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LARGE SIGNAL ELECTRON BUNCHING

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1. Introduction

Electron bunching is the basis of many microwave devices in which the interaction between the electron beam and electromagnetic wave plays an essential role as for instance in klystrons or linear accelerators. Consequently, the problem of electron bunching was analysed by many authors. Usually this analysis is based upon some simplifying assumptions. The most important of them are:

i) The electron transit times through the buncher are assumed to be negligible in comparison with the period of electromagnetic oscillations.

ii) The alternating voltage Va between buncher electrodes is assumed to be small in comparison with the DC gun accelerating voltage Vg.

iii) Space charge effects are ignored.

The measurements made at LAL [1] for the assembly prebuncher - buncher of the linac V-pre-injector for LEP, have shown that good conditions for electron bunching exist also for the case when the first two of the above conditions are not fulfilled, e.g. instead of $(Va/Vg) \ll 1$ one can have $Va \approx (2 - 3)Vg$. Also the transit times of electrons through the cavity of prebuncher can be of the order of the period of field oscillations in the cavity. To take these effects into account it is necessary to solve the equations of motion of electrons in the prebuncher without the above-mentioned assumptions 1 and 2.

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The non-relativistic case was treated by Shevtchik [7], who found that the "ideal bunching" (the current of bunched electrons tends to infinity) is obtained for Va = 2Vg and that the maximum of the first harmonic of current equal to $I_1 = 1.16\ I_0$ (I_0 - incoming current) is obtained for Vs ≈ 3.68 Vg. Below we shall solve the relativistic equations of motion of electrons in strong electromagnetic fields. Two kinds of axial field distribution in the prebuncher will be considered:

1. Constant field amplitude for which case the analytical solution is possible.

2. Gaussian field distribution along the axis of the bunching cavity.

In both cases a broad maximum for bunched electrons was found for

$$1.5 < \frac{Va}{Vg} \lesssim 3.5$$

This is in good agreement with measurements made by R. Belbéoch [1] and with numerical calculations made at LAL for Gaussian field distribution [2]. It agrees also with the results obtained by Shevtchik [7].

2. Equations of Motion of Electrons in the Prebuncher

We shall consider the axial motion of electrons in the prebuncher in the approximation of zero space charge. Equations of motion can then be written in the form of:

$$\frac{d\gamma}{ds} = A \cos \phi \qquad (1)$$

$$\frac{d\phi}{ds} = 2\pi \frac{\gamma}{\sqrt{\gamma^2-1}} \qquad (2)$$

where:

$$\gamma = \frac{mc^2}{m_0 c^2}$$

m, m_0 = mass and rest mass of electron

c = the velocity of light in vacuum,

$\phi = \phi_0 + 2\pi f t$ - phase of the electromagnetic field in the prebuncher

f = frequency of the electromagnetic field

$s = \frac{z}{\lambda}$, z - distance along the axis

λ = wave-length of the electromagnetic field

$$A = \frac{q E \lambda}{W_0}$$

$W_0 = m_0 c^2$ = 0,5110041 MeV - rest energy of electron

q = charge of electron

E = amplitude of the axial component of electric field intensity in the prebuncher

Generally E is a function at least of z and Eqs. (2) and (3) can be integrated only numerically. However, as it will be shown later the physical picture is similar for E(z) = const. and e.g. for gaussian field dependence. Since for E(z) = const. some analytic results are possible, we shall begin our considerations with this case.

3. Solution to the Equations of Motion for the Case E = const.

By elimination of ds from Eqs. (1) and (2) we obtain the equation

$$\frac{A}{2\pi} \cos\phi \, d\phi = \frac{\gamma \, d\gamma}{\sqrt{\gamma^2 - 1}} \tag{3}$$

which can easily be integrated to give

$$\sqrt{\gamma^2 - 1} = A_1 (\sin\phi + K) \tag{4}$$

where $A_1 = \frac{A}{2\pi}$, and K is a constant which can be defined by initital conditions.

The left-hand side of the Eq. (4) is a normalized momentum of a particle:

$$p = \frac{mv}{m_0 c} = \gamma\beta = \sqrt{\gamma^2 - 1} = A_1 (\sin\phi + K) \tag{5}$$

where $\beta = \frac{v}{c}$, v = the velocity of an electron.

For further considerations it will be convenient to take p as a new variable instead of γ. Equation (5) gives then the phase trajectories in the phase space (p, ϕ), and is frequently used to calculate the separatrix and phase acceptance when analysing, e.g. the linear accelerator. However, it does not give directly the dependences $\gamma = \gamma(s)$ and $\phi = \phi(s)$, which are necessary to describe the axial motion. To obtain these relations we will follow the method given in [4,5].

