\
&+\Biggl(-\frac{32S_1^2P_{24}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{32S_{-1}P_{81}}{(N-2)(N-1)N^3(N+1)^2(N+2)}\\
&+\frac{16S_1P_{147}}{3(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}-\frac{16P_{164}}{3(N-2)(N-1)^2N^3(N+1)^4(N+2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{32\left(2-15N-12N^2+N^3\right)S_2}{3\left(N-1\right)N^2\left(N+1\right)^2}\Bigg)S_{-2}-\frac{32\left(-13+2N+2N^2\right)S_{-2}^2}{3N^2\left(N+1\right)^2}\\
&+\left(-\frac{32S_1P_9}{3\left(N-1\right)N^2\left(N+1\right)^2\left(N+2\right)}+\frac{8P_{124}}{3\left(N-2\right)\left(N-1\right)N^3\left(N+1\right)^3\left(N+2\right)}\right)S_{-3}\\
&+\left(\frac{4S_1P_{101}}{3N^3\left(N+1\right)^3}+\frac{4P_{138}}{9N^4\left(N+1\right)^4}-\frac{4\left(N-1\right)\left(N+2\right)S_1^2}{N^2\left(N+1\right)^2}-\frac{12\left(N-1\right)\left(N+2\right)S_2}{N^2\left(N+1\right)^2}\right.\\
&\left.-\frac{24\left(N-1\right)\left(N+2\right)S_{-2}}{N^2\left(N+1\right)^2}\right)\zeta_2+\left(-\frac{8S_1P_{110}}{9\left(N-1\right)N^2\left(N+1\right)^2\left(N+2\right)}\right.\\
&\left.+\frac{P_{170}}{36\left(N-2\right)\left(N-1\right)N^3\left(N+1\right)^3\left(N+2\right)}\right)\zeta_3\Bigg)+T_F^2\Bigg(\frac{4S_2P_{48}}{135N^2\left(N+1\right)^2}\\
&+\frac{\left[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3\right]2^{2-2N}\binom{2N}{N}P_{109}}{45N\left(N+1\right)^2\left(2N-3\right)\left(2N-1\right)}+\frac{P_{175}}{3645N^4\left(N+1\right)^4\left(2N-3\right)\left(2N-1\right)}\\
&+N_F\Bigg(\frac{4S_2P_{44}}{27N^2\left(N+1\right)^2}-\frac{8S_1P_{113}}{729N^3\left(N+1\right)^3}-\frac{2P_{145}}{729N^4\left(N+1\right)^4}+\frac{4\left(16-9N-25N^2-16N^3\right)S_1^2}{27N^2\left(N+1\right)^2}\\
&+\left(\frac{4P_{33}}{27N^2\left(N+1\right)^2}-\frac{160S_1}{27}\right)\zeta_2+\left(-\frac{896}{27N\left(N+1\right)}+\frac{448S_1}{27}\right)\zeta_3\Bigg)-\frac{64\left(N+2\right)S_3}{15\left(N+1\right)}\\
&-\frac{8S_1P_{146}}{3645N^3\left(N+1\right)^3\left(2N-3\right)\left(2N-1\right)}-\frac{4\left(89-108N+25N^2+70N^3\right)S_1^2}{135N^2\left(N+1\right)^2}+\frac{64\left(N+2\right)S_{2,1}}{15\left(N+1\right)}\\
&+\left(\frac{4P_{45}}{27N^2\left(N+1\right)^2}-\frac{560S_1}{27}\right)\zeta_2+\left(-\frac{7\left(-3200+2439N+1287N^2\right)}{270N\left(N+1\right)}-\frac{1120}{27}S_1\right)\zeta_3\Bigg)\Bigg]\\
&+C_FT_F^2\Bigg[-\frac{2P_{194}}{243N^5\left(N+1\right)^5\left(2N-3\right)\left(2N-1\right)}+N_F\Bigg(-\frac{2P_{171}}{243N^5\left(N+1\right)^5}\\
&-\left(\frac{32\left(N-1\right)\left(N+2\right)P_{46}}{81N^4\left(N+1\right)^4}+\frac{16\left(N-1\right)\left(N+2\right)S_2}{3N^2\left(N+1\right)^2}\right)S_1+\frac{160\left(N-1\right)\left(N+2\right)S_3}{27N^2\left(N+1\right)^2}\\
&+\frac{16\left(N-1\right)\left(N+2\right)\left(6+20N+29N^2\right)S_1^2}{27N^3\left(N+1\right)^3}-\frac{112\left(N-1\right)\left(N+2\right)S_1^3}{27N^2\left(N+1\right)^2}\\
&-\frac{16\left(N-1\right)\left(N+2\right)\left(-6-8N+N^2\right)S_2}{9N^3\left(N+1\right)^3}-\left(\frac{4P_{95}}{9N^3\left(N+1\right)^3}+\frac{16\left(N-1\right)\left(N+2\right)S_1}{3N^2\left(N+1\right)^2}\right)\zeta_2\\
&-\frac{448\left(N-1\right)\left(N+2\right)\zeta_3}{9N^2\left(N+1\right)^2}\Bigg)-\frac{\left[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3\right]\left(N-1\right)\left(N+2\right)\left(8+9N-9N^2\right)2^{4-2N}\binom{2N}{N}}{3N\left(N+1\right)^2\left(2N-3\right)\left(2N-1\right)}\\
&+\left(\frac{32\left(N-1\right)\left(N+2\right)P_{103}}{81N^4\left(N+1\right)^4\left(2N-3\right)\left(2N-1\right)}-\frac{16\left(N-1\right)\left(N+2\right)S_2}{3N^2\left(N+1\right)^2}\right)S_1\\
&+\frac{16\left(N-1\right)\left(N+2\right)\left(-6-8N+N^2\right)S_1^2}{27N^3\left(N+1\right)^3}+\frac{16\left(N-1\right)\left(N+2\right)S_1^3}{27N^2\left(N+1\right)^2}\\
&-\frac{16\left(N-1\right)\left(N+2\right)\left(-6-8N+N^2\right)S_2}{9N^3\left(N+1\right)^3}-\frac{352\left(N-1\right)\left(N+2\right)S_3}{27N^2\left(N+1\right)^2}+\frac{64\left(N-1\right)\left(N+2\right)S_{2,1}}{3N^2\left(N+1\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{8P_{102}}{9N^3(N+1)^3}+\frac{16(N-1)(N+2)S_1}{3N^2(N+1)^2}\right)\zeta_2+\frac{P_1}{9N^2(N+1)^2}\zeta_3\Bigg]\\
&+C_F^2T_F\Bigg[\frac{32[\mathsf{B}_4+\mathsf{BS}_7-\mathsf{BS}_9+7\zeta_3\mathsf{BS}_3](2+N+N^2)}{N^2(N+1)^2}-\frac{144(2+N+N^2)\zeta_4}{N^2(N+1)^2}+\frac{4S_1^3P_{15}}{9N^3(N+1)^3}\\
&-\frac{64[S_{-1}S_2-S_{2,-1}+S_{-2,-1}]P_{60}}{3(N-2)(N-1)N^3(N+1)(N+2)}-\frac{[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3]2^{5-2N}\binom{2N}{N}P_{85}}{3(N-2)(N-1)N^3(N+1)^2}\\
&+\frac{32S_{-2,1}P_{89}}{3(N-2)(N-1)N^3(N+1)^3}+\frac{16S_3P_{119}}{9(N-2)(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{16S_{2,1}P_{134}}{3(N-2)(N-1)N^3(N+1)^3(N+2)}+\frac{4S_2P_{151}}{3(N-2)(N-1)N^4(N+1)^4(N+2)}\\
&+\frac{2P_{210}}{9(N-2)(N-1)^2N^6(N+1)^6(N+2)^2}+\left(-\frac{4S_2P_{51}}{3N^3(N+1)^3(N+2)}\right.\\
&+\frac{8P_{163}}{3(N-2)N^5(N+1)^5(N+2)}+\frac{16(46+13N+13N^2)S_3}{9N^2(N+1)^2}+\frac{32(2+3N+3N^2)S_{2,1}}{3N^2(N+1)^2}\\
&\left.-\frac{256S_{-2,1}}{N^2(N+1)^2}\right)S_1+\left(-\frac{4P_{90}}{3N^4(N+1)^4}+\frac{4(22+5N+5N^2)S_2}{3N^2(N+1)^2}\right)S_1^2+\frac{2(N-1)(N+2)S_1^4}{9N^2(N+1)^2}\\
&-\frac{2(-30+31N+31N^2)S_2^2}{3N^2(N+1)^2}-\frac{4(-54+11N+11N^2)S_4}{3N^2(N+1)^2}\\
&+\left(\frac{64S_{-1}P_{60}}{3(N-2)(N-1)N^3(N+1)(N+2)}+\frac{32P_{65}}{3(N-2)(N-1)^2(N+1)^2(N+2)^2}\right.\\
&\left.-\frac{32S_1P_{89}}{3(N-2)(N-1)N^3(N+1)^3}+\frac{128S_1^2}{N^2(N+1)^2}+\frac{256S_2}{3N^2(N+1)^2}\right)S_{-2}-\frac{64S_{-2}^2}{N^2(N+1)^2}\\
&+\left(-\frac{16P_{118}}{3(N-2)(N-1)N^3(N+1)^3(N+2)}+\frac{128S_1}{3N^2(N+1)^2}\right)S_{-3}-\frac{128S_{-4}}{3N^2(N+1)^2}\\
&+\frac{128(1+N+N^2)S_{3,1}}{3N^2(N+1)^2}-\frac{256S_{-2,2}}{3N^2(N+1)^2}-\frac{128S_{-3,1}}{3N^2(N+1)^2}-\frac{32(10+7N+7N^2)S_{2,1,1}}{3N^2(N+1)^2}\\
&+\frac{256S_{-2,1,1}}{N^2(N+1)^2}+\left(-\frac{2(N-1)(N+2)P_{42}}{N^4(N+1)^4}+\frac{4(N-1)(N+2)(-4-3N+3N^2)S_1}{N^3(N+1)^3}\right.\\
&\left.+\frac{4(N-1)(N+2)S_1^2}{N^2(N+1)^2}-\frac{12(N-1)(N+2)S_2}{N^2(N+1)^2}\right)\zeta_2+\left(\frac{P_{154}}{3(N-2)(N-1)N^3(N+1)^3(N+2)}\right.\\
&\left.-\frac{16(118+N+N^2)S_1}{3N^2(N+1)^2}\right)\zeta_3\Bigg]+C_A^2T_F\Bigg[\frac{256S_{-2,2}P_4}{9(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{8[\mathsf{BS}_7-\mathsf{BS}_9+7\zeta_3\mathsf{BS}_3](N-1)(18+21N+16N^2+4N^3)}{3N^2(N+1)^2}\\
&+\frac{32S_{-2,1,1}P_{53}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{16S_{-3,1}P_{63}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{8S_2^2P_{68}}{9N^2(N+1)^2(N+2)}\\
&-\frac{16S_{2,1,1}P_{86}}{3(N-1)N^2(N+1)^2(N+2)}-\frac{16[S_{-1}S_2-S_{2,-1}+S_{-2,-1}]P_{87}}{3(N-2)(N-1)N^3(N+1)^2(N+2)}
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{4S_4P_{88}}{9(N-1)N^2(N+1)^2(N+2)}-\frac{8S_{3,1}P_{94}}{9(N-1)N^2(N+1)^2(N+2)}\\
&-\frac{[\mathsf{BS}_8-\mathsf{BS}_4S_1+7\zeta_3]\,2^{1-2N}\binom{2N}{N}P_{131}}{3(N-2)(N-1)N^3(N+1)^3}-\frac{16S_{-2,1}P_{155}}{81(N-2)(N-1)N^3(N+1)^3(N+2)}\\
&-\frac{2S_{2,1}P_{168}}{9(N-2)(N-1)N^3(N+1)^3(N+2)}-\frac{4S_3P_{169}}{81(N-2)(N-1)N^3(N+1)^3(N+2)}\\
&+\frac{P_{212}}{2916(N-2)(N-1)^2N^6(N+1)^6(N+2)^2}+\mathsf{B}_4\left(-\frac{8(-18+5N+5N^2)}{3N^2(N+1)^2}+\frac{32}{3}S_1\right)\\
&+\zeta_4\left(\frac{96(-3+2N+2N^2)}{N^2(N+1)^2}-96S_1\right)+\Bigg(-\frac{32S_{-2,1}P_{57}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{8S_{2,1}P_{97}}{9(N-1)N^2(N+1)^2(N+2)}-\frac{16S_3P_{104}}{27(N-1)N^2(N+1)^2(N+2)}+\frac{16}{9}S_2^2+\frac{592}{9}S_4-\frac{832}{9}S_{3,1}\\
&+\frac{4S_2P_{143}}{81(N-1)N^3(N+1)^3(N+2)}+\frac{2P_{206}}{729(N-2)(N-1)^2N^5(N+1)^5(N+2)^2}\\
&-\frac{128}{9}S_{-2,2}-\frac{128}{9}S_{-3,1}+\frac{32}{3}S_{2,1,1}+\frac{256}{9}S_{-2,1,1}\Bigg)S_1+\Bigg(\frac{4S_2P_{55}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{P_{142}}{27(N-1)N^4(N+1)^3(N+2)}+\frac{272}{9}S_3-\frac{32}{9}S_{2,1}-\frac{128}{9}S_{-2,1}\Bigg)S_1^2+\Bigg(-\frac{64(2N+1)}{27N^2(N+1)^2}\\
&+\frac{64}{27}S_2\Bigg)S_1^3+\Bigg(\frac{P_{193}}{81(N-2)(N-1)^2N^4(N+1)^4(N+2)}-\frac{16(-108+175N+175N^2)S_3}{27N(N+1)}\\
&-\frac{80}{9}S_{2,1}-\frac{64(-3+N+N^2)S_{-2,1}}{3N(N+1)}\Bigg)S_2-\frac{16(-54+55N+55N^2)S_5}{9N(N+1)}\\
&+\Bigg(\frac{16S_1^2P_{59}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{16S_2P_{80}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{16S_{-1}P_{87}}{3(N-2)(N-1)N^3(N+1)^2(N+2)}+\frac{16S_1P_{179}}{81(N-2)(N-1)^2N^3(N+1)^3(N+2)^2}\\
&+\frac{16P_{187}}{81(N-2)(N-1)^2N^4(N+1)^4(N+2)^2}+\frac{128}{27}S_1^3-\frac{320}{27}S_3+\frac{64(-9+2N+2N^2)S_{2,1}}{9N(N+1)}\\
&+\frac{64}{9}S_{-2,1}\Bigg)S_{-2}+\left(\frac{16P_{71}}{27N^2(N+1)^2(N+2)}-\frac{64}{9}S_1\right)S_{-2}^2+\Bigg(-\frac{16S_1P_{61}}{9(N-1)N^2(N+1)^2(N+2)}\\
&+\frac{8P_{156}}{81(N-2)(N-1)N^3(N+1)^3(N+2)}+\frac{64}{9}S_1^2-\frac{32(9+2N+2N^2)S_2}{9N(N+1)}\\
&-\frac{16(-18+19N+19N^2)S_{-2}}{9N(N+1)}\Bigg)S_{-3}+\left(\frac{16P_{78}}{9(N-1)N^2(N+1)^2(N+2)}+\frac{64}{9}S_1\right)S_{-4}\\
&-\frac{32(9+5N+5N^2)S_{-5}}{9N(N+1)}+\frac{16(-54+77N+77N^2)S_{2,3}}{9N(N+1)}+\frac{16(-18+17N+17N^2)S_{2,-3}}{9N(N+1)}\\
&-96S_{4,1}-\frac{16(N-2)(N+3)S_{-2,3}}{3N(N+1)}+\frac{64(-18+11N+11N^2)S_{-2,-3}}{9N(N+1)}
\end{aligned}
$$

N	complete expression	rel. asymp. accuracy
3	–429.345090408771279	−9.39608D–4
5	–264.713704879676430	−7.20184D–6
7	–195.879611926179533	−2.77043D–7
9	–158.086523949063478	−2.36920D–8
11	–133.968747075663141	−3.28594D–9
21	–80.4139205966345783	−5.45018D–12
41	–48.2243776140486971	−7.23295D–15
101	–24.2616935237270326	−1.03115D–18

Table 2. Numerical comparison of $\Delta\tilde{a}_{gg,Q}^{(3)}(N)$ in QCD with its asymptotic representation for $N_F = 0$ retaining 10 terms of the asymptotic expansion.

$$-\frac{16(N-1)(N+2)S_{-4,1}}{N(N+1)}-\frac{32(-18+5N+5N^2)S_{2,1,-2}}{9N(N+1)}+16S_{2,2,1}+\frac{416}{3}S_{3,1,1}$$
$$-\frac{32(-18+13N+13N^2)S_{-2,1,-2}}{9N(N+1)}+\frac{128}{9}S_{-2,2,1}+\frac{128}{9}S_{-3,1,1}-\frac{160}{9}S_{2,1,1,1}$$
$$-\frac{256}{9}S_{-2,1,1,1}+\Biggl[-\frac{4P_{99}}{27N^3(N+1)^3}+\Biggl(-\frac{16(36+72N+N^2+2N^3+N^4)}{27N^2(N+1)^2}$$
$$+\frac{32}{3}S_2\Biggr)S_1-\frac{64S_2}{3N(N+1)}+\frac{16}{3}S_3+\Biggl(-\frac{64}{3N(N+1)}+\frac{32S_1}{3}\Biggr)S_{-2}+\frac{16}{3}S_{-3}$$
$$-\frac{32}{3}S_{-2,1}\Biggr]\zeta_2+\Biggl(-\frac{32(-3+N+N^2)S_2}{3N(N+1)}+\frac{P_{173}}{216(N-2)(N-1)N^3(N+1)^3(N+2)}$$
$$+\frac{4S_1P_{112}}{27(N-1)N^2(N+1)^2(N+2)}-\frac{80}{3}S_1^2-\frac{32(N-2)(N+3)S_{-2}}{3N(N+1)}\Biggr)\zeta_3\Biggr]+\frac{64}{27}T_F^3\zeta_3\Biggr\}. \tag{3.5}$$

One may represent $\Delta a_{gg,Q}^{(3)}(N)$ in an analogous way to eq. (3.3). As in the unpolarized case, the expansion converges very quickly for the positive odd integer values, cf. table 2, with a reasonable description down to $N = 3$.

4 The x-space representation

We perform an analytic inverse Mellin transform to x-space by using algorithms implemented in HarmonicSums. In momentum fraction space, the quantities $(\Delta)a_{gg,Q}(x)$ depend besides harmonic polylogarithms, $\mathrm{H}_{\vec{a}}(x)$, on G-functions up to weight $\mathsf{w} = 5$ at arguments x and 1 over the alphabet (2.8), e.g.

$$\mathrm{G}\left(\left\{\sqrt{(1-y)y},\frac{1}{y},\sqrt{(1-y)y},\frac{1}{y},\frac{1}{1-y}\right\},x\right)=\mathrm{G}(\{5,1,5,1,2\},x), \tag{4.1}$$

and 17 similar functions both in the unpolarized and polarized case. The appearing constants can all be calculated analytically by using HarmonicSums and only multiple zeta values remain at the end. In addition to this, the following 48 harmonic polylogarithms contribute

$$\Big\{\mathrm{H}_0,\mathrm{H}_{-1},\mathrm{H}_1,\mathrm{H}_{-1,1},\mathrm{H}_{0,-1},\mathrm{H}_{0,1},\mathrm{H}_{0,-1,-1},\mathrm{H}_{0,-1,1},\mathrm{H}_{0,0,-1},\mathrm{H}_{0,0,1},\mathrm{H}_{0,1,-1},\mathrm{H}_{0,1,1},\mathrm{H}_{0,-1,-1,-1},$$
$$\mathrm{H}_{0,-1,-1,1},\mathrm{H}_{0,-1,0,1},\mathrm{H}_{0,-1,1,-1},\mathrm{H}_{0,-1,1,1},\mathrm{H}_{0,0,-1,-1},\mathrm{H}_{0,0,-1,1},\mathrm{H}_{0,0,0,-1},\mathrm{H}_{0,0,0,1},\mathrm{H}_{0,0,1,-1},\mathrm{H}_{0,0,1,1},$$
$$\mathrm{H}_{0,1,-1,-1},\mathrm{H}_{0,1,-1,1},\mathrm{H}_{0,1,1,-1},\mathrm{H}_{0,1,1,1},\mathrm{H}_{0,-1,-1,0,1},\mathrm{H}_{0,-1,0,-1,-1},\mathrm{H}_{0,-1,0,1,1},\mathrm{H}_{0,-1,1,0,1},$$
$$\mathrm{H}_{0,0,-1,-1,-1},\mathrm{H}_{0,0,-1,0,-1},\mathrm{H}_{0,0,-1,0,1},\mathrm{H}_{0,0,-1,1,1},\mathrm{H}_{0,0,0,-1,-1},\mathrm{H}_{0,0,0,-1,1},\mathrm{H}_{0,0,0,0,-1},\mathrm{H}_{0,0,0,0,1},$$
$$\mathrm{H}_{0,0,0,1,-1},\mathrm{H}_{0,0,0,1,1},\mathrm{H}_{0,0,1,0,-1},\mathrm{H}_{0,0,1,0,1},\mathrm{H}_{0,0,1,1,-1},\mathrm{H}_{0,0,1,1,1},\mathrm{H}_{0,1,0,1,-1},\mathrm{H}_{0,1,0,1,1},\mathrm{H}_{0,1,1,1,1}\Big\}, \tag{4.2}$$

both after algebraic reduction for the G- and H-functions [110], where we suppressed the argument x of the harmonic polylogarithms.

Furthermore, denominator structures of the kind

$$\frac{1}{(1\pm x)^k}, \quad k=2,3 \tag{4.3}$$

appear, which are also known from other massive calculations [160, 161]. Since the corresponding expressions are very lengthy, we present them only in a file in computer-readable form in the supplementary material attached to this paper. Here we discuss their principal structure. The expressions for $(\Delta)a_{gg,Q}^{(3)}(x)$ have the following form

$$(\Delta)\,a_{gg,Q}^{(3)}\,(x) = (\Delta)\,a_{gg,Q,\delta}^{(3)}\delta\,(1-x) + \Big[(\Delta)\,a_{gg,Q,\mathrm{plus}}^{(3)}\,(x)\Big]_+ + (\Delta)\,a_{gg,Q,\mathrm{reg}}^{(3)}(x), \tag{4.4}$$

where

$$\int_0^1 dx g(x)[f(x)]_+ := \int_0^1 [g(x)-g(1)]f(x). \tag{4.5}$$

For the $\delta(1-x)$ and +-function contributions we obtain

$$\begin{aligned}(\Delta)a_{gg,Q,\delta}^{(3)} = T_F\Big\{&C_F\Big[C_A\Big(\frac{16541}{162}-\frac{64\mathrm{B}_4}{3}+\frac{128\zeta_4}{3}+52\zeta_2-\frac{2617\zeta_3}{12}\Big)+T_F\Big(-\frac{1478}{81}\\&+N_F\Big(-\frac{1942}{81}-\frac{20\zeta_2}{3}\Big)-\frac{88\zeta_2}{3}-7\zeta_3\Big)\Big]+C_A^2\Big[\frac{34315}{324}+\frac{32\mathrm{B}_4}{3}-\frac{3778\zeta_4}{27}\\&+\frac{992}{27}\zeta_2+\Big(\frac{20435}{216}+24\zeta_2\Big)\zeta_3-\frac{304}{9}\zeta_5\Big]+C_AT_F\Big[\frac{2587}{135}+N_F\Big(-\frac{178}{9}+\frac{196\zeta_2}{27}\Big)\\&+\frac{572\zeta_2}{27}-\frac{291\zeta_3}{10}\Big]+C_F^2\Big[\frac{274}{9}+\frac{95\zeta_3}{3}\Big]+\frac{64}{27}T_F^2\zeta_3\Big\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}(\Delta)a_{gg,Q,\mathrm{plus}}^{(3)} = \frac{T_F}{1-x}\Big\{&C_AT_F\Big[\frac{35168}{729}+N_F\Big(\frac{55552}{729}+\frac{160\zeta_2}{27}-\frac{448\zeta_3}{27}\Big)+\frac{560}{27}\zeta_2+\frac{1120}{27}\zeta_3\Big]\\&+C_A^2\Big[-\frac{32564}{729}-\frac{32\mathrm{B}_4}{3}+104\zeta_4-\frac{3248\zeta_2}{81}-\frac{1796\zeta_3}{27}\Big]+C_AC_F\Big[-\frac{6152}{27}+\frac{64\mathrm{B}_4}{3}\\&-96\zeta_4-40\zeta_2+\frac{1208\zeta_3}{9}\Big]\Big\}.\end{aligned} \tag{4.7}$$

They are the same in both cases. The regular parts $(\Delta)a_{gg,Q,\mathrm{reg}}^{(3)}(x)$ are given in a file in the supplementary material attached to the present paper.

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We further expand $(\Delta)a_{gg,Q}(x)$ in the small x and large x regions, where much simpler structures ruled by logarithms are obtained. The principal x-space structure to higher powers in x is

$$\begin{aligned}(\Delta)a_{gg,Q}^{x\to 0}(x) = {}& c_1\frac{\ln(x)}{x} + c_2\frac{1}{x} + \sum_{k=0}^{\infty}\Big[c_{3,k} + c_{4,k}\ln(x) + c_{5,k}\ln^6(x) + c_{7,k}\ln^3(x) + c_{8,k}\ln^4(x) \\ & + c_{9,k}\ln^5(x)\Big]x^k \end{aligned} \tag{4.8}$$

$$\begin{aligned}(\Delta)a_{gg,Q}^{x\to 1}(x) = {}& d_1\delta(1-x) + \frac{d_2}{1-x} + \sum_{k=0}^{\infty}\Big[d_{3,k} + d_{4,k}\ln(1-x) + d_{5,k}\ln^2(1-x) \\ & + d_{6,k}\ln^3(1-x) + d_{7,k}\ln^4(1-x)\Big]x^k, \end{aligned} \tag{4.9}$$

where some of the coefficients can be zero. The leading behavior for the two expansions around $x=0$ and $x=1$ in the unpolarized case are given by

$$\begin{aligned}
& a_{gg,Q}^{x\to 0}(x) \propto \\
& \frac{1}{x}\Bigg\{\ln(x)\Bigg[C_A^2T_F\left(-\frac{11488}{81}+\frac{224\zeta_2}{27}+\frac{256\zeta_3}{3}\right)+C_AC_FT_F\Bigg(-\frac{15040}{243}-\frac{1408\zeta_2}{27} \\
& -\frac{1088\zeta_3}{9}\Bigg)\Bigg]+C_AT_F^2\Bigg[\frac{112016}{729}+\frac{1288}{27}\zeta_2+\frac{1120}{27}\zeta_3+\left(\frac{108256}{729}+\frac{368\zeta_2}{27}-\frac{448\zeta_3}{27}\right) \\
& \times N_F\Bigg]+C_F\Bigg[T_F^2\Bigg(-\frac{107488}{729}-\frac{656}{27}\zeta_2+\frac{3904}{27}\zeta_3+\left(\frac{116800}{729}+\frac{224\zeta_2}{27}-\frac{1792\zeta_3}{27}\right)N_F\Bigg) \\
& +C_AT_F\left(-\frac{5538448}{3645}+\frac{1664\mathrm{B}_4}{3}-\frac{43024\zeta_4}{9}+\frac{12208}{27}\zeta_2+\frac{211504}{45}\zeta_3\right)\Bigg] \\
& +C_A^2T_F\left(-\frac{4849484}{3645}-\frac{352\mathrm{B}_4}{3}+\frac{11056\zeta_4}{9}-\frac{1088}{81}\zeta_2-\frac{84764}{135}\zeta_3\right) \\
& +C_F^2T_F\left(\frac{10048}{5}-640\mathrm{B}_4+\frac{51104\zeta_4}{9}-\frac{10096}{9}\zeta_2-\frac{280016}{45}\zeta_3\right)\Bigg\} \\
& +\left[-\frac{4}{3}C_FC_AT_F+\frac{2}{15}C_F^2T_F\right]\ln^5(x)+\Bigg[-\frac{40}{27}C_A^2T_F+\frac{4}{9}C_F^2T_F+C_F\Bigg(-\frac{296}{27}C_AT_F \\
& +\left(\frac{28}{27}+\frac{56}{27}N_F\right)T_F^2\Bigg)\Bigg]\ln^4(x)+\Bigg[\frac{112}{81}C_A(1+2N_F)T_F^2+C_F\Bigg(\left(\frac{1016}{81}+\frac{496}{81}N_F\right)T_F^2 \\
& +C_AT_F\left(-\frac{10372}{81}-\frac{328\zeta_2}{9}\right)\Bigg)+C_F^2T_F\left[-\frac{2}{3}+\frac{4\zeta_2}{9}\right]+C_A^2T_F\left[-\frac{1672}{81}+8\zeta_2\right]\Bigg]\ln^3(x) \\
& +\Bigg[\frac{8}{81}C_A(155+118N_F)T_F^2+C_F\Bigg[T_F^2\left(-\frac{32}{81}+N_F\left(\frac{3872}{81}-\frac{16\zeta_2}{9}\right)+\frac{232\zeta_2}{9}\right) \\
& +C_AT_F\left(-\frac{70304}{81}-\frac{680\zeta_2}{9}+\frac{80\zeta_3}{3}\right)\Bigg]+C_A^2T_F\left[\frac{4684}{81}+\frac{20\zeta_2}{3}\right]+C_F^2T_F\Bigg[56 \\
& +\frac{8\zeta_2}{3}-40\zeta_3\Bigg]\Bigg]\ln^2(x)+\Bigg[C_F\Bigg[T_F^2\Bigg(\frac{140992}{243}+N_F\left(\frac{182528}{243}-\frac{400\zeta_2}{27}-\frac{640\zeta_3}{9}\right) \\
& -\frac{728}{27}\zeta_2-\frac{224}{9}\zeta_3\Bigg)+C_AT_F\left(-\frac{514952}{243}+\frac{152\zeta_4}{3}-\frac{21140\zeta_2}{27}-\frac{2576\zeta_3}{9}\right)\Bigg] \\
& +C_AT_F^2\left[\frac{184}{27}+N_F\left(\frac{656}{27}-\frac{32\zeta_2}{27}\right)+\frac{464\zeta_2}{27}\right]+C_A^2T_F\Bigg[-\frac{42476}{81}-92\zeta_4+\frac{4504\zeta_2}{27} \\
& +\frac{64\zeta_3}{3}\Bigg]+C_F^2T_F\left[-\frac{1036}{3}-\frac{976\zeta_4}{3}-\frac{58\zeta_2}{3}+\frac{416\zeta_3}{3}\right]\Bigg]\ln(x),
\end{aligned} \tag{4.10}$$

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and

$$a_{gg,Q}^{(3),x\to1}(x) \propto a_{gg,Q,\delta}^{(3)}\delta(1-x)+a_{gg,Q,\text{plus}}^{(3)}(x)+\Biggl[-\frac{32}{27}C_AT_F^2(17+12N_F)+C_AC_FT_F\left(56-\frac{32\zeta_2}{3}\right)$$
$$+C_A^2T_F\left(\frac{9238}{81}-\frac{104\zeta_2}{9}+16\zeta_3\right)\Biggr]\ln(1-x)+\Biggl[-\frac{8}{27}C_AT_F^2(7+8N_F)$$
$$+C_A^2T_F\left(\frac{314}{27}-\frac{4\zeta_2}{3}\right)\Biggr]\ln^2(1-x)+\frac{32}{27}C_A^2T_F\ln^3(1-x). \tag{4.11}$$