From Eq. (5) we can express γ and $\cos\phi$ as function of p:

$$\gamma = \sqrt{1 + p^2}, \qquad d\gamma = \frac{p \, dp}{\sqrt{1 + p^2}} \tag{6}$$

$$\cos \phi = \pm \sqrt{1 - \left(\frac{p}{A_1} - K\right)^2} \tag{7}$$

Inserting (6) and (7) into (1) gives

$$ds = \pm \frac{1}{A} \frac{p\,d\,p}{\sqrt{2 + p^2}\sqrt{1 - \left(\frac{p}{A_1} - K\right)^2}}$$

or after integration

$$s = \pm \frac{1}{2\,\pi} \int \frac{p\,d\,p}{\sqrt{1 + p^2}\sqrt{A_1^2 \quad - (p - A_1K)^2}} \tag{8}$$

The sign (±) in the integral (8) depends upon the sign of cos ϕ as given by (7).

The integral given by Eq. (8) can be expressed in terms of incomplete elliptic integrals of the first and third kinds. The suitable expressions are given in Appendix I. Since these expressions are rather complicated, for numerical results we will prefer to integrate directly using either Eq. (8) or Eqs. (1) and (3). However, we shall see below that some interesting results can be obtained by considering the analytical expressions (5) and (8).

4. Condition for Transmission and Reflection of Electrons by Bunchers

In the case when the alternating voltage amplitude Va is much smaller than the gun accelerating voltage Vg, all electrons entering the bunching cavity pass through it with some small velocity modulation and are bunched in the drift space which follows the buncher. In the case of Va > Vg the picture is much more complicated. Generally, depending upon the ratio of Va/Vg > 1, the width of the bunching cavity in comparison with the wave-length and the phase of electron, 3 cases are possible:

i) Electron phases through the cavity without oscillations, i.e. its velocity does not change the sign inside the cavity.

ii) Electron passes through the cavity after one or more oscillations.

iii) Electron is reflected by the cavity.

We shall now discuss the conditions for which these 3 cases can be realized.

4.1 Conditions for Electron Transmission without oscillations

We begin with Eq. (5)

$$p = \sqrt{\gamma^2 - 1} = A_1 (\sin \phi + K)$$

In this equation A_1 is defined by the electric field intensity in the cavity (see Eqs. (1) and (3)). K can be obtained from initial values of p and ϕ.

Assume that we know energy γ_0 (corresponding to Vg) and phase ϕ_0 of an electron entering the bunching cavity. We also know then

$p_0 = \sqrt{\gamma_0^2 - 1}$ and we obtain for K

$$K = \frac{p_0}{A_1} - \sin \phi_0 \tag{9}$$

Inserting (9) into (5) yields the following relation between p, p_0, ϕ, ϕ_0

$$\frac{p - p_0}{A_1} = \sin \phi - \sin \phi_0$$

during the motion. For momentum p we have

$$p = p_0 + A_1 (\sin \phi - \sin \phi_0) \tag{10}$$

According to Eq. (10) p is a periodic function of ϕ. Assume for a moment that the bunching cavity is sufficiently long so that electron spends at least one period in the cavity (the case of shorter times will be discussed below). Since p is periodic it attains its minimum and maximum values given correspondingly by:

$$p_{min} = p_0 - A_1 (1 + \sin \phi_0) \qquad (11)$$

$$p_{max} = p_0 + A_1 (1 - \sin \phi_0) \qquad (12)$$

Depending upon the values of p_0 and A_1 we can have

i) $p_{min} > 0$ for all values of ϕ_0

ii) $p_{min} = 0$ for some values of ϕ_0

iii) $p_{min} < 0$ for some values of ϕ_0

It is obvious that the condition for electron transmission without oscillations is $p_{min} > 0$. For the case $p_{min} < 0$ the electron can oscillate and both transmission and reflection are possible. Applying condition $p_{min} > 0$ to Eq. (11) yields

$$p_0 - A_1 (1 + \sin \phi_0) > 0$$

or

$$A_1 < \frac{p_0}{1 + \sin P_0} = A_{1T} \qquad (13)$$

Relation (13) then defines the limits for A_1 and for corresponding values of electric field intensity in the bunching cavity for which the electron with momentum p_0 and phase ϕ_0 pass through the cavity without oscillations.