In the polarized case one has

$$\Delta a_{gg,Q}^{x\to0}(x) \propto \left[-\frac{4}{15}C_A^2T_F+\frac{12}{5}C_AC_FT_F+\frac{2}{15}C_F^2T_F\right]\ln^5(x)+\Biggl[\frac{34}{27}C_A^2T_F+\frac{14}{9}C_F^2T_F$$
$$+C_F\left(\frac{760}{27}C_AT_F+T_F^2\left(\frac{28}{27}+\frac{56N_F}{27}\right)\right)\Biggr]\ln^4(x)+\Biggl[\frac{112}{81}C_A(1+2N_F)T_F^2$$
$$+C_F\left(T_F^2\left(\frac{968}{81}+\frac{1552}{81}N_F\right)+C_AT_F\left(\frac{13484}{81}-\frac{184\zeta_2}{9}\right)\right)+C_F^2T_F\left(-\frac{70}{9}+\frac{4\zeta_2}{9}\right)$$
$$+C_A^2T_F\left(\frac{2848}{81}+8\zeta_2\right)\Biggr]\ln^3(x)+\Biggl[\frac{16}{81}C_A(85+146N_F)T_F^2+C_F\Biggl(T_F^2\Biggl(\frac{2680}{81}$$
$$+N_F\left(\frac{6704}{81}-\frac{16\zeta_2}{9}\right)+\frac{232\zeta_2}{9}\Biggr)+C_AT_F\left(\frac{44476}{81}-\frac{1184\zeta_2}{9}-\frac{544\zeta_3}{3}\right)\Biggr)$$
$$+C_F^2T_F\left(-\frac{358}{3}-\frac{412\zeta_2}{3}+\frac{8\zeta_3}{3}\right)+C_A^2T_F\left(\frac{2588}{27}+\frac{244\zeta_2}{9}+112\zeta_3\right)\Biggr]\ln^2(x)$$
$$+\Biggl[C_F\Biggl(T_F^2\left(\frac{53920}{243}+N_F\left(\frac{121280}{243}-\frac{304\zeta_2}{27}-\frac{640\zeta_3}{9}\right)+\frac{2776}{27}\zeta_2-\frac{224}{9}\zeta_3\right)$$
$$+C_AT_F\left(-\frac{638672}{243}+\frac{4400\zeta_4}{3}-\frac{16484\zeta_2}{27}+\frac{5968\zeta_3}{9}\right)\Biggr)+C_AT_F^2\Biggl(\frac{4880}{81}$$
$$+N_F\left(\frac{9760}{81}-\frac{32\zeta_2}{27}\right)+\frac{464\zeta_2}{27}\Biggr)+C_F^2T_F\left(-\frac{2152}{3}-\frac{1744\zeta_4}{3}-\frac{530\zeta_2}{3}-\frac{3136\zeta_3}{3}\right)$$
$$+C_A^2T_F\left(\frac{964}{9}-\frac{452\zeta_4}{3}+\frac{2240\zeta_2}{27}+\frac{1904\zeta_3}{9}\right)\Biggr]\ln(x), \tag{4.12}$$

and

$$\Delta a_{gg,Q}^{x\to1}(x) = a_{gg,Q}^{x\to1}(x) \tag{4.13}$$

for all terms up to $\propto \ln(1-x)$. There are no predictions for these limits in the literature. As will be shown in section 5, both the formally most leading small x and large x contributions are insufficient approximations to $(\Delta)a_{gg,Q}(x)$, as expected from other results [27, 162–165]. This also applies to the leading large x result due to its limited reach.

5 Numerical results

For the numerical representations one needs to calculate the G-functions in a new way, while for the harmonic polylogarithms different numerical codes exist, cf. e.g. refs. [166–168]. One way to proceed in the present case would be to rationalize the letters of the G-functions by a formalism available in the package `HarmonicSums` leading to the cyclotomic letters [98]

$$\left\{\frac{1}{1+t^2}, \frac{t}{1+t^2}\right\}. \tag{5.1}$$

The corresponding functions can also be decomposed into letters belonging to generalized harmonic polylogarithms [96, 97],

$$\left\{\frac{1}{t}, \frac{1}{1-t}, \frac{1}{1+t}, \frac{1}{1+it}, \frac{1}{1-it}\right\}. \tag{5.2}$$

Iterated integrals over the latter letters can be calculated numerically using the Hölder convolution [169][3] implemented in ref. [167].

Since the above representation is still somewhat slow, we have alternatively expanded the final expressions for $a_{gg,Q}^{(3)}(x)$ and $\Delta a_{gg,Q}^{(3)}(x)$ in series around $x = 0$ and $x = 1$ analytically to 50 terms. Both representations are matched in the middle of the x range $]0, 1]$ at an accuracy of $5 \cdot 10^{-15}$. The corresponding routines are found in an attachment to the present paper. Since our expansions are fully analytic, they can be extended to an even higher accuracy if needed.

We split the discussion of $(\Delta)a_{gg,Q}^{(3)}(x)$ into the terms free of N_F and the linear N_F term, since the parameter N_F is arbitrary. In the unpolarized case the corresponding results are given in figures 1 and 2, where we have used the rescaling factor $x(1-x)^2$ for better visibility.

The formally leading small x term $\propto \ln(x)/x$ of the N_F-independent part deviates from the complete result everywhere, cf. figure 1.[4] Adding the $1/x$ allows to describe the region $x < 0.001$. Adding also the remaining terms of the expansion around $x = 0$ up to the constant term one covers the region $x < 0.1$. The large x singular and logarithmic contributions $\propto \ln^l(1-x)$ up to the constant term stop to agree with the complete result at $x \sim 0.7$. The matching between the small x and large x expansions, taking into account 50 expansion terms in both cases, can be performed with a relative accuracy of $O(10^{-14})$ and finally allows to describe the whole region $x \in]0, 1[$.

Both the N_F-dependent terms in the unpolarized and polarized cases are smaller than the corresponding N_F-independent parts. The unpolarized N_F-dependent part of $a_{gg,Q}^{(3)}(x)$ behaves like $1/x$ in the small x limit, as shown in figure 2. The expansions around $x = 0$ and $x = 1$ to $O(x^{50})$ can be matched at $x \sim 0.5$ at a relative accuracy of $O(10^{-14})$. Just keeping the divergent and $\ln^l(1-x)$ up to the constant term in the large x region yields agreement with the complete result for $x \gtrsim 0.7$. The leading small x term, which in view of

[3]This is a particular convolution in which the Hölder condition [170] and Hölder mean [171, 172] are used.
[4]This has also been observed before for other OMEs in refs. [27, 173].

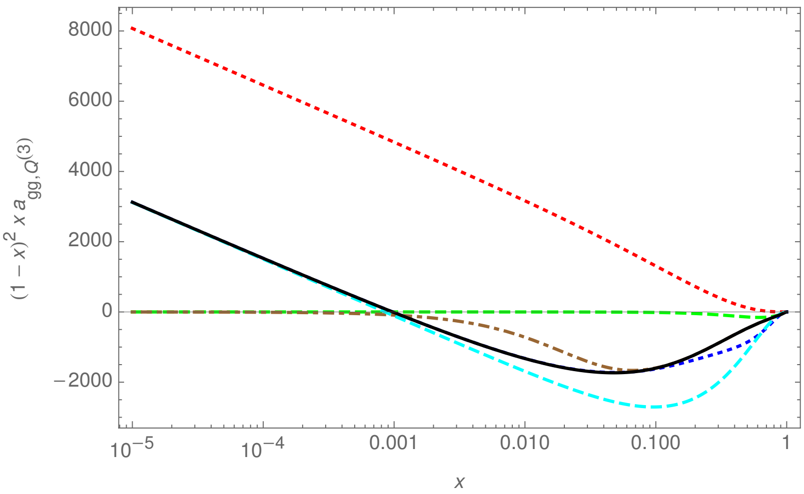

Figure 1. The non-N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; lower dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up to the constant term; upper dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): complete large x contribution.

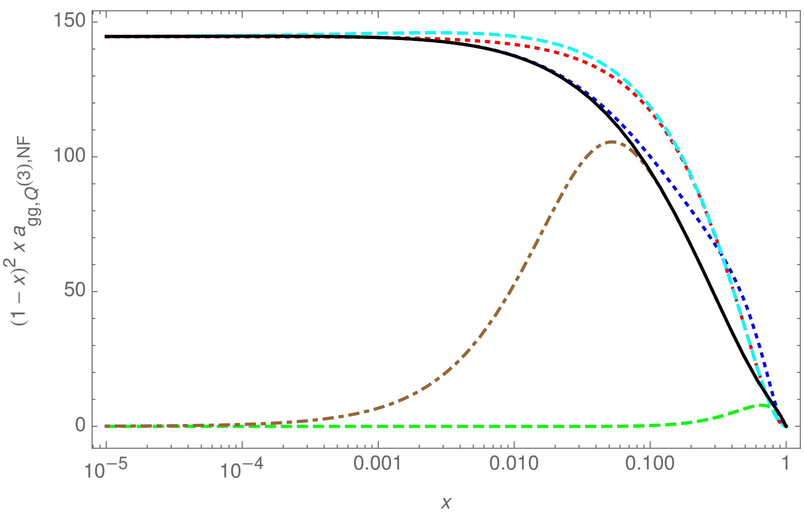

Figure 2. The N_F terms of $a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln(x)/x$; upper dashed line (cyan): small x terms $\propto 1/x$; lower dotted line (blue): small x terms including all $\ln(x)$ terms up the constant term; lower dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution.

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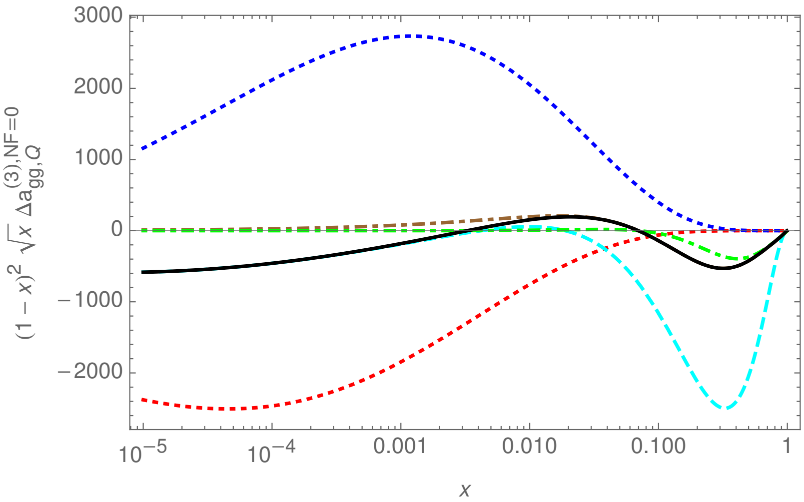

Figure 3. The non-N_F terms of $\Delta a_{gg,Q}^{(3)}(N)$ (rescaled) as a function of x. Full line (black): complete result; lower dotted line (red): term $\ln^5(x)$; upper dotted line (blue): small x terms $\propto \ln^5(x)$ and $\ln^4(x)$; upper dashed line (cyan): small x terms including all $\ln(x)$ terms up to the constant term; lower dash-dotted line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution.

the complete expression for $a_{gg,Q}^{(3)}(x)$ is subleading, starts to deviate for values $x > 0.002$ from the complete expression. Accounting in addition for the next term $\propto \ln^4(x)$ the range is even only $x < 0.0002$. Including all small x logarithmic terms up to the constant term cover the region $x < 0.02$.

In the polarized case, we rescale $\Delta a_{gg,Q}^{(3)}(N)$ by the factor $\sqrt{x}(1-x)^2$ for better visibility, which is different from the one in the unpolarized case, cf. figures 3, 4. In the N_F-independent case, the leading small x result $\propto \ln^5(x)$ deviates from the complete result significantly, which is also the case adding one subleading term. Taking all small x logarithms into account one covers the region $x < 0.002$. $x \sim 0.5$ is the ideal matching point of the 50-term small x and large x expansions at a relative accuracy of $\lesssim 10^{-14}$. The large x expansion of the leading terms $\propto 1/(1-x)$ and $\ln^k(1-x)$ down to the constant term range to $0.7 < x$, while the 50-term expansion covers $0.02 < x$ and does then deviate from the complete result.

A similar behavior of the different curves in the N_F-dependent part in the polarized case to the N_F-independent part is observed in figure 4. Again, the leading small x term $\propto \ln^4(x)$ does not describe the complete expression anywhere, as well as taking into account the next logarithmic order, while the complete set of the small x logarithms $\ln^k(x)$ covers the range $x < 0.008$. The 50-term large x expansion describes the complete result for $x \gtrsim 0.3$, while the leading large x singular contributions agree with it only for $x > 0.8$.

For the numerical representation used we present the regular parts expanded in terms of power series to $O(x^{50})$ around $x = 0$ and $x = 1$ in a `Mathematica` .m file attached to this paper. Since the corresponding expansions are performed analytically, it can be extended to work at an even higher precision, if needed.

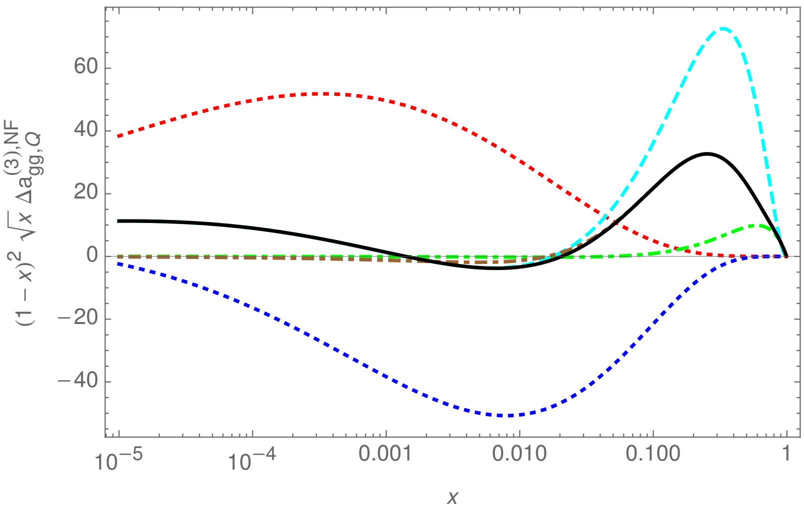

Figure 4. The $\Delta a_{gg,Q}^{(3)}(N)$ term $\propto N_F$ (rescaled) as a function of x. Full line (black): complete result; upper dotted line (red): term $\propto \ln^4(x)$; lower dotted line (blue): small x terms $\propto \ln^4(x)$ and $\ln^3(x)$; upper dashed line (cyan): small x terms including all $\ln(x)$ terms up to the constant term; lower dashed line (green): large x contribution up to the constant term; dash-dotted line (brown): full large x contribution.

6 Conclusions

We have calculated the single heavy quark mass OMEs $A_{gg,Q}^{(3)}$ and $\Delta A_{gg,Q}^{(3)}$ in the unpolarized and polarized cases, and the previously known logarithmic contributions are now supplemented by the functions $a_{gg,Q}^{(3)}$ and $\Delta a_{gg,Q}^{(3)}$. In Mellin N-space, they contain besides nested harmonic sums also nested finite binomial sums. In momentum fraction x-space, theses quantities are represented by iterated integrals, the G-functions, over an alphabet containing also square root-valued letters. We provided recursions and asymptotic expansions in Mellin N-space for these quantities, allowing also the analytic continuation of $a_{gg,Q}^{(3)}$ and $\Delta a_{gg,Q}^{(3)}$ from the even (odd) moments to $N \in \mathbb{C}$. In x-space it is convenient to use highly accurate Taylor series expansions. Large expressions both in N- and x-space are provided in computer-readable files in the supplementary material attached to the present paper. We find that leading small and large x expansions of $a_{gg,Q}^{(3)}$ and $\Delta a_{gg,Q}^{(3)}$ are of limited use, since they cover rather small regions in x only.

The present results complete the transition matrix elements in the single- and double-mass variable flavor number scheme for the gluon distributions in the unpolarized and polarized cases at three-loop order.

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A The contributing polynomials in N-space

The polynomials P_i occurring in the expressions of $(\Delta)a_{gg,Q}^{(3)}$ in section 3 are given by

$$P_1 = -63N^4-126N^3-431N^2-368N+736, \tag{A.1}$$
$$P_2 = N^4+2N^3-61N^2-62N-8, \tag{A.2}$$
$$P_3 = N^4+2N^3-23N^2-24N-4, \tag{A.3}$$
$$P_4 = N^4+2N^3-16N^2-17N+3, \tag{A.4}$$
$$P_5 = N^4+2N^3-N^2-2N+3, \tag{A.5}$$
$$P_6 = N^4+2N^3+3N^2+2N-2, \tag{A.6}$$
$$P_7 = N^4+2N^3+17N^2+16N+3, \tag{A.7}$$
$$P_8 = N^4+2N^3+25N^2+24N-4, \tag{A.8}$$
$$P_9 = N^4+2N^3+63N^2+62N-8, \tag{A.9}$$
$$P_{10} = N^4+2N^3+75N^2+74N-8, \tag{A.10}$$
$$P_{11} = 2N^4+4N^3-3N^2-5N+6, \tag{A.11}$$
$$P_{12} = 2N^4+4N^3-N^2-3N+6, \tag{A.12}$$
$$P_{13} = 2N^4+4N^3+7N^2+5N+6, \tag{A.13}$$
$$P_{14} = 2N^4+6N^3+N^2-3N-12, \tag{A.14}$$
$$P_{15} = 3N^4-12N^3-37N^2-10N+8, \tag{A.15}$$
$$P_{16} = 3N^4+6N^3-11N^2-14N+9, \tag{A.16}$$
$$P_{17} = 3N^4+6N^3-8N^2-11N+9, \tag{A.17}$$
$$P_{18} = 3N^4+6N^3-4N^2-7N+9, \tag{A.18}$$
$$P_{19} = 5N^4-8N^3-23N^2-22N-8, \tag{A.19}$$
$$P_{20} = 5N^4+4N^3+N^2-10N-8, \tag{A.20}$$
$$P_{21} = 5N^4+10N^3-65N^2-70N-16, \tag{A.21}$$
$$P_{22} = 5N^4+10N^3-29N^2-34N-16, \tag{A.22}$$
$$P_{23} = 5N^4+10N^3-N^2-6N-16, \tag{A.23}$$
$$P_{24} = 5N^4+10N^3+11N^2+6N-16, \tag{A.24}$$
$$P_{25} = 5N^4+10N^3+25N^2+20N+36, \tag{A.25}$$
$$P_{26} = 5N^4+10N^3+39N^2+34N-16, \tag{A.26}$$
$$P_{27} = 5N^4+10N^3+75N^2+70N-16, \tag{A.27}$$
$$P_{28} = 6N^4+12N^3-7N^2-13N+18, \tag{A.28}$$
$$P_{29} = 6N^4+12N^3-N^2-7N+18, \tag{A.29}$$
$$P_{30} = 6N^4+12N^3+N^2-5N+18, \tag{A.30}$$

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$$P_{31} = 6N^4+12N^3+5N^2-N+18, \tag{A.31}$$
$$P_{32} = 6N^4+12N^3+7N^2+N+18, \tag{A.32}$$
$$P_{33} = 9N^4+18N^3+113N^2+104N-24, \tag{A.33}$$
$$P_{34} = 11N^4+4N^3-59N^2-88N-12, \tag{A.34}$$
$$P_{35} = 11N^4+22N^3-5N^2-16N+68, \tag{A.35}$$
$$P_{36} = 11N^4+22N^3+27N^2+16N+68, \tag{A.36}$$
$$P_{37} = 13N^4+62N^3+63N^2+14N+88, \tag{A.37}$$
$$P_{38} = 18N^4+36N^3+19N^2+N+54, \tag{A.38}$$
$$P_{39} = 18N^4+36N^3+41N^2+23N+54, \tag{A.39}$$
$$P_{40} = 29N^4+58N^3+9N^2-20N+20, \tag{A.40}$$
$$P_{41} = 29N^4+58N^3+49N^2+20N+20, \tag{A.41}$$
$$P_{42} = 35N^4+64N^3+28N^2-13N-6, \tag{A.42}$$
$$P_{43} = 36N^4+72N^3+103N^2+67N+108, \tag{A.43}$$
$$P_{44} = 40N^4+72N^3+5N^2-27N+48, \tag{A.44}$$
$$P_{45} = 99N^4+198N^3+463N^2+364N-84, \tag{A.45}$$
$$P_{46} = 130N^4+269N^3+142N^2-24N-18, \tag{A.46}$$
$$P_{47} = 131N^4+454N^3+471N^2+148N+236, \tag{A.47}$$
$$P_{48} = 220N^4+330N^3-25N^2-198N+249, \tag{A.48}$$
$$P_{49} = 1287N^4+3726N^3-3047N^2-7214N-2624, \tag{A.49}$$
$$P_{50} = N^5-12N^4-11N^3-54N^2-52N-8, \tag{A.50}$$
$$P_{51} = N^5+46N^4+305N^3+484N^2+156N-16, \tag{A.51}$$
$$P_{52} = 3N^5-7N^4-25N^3-269N^2-254N-72, \tag{A.52}$$
$$P_{53} = 3N^5-7N^4-25N^3+259N^2+274N-72, \tag{A.53}$$
$$P_{54} = 3N^5-5N^4-21N^3-79N^2-66N-24, \tag{A.54}$$
$$P_{55} = 3N^5-5N^4-21N^3+89N^2+102N-24, \tag{A.55}$$
$$P_{56} = 3N^5-4N^4-19N^3-146N^2-134N-60, \tag{A.56}$$
$$P_{57} = 3N^5-4N^4-19N^3+142N^2+154N-60, \tag{A.57}$$
$$P_{58} = 3N^5-N^4-13N^3-47N^2-38N-48, \tag{A.58}$$
$$P_{59} = 3N^5-N^4-13N^3+49N^2+58N-48, \tag{A.59}$$
$$P_{60} = 3N^5+7N^4+8N^3+56N^2-32N+48, \tag{A.60}$$
$$P_{61} = 3N^5+16N^4+21N^3-190N^2-198N-12, \tag{A.61}$$
$$P_{62} = 3N^5+16N^4+21N^3+194N^2+186N-12, \tag{A.62}$$
$$P_{63} = 3N^5+25N^4+39N^3-253N^2-270N+24, \tag{A.63}$$
$$P_{64} = 3N^5+25N^4+39N^3+275N^2+258N+24, \tag{A.64}$$
$$P_{65} = 4N^5+15N^4+102N^3+223N^2-148N-340, \tag{A.65}$$

$$P_{66} = 8N^5+16N^4-25N^3-40N^2-69N+62, \quad (A.66)$$
$$P_{67} = 9N^5+18N^4-19N^3-30N^2-20N+18, \quad (A.67)$$
$$P_{68} = 9N^5+45N^4+62N^3-6N^2-50N-36, \quad (A.68)$$
$$P_{69} = 13N^5+36N^4+55N^3+60N^2+116N-176, \quad (A.69)$$
$$P_{70} = 16N^5+41N^4+2N^3+47N^2+70N+32, \quad (A.70)$$
$$P_{71} = 29N^5+107N^4+160N^3+142N^2-30N-180, \quad (A.71)$$
$$P_{72} = 70N^5+95N^4-223N^3-751N^2-629N-142, \quad (A.72)$$
$$P_{73} = 131N^5+192N^4+35N^3+270N^2+532N-472, \quad (A.73)$$
$$P_{74} = -63N^6-189N^5-431N^4-547N^3-1714N^2-1472N-1472, \quad (A.74)$$
$$P_{75} = 2N^6+8N^5+3N^4-14N^3-5N^2+6N+24, \quad (A.75)$$
$$P_{76} = 2N^6+8N^5+9N^4-2N^3-17N^2-12N+36, \quad (A.76)$$
$$P_{77} = 2N^6+8N^5+9N^4-2N^3+7N^2+12N+36, \quad (A.77)$$
$$P_{78} = 3N^6+3N^5-11N^4-19N^3+86N^2+94N+60, \quad (A.78)$$
$$P_{79} = 3N^6+3N^5-5N^4+17N^3-64N^2-86N+60, \quad (A.79)$$
$$P_{80} = 3N^6+9N^5-9N^4-73N^3-84N^2-26N-36, \quad (A.80)$$
$$P_{81} = 3N^6+9N^5+10N^4+40N^3-12N^2+8N+32, \quad (A.81)$$
$$P_{82} = 4N^6+3N^5-50N^4-129N^3-100N^2-56N-24, \quad (A.82)$$
$$P_{83} = 4N^6+16N^5-53N^4-218N^3-217N^2-64N-44, \quad (A.83)$$
$$P_{84} = 4N^6+16N^5+9N^4-22N^3-7N^2+12N+36, \quad (A.84)$$
$$P_{85} = 5N^6-2N^5-22N^4+138N^3+5N^2-40N+12, \quad (A.85)$$
$$P_{86} = 5N^6+20N^5+25N^4+8N^3+24N^2+26N-36, \quad (A.86)$$
$$P_{87} = 6N^6+17N^5+15N^4+56N^3-60N^2+8N+48, \quad (A.87)$$
$$P_{88} = 6N^6+45N^5-419N^4-1287N^3-583N^2+246N-168, \quad (A.88)$$
$$P_{89} = 7N^6+11N^5+55N^4+69N^3-54N^2+40N+64, \quad (A.89)$$
$$P_{90} = 8N^6+27N^5-64N^4-309N^3-334N^2-104N+8, \quad (A.90)$$
$$P_{91} = 9N^6-15N^5-89N^4-177N^3+36N^2+28N-16, \quad (A.91)$$
$$P_{92} = 9N^6+9N^5-53N^4+47N^3+44N^2-104N-80, \quad (A.92)$$
$$P_{93} = 9N^6+27N^5+161N^4+277N^3+358N^2+224N+48, \quad (A.93)$$
$$P_{94} = 9N^6+60N^5-259N^4-900N^3-632N^2-42N+36, \quad (A.94)$$
$$P_{95} = 15N^6+45N^5+13N^4-13N^3+80N^2+4N-24, \quad (A.95)$$
$$P_{96} = 15N^6+60N^5+43N^4-76N^3-112N^2-38N-132, \quad (A.96)$$
$$P_{97} = 15N^6+60N^5+43N^4-76N^3-64N^2+10N-132, \quad (A.97)$$
$$P_{98} = 17N^6+33N^5-27N^4+59N^3+130N^2-44N-24, \quad (A.98)$$
$$P_{99} = 27N^6+81N^5+148N^4+161N^3+253N^2-390N-144, \quad (A.99)$$
$$P_{100} = 29N^6+78N^5+71N^4+90N^3+206N^2+138N+180, \quad (A.100)$$

$$P_{101} = 30N^6+90N^5+79N^4+8N^3+23N^2+70N+12, \tag{A.101}$$
$$P_{102} = 33N^6+99N^5+88N^4+47N^3+68N^2-29N-42, \tag{A.102}$$
$$P_{103} = 37N^6+18N^5-15N^4-323N^3-174N^2-36N-54, \tag{A.103}$$
$$P_{104} = 38N^6+96N^5+233N^4+426N^3+35N^2-216N+108, \tag{A.104}$$
$$P_{105} = 38N^6+108N^5+151N^4+106N^3+21N^2-28N-12, \tag{A.105}$$
$$P_{106} = 40N^6+112N^5-3N^4-166N^3-301N^2-210N-96, \tag{A.106}$$
$$P_{107} = 44N^6+123N^5+386N^4+543N^3+520N^2+248N+24, \tag{A.107}$$
$$P_{108} = 99N^6+297N^5+631N^4+767N^3+1118N^2+784N+168, \tag{A.108}$$
$$P_{109} = 100N^6+439N^5+8N^4-1286N^3-858N^2+1519N-498, \tag{A.109}$$
$$P_{110} = 199N^6+645N^5+505N^4-81N^3-1052N^2-912N+2712, \tag{A.110}$$
$$P_{111} = 220N^6+550N^5-135N^4-883N^3-1621N^2-1329N-462, \tag{A.111}$$
$$P_{112} = 421N^6+831N^5+637N^4+1473N^3-626N^2-1872N+5184, \tag{A.112}$$
$$P_{113} = 6944N^6+19536N^5+17781N^4+5108N^3+1791N^2+576N-432, \tag{A.113}$$
$$P_{114} = 3N^7+3N^6-21N^5-31N^4-64N^3-122N^2-104N+72, \tag{A.114}$$
$$P_{115} = 3N^7+41N^6+335N^5+471N^4-302N^3-740N^2+552N+288, \tag{A.115}$$
$$P_{116} = 4N^7+8N^6-85N^5-112N^4+59N^3-46N^2-172N+88, \tag{A.116}$$
$$P_{117} = 5N^7+10N^6-15N^5-42N^4-4N^3+86N^2+32N+72, \tag{A.117}$$
$$P_{118} = 5N^7+23N^6+63N^5+189N^4+148N^3-28N^2+144N+128, \tag{A.118}$$
$$P_{119} = 6N^7+3N^6-17N^5+843N^4+1463N^3+218N^2-348N+136, \tag{A.119}$$
$$P_{120} = 6N^7+33N^6-533N^5-545N^4+767N^3+740N^2-180N+336, \tag{A.120}$$
$$P_{121} = 8N^7-21N^6-41N^5+387N^4+373N^3-526N^2-428N+56, \tag{A.121}$$
$$P_{122} = 8N^7+16N^6-57N^5-104N^4+87N^3-26N^2-28N-248, \tag{A.122}$$
$$P_{123} = 9N^7+42N^6-403N^5-478N^4+760N^3+310N^2-456N-72, \tag{A.123}$$
$$P_{124} = 9N^7+43N^6+277N^5+477N^4-106N^3-844N^2+408N+288, \tag{A.124}$$
$$P_{125} = 15N^7-25N^6-192N^5+442N^4+107N^3+2391N^2+1030N+1032, \tag{A.125}$$
$$P_{126} = 38N^7+20N^6+77N^5+104N^4-385N^3-466N^2+36N-216, \tag{A.126}$$
$$P_{127} = 109N^7+877N^6+2129N^5+2215N^4+908N^3-620N^2-878N-276, \tag{A.127}$$
$$P_{128} = 199N^7+247N^6-785N^5-1091N^4+1510N^3-56N^2+888N-5424, \tag{A.128}$$
$$P_{129} = 421N^7-11N^6-881N^5+775N^4+172N^3-3356N^2+2880N-10368, \tag{A.129}$$
$$P_{130} = N^8+4N^7+2N^6+64N^5+173N^4+292N^3+256N^2-72N-72, \tag{A.130}$$
$$P_{131} = 4N^8+25N^7+32N^6-152N^5+678N^4+359N^3-882N^2-112N+144, \tag{A.131}$$
$$P_{132} = 4N^8+45N^7+44N^6-248N^5+1142N^4+931N^3-1022N^2-224N+192, \tag{A.132}$$
$$P_{133} = 5N^8+41N^7+41N^6+25N^5-14N^4-54N^3-84N^2-72N-16, \tag{A.133}$$
$$P_{134} = 8N^8+21N^7-33N^6+5N^5+601N^4+698N^3-156N^2-88N+96, \tag{A.134}$$
$$P_{135} = 15N^8-252N^6+228N^5+631N^4+1780N^3+3822N^2+1032N+744, \tag{A.135}$$