Since we are interested in a minimum value of A_1 it is obtained for $\sin \phi_0 = 1$ and it is equal to

$$A_{1Tmin} < \frac{p_0}{2}$$

Of course this value of A_{1Tmin} corresponds to a maximum value of electric field intensity E_T for which all electrons still pass through the cavity without oscillations. Taking into account definitions of A and A_1 we obtain the condition for electric field intensity E_T

$$E_T = \frac{2\pi W_0}{q\lambda} A_{1Tmin} < \frac{\pi W_0}{q\lambda} p_0 = \frac{0.511041 . \pi}{\lambda} \sqrt{\gamma_0^2 -1} \qquad (14)$$

E_T is in MV/m if λ is in meters. The physical meaning of the conditions (13) or (14) is clear. It states that the maximum deceleration which according to Eq. (1) occurs if electron enters the field at phase $\phi_0 = \pi/2$ and leaves it at phase the $\phi = 3/2\ \pi$ is such that change of electron kinetic energy is just equal to the kinetic energy it had when entering the electromagnetic field. Notice that condition given by Eq. (14) is the sufficient condition for an electron to pass through the cavity without oscillations. This condition depends only on the electric field intensity and does not depend on the cavity length. Generally, however, three different cases can occur depending upon the cavity length in comparison with the path traversed by electron during the time when phase changes from $\pi/2$ to $3/2\ \pi$, i.e. momentum changes from p_0 to $p_{min} > 0$. According to Eq. (8) the path traversed by the electron which changes the momentum from p_0 to p_{min} is given by

$$\Delta S = \frac{1}{2\pi} \int_{p_{min}}^{p_0} \frac{p\ d\ p}{\sqrt{(1 + p^2)\ (A_1^2 \quad - (p - A_1\ K)^2)}} \qquad (15)$$

Now we can have:

i) $\Delta S > \frac{L}{\lambda}$ (L = cavity length) electron leaves the cavity before the phase attains value $\phi = 3/2\ \pi$, so that its momentum $p > p_{min}$

ii) $\Delta S = \frac{L}{\lambda}$ electron leaves the cavity with $p = p_{min}$

iii) $\Delta S < \frac{L}{\lambda}$ electron leaves the cavity with p such that $p_{min} < p < p_{max}$ depending on the moment it arrives to the end.

We can also find the maximum energy spread for electrons leaving the bunching cavity. According to Eq. (11) and (12) the minimum value of p_{min} is obtained for $\phi_0 = \pi/2$ an equals to $p_{min} = p_0 - 2A_1$, whereas the maximum value of p_{max} is obtained for $\phi_0 = -\pi/2$ and is equal to $p_{max} = p_0 + 2\ A_1$. The corresponding values of γ_{min} and γ_{max} are then according to Eq. (6)

$$\gamma_{min} = \sqrt{1 + p_{min}^2} \tag{16}$$

$$\gamma_{max} = \sqrt{1 + p_{max}^2} \tag{17}$$

The change of kinetic energy is then

$$\Delta E_{kin} = m_0 c^2 (\gamma_{max} - \gamma_{min}) = \frac{8\, p_0\, A_1}{\sqrt{1 + p_{min}^2} + \sqrt{1 + p_{max}^2}} \tag{18}$$

For the limiting case of $A_1 = 1/2\ p_0$ we have $p_{min} = 0$, $\gamma_{min} = 1$, $p_{max} = 2\ p_0$, $\gamma_{max} = \sqrt{1 + (2\ P_0)^2} = \sqrt{4\ \gamma_0^2 - 3}$

and maximum energy spread is equal to

$$\Delta E_{kin\ max} = (\gamma_{max} - 1) \cdot m_0 c^2 = (\sqrt{4\ \gamma_0^2 - 3} - 1) \cdot 511.0041\ \text{keV}$$

4.2 Conditions for Electron Oscillations and Reflection

Above the value E_T was found for the electric field intensity in the bunching cavity below which all electrons pass through without being stopped or returned. Now we will assume that $E > E_T$ and we will look for $E = E_R > E_T$ such that electrons can be stopped and reverse the direction of their motion at leat twice (generally an even number of cases) being still transmitted through the cavity.

According to the above considerations we will define E_R as a minimum value of electric field intensity $E > E_T$ such that an electron entering the field region with some initial momentum p_0 and initial phase ϕ_0, after being returned in the bunching cavity, will arrive again at the point of departure with zero velocity and accelerating phase ϕ. We can also say that this condition requires equality of paths passed by an electron from the entrance into the cavity to the first turning point - path of ΔS_1, and from this point to the second turning point - path ΔS_2. Changes of momentum p and path S for the case $\phi_0 = 70^0$ and for different values of E are presented on Figs. 1.

The general expression for the path ΔS traversed by an electron is given by Eq. (15) so we should find only the corresponding values of momentum changes for both paths. For the first part ΔS_1 of the trajectory we have initial values: p_0 and ϕ_0 and we can find the constant $K = \frac{p_0}{A_1} - \sin \phi_0$.