$$P_{136} = 15N^8+60N^7+4N^6-162N^5-311N^4-186N^3-220N^2-80N+48, \tag{A.136}$$
$$P_{137} = 30N^8-5N^7-514N^6+626N^5+902N^4+2735N^3+5654N^2+4N+168, \tag{A.137}$$
$$P_{138} = 33N^8+132N^7+46N^6-225N^5-296N^4-285N^3-185N^2-456N-108, \tag{A.138}$$
$$P_{139} = 33N^8+132N^7+106N^6-108N^5-74N^4+282N^3+245N^2+148N+84, \tag{A.139}$$
$$P_{140} = 100N^8+539N^7+283N^6-2094N^5+452N^4+219N^3-1495N^2+712N \\ +996, \tag{A.140}$$
$$P_{141} = 205N^8+856N^7+3169N^6+6484N^5+7310N^4+4722N^3+1534N^2 \\ +48N-72, \tag{A.141}$$
$$P_{142} = 266N^8+717N^7-697N^6-4325N^5-4481N^4+560N^3+1120N^2 \\ +1512N+864, \tag{A.142}$$
$$P_{143} = 1720N^8+6898N^7+6007N^6-4079N^5-5207N^4-335N^3-252N^2 \\ -648N-1512, \tag{A.143}$$
$$P_{144} = 6944N^8+26480N^7+23321N^6-15103N^5-39319N^4-27001N^3 \\ -11178N^2-2016N+864, \tag{A.144}$$
$$P_{145} = 7209N^8+28836N^7-39838N^6-199272N^5-187779N^4-51844N^3 \\ -3888N^2+3168N+6048, \tag{A.145}$$
$$P_{146} = 96020N^8+84383N^7-200790N^6-241078N^5-14299N^4+60396N^3 \\ +35730N^2-12960N+6480, \tag{A.146}$$
$$P_{147} = 3N^9+32N^8+302N^7+584N^6-377N^5-1224N^4+1176N^3+2144N^2 \\ -1104N-768, \tag{A.147}$$
$$P_{148} = 5N^9+3N^8-66N^7-82N^6+469N^5+1099N^4+2392N^3+1092N^2 \\ +656N+192, \tag{A.148}$$
$$P_{149} = 7N^9-3N^8-78N^7-46N^6+439N^5+1285N^4+2112N^3+1068N^2 \\ +592N+384, \tag{A.149}$$
$$P_{150} = 27N^9+36N^8-1166N^7-1760N^6+4331N^5+88N^4-10864N^3 \\ +2740N^2+10192N-1104, \tag{A.150}$$
$$P_{151} = 30N^9+109N^8+121N^7+939N^6+2417N^5+1188N^4-932N^3 \\ -32N^2-64N-128, \tag{A.151}$$
$$P_{152} = 35N^9-120N^8-689N^7+2546N^6+3317N^5+7020N^4+36669N^3 \\ +27874N^2+13468N-3720, \tag{A.152}$$
$$P_{153} = 40N^9-125N^8-245N^7+542N^6+1938N^5-2977N^4+9079N^3 \\ +9040N^2+1188N+720, \tag{A.153}$$
$$P_{154} = 95N^9+446N^8+344N^7-1122N^6+5165N^5+29844N^4+27308N^3 \\ -8512N^2-896N+7232, \tag{A.154}$$

$$P_{155} = 448N^9+788N^8-1990N^7-4453N^6-2185N^5+551N^4+3637N^3 \\ +8244N^2-2988N-3024, \tag{A.155}$$

$$P_{156} = 448N^9+860N^8-1558N^7-4381N^6-5443N^5-601N^4+8767N^3 \\ +9036N^2-3348N-2160, \tag{A.156}$$

$$P_{157} = N^{10}+37N^9-10N^8-634N^7+81N^6+5157N^5+12472N^4+9408N^3 \\ +896N^2+4272N+2880, \tag{A.157}$$

$$P_{158} = 3N^{10}-29N^9-62N^8+538N^7+251N^6-4533N^5-13200N^4-11384N^3 \\ -432N^2-3408N-2304, \tag{A.158}$$

$$P_{159} = 8N^{10}-51N^9-96N^8+508N^7+458N^6-1601N^5-2194N^4+152N^3 \\ -464N^2-976N+224, \tag{A.159}$$

$$P_{160} = 8N^{10}+27N^9+4N^8-176N^7-322N^6-75N^5-690N^4-672N^3+1000N^2 \\ +368N-384, \tag{A.160}$$

$$P_{161} = 15N^{10}+10N^9-238N^8+88N^7+2647N^6+9610N^5+17712N^4 \\ +13108N^3+5128N^2-1872N-1440, \tag{A.161}$$

$$P_{162} = 15N^{10}+19N^9-238N^8-358N^7+1087N^6+4483N^5+10400N^4 \\ +9536N^3+2176N^2+5328N+2880, \tag{A.162}$$

$$P_{163} = 16N^{10}+68N^9-13N^8-656N^7-1581N^6-974N^5+1800N^4+3412N^3 \\ +2008N^2+336N-32, \tag{A.163}$$

$$P_{164} = 16N^{10}+89N^9+294N^8+554N^7+48N^6-835N^5+330N^4+1776N^3 \\ +688N^2+336N-32, \tag{A.164}$$

$$P_{165} = 23N^{10}+136N^9-221N^8+388N^7+1470N^6+2206N^5+2192N^4+2564N^3 \\ +2082N^2+1008N+216, \tag{A.165}$$

$$P_{166} = 25N^{10}-35N^9-295N^8+185N^7+1615N^6+897N^5+5981N^4+13197N^3 \\ +5802N^2+1068N+360, \tag{A.166}$$

$$P_{167} = 30N^{10}+150N^9+163N^8-248N^7-562N^6-296N^5+33N^4-30N^3-48N^2 \\ +184N+48, \tag{A.167}$$

$$P_{168} = 102N^{10}+309N^9-238N^8-1822N^7+106N^6+4733N^5+1294N^4-3580N^3 \\ -3976N^2-864N+2496, \tag{A.168}$$

$$P_{169} = 270N^{10}-221N^9-2926N^8-1180N^7+5054N^6+4823N^5-2578N^4-2942N^3 \\ +15732N^2+15288N-3024, \tag{A.169}$$

$$P_{170} = 1536N^{10}-2955N^9-16182N^8+14924N^7+34190N^6-155541N^5-442072N^4 \\ -107436N^3+390656N^2-26624N-227328, \tag{A.170}$$

$$P_{171} = 2913N^{10}+14565N^9+4234N^8-60374N^7-54875N^6+68545N^5+112000N^4 \\ +39280N^3-10752N^2-16272N-6048, \tag{A.171}$$

$$P_{172} = 7209N^{10}+36045N^9-52924N^8-417598N^7-794647N^6-770095N^5 \\ -388726N^4-63040N^3-576N^2-25344N-12096, \tag{A.172}$$

$$P_{173} = 12672N^{10}+42963N^9+6N^8-264652N^7-183166N^6+673325N^5+703736N^4 \\ -277684N^3-833920N^2+133632N+525312, \tag{A.173}$$

$$P_{174} = 96020N^{10}+180403N^9-293651N^8-563492N^7+196513N^6+478087N^5 \\ -194200N^4-207066N^3-7470N^2-38880N-12960, \tag{A.174}$$

$$P_{175} = 149796N^{10}+331992N^9+2242307N^8+877052N^7-6336162N^6-4554532N^5 \\ +1462595N^4+1113864N^3+133200N^2+246240N-90720, \tag{A.175}$$

$$P_{176} = 27N^{11}+189N^{10}+631N^9+1356N^8+2155N^7+2207N^6+211N^5-4984N^4 \\ -8400N^3-5824N^2-2544N-576, \tag{A.176}$$

$$P_{177} = 75N^{11}-35N^{10}+624N^9-7558N^8+12763N^7+46561N^6+91954N^5 \\ +198152N^4+119160N^3+5280N^2-4256N-1920, \tag{A.177}$$

$$P_{178} = 76N^{11}+875N^{10}+3212N^9+4756N^8+1408N^7-5169N^6-12976N^5 \\ -12806N^4+112N^3+3392N^2-1984N-1632, \tag{A.178}$$

$$P_{179} = 448N^{11}+1236N^{10}-2116N^9-7857N^8-1560N^7+9270N^6+6398N^5 \\ -237N^4-12098N^3-18612N^2+6984N+7776, \tag{A.179}$$

$$P_{180} = 475N^{11}+330N^{10}-6255N^9-4360N^8-6703N^7+109282N^6+63439N^5 \\ +360220N^4+1376628N^3+1002368N^2+175616N+154560, \tag{A.180}$$

$$P_{181} = 502N^{11}-1112N^{10}-4284N^9+6519N^8+14409N^7-12978N^6-17866N^5 \\ -12913N^4-38013N^3+7524N^2-6588N-12960, \tag{A.181}$$

$$P_{182} = 502N^{11}-1112N^{10}-4248N^9+6609N^8+13113N^7-11466N^6-14842N^5 \\ -12427N^4-51441N^3+8028N^2-2700N-7776, \tag{A.182}$$

$$P_{183} = 1936N^{11}+5826N^{10}-8779N^9-34974N^8+5532N^7+59112N^6-4333N^5 \\ -41196N^4+21988N^3+34344N^2+6336N-4320, \tag{A.183}$$

$$P_{184} = 20N^{12}+125N^{11}-20N^{10}-1835N^9-3590N^8+10755N^7+23456N^6 \\ +33335N^5+108782N^4+67828N^3-34184N^2-5952N+8640, \tag{A.184}$$

$$P_{185} = 20N^{12}+225N^{11}+40N^{10}-3295N^9-5210N^8+18695N^7+39964N^6+64435N^5 \\ +209418N^4+196612N^3+16504N^2+4032N+11520, \tag{A.185}$$

$$P_{186} = 30N^{12}+25N^{11}+120N^{10}-1204N^9-2904N^8-8041N^7-11950N^6-3528N^5 \\ +6536N^4+4620N^3-520N^2-1520N-480, \tag{A.186}$$

$$P_{187} = 54N^{12}-983N^{11}-2940N^{10}+8147N^9+25836N^8-7266N^7-51705N^6 \\ -1540N^5+55887N^4+26122N^3+300N^2+792N-864, \tag{A.187}$$

$$P_{188} = 84N^{12}+192N^{11}-661N^{10}-2146N^9+452N^8+3894N^7+303N^6-3234N^5 \\ -5930N^4-4370N^3+2560N^2+3864N-1920, \tag{A.188}$$

$$P_{189} = 135N^{12} - 90N^{11} - 7255N^{10} + 1000N^9 + 67581N^8 - 2322N^7 - 181501N^6 \\ - 48456N^5 - 137736N^4 - 612388N^3 - 150184N^2 + 174096N + 113760, \tag{A.189}$$

$$P_{190} = 266N^{12} + 983N^{11} - 1576N^{10} - 9928N^9 - 6696N^8 + 7669N^7 - 954N^6 - 5380N^5 \\ + 16080N^4 + 10832N^3 - 2656N^2 + 8640N + 6912, \tag{A.190}$$

$$P_{191} = 394N^{12} - 36N^{11} - 4636N^{10} - 2733N^9 + 18141N^8 + 12348N^7 - 27010N^6 \\ - 32985N^5 - 61373N^4 - 57162N^3 - 18828N^2 - 49032N - 20736, \tag{A.191}$$

$$P_{192} = 2913N^{12} + 17478N^{11} + 6253N^{10} - 121030N^9 - 399973N^8 - 664606N^7 \\ - 829641N^6 - 867778N^5 - 563504N^4 - 110240N^3 + 67728N^2 \\ + 45504N + 12096, \tag{A.192}$$

$$P_{193} = 3300N^{12} + 5838N^{11} - 26849N^{10} - 35867N^9 + 83580N^8 + 69996N^7 - 113901N^6 \\ + 16689N^5 + 167318N^4 - 21088N^3 - 35688N^2 + 5904N + 5184, \tag{A.193}$$

$$P_{194} = 8868N^{12} + 26604N^{11} - 12901N^{10} - 61985N^9 + 24562N^8 - 16310N^7 - 140777N^6 \\ - 67997N^5 + 55040N^4 + 63048N^3 + 55008N^2 + 4968N - 9072, \tag{A.194}$$

$$P_{195} = 149796N^{12} + 481788N^{11} + 4037555N^{10} + 6431215N^9 - 710852N^8 - 14957774N^7 \\ - 21164117N^6 - 11167685N^5 + 2360450N^4 + 2452488N^3 - 1225440N^2 \\ - 518400N + 181440, \tag{A.195}$$

$$P_{196} = 33N^{13} + 264N^{12} + 574N^{11} - 470N^{10} - 2978N^9 - 912N^8 + 8524N^7 + 14408N^6 \\ + 9543N^5 + 4750N^4 + 4440N^3 + 3344N^2 + 2544N + 864, \tag{A.196}$$

$$P_{197} = 40N^{13} + 55N^{12} - 450N^{11} - 1455N^{10} + 1040N^9 + 10269N^8 + 1382N^7 - 28529N^6 \\ - 34324N^5 - 70484N^4 - 96232N^3 - 16416N^2 + 4704N - 11520, \tag{A.197}$$

$$P_{198} = 270N^{13} - 4885N^{12} + 990N^{11} + 51475N^{10} - 25020N^9 - 165453N^8 + 159306N^7 \\ + 313033N^6 - 458562N^5 - 819322N^4 - 269976N^3 + 512832N^2 \\ + 597312N + 142560, \tag{A.198}$$

$$P_{199} = 510N^{13} + 765N^{12} - 6830N^{11} - 15855N^{10} + 33390N^9 + 79359N^8 - 86978N^7 \\ - 140429N^6 + 273876N^5 + 415216N^4 + 51568N^3 - 232496N^2 - 95616N \\ + 46080, \tag{A.199}$$

$$P_{200} = 7680N^{13} - 30135N^{12} - 89760N^{11} + 339115N^{10} + 393290N^9 - 770421N^8 \\ - 1106268N^7 - 1744699N^6 - 9260454N^5 - 32119364N^4 - 35024552N^3 \\ - 4473856N^2 - 2520576N - 5575680, \tag{A.200}$$

$$P_{201} = 51840N^{13} + 93855N^{12} - 531360N^{11} - 2157395N^{10} + 1567910N^9 \\ + 11256237N^8 - 3136644N^7 - 17459037N^6 + 30905238N^5 + 68574308N^4 \\ + 30890344N^3 - 30448128N^2 - 5833728N + 14100480, \tag{A.201}$$

$$P_{202} = 95N^{14} + 125N^{13} - 1134N^{12} - 2050N^{11} + 6499N^{10} + 26893N^9 + 38472N^8 \\ - 55832N^7 - 205660N^6 - 56080N^5 + 145792N^4 + 53472N^3 - 6144N^2 \\ - 88448N - 42240, \tag{A.202}$$