To calculate ΔS_1 we should consider two cases:

1) $\phi_0 < \frac{\pi}{2}$

For this case the momentum p at the beginning increases and attains maximum $p_{max} = p_0 + A_1 (1 - \sin \phi_0)$ for $\phi = \pi/2$ and then decreases to zero. According to this $\Delta S1$ is given by

$$\Delta S1 = \frac{1}{2\pi} \int_{p_0}^{p_{max}} \frac{p\, d p}{\sqrt{1 + p^2} \sqrt{A_1^2 - (p - p_0 + A_1 \sin \phi_0)^2}} +$$

$$+ \frac{1}{2\pi} \int_{0}^{p_{max}} \frac{p\, d p}{\sqrt{2 + p^2} \sqrt{A_1^2 - (p - p_0 + A_1 \sin \phi_0)^2}} \quad (19)$$

2) $\phi_0 > \frac{\pi}{2}$

Now p decreases from p_0 to 0 and $\Delta S1$ is

$$\Delta S1 = \frac{1}{2\pi} \int_{0}^{p_0} \frac{p\, d p}{\sqrt{1 + p^2} \sqrt{A_1^2 - (p - p_0 + A_1 \sin \phi_0)^2}} \quad (20)$$

If we knew the value of A_1 we could also calculate the phase ϕ_{1R} for the first turning point by putting p = 0 in Eq. (18). We should then have

$$\sin \phi_{1R} = \sin \phi_0 - \frac{p_0}{A_1} \quad (21)$$

Usually ϕ_{1R} will be between $\frac{\pi}{2}$ and $\frac{3}{2}\pi$.

The branch ΔS_2 of the trajectory is composed of two parts: firstly the electron is accelerated in opposite direction and its momentum attains a minimum

$$p_{min} = p_0 - A_1 (1 + \sin \phi_0) < 0 \quad \text{at } \phi = \frac{3}{2}\pi$$

Secondly the motion in the opposite direction is decelerated and momentum again passes through zero. It can be verified that these two parts are symmetric in respect to $p = p_{min}$ so that ΔS_2 is given by:

$$\Delta S_2 = \frac{2}{2\pi} \int_0^{p_{min}} \frac{p\, d\, p}{\sqrt{(1 + p^2)\, (A_1^2 \quad - (p - p_0 + A_1 \sin \phi_0)^2)}} \tag{22}$$

Equation: $$\Delta S_1 - \Delta S_2 = 0 \tag{23}$$

determines the values of $A_1 = A_{1R}$ and the corresponding values of E_R above which electrons are reflected. Generally, as it is seen from expressions for ΔS_1, ΔS_2 and from Eq. (23), the values of A_1R depend on p_0 and ϕ_0, i.e. on input energy and input phase. Solving Eq. (23) for a few interesting values of p_0 and for decelerating phases

$$\frac{\pi}{2} < \phi_0 < \frac{3}{2}\pi$$

We can obtain A_{1R} and E_R as a function of p_0 and ϕ_0. A special program REFLEC has been written to solve this problem. A short description of this program is given in Appendix 2.

The main results of numercial calculations will be given below, but before going into the details of these calculations, we would like to explain what happens in the case $E_T < E < E_R$ with these electrons which after being returned twice begin to move forward again. Generally it is not evident that the relation $E < E_R$ is sufficient for transmission of these electrons through the bunching field, since now the initial conditions are different. The boundary $E = E_R$ was established by taking into account the initial values $p = p_0 > 0$, and $\pi/2 < \phi_0 < 3/2\ \pi$ so that $E_R = f(p_0, \phi_0)$.

The initial conditions at the moment of the second turning point are:

$$p = p_0, \quad \Phi = \Phi_{2R}$$

with additional restriction on Φ_{2R} stating that $\cos\phi_{2R} > 0$ (acceleration in forward direction).

It is obvious that the condition for electrons to pass depends now only on the value of phase Φ at the moment of the second turning point (generally even turning points). To find this condition we will analyse the changes of momentum p starting from any turning point. According to Eq. (10) the momentum p is given by:

$$p = p_0 + A_1 \ (\sin\Phi - \sin\Phi_0)$$

For any turning point we have:

$$p_0 = A_1 \ (\sin\Phi - \sin\Phi_R) \qquad (24)$$

starting from any even turning point the motion goes as follows: at first the electron is accelerated and its momentum p attains the maximum value equal to

$$p_{max} = A_1 \ (1 - \sin\Phi_R) \qquad (25)$$

Then moving always in the same direction the electron is decelerated and its momentum passes again through zero (odd turnig point). The path traversed by this electron in the forward direction is then

$$\Delta S_1 = \frac{2}{2\pi} \int_0^{p_{max}} \frac{p \, d \, p}{\sqrt{1 + p^2} \sqrt{A_1^2 - (p + A_1 \sin\Phi_R)^2}} \qquad (26)$$

From this moment the electron begins to move in the opposite direction. The path traversed in the opposite direction is equal to

$$\Delta S_2 = \int_{p_{min}}^{0} \frac{p \, d \, p}{\sqrt{1 + p^2} \sqrt{A_1^2 - (p + A \sin\Phi_R)^2}} \qquad (27)$$

where $p_{min} = - A_1 \ (1 + \sin\Phi_R)$ (28)

The condition for electron transmission is then

$$\Delta S_1 > \Delta S_2$$

from which it follows

$$p_{max} > |p_{min}|$$

or

$$- \sin\Phi_R > \sin\Phi_R$$

so that

$$\sin\Phi R < 0 \qquad (29)$$

for any turning point.

Since the motion is periodic it is enough to consider only the first period and phases of the first and second turning points. We obtain then

$$\frac{3}{2}\pi > \Phi_{1R} > \pi \tag{30}$$

for the first turning point.

Taking into account that the phase Φ_{2R} of the second turning point is placed symmetrically to the phase Φ_{1R} on the other side of the phase $\Phi = \frac{3}{2}\pi$ we obtain

$$\Phi_{2R} = \frac{3}{2}\pi + \left(\frac{3}{2}\pi - \Phi_{1R}\right) = 3\pi - \Phi_{1R} < 2\pi \tag{31}$$

Combining Eqs. (21) and (25) we can also find the upper limit A_{1To} as a function of p_0 and Φ_0 for the transmission of oscillating electrons:

$$\sin\Phi_R = \sin\Phi_0 - \frac{p_0}{A_1} < 0 \qquad \text{or}$$

$$A_1 < A_{1To} = p_0/\sin\Phi_0 \tag{32}$$

Equation (32) should be considered only for $\sin\Phi_0 > 0$ since for $\sin\Phi_0 < 0$ relation (29) is always fulfilled.

The numerical calculations presented, e.g. in Table II have shown that we have always the relation

$$A_{1R} < A_{1To}$$

It means then that all oscillating electrons are transmitted through the bunching field and that the relation $A_1 < A_{1R}$ is the sufficient condition for electrons transmission.

We can now discuss some numerical results obtained with the aid of program REFLEC. They are given in Table I and Table II and are also presented on Figures 1, 2.

In Table I some limiting values of electric field intensity E_T, E_R, and E_{T_0} are given for few values of Vg between 50 and 100 kV. Note that always $E_T < E_R < E_{T_0}$. The phase angle ϕ_{0Rmin} is that entrance phase of electromagnetic field for which the minimum amplitude of electric field is needed to reflect the electron entering at this phase. The other quantities with subscripts $_{Rmin}$ correspond to that value of entrance phase $\phi_{0_{Rmin}}$.

It is interesting to note that:

1) Φ_{0Rmin} practically does not depend on the gun voltage Vg, i.e. on the input energy of electrons. It changes only from 107.75° for Vg = 50 kV to 107.50° for Vg = 100 kV.

2) Φ_{0Rmin} is slightly higher than $\Phi_0 = \pi/2$ corresponding to the maximum of electron deceleration which is obtained for the case when the electron enters the field at $\Phi_0 = \pi/2$ and leaves it at $\phi = \frac{3}{2}\pi$.

3) On the other hand the change of E_{Rmin} is roughly proportional to square root of Vg: E_{Rmin} = 11.46 for Vg = 50 kV and E_{Rmin} = 16.46 for Vg = 100 kV. The ratio is then 1.435 ≈ $\sqrt{2}$. The behaviour of E_R is similar to E_{Rmin}. Note that the voltage V_{Rmin} which is the voltage necessary for reflection is more than 3 times larger than Vg for Vg = 50 kV and about 2.5 times larger for Vg = 100 kV.

Below we shall see that the optimum bunching occurs for field values in the vicinity of $E = E_{Rmin}$.

The phase ϕ_{1Rmin} corresponding to the phase at the first returning point, behaves similarly to ϕ_{0Rmin}: it changes from 198.51° to 198.98° for Vg between 50 and 100 V.

Table II gives the values of E_T, E_R, E_{T0}, ϕ_{1R} as a function of initial phase ϕ_0 for gun voltage equal to 60, 80 and 100 kV.

One can draw now two curves $E_T = E_T(\phi_0)$ and $E_R = E_R(\phi_0)$. They divide the space (E, ϕ_0) into three regions:

i) $E < E_T$ - all electrons are transmitted without oscillations

ii) $E_T < E < E_R$ electrons pass through the cavity after two or more turning points.

iii) $E > E_R$ electrons are reflected. Figs. 2, 3, 4 show these regions for Vg = 60, 80 and 100 kV correspondingly.