$$
\begin{aligned}
P_{203} = {} & 8868N^{14}+35472N^{13}-9409N^{12}-152862N^{11}+61883N^{10}+593774N^{9} \\
& -379547N^{8}-1672874N^{7}-807075N^{6}+89818N^{5}-325576N^{4}-407328N^{3} \\
& -167688N^{2}-21600N+18144, \qquad \text{(A.203)} \\
P_{204} = {} & 540N^{15}-6940N^{14}-6255N^{13}+92984N^{12}+99855N^{11}-389419N^{10}-647943N^{9} \\
& +663238N^{8}+1833777N^{7}-126095N^{6}-1116630N^{5}+69928N^{4}-480432N^{3} \\
& -718416N^{2}-1212192N-570240, \qquad \text{(A.204)} \\
P_{205} = {} & 2091N^{15}+9807N^{14}+3454N^{13}-50522N^{12}-77079N^{11}+55082N^{10}+164624N^{9} \\
& -48090N^{8}-195178N^{7}+109703N^{6}+248568N^{5}+38380N^{4}-49640N^{3} \\
& -5280N^{2}-26208N-13824, \qquad \text{(A.205)} \\
P_{206} = {} & 8020N^{15}+11831N^{14}-92283N^{13}-365351N^{12}-433033N^{11}+1065225N^{10} \\
& +3874583N^{9}+2278519N^{8}-4959567N^{7}-7109812N^{6}-895688N^{5}+2185524N^{4} \\
& +697824N^{3}+281664N^{2}+388800N+77760, \qquad \text{(A.206)} \\
P_{207} = {} & 420N^{16}+540N^{15}-8300N^{14}-15615N^{13}+49927N^{12}+148830N^{11}-80392N^{10} \\
& -672719N^{9}-625021N^{8}+156216N^{7}+823430N^{6}+3125340N^{5}+4621504N^{4} \\
& +2625824N^{3}+429792N^{2}+87744N+46080, \qquad \text{(A.207)} \\
P_{208} = {} & 685N^{16}+1370N^{15}-9010N^{14}-19290N^{13}+26146N^{12}+91966N^{11}-14748N^{10} \\
& -149230N^{9}+45035N^{8}+174316N^{7}-271314N^{6}-505068N^{5}-281130N^{4} \\
& -52080N^{3}+16080N^{2}+17280N+4320, \qquad \text{(A.208)} \\
P_{209} = {} & 12180N^{16}+8370N^{15}-256195N^{14}-157950N^{13}+1778081N^{12}+1177830N^{11} \\
& -4307281N^{10}-3049362N^{9}+3710647N^{8}+11089008N^{7}+11202520N^{6} \\
& -23576760N^{5}-52089008N^{4}-32240448N^{3}-12305664N^{2} \\
& -7993728N-1866240, \qquad \text{(A.209)} \\
P_{210} = {} & 137N^{17}+822N^{16}+424N^{15}-5764N^{14}-8196N^{13}+16720N^{12}+41536N^{11} \\
& +15066N^{10}-25651N^{9}-42278N^{8}-54216N^{7}-37786N^{6}+78N^{5} \\
& -396N^{4}-8496N^{3}-4416N^{2}+1248N+576, \qquad \text{(A.210)} \\
P_{211} = {} & 39255N^{17}+240282N^{16}+41728N^{15}-2061700N^{14}-2654870N^{13}+5644496N^{12} \\
& +10599172N^{11}-8903908N^{10}-24858601N^{9}+2221990N^{8}+27458420N^{7} \\
& +6952696N^{6}-13270864N^{5}-14011424N^{4}-5396352N^{3} \\
& -1497600N^{2}+2412288N+1119744, \qquad \text{(A.211)} \\
P_{212} = {} & 242739N^{17}+1505034N^{16}+1803872N^{15}-6299252N^{14}-18083694N^{13} \\
& -3855808N^{12}+41147860N^{11}+41085852N^{10}-48824861N^{9}-83231194N^{8} \\
& +12645540N^{7}+64164904N^{6}+31839104N^{5}+22953024N^{4}+15952896N^{3} \\
& -2543616N^{2}-4064256N-746496, \qquad \text{(A.212)} \\
P_{213} = {} & 10455N^{18}+59490N^{17}-15790N^{16}-741440N^{15}-1390120N^{14}+1428397N^{13} \\
& +7150867N^{12}+5281630N^{11}-7138741N^{10}-8816137N^{9}+10256689N^{8}+
\end{aligned}
$$

$$+16683860N^7-3484864N^6-12146200N^5-502576N^4+5909760N^3$$
$$+4302720N^2+2851200N+829440, \tag{A.213}$$
$$P_{214}=40100N^{18}+99255N^{17}-727380N^{16}-3314430N^{15}-1772040N^{14}+14821596N^{13}$$
$$+48888776N^{12}+98029290N^{11}+59157432N^{10}-330544971N^9-879181188N^8$$
$$-779917140N^7-105697492N^6+173719200N^5+551952N^4-97381440N^3$$
$$-68610240N^2-34525440N-8398080, \tag{A.214}$$
$$P_{215}=1213695N^{20}+7525170N^{19}-6722900N^{18}-132732760N^{17}-180657906N^{16}$$
$$+706987500N^{15}+1986194496N^{14}-505023504N^{13}-7245869189N^{12}$$
$$-7460329438N^{11}+6529524348N^{10}+22209128904N^9+18794760144N^8$$
$$-4187656992N^7-23855002304N^6-26274133120N^5-18561813504N^4$$
$$-9634314240N^3-2690703360N^2+136028160N+199065600, \tag{A.215}$$
$$P_{216}=196275N^{22}+1397685N^{21}-1454770N^{20}-30923820N^{19}-41291522N^{18}$$
$$+225466098N^{17}+630395612N^{16}-373372336N^{15}-3171331361N^{14}$$
$$-2650077679N^{13}+4908510270N^{12}+10281951044N^{11}+3751227016N^{10}$$
$$-6343664096N^9-8882356992N^8-5272720448N^7+530329472N^6$$
$$+4243436032N^5+2879400960N^4+137687040N^3-705024000N^2-525864960N$$
$$-149299200. \tag{A.216}$$

B Special constants

In the following, we present some examples for G-functions over the alphabet $\mathfrak{A}$ at $x=1$, which appear in the present calculation. These constants can be mapped to cyclotomic numbers. They finally reduce to multiple zeta values.

$$\mathrm{G}\left(\left\{\frac{\sqrt{1-x}}{x}\right\},1\right) = -2+2\ln(2), \tag{B.1}$$
$$\mathrm{G}\left(\left\{\sqrt{1-x}\sqrt{x}\right\},1\right) = \frac{\pi}{8}, \tag{B.2}$$
$$\mathrm{G}\left(\left\{\frac{\sqrt{1-x}}{x},\frac{\sqrt{1-x}}{x}\right\},1\right) = 2\left(1-\ln(2)\right)^2, \tag{B.3}$$
$$\mathrm{G}\left(\left\{\sqrt{1-x}\sqrt{x},\sqrt{1-x}\sqrt{x}\right\},1\right) = \frac{\pi^2}{128}, \tag{B.4}$$
$$G\left(\left\{\frac{1}{x},\frac{\sqrt{1-x}}{x},\frac{\sqrt{1-x}}{x}\right\},1\right) = 6-8\ln(2)+8\ln^2(2)-\frac{8}{3}\ln^3(2)-2\zeta_2 + 2\ln(2)\zeta_2-\frac{3}{2}\zeta_3, \tag{B.5}$$
$$G\left(\left\{\frac{1}{x},\frac{\sqrt{1-x}}{x},\frac{1}{x},\frac{1}{x}\right\},1\right) = -48+48\ln(2)-24\ln^2(2)+8\ln^3(2)-2\ln^4(2) + 12\zeta_2-12\ln(2)\zeta_2+6\ln^2(2)\zeta_2+\frac{27}{10}\zeta_2^2+12\zeta_3 - 12\ln(2)\zeta_3, \tag{B.6}$$

$$G\left(\left\{\frac{1}{x},\frac{1}{x},\frac{1}{1-x},\frac{\sqrt{1-x}}{x},\frac{1}{1-x}\right\},1\right) = 128-64\mathrm{Li}_5\left(\frac{1}{2}\right)-96\ln(2)-64\mathrm{Li}_4\left(\frac{1}{2}\right)\ln(2)$$
$$+32\ln^2(2)-\frac{32}{15}\ln^5(2)-40\zeta_2+24\ln(2)\zeta_2$$
$$+\frac{32}{3}\ln^3(2)\zeta_2-32\zeta_3-28\ln^2(2)\zeta_3+\frac{55}{2}\zeta_2\zeta_3$$
$$+\frac{93}{8}\zeta_5. \tag{B.7}$$

C Mellin inversion of finite binomial sums

The following Mellin inversions are obtained for the nested finite binomial sums occurring in the present paper. We define

$$\mathbf{M}[f(x)](N) = \int_0^1 dx\ x^N f(x) \quad \text{and} \quad \mathbf{M}[[g(x)]_+](N) = \int_0^1 dx\ (x^N-1)g(x). \tag{C.1}$$

Let

$$w=\sqrt{1-x} \quad \text{and} \quad r=\sqrt{x(1-x)}. \tag{C.2}$$

One obtains

$$\mathbf{M}^{-1}[\mathsf{BS}_0(N)](x)=\frac{1}{2x^{l+3/2}}, \tag{C.3}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_1(N)](x)=\delta(1-x)-\frac{1}{2}\left[\frac{1}{(1-x)^{3/2}}\right]_+, \tag{C.4}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_2(N)](x)=\frac{1}{\pi}\frac{1}{\sqrt{x}\sqrt{1-x}}, \tag{C.5}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_3(N)](x)=-\left[\frac{\left(\pi(1-x)^{3/2}-2\sqrt{x}+8x^{3/2}-10x^{5/2}+4x^{7/2}\right)}{\pi(1-x)^{5/2}}-\frac{8\mathrm{G}(\{5\},x)}{\pi(1-x)}\right]_+, \tag{C.6}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_4(N)](x)=\left[-\frac{2\ln(2)}{1-x}+2\left(-1+\sqrt{1-x}+x\right)\frac{1}{(1-x)^{3/2}}+\frac{\mathrm{G}(\{4\},x)}{1-x}\right]_+, \tag{C.7}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_5(N)](x)=\left[\frac{-2\ln^2(2)-2(1-\ln(2))\mathrm{H}_0(x)+\zeta_2}{1-x}+\frac{2\mathrm{G}(\{4\},x)}{1-x}-\frac{\mathrm{G}(\{1,4\},x)}{1-x}\right]_+, \tag{C.8}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_6(N)](x)=\Bigg[\mathrm{G}(\{5\},x)\left[-\frac{4\pi}{1-x}+16(1-2x)\frac{\sqrt{x}}{\sqrt{1-x}}\right]-\pi(1-2x)\frac{\sqrt{x}}{\sqrt{1-x}}$$
$$+2x(1-2x)^2+\frac{64\mathrm{G}(\{5,5\},x)}{1-x}\Bigg]_+, \tag{C.9}$$

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$$\mathbf{M}^{-1}[\mathsf{BS}_7(N)](x) = \Biggl[\frac{x(18-33x+40x^2-18x^3)}{3(1-x)} + \mathrm{G}(\{5\},x)\biggl[8\frac{\sqrt{x}-3x^{3/2}+2x^{5/2}}{(1-x)^{3/2}} - \frac{8\pi\ln(2)}{1-x} - \frac{28\zeta_3}{\pi(1-x)}\biggr] - 2\pi\ln(2)\frac{\sqrt{x}-3x^{3/2}+2x^{5/2}}{(1-x)^{3/2}} - 2x(1-2x)^2\mathrm{H}_0(x) - \frac{7\left(\sqrt{x}-3x^{3/2}+2x^{5/2}\right)\zeta_3}{\pi(1-x)^{3/2}} + 16(1-x)(-1+2x)\frac{\sqrt{x}}{(1-x)^{3/2}}\mathrm{G}(\{5,1\},x) + \frac{32\mathrm{G}(\{5,5\},x)}{1-x} - \frac{64\mathrm{G}(\{5,5,1\},x)}{1-x}\Biggr]_+, \tag{C.10}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_8(N)](x) = \Biggl[-\frac{4(1-\sqrt{1-x})}{1-x} + \left(\frac{2(1-\ln(2))}{1-x} + \frac{\mathrm{H}_0(x)}{\sqrt{1-x}}\right)\mathrm{H}_1(x) - \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}} + \frac{\mathrm{H}_1(x)\mathrm{G}(\{6,1\},x)}{2(1-x)} - \frac{\mathrm{G}(\{6,1,2\},x)}{2(1-x)}\Biggr]_+, \tag{C.11}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_9(N)](x) = \Biggl[\mathrm{G}(\{5\},x)\left(-\frac{8\ln(2)\pi}{1-x} + 8(1-2x)\frac{\sqrt{x}}{\sqrt{1-x}} + \frac{28\zeta_3}{\pi(1-x)}\right) - \frac{1}{3\pi(1-x)}\Bigl[\pi x^2(21+2x(-16+9x)) + 36\ln(2)(1-2x)\sqrt{(1-x)x}\zeta_2 - 21(1-2x)\sqrt{(1-x)x}\zeta_3\Bigr] + 2(1-2x)^2 x\mathrm{H}_1(x) + \frac{32\mathrm{G}(\{5,5\},x)}{1-x} + \frac{64\mathrm{G}(\{5,5,2\},x)}{1-x} + 16(1-2x)\frac{\sqrt{x}}{\sqrt{1-x}}\mathrm{G}(\{5,2\},x)\Biggr]_+, \tag{C.12}$$

$$\mathbf{M}^{-1}[\mathsf{BS}_{10}(N)](x) = \Biggl[-\frac{1}{1-x}\left[-4-4\ln(2)\left(-1+\sqrt{1-x}\right)+4\sqrt{1-x}+\zeta_2\right] + 2(-1+\ln(2))\left(-1+\sqrt{1-x}+x\right)\frac{\mathrm{H}_0(x)}{(1-x)^{3/2}} - 2\frac{\mathrm{H}_1(x)}{\sqrt{1-x}} + \frac{\mathrm{H}_{0,1}(x)}{\sqrt{1-x}} - \frac{(-2+\ln(2))\mathrm{G}(\{6,1\},x)}{1-x} + \frac{\mathrm{G}(\{6,1,2\},x)}{2(1-x)} - \frac{\mathrm{G}(\{1,6,1\},x)}{2(1-x)}\Biggr]_+. \tag{C.13}$$

At lower weight, the G-functions over the alphabet (2.8) can be expressed in terms of known functions, i.e. by elementary functions and polylogarithms with involved arguments. This is, however, not possible in general at higher weight. For the simplest cases one finds.