Numerical Calculations Bunching

The main aim of the above analytical considerations was to find the limits for electrical field intensity in the bunching cavity for which the transmission of all electrons through the cavity was possible. Now we will analyse the bunching properties of RF cavities as a function of their electric field intensities. Since we would like to treat also the spatially dependent fields, the direct numerical solution of Eqs. (1) and (2) appeared to be more convenient.

Generally the z-dependence of the amplitude of electric field intensity can be written in the form

$$E(z) = E_0 \, f\,(z) \tag{30}$$

Here E_0 is the maximum value of electric field amplitude usually attained in the middle of the cavity, where we place the origin of the z-axis. Since $E(o) = E_0$, then $f(o)$ is 1.

Two different functions $f(z)$ were used for calculations:

i) $f(z) = \text{const.}$

ii) $f(z) \sim e^{-0.5(\frac{z}{\delta})^2}$ - gaussian shape of electric field intensity. This type of field dependence was proposed by R. Belbéoch [2] on the basis of numerical calculation of field distribution in the cavity using the program SUPERFISH. The least square fitting of calculated field and gaussian curve gave for $\delta \approx 0.0071$ m. The half length of the cavity was equal to $\frac{L}{2} = 0.00742$, so that $\delta \approx \frac{L}{2}$. Since the E_z field did not fall abruptly outside the cavity the integration was performed for $|z| < 3\,\delta$. The problem is then the following: to solve the quations of motion (1) and (2), i.e.

$$\frac{d\gamma}{ds} = A\, f(s) \cos\phi \tag{1}$$

$$\frac{d\phi}{ds} = 2\,\pi \frac{\gamma}{\sqrt{\gamma^2 - 1}} \tag{2}$$

where now $A = q\, A_0 \lambda / W_0$. For the geometry shown on Fig. 5 corresponding to that of

LEP V prebuncher - buncher assembly. Find then the energy phase distribution of electrons at the entrance of the 30 MeV buncher. To solve the above set equations we used the fourth order Runge-Kutta-Merson method with the automatic variation of the step length in order to obtain the desired accuracy on a given interval of integration. The equations of motion enter into the program as a special procedure required by the main procedure Merson. Any physically reasonable function F(z) either analytical or experimental can be used without restrictions for these calculations.

Since according to previous considerations we do not exclude the possibility of electron oscillations in the cavity some problems can arise in the case when electron is stopped so that $\gamma = 1$ and the right hand side of the second of equations of motion will become infinite. To overcome this difficulty some modifications in the procedure Merson were made which allowed for solving numerically the equations up to the place where the value of momentum $p = \sqrt{\gamma^2 - 1}$ becomes smaller than some chose value η so that the kinetic energy of electrons is maller than $m_o c^2 (\sqrt{1 + \eta^2} - 1)$, e.g. for $\eta = 10^{-2}$ kinetic energy of electrons is smaller than 25.5 eV. The set of Eq(s) (1), (2) is then solved analytically for $|p|<\eta$ assuming that f(z) = const. chosing properly the value η and the precision ε with which the set of equations is solved, the desired accuracy of solution can be obtained. To obtain the bunching as a function of electric field intensity in the prebunching cavity several values of field intensity EpB were used for calculations. They were chosen to be equal to E_{pB} = 2.2, 5, 7.5, 10, 15, 18 and 20 MB/m for the gaussian shape of field and also EpB equal to 2.2 and 15 MV/m for square field shape for aim of comparison.

To see the evolution of phase bunching in space, the phase was calculated in 3 places:

i) at the end of the prebuncher

ii) in the middle between prebuncher and buncher

iii) at the input to the buncher

The results of calculations are presented in the form of suitable tables, curves in the energy-phase space and histograms showing the efficiency of bunching at different positions.

Remark Usually in the calculations we use the electric field intensity, which is always well defined quantity. However, sometimes for illustration it is convenient to give also the values of r.f. voltage existing on the electrodes of bunching cavity. Since in a general case the transit time through the cavity can be of the order of the period of electromagnetic field, it is impossible to calculate analytically the effective potential of the cavity which could take into account the transit time and variations of velocity of electrons. This value can only be calculated numerically by the solution of equations of motion and finding the energy exchange between the field and electrons. However, to get some idea about the potential drop we will use the static field approximation given by:

$$v = \int_L E_z(z)\, dz$$

where ℓ is the length of the region where E_z=0. We then have

$V_{pbS} = E_0 \cdot L$ for the "square wave" field and

$$V_{pBG} = E_0 \int_{-3\sigma}^{3\sigma} f(z)dz$$

for the gaussian form of Ez(z) distribution. In all cases when we will speak about the potential difference V_{pB} we will use either V_{pBS} or V_{pBG} correspondingly.

We will discuss now the results obtained. Since most of the calculations were done for the gaussian field distribution then if nothing has been mentioned to the contrary the results are for this form of electric field in the cavity. The value of electric field intensity E_{pB} then corresponds to the maximum value in the middle of the cavity.

Fig. 3A shows the distribution of electrons in the energy-phase space at the three above defined places whereas the Figs. 3B, 3C and 3D give the histograms of phase distribution at these place for E_{pB} = 2.2 MV/m ($V_{pB} \approx$ 30 kV). It is seen from these figures that in the case of $V_{pB} \ll V_g$ we have the so called velocity bunching at the end of the bunching cavity, the phase distribution is almost not changed but there is a velocity modulation which in the drift space after the cavity produces the phase bunching. The process of bunching is already well visible in the middle between prebuncher and buncher, but its is much more pronounced at the buncher input.

Figures 4A, 4B, 4C, 4D until 5A, 5B, 5C, 5D present the same pictures as those of Figs.3A, 3B, 3C, 3D but for E_{bP} corresponding to: 7.5, and 15 MV/m.

It is seen from these figures that already for E_{pB} = 7.5 MV/m (V_{pB} ≈ 111 kV) the picture is radically changed. Now, at the end of prebuncher the electrons are already fairly bunched and there are only small changes in electron phase distribution in the drift space. With increasing value of the electric field intensity E_{pB} this behaviour becomes more and more pronounced and in the limit of high electric fields e.g. E_{pB} > 10 MV/m (V_{pB} ≈ 150 kV) the best bunching is produced already at the end of the prebuncher. It means then that the bunching is now produced not in the drift space as for small E_{pB} but in the region of strong electric field.

Another interesting feature of bunching with high electric field intensity is that, although there are some electrons with very small energies well below the output energy from the gun, most of them have energy higher than their energy when entering the cavity. This creates a good chance for electrons to be accepted by the accelerating structure which follows the bunching cavity. As shown by the calculations this occurs in fact (it will be a subject of the second note).

The results for the case of "square wave" field distribution for two values of E_{pB}: E_{pB} = 2.2 MV/m and 15 MV/m are presented on Figs. 6A,B, C, D and 7A, B, C, D, Also the cases of Vg = 60 kV, E_{pB} = 2.2 MV/m and 15 MV/m (gaussian distribution) were calculated. The results are similar to the corresponding previous ones.

DISCUSSION

The analytical considerations made for the case of constant field have shown that even for the case of V_{pB} = (2.5 - 3)Vg all electrons pass through the bunching cavity. These values can be regarded as the lower limits since for the case of non uniform fields the time of transit through the field region will be as a rule longer and energy exchange less effective, e.g. the calculations made for the gaussian type have shown that the limit for reflection is higher by more than 10%, e.g. for Vg = 80 kV the momentum p of eletrons can become negative when E_{pB} > 10.75 for Gaussian field distribution whereas it occurs already for E_{pB}> 9.3 MV/m for the constant field.

The results of numerical calculations presented on Figs. 3A, B, C, D up to 7A, B, C, D have shown that there exists a very good bunching for electric field as high as 15 MV/m and more which corresponds to $V_{pB}/Vg \approx 3$ and V_{pB} of the order of 200 kV (for Vg ≃ 80 kV). An important thing obtained in the case of high fields is that now the electrons are fairly bunched already at the exit of the bunching cavity so that no additional drift space is needed for bunching.

ACKNOWLEDGEMENTS

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APPENDIX I

According to Eq. (14) the path traversed by the electron which changes the momentum from p_0 to p_{min} is given by:

$$\Delta S = \frac{1}{2\pi} \int_{p_{min}}^{p_0} \frac{x \quad dx}{\sqrt{(1 + x^2)\left(A_1^2 - (x - A_1 K)^2\right)}} \qquad (A1)$$

Taking into account the expression (9) for K and exp. (10) for p the integral of (A1) can be written in the form of

$$I = \int_{p_{min}}^{p_0} \frac{x \quad dx}{\sqrt{(1 + x^2)(p_{max} - x)(x - p_{min})}} \qquad (A2)$$

where:

$$p_{min} = p_0 - A_1 (1 + \sin \phi_0) \qquad (A3)$$

$$p_{max} = p_0 + A_1 (1 - \sin \phi_0) \qquad (A4)$$

This integral is of the type [3]

$$\sqrt{(a - x)(x - b)(x - c)(x - \bar{c})} = \sqrt{(a - x)(x - b)\left[(x - b_1)^2 + a_1^2\right]} \qquad (A5)$$