$$\mathrm{G}(\{4\},x) = 2(-1+\ln(2)+w) + w\mathrm{H}_0(x) - (1+w)\mathrm{H}_{-1}(w) - (1-w)\mathrm{H}_1(w), \tag{C.14}$$

$$
\begin{aligned}
\mathrm{G}(\{1,4\},x) = {} & -4+4\ln(2)-2\ln^2(2)+4w+\frac{1}{2}w\mathrm{H}_0(x)^2-2\ln(2)\mathrm{H}_1(w)+\frac{1}{2}(1-w)\mathrm{H}_1^2(w) \\
& +(2(-2+\ln(2))+(1+w)\mathrm{H}_1(w))\mathrm{H}_{-1}(w)-\frac{1}{2}(1+w)\mathrm{H}_{-1}^2(w) \\
& -2\mathrm{H}_{-1,1}(w)+\zeta_2, \qquad \text{(C.15)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{G}(\{2,4\},x) = {} & 4-w(4-\zeta_2)+(4w-2(1-w)\mathrm{H}_1(w)-2(1+w)\mathrm{H}_{-1}(w))\mathrm{H}_0(w) \\
& +(2(-1+\ln(2)+w)+w\mathrm{H}_0(x))\mathrm{H}_1(x)-(1-w)\mathrm{H}_1(w)\mathrm{H}_1(x) \\
& -(1+w)\mathrm{H}_{-1}(w)\mathrm{H}_1(x)+2(1-w)\mathrm{H}_{0,1}(w)-w\mathrm{H}_{0,1}(x)+2(1+w)\mathrm{H}_{0,-1}(w) \\
& -3\zeta_2, \qquad \text{(C.16)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{G}(\{4,4\},x) = {} & 2\left(2+\ln^2(2)-2\ln(2)(1-w)-2w-x\right)-x\mathrm{H}_0(x)+\frac{1}{2}\mathrm{H}_0(x)^2+(2 \\
& -2\ln(2)-2w-x)\mathrm{H}_1(w)+(2-2\ln(2)-2w+x+2\mathrm{H}_1(w))\mathrm{H}_{-1}(w), \qquad \text{(C.17)}
\end{aligned}
$$

$$
\mathrm{G}(\{5\},x) = \frac{1}{4}\arcsin(\sqrt{x})-\frac{1}{4}w(1-2x)\sqrt{x}, \qquad \text{(C.18)}
$$

$$
\begin{aligned}
\mathrm{G}(\{5,1\},x) = {} & \frac{C}{4}+\frac{1}{32}(1-6\ln(2))\pi-\frac{i\pi^2}{48}-\frac{1}{4}i\arctan^2\left(\frac{1-2r}{1-2x}\right) \\
& +\ln(2)\left[\frac{1}{4}\arctan\left(\frac{1-2r}{1-2x}\right)+\frac{(-1+2x)(-r-4(-1+r)x+4(-1+r)x^2)}{4(1-2r)^2}\right] \\
& +\frac{1}{8(1-2r)^2}\Bigg[\left(-r-4(-1+r)x+4(-1+r)x^2\right)\Bigg[-3+2x \\
& +2(-1+2x)\ln\left(\frac{2-4r}{(1-2x)^2}\right)-4(-1+2x)\ln\left(-\frac{2(r-x)}{-1+2x}\right)\Bigg]\Bigg] \\
& +\frac{1}{8}\arctan\left(\frac{1-2r}{1-2x}\right)\Bigg[-1+4\ln\left(\frac{(1+i)(-1+2x)}{-1+(1-i)r+(1+i)x}\right) \\
& +2\ln\left(\frac{2-4r}{(1-2x)^2}\right)\Bigg]-\frac{1}{2}\arctan\left(\frac{r-x}{-1+r+x}\right)\ln\left(-\frac{2(r-x)}{-1+2x}\right) \\
& +\frac{i}{4}\mathrm{Li}_2\left[-\frac{(1+i)(r-x)}{-1+2x}\right]-\frac{i}{4}\mathrm{Li}_2\left[-\frac{(1-i)(r-x)}{-1+2x}\right] \\
& -\frac{i}{4}\mathrm{Li}_2\left[\frac{-1+(1+i)r+(1-i)x}{-i+(1+i)r+(-1+i)x}\right] \\
& \qquad \text{(C.19)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{G}(\{5,2\},x) = {} & \frac{1}{8}\Bigg[\frac{i\pi^2}{3}+2i\arccos^2(\sqrt{x})+\arccos(\sqrt{x})\Bigg(1-4\ln\left(2-2x+2i\sqrt{x(1-x)}\right) \\
& +2\ln(1-x)\Bigg)+(1+2x+2(1-2x)\ln(1-x))\sqrt{x(1-x)} \\
& -\frac{1}{2}\pi(1-4\ln(2))-2i\mathrm{Li}_2\left(\frac{1}{(i\sqrt{1-x}+\sqrt{x})^2}\right)\Bigg], \qquad \text{(C.20)}
\end{aligned}
$$

$$\mathrm{G}\left(\{5,5\},x\right)=\frac{1}{32}\Bigg[(1-2x)^2(1-x)x+\arcsin^2\left(\sqrt{x}\right)-2(1-2x)\arcsin\left(\sqrt{x}\right)$$
$$\times\sqrt{(1-x)x}\Bigg], \tag{C.21}$$

$$\mathrm{G}\left(\{6,1\},x\right)=4\left(-1+\ln(2)+w\right)-2(1-w)\mathrm{H}_1(w)-2(1+w)\mathrm{H}_{-1}(w), \tag{C.22}$$

$$\begin{aligned}\mathrm{G}\left(\{6,1,2\},x\right)=&-8-2w(-4+\zeta_2)+(-8w+4(1-w)\mathrm{H}_1(w)+4(1+w)\mathrm{H}_{-1}(w))\\&\times\mathrm{H}_0(w)-4(1-w)\mathrm{H}_{0,1}(w)-4(1+w)\mathrm{H}_{0,-1}(w)+6\zeta_2,\end{aligned} \tag{C.23}$$

$$\begin{aligned}\mathrm{G}\left(\{1,6,1\},x\right)=&4\left(-4+4\ln(2)-2\ln^2(2)+4w+\zeta_2\right)-4(1+\ln(2)-w)\mathrm{H}_1(w)\\&+(4(-3+\ln(2)-w)+4\mathrm{H}_1(w))\mathrm{H}_{-1}(w)-8\mathrm{H}_{-1,1}(w),\end{aligned} \tag{C.24}$$

where C denotes Catalan's constant [174]

$$C=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}. \tag{C.25}$$

Furthermore, one has

$$\arctan(x)=\frac{i}{2}\ln\left[\frac{1-ix}{1+ix}\right] \tag{C.26}$$
$$\arcsin\left(\sqrt{x}\right)=-i\ln\left(w+i\sqrt{x}\right) \tag{C.27}$$
$$\arccos\left(\sqrt{x}\right)=\frac{\pi}{2}+i\ln\left(w+i\sqrt{x}\right) \tag{C.28}$$
$$\mathrm{arctanh}(w)=\frac{1}{2}\left[\mathrm{H}_1(w)+\mathrm{H}_{-1}(w)\right] \tag{C.29}$$
$$\mathrm{Li}_2\left(\frac{1}{u}\right)=2\zeta_2-\frac{1}{2}\mathrm{H}_0^2(u)-\mathrm{H}_{0,1}(u)-i\pi\mathrm{H}_0(u) \tag{C.30}$$

and in case, one has to consider the analytic continuation of the above quantities.

D Asymptotic expansion of $\tilde{a}_{gg,Q}^{(3)}(N)$ and $\Delta\tilde{a}_{gg,Q}^{(3)}(N)$

Here we present the asymptotic expansion of the finite binomial sums used and for $\tilde{a}_{gg,Q}^{(3)}(N)$ and $\Delta\tilde{a}_{gg,Q}^{(3)}(N)$ to $O(1/N^{10})$ for QCD, by specifying the color factors to SU(3). One obtains

$$\mathsf{BS}_0(N)\propto\frac{1}{2N}\sum_{k=0}^{\infty}\left(\frac{2l+1}{2N}\right)^k, \tag{D.1}$$

$$\begin{aligned}\mathsf{BS}_1(N)\propto\sqrt{\pi}\sqrt{N}\Bigg[&1+\frac{1}{8N}+\frac{1}{128N^2}-\frac{5}{1024N^3}-\frac{21}{32768N^4}+\frac{399}{262144N^5}+\frac{869}{4194304N^6}\\&-\frac{39325}{33554432N^7}-\frac{334477}{2147483648N^8}+\frac{28717403}{17179869184N^9}+\frac{59697183}{274877906944N^{10}}\Bigg],\end{aligned} \tag{D.2}$$

$$\mathsf{BS}_2(N)\propto\frac{1}{\sqrt{\pi}\sqrt{N}}\left[1-\frac{1}{8N}+\frac{1}{128N^2}+\frac{5}{1024N^3}-\frac{21}{32768N^4}-\frac{399}{262144N^5}+\frac{869}{4194304N^6}\right.$$
$$\left.+\frac{39325}{33554432N^7}-\frac{334477}{2147483648N^8}-\frac{28717403}{17179869184N^9}+\frac{59697183}{274877906944N^{10}}\right],\tag{D.3}$$

$$\mathsf{BS}_3(N)\propto 2\ln(2)+\frac{1}{\sqrt{N}\sqrt{\pi}}\left[-2+\frac{7}{12N}-\frac{61}{320N^2}+\frac{307}{10752N^3}+\frac{911}{49152N^4}-\frac{12559}{1441792N^5}\right.$$
$$-\frac{1404237}{136314880N^6}+\frac{386621}{50331648N^7}+\frac{223373117}{18253611008N^8}-\frac{42971056687}{3427383902208N^9}$$
$$\left.-\frac{50702821769}{2061584302080N^{10}}\right],\tag{D.4}$$

$$\mathsf{BS}_4(N)\propto 3\zeta_2+\frac{\sqrt{\pi}}{\sqrt{N}}\left[-2+\frac{5}{12N}-\frac{21}{320N^2}-\frac{223}{10752N^3}+\frac{671}{49152N^4}+\frac{11635}{1441792N^5}\right.$$
$$-\frac{1196757}{136314880N^6}-\frac{376193}{50331648N^7}+\frac{201980317}{18253611008N^8}+\frac{42437231395}{3427383902208N^9}$$
$$\left.-\frac{47256733409}{2061584302080N^{10}}\right],\tag{D.5}$$

$$\mathsf{BS}_5(N)\propto-\frac{7}{2}\zeta_3+6\ln(2)\zeta_2+\frac{\sqrt{\pi}}{\sqrt{N}}\left[-\frac{2}{3N}+\frac{9}{20N^2}-\frac{199}{1344N^3}-\frac{145}{4608N^4}+\frac{8909}{180224N^5}\right.$$
$$+\frac{427409}{25559040N^6}-\frac{1375417}{31457280N^7}-\frac{17099251}{855638016N^8}+\frac{30183885889}{428422987776N^9}$$
$$\left.+\frac{110144579981}{2705829396480N^{10}}\right],\tag{D.6}$$

$$\mathsf{BS}_6(N)\propto\frac{7}{2}\zeta_3+\frac{2}{N}-\frac{4}{3N^2}+\frac{32}{45N^3}-\frac{8}{35N^4}-\frac{8}{225N^5}+\frac{248}{3465N^6}+\frac{6856}{315315N^7}$$
$$-\frac{3176}{45045N^8}-\frac{11304}{425425N^9}+\frac{615256}{4849845N^{10}}+\frac{8597752}{160044885N^{11}}+\frac{\zeta_2}{\sqrt{N}\sqrt{\pi}}\left[-6+\frac{7}{4N}\right.$$
$$-\frac{183}{320N^2}+\frac{307}{3584N^3}+\frac{911}{16384N^4}-\frac{37677}{1441792N^5}-\frac{4212711}{136314880N^6}$$
$$\left.+\frac{386621}{16777216N^7}+\frac{670119351}{18253611008N^8}-\frac{42971056687}{1142461300736N^9}-\frac{50702821769}{687194767360N^{10}}\right],\tag{D.7}$$

$$\mathsf{BS}_7(N)\propto-16\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{2}{3}\ln^4(2)+4\ln^2(2)\zeta_2-7\ln(2)\zeta_3+\frac{53}{10}\zeta_2^2+\frac{1}{3N^2}-\frac{23}{45N^3}+\frac{17}{35N^4}$$
$$-\frac{7}{25N^5}+\frac{169}{10395N^6}+\frac{12737}{105105N^7}-\frac{719}{27027N^8}-\frac{1755653}{11486475N^9}+\frac{4601449}{72747675N^{10}}$$
$$+\frac{1}{\sqrt{N}\sqrt{\pi}}\left\{\ln(2)\zeta_2\left[-12+\frac{7}{2N}-\frac{183}{160N^2}+\frac{307}{1792N^3}+\frac{911}{8192N^4}-\frac{37677}{720896N^5}\right.\right.$$

$$
-\frac{4212711}{68157440N^6}+\frac{386621}{8388608N^7}+\frac{670119351}{9126805504N^8}-\frac{42971056687}{571230650368N^9}
-\frac{50702821769}{343597383680N^{10}}\Bigg]+\zeta_3\Bigg[7-\frac{49}{24N}+\frac{427}{640N^2}-\frac{307}{3072N^3}-\frac{6377}{98304N^4}
+\frac{87913}{2883584N^5}+\frac{9829659}{272629760N^6}-\frac{2706347}{100663296N^7}-\frac{1563611819}{36507222016N^8}
+\frac{42971056687}{979252543488N^9}+\frac{354919752383}{4123168604160N^{10}}\Bigg]\Bigg\}, \tag{D.8}
$$

$$
\mathsf{BS}_8(N)\propto -7\zeta_3+\Bigg[+3(\ln(N)+\gamma_E)+\frac{3}{2N}-\frac{1}{4N^2}+\frac{1}{40N^4}-\frac{1}{84N^6}+\frac{1}{80N^8}-\frac{1}{44N^{10}}\Bigg]\zeta_2
+\sqrt{\frac{\pi}{N}}\Bigg[4-\frac{23}{18N}+\frac{1163}{2400N^2}-\frac{64177}{564480N^3}-\frac{237829}{7741440N^4}+\frac{5982083}{166526976N^5}
+\frac{5577806159}{438593126400N^6}-\frac{12013850977}{377864847360N^7}-\frac{1042694885077}{90766080737280N^8}
+\frac{6663445693908281}{127863697547722752N^9}+\frac{23651830282693133}{1363413316298342400N^{10}}\Bigg], \tag{D.9}
$$

$$
\mathsf{BS}_9(N)\propto 7\ln(2)\zeta_3+\frac{6}{N}-\frac{20}{9N^2}+\frac{203}{450N^3}+\frac{2752}{11025N^4}-\frac{8647}{31500N^5}+\frac{587381}{24012450N^6}
+\frac{1143813683}{9468909450N^7}-\frac{12378799}{2029052025N^8}-\frac{3519473211853}{23455841409000N^9}
-\frac{2992510324519}{282251958288300N^{10}}+\frac{1509837197342693}{4657157311756950N^{11}}+(\ln(N)+\gamma_E)\Bigg[\frac{2}{N}-\frac{4}{3N^2}
+\frac{32}{45N^3}-\frac{8}{35N^4}-\frac{8}{225N^5}+\frac{248}{3465N^6}+\frac{6856}{315315N^7}-\frac{3176}{45045N^8}-\frac{11304}{425425N^9}
+\frac{615256}{4849845N^{10}}+\frac{8597752}{160044885N^{11}}\Bigg]+\frac{1}{\sqrt{\pi N}}\Bigg\{\ln(2)\zeta_2\Bigg[-12+\frac{7}{2N}-\frac{183}{160N^2}
+\frac{307}{1792N^3}+\frac{911}{8192N^4}-\frac{37677}{720896N^5}-\frac{4212711}{68157440N^6}+\frac{386621}{8388608N^7}
+\frac{670119351}{9126805504N^8}-\frac{42971056687}{571230650368N^9}-\frac{50702821769}{343597383680N^{10}}\Bigg]+\zeta_3\Bigg[-7+\frac{49}{24N}
-\frac{427}{640N^2}+\frac{307}{3072N^3}+\frac{6377}{98304N^4}-\frac{87913}{2883584N^5}-\frac{9829659}{272629760N^6}+\frac{2706347}{100663296N^7}
+\frac{1563611819}{36507222016N^8}-\frac{42971056687}{979252543488N^9}-\frac{354919752383}{4123168604160N^{10}}\Bigg]\Bigg\}, \tag{D.10}
$$

$$
\mathsf{BS}_{10}(N)\propto 6\ln(2)\zeta_2+\frac{7}{2}\zeta_3+\sqrt{\frac{\pi}{N}}\Bigg\{-4-\frac{7}{18N}+\frac{817}{2400N^2}-\frac{3835}{37632N^3}-\frac{24677}{1105920N^4}
+\frac{1822519}{71368704N^5}+\frac{797783941}{55820943360N^6}-\frac{149172521}{7927234560N^7}-\frac{45292690253}{2327335403520N^8}
+\frac{7730705027207}{293041323638784N^9}+\frac{51644847163033}{1190564934451200N^{10}}-\frac{286475788323757}{1809658700365824 0N^{11}}
$$

$$
+(\ln(N)+\gamma_E)\Biggl[-2+\frac{5}{12N}-\frac{21}{320N^2}-\frac{223}{10752N^3}+\frac{671}{49152N^4}+\frac{11635}{1441792N^5}
-\frac{1196757}{136314880N^6}-\frac{376193}{50331648N^7}+\frac{201980317}{18253611008N^8}+\frac{42437231395}{3427383902208N^9}
-\frac{47256733409}{2061584302080N^{10}}\Biggr]\Biggr\}. \qquad (D.11)
$$

The asymptotic expansions of $\tilde{a}_{gg,Q,N_F=0}^{(3)}(N)$ and $\Delta\tilde{a}_{gg,Q,N_F=0}^{(3)}(N)$ are given by

$$
\tilde{a}_{gg,Q,N_F=0}^{(3)} \propto \frac{1}{2}(1+(-1)^N)
\times\Biggl\{\frac{1}{N}\Biggl[-\frac{16L^3}{3}+L^2\left(\frac{457}{9}-6\zeta_2\right)+L\left(-\frac{5491}{9}+\frac{172\zeta_2}{3}-72\zeta_3\right)+\frac{1545977}{2430}+128\text{Li}_4\left(\frac{1}{2}\right)
+\frac{16\ln^4(2)}{3}-\frac{2501\zeta_4}{3}+\frac{569}{3}\zeta_2-32\ln^2(2)\zeta_2+\frac{10261}{45}\zeta_3\Biggr]+\frac{1}{N^2}\Biggl[-\frac{20L^4}{81}-\frac{16L^3}{81}
+L^2\left(-\frac{11323}{162}-\frac{284\zeta_2}{27}\right)+L\left(\frac{346327}{486}-\frac{9604\zeta_2}{27}-\frac{6280\zeta_3}{81}\right)-\frac{40929283}{14580}-\frac{2368}{9}\text{Li}_4\left(\frac{1}{2}\right)
-\frac{296\ln^4(2)}{27}+\frac{12044\zeta_4}{9}-\frac{23179}{54}\zeta_2-\frac{40}{3}\ln(2)\zeta_2+\frac{592}{9}\ln^2(2)\zeta_2-\frac{596741}{810}\zeta_3\Biggr]
+\frac{1}{N^3}\Biggl[\frac{20L^4}{81}-\frac{584L^3}{27}+L^2\left(\frac{110639}{810}+\frac{68\zeta_2}{27}\right)+L\left(-\frac{348873919}{85050}+\frac{22421\zeta_2}{27}-\frac{1496\zeta_3}{81}\right)
+\frac{70746883829}{17860500}+256\text{Li}_4\left(\frac{1}{2}\right)+\frac{32\ln^4(2)}{3}-\frac{4021\zeta_4}{3}+\frac{36481}{135}\zeta_2-\frac{172}{9}\ln(2)\zeta_2
-64\ln^2(2)\zeta_2+\frac{41777}{27}\zeta_3\Biggr]+\frac{1}{N^4}\Biggl[-\frac{140L^4}{81}+\frac{5012L^3}{243}+L^2\left(-\frac{57859}{810}-100\zeta_2\right)
+L\left(\frac{84447577}{18900}-\frac{91496\zeta_2}{81}-\frac{69208\zeta_3}{81}\right)-\frac{102450206417}{8930250}-\frac{34528}{45}\text{Li}_4\left(\frac{1}{2}\right)
-\frac{4316\ln^4(2)}{135}+\frac{901591\zeta_4}{270}-\frac{983209}{405}\zeta_2+\frac{728}{9}\ln(2)\zeta_2+\frac{8632}{45}\ln^2(2)\zeta_2
-\frac{645605}{972}\zeta_3\Biggr]+\frac{1}{N^5}\Biggl[\frac{260L^4}{81}-\frac{237004L^3}{1215}+L^2\left(\frac{1889789}{4050}+\frac{4900\zeta_2}{27}\right)
+L\left(-\frac{52226399557}{2338875}+\frac{1231751\zeta_2}{810}+\frac{124360\zeta_3}{81}\right)+\frac{342439924906957}{8644482000}
+1280\text{Li}_4\left(\frac{1}{2}\right)+\frac{160\ln^4(2)}{3}-\frac{145781\zeta_4}{27}+\frac{9055723\zeta_2}{5400}-\frac{2788}{9}\ln(2)\zeta_2-320\ln^2(2)\zeta_2
-\frac{2976962\zeta_3}{1215}\Biggr]+\frac{1}{N^6}\Biggl[-\frac{700L^4}{81}+\frac{35950L^3}{81}+L^2\left(-\frac{1942289}{1215}-\frac{13340\zeta_2}{27}\right)
+L\left(\frac{354034895677}{6949800}-\frac{438671\zeta_2}{405}-\frac{341240\zeta_3}{81}\right)-\frac{3512899304529779}{35777570400}
$$

$$
\begin{aligned}
&-\frac{532288}{189}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{66536\ln^4(2)}{567}+\frac{739532\zeta_4}{63}-\frac{469123591\zeta_2}{34020}+\frac{6128}{9}\ln(2)\zeta_2\\
&+\frac{133072}{189}\ln^2(2)\zeta_2+\frac{75254303\zeta_3}{5670}\Bigg]+\frac{1}{N^7}\Bigg[\frac{1460L^4}{81}-\frac{772130L^3}{567}+L^2\Bigg(\frac{326977031}{59535}\\
&+\frac{9052\zeta_2}{9}\Bigg)+L\Bigg(-\frac{366567325520251}{2383781400}-\frac{735578\zeta_2}{405}+\frac{688744\zeta_3}{81}\Bigg)\\
&+\frac{48350302011570733597}{143169910884000}+5376\mathrm{Li}_4\left(\frac{1}{2}\right)+224\ln^4(2)-\frac{606637\zeta_4}{27}\\
&+\frac{4619767781\zeta_2}{476280}-\frac{22108}{9}\ln(2)\zeta_2-1344\ln^2(2)\zeta_2-\frac{37472116}{945}\zeta_3\Bigg]\\
&+\frac{1}{N^8}\Bigg[-\frac{1060L^4}{27}+\frac{16654490L^3}{5103}+L^2\Bigg(-\frac{294540853}{17010}-\frac{58828\zeta_2}{27}\Bigg)\\
&+L\Bigg(\frac{20298162797504269}{52102650600}+\frac{13779523\zeta_2}{1701}-\frac{165592\zeta_3}{9}\Bigg)-\frac{495344}{45}\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&-\frac{314238928512294227009}{354652322104080}-\frac{61918\ln^4(2)}{135}+\frac{24921641\zeta_4}{540}-\frac{107294041\zeta_2}{1701}\\
&+\frac{14408}{9}\ln(2)\zeta_2+\frac{123836}{45}\ln^2(2)\zeta_2+\frac{22360477051\zeta_3}{204120}\Bigg]+\frac{1}{N^9}\Bigg[\frac{6500L^4}{81}\\
&-\frac{206201494L^3}{25515}+L^2\Bigg(\frac{4303709917}{85050}+\frac{119108\zeta_2}{27}\Bigg)+L\Bigg(-\frac{2535160339716913903}{2474875903500}\\
&-\frac{739828937\zeta_2}{34020}+\frac{3007240\zeta_3}{81}\Bigg)+\frac{2438276132171940601235471 9}{9144963448540920000}+21760\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&+\frac{2720\ln^4(2)}{3}-\frac{824183\zeta_4}{9}+\frac{747533966657\zeta_2}{1360800}-\frac{139924}{9}\ln(2)\zeta_2-5440\ln^2(2)\zeta_2\\
&-\frac{2513889701\zeta_3}{10206}\Bigg]+\frac{1}{N^{10}}\Bigg[-\frac{13340L^4}{81}+\frac{13615043L^3}{729}+L^2\Bigg(-\frac{4157236511}{29160}-\frac{81236\zeta_2}{9}\Bigg)\\
&+L\Bigg(\frac{2035166415603071}{814438800}+\frac{487601393\zeta_2}{5670}-\frac{6147640\zeta_3}{81}\Bigg)-\frac{4333888}{99}\mathrm{Li}_4\left(\frac{1}{2}\right)\\
&-\frac{2408250929100519977653159}{347591370894768000}-\frac{541736\ln^4(2)}{297}+\frac{54840506\zeta_4}{297}-\frac{8559461498321\zeta_2}{22453200}\\
&-\frac{63136}{9}\ln(2)\zeta_2+\frac{1083472}{99}\ln^2(2)\zeta_2+\frac{186728120056\zeta_3}{280665}\Bigg]\Bigg\},
\end{aligned}
\tag{D.12}
$$

where L is defined in eq. (3.4) and

$$
\begin{aligned}
&\Delta\tilde{a}_{gg,Q,N_F=0}^{(3)}\propto\frac{1}{2}(1-(-1)^N)\\
&\times\Bigg\{\frac{1}{N}\Bigg[-\frac{16L^3}{3}+L^2\left(\frac{457}{9}-6\zeta_2\right)+L\left(-\frac{5491}{9}+\frac{172\zeta_2}{3}-72\zeta_3\right)+\frac{1545977}{2430}+128\mathrm{Li}_4\left(\frac{1}{2}\right)
\end{aligned}
$$

$$+\frac{16\ln^4(2)}{3}+\frac{569}{3}\zeta_2-32\ln^2(2)\zeta_2-\frac{5002}{15}\zeta_2^2+\frac{10261}{45}\zeta_3\Big]+\frac{1}{N^2}\Big[-\frac{20L^4}{81}+\frac{64L^3}{81}$$

$$+L^2\left(-\frac{10411}{162}-\frac{284\zeta_2}{27}\right)+L\left(\frac{348367}{486}-332\zeta_2-\frac{6280\zeta_3}{81}\right)-\frac{31616491}{14580}-\frac{2368}{9}\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$-\frac{296\ln^4(2)}{27}-\frac{12611}{54}\zeta_2-\frac{40}{3}\ln(2)\zeta_2+\frac{592}{9}\ln^2(2)\zeta_2+\frac{24088}{45}\zeta_2^2-\frac{686773}{810}\zeta_3\Big]+\frac{1}{N^3}\Big[\frac{20L^4}{81}$$

$$-\frac{1832L^3}{81}+L^2\left(\frac{111599}{810}+\frac{68\zeta_2}{27}\right)+L\left(-\frac{328566919}{85050}+\frac{21781\zeta_2}{27}-\frac{1496\zeta_3}{81}\right)$$

$$+\frac{62031195029}{17860500}+\frac{32\ln^4(2)}{3}+256\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{4781}{135}\zeta_2-\frac{172}{9}\ln(2)\zeta_2-64\ln^2(2)\zeta_2$$

$$-\frac{8042}{15}\zeta_2^2+\frac{675559}{405}\zeta_3\Big]+\frac{1}{N^4}\Big[\frac{20L^4}{81}+\frac{9860L^3}{243}+L^2\left(-\frac{50743}{162}+\frac{332\zeta_2}{9}\right)+L\Big(\frac{43674185}{6804}$$

$$-\frac{83456\zeta_2}{81}+\frac{31528\zeta_3}{81}\Big)-\frac{1094756651}{198450}-\frac{11488}{45}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{1436\ln^4(2)}{135}-\frac{2261}{5}\zeta_2$$

$$-\frac{488}{9}\ln(2)\zeta_2+\frac{2872}{45}\ln^2(2)\zeta_2+\frac{345311}{675}\zeta_2^2-\frac{12599017\zeta_3}{4860}\Big]+\frac{1}{N^5}\Big[-\frac{20L^4}{27}-\frac{51964L^3}{1215}$$

$$+L^2\left(\frac{3020069}{4050}-\frac{2492\zeta_2}{27}\right)+L\left(-\frac{3048134171}{334125}+\frac{1160471\zeta_2}{810}-952\zeta_3\right)$$

$$+\frac{16455625566683}{1234926000}+256\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{32\ln^4(2)}{3}+\frac{1333649\zeta_2}{16200}-\frac{940}{9}\ln(2)\zeta_2-64\ln^2(2)\zeta_2$$

$$-\frac{13810}{27}\zeta_2^2+\frac{1101248}{243}\zeta_3\Big]+\frac{1}{N^6}\Big[\frac{100L^4}{81}+\frac{8858L^3}{243}+L^2\left(-\frac{353827}{243}+\frac{5140\zeta_2}{27}\right)$$

$$+L\left(\frac{532290942707}{48648600}-\frac{684791\zeta_2}{405}+\frac{162440\zeta_3}{81}\right)-\frac{48448}{189}\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$-\frac{224935849458230479}{8765504748000}-\frac{6056\ln^4(2)}{567}-\frac{34195783\zeta_2}{34020}+\frac{1048}{9}\ln(2)\zeta_2+\frac{12112}{189}\ln^2(2)\zeta_2$$

$$+\frac{543232}{945}\zeta_2^2-\frac{124918583\zeta_3}{17010}\Big]+\frac{1}{N^7}\Big[-\frac{140L^4}{81}-\frac{36406L^3}{1701}+L^2\left(\frac{147944849}{59535}-\frac{3268\zeta_2}{9}\right)$$

$$+L\left(-\frac{27141581812291}{2383781400}+\frac{712342\zeta_2}{405}-\frac{318616\zeta_3}{81}\right)+\frac{1233346073528967337}{28633982176800}+256\mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$+\frac{32\ln^4(2)}{3}+\frac{680314781\zeta_2}{476280}-\frac{6796}{9}\ln(2)\zeta_2-64\ln^2(2)\zeta_2-\frac{100714}{135}\zeta_2^2+\frac{21870544\zeta_3}{1701}\Big]$$

$$+\frac{1}{N^8}\Big[+\frac{20L^4}{9}-\frac{13126L^3}{5103}+L^2\left(-\frac{67038673}{17010}+\frac{18788\zeta_2}{27}\right)+L\Big(\frac{36270418853929}{4007896200}$$

$$-\frac{15210661\zeta_2}{8505}+\frac{208376\zeta_3}{27}\Big)-\frac{13106933647387977503}{227341232118000}-\frac{11504}{45}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{1438\ln^4(2)}{135}$$

$$-\frac{1097947\zeta_2}{1215}+\frac{9784}{9}\ln(2)\zeta_2+\frac{2876}{45}\ln^2(2)\zeta_2+\frac{1557881\zeta_2^2}{1350}-\frac{4263728837\zeta_3}{204120}\Big]+\frac{1}{N^9}\Big[-\frac{220L^4}{81}$$

$$+\frac{900506L^3}{25515}+L^2\left(\frac{57444913}{9450}-\frac{36124\zeta_2}{27}\right)+L\Biggl(-\frac{358561874254051}{190375069500}+\frac{75800311\zeta_2}{34020}$$
$$-\frac{1223672\zeta_3}{81}\Biggr)+\frac{3796963008541133583652 3}{703458726810840000}+256\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{32\ln^4(2)}{3}-\frac{3288591983\zeta_2}{1360800}$$
$$-\frac{32812}{9}\ln(2)\zeta_2-64\ln^2(2)\zeta_2-\frac{90782}{45}\zeta_2^2+\frac{1953040931\zeta_3}{51030}\Biggr]+\frac{1}{N^{10}}\Biggl[\frac{260L^4}{81}-\frac{389675L^3}{5103}$$
$$+L^2\left(-\frac{1908055889}{204120}+\frac{7828\zeta_2}{3}\right)+L\left(-\frac{393614680461559 79}{1979900722800}-\frac{1383047\zeta_2}{5670}+\frac{2414920\zeta_3}{81}\right)$$
$$-\frac{23282982159571165151037 23}{768176929677437280000}-\frac{25408}{99}\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{3176\ln^4(2)}{297}+\frac{47691873199\zeta_2}{22453200}$$
$$+\frac{47320}{9}\ln(2)\zeta_2+\frac{6352}{99}\ln^2(2)\zeta_2+\frac{5656652\zeta_2^2}{1485}-\frac{2480325346\zeta_3}{40095}\Biggr]\Biggr\}. \quad \text{(D.13)}$$

Furthermore, one finds that

$$\frac{\tilde{a}_{gg,Q,N_F=0}^{(3)}}{[1+(-1)^N]}-\frac{\Delta\tilde{a}_{gg,Q,N_F=0}^{(3)}}{[1-(-1)^N]}\propto\frac{1}{N^2}. \quad \text{(D.14)}$$

This explains the close agreement of the numbers in column 2 of tables 1 and 2.

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