where in our case we have

$$a = p_{max}$$

$$b = p_{min}$$

$$c = i = \sqrt{-1}\ , \quad \bar{c} = -i$$

$$b_1 = 0, \quad a_1^2 = 1$$

denoting also

$$A^2 = (a - b_1)^2 + a_1^2 = 1 + p_{max}^2 = \gamma_{max}^2,$$

$$B^2 = (b - b^1)^2 + a_1^2 = 1 + p_{min}^2 = \gamma_{min}^2$$

$$g = \frac{1}{\sqrt{AB}} = \frac{1}{\sqrt{\gamma_{max}\, \gamma_{min}}}$$

$$k^2 = \frac{(a-b)^2 - (A-B)^2}{4\ AB} = \frac{(p_{max} - p_{min})^2 - (\gamma_{max} - \gamma_{min})^2}{4\ \gamma_{max}\ \gamma_{min}},$$

$$cnu = \frac{(p_{max} - x)\ \gamma_{min} - (x - p_{min})\ \gamma_{max}}{(p_{max} - x)\ \gamma_{min} + (x - p_{min})\ \gamma_{max}}$$

cnu = cosine amplitude function;
(Jacobian elliptic function)

$$cnu = \cos\gamma,$$
$$\gamma = amu_1 = \arccos\left(- \frac{(p_{max} - p)\ \gamma_{min} - (p - p_{min})\ \gamma_{max}}{(p_{max} - p)\ \gamma_{min} + (p - p_{min})\ \gamma_{max}}\right)$$

amu = amplitude u

Now according to [3] (see 259.03, 341.02, 341.03 and 361.54 of this book) we have

$$I = \int_b^p \frac{x\ d\ x}{\sqrt{(a - x)\ (x - b)\ (x - c)\ (x - c)}} = \frac{g\ (aB + bA)}{A - B} *$$

$$\left[\ \alpha_2\ F\ (\gamma,\ k) + \frac{\alpha - \alpha_2}{1 - \alpha_2}\ \left(\Pi\ (\gamma,\ \frac{\alpha_2}{1 - \alpha_2},\ k) - \alpha\ f1\right)\ \right] \qquad (A6)$$

$$\alpha^2 \neq 1$$

where $\alpha = \frac{A - B}{A + B} = \frac{\gamma_{max} - \gamma_{min}}{\gamma_{max} - \gamma_{min}}$

$F(\gamma, k)$ - incomplete elliptic integral of the first kind

$\Pi(\gamma, \alpha^2, k)$ - Legendre - incomplete elliptic integral of the third kind

k - modulus of Jacobian elliptic functions and integrals

$k' = \sqrt{1 - k^2}$ - complementary modules

$$f1 = \sqrt{\frac{1 - \alpha^2}{k^2+k'\alpha^2}} \text{ arctg } \left(\sqrt{\frac{k^2 + k'^2\alpha^2}{1 - \alpha^2}} \text{ sdu }\right), \text{ if } \frac{\alpha^2}{\alpha^2-1} < k^2,$$

$$= \text{sdu}, \quad \text{if } \frac{\alpha^2}{\alpha^2-1} = k^2;$$

$$= \frac{1}{2}\sqrt{\frac{\alpha^2-1}{k^2+k'^2\alpha^2}} \, \ln\left(\frac{\sqrt{k^2+k'^2\alpha^2} \text{ dnu} + \sqrt{\alpha^2 - 1} \text{ snu}}{\sqrt{k^2+k'^2\alpha^2} \text{ dnu} - \sqrt{\alpha^2 - 1} \text{ snu}}\right)$$

$$\text{if } \frac{\alpha^2}{\alpha^2-1} > k^2$$

dnu, snu, sdu = $\frac{\text{snu}}{\text{dnu}}$ - Jacobian elliptic functions

In our case we have

$$\frac{\alpha^2}{\alpha^2-1} = \frac{(A - B)^2}{- 4AB}$$

$$k^2 = \frac{(a - b)^2}{4AB} - \frac{(A - B)^2}{4AB} = \frac{(a - b)^2}{4AB} + \frac{\alpha^2}{\alpha^2-1} > \frac{\alpha^2}{\alpha^2-1}$$

Then according to (A7) we should take for f1:

$$f1 = \sqrt{\frac{1 - \alpha^2}{k^2 + k'^2 \alpha^2}} \text{ arctg } \left(\sqrt{\frac{k^2 + k'^2\alpha^2}{1 - \alpha^2}} \cdot \text{sdu} \right)$$

The integral I as given by (A6) is now completely defined. The values of the incomplete elliptic integral of the third kind can be found in reference [6].

APPENDIX 2

A short description of program REFLEC

It was shown above that the limiting value of electric field intensity E_R where electrons are reflected is defined by the Eq. (23).

$$\Delta S_1 - \Delta S_2 = 0 \qquad A2(1)$$

ΔS_1 and ΔS_2 are given by Eqs. (19), (20) and (22) which can be expressed in terms of elliptic integrals of the first and third kinds (see Appendix 1).

Electric field intensity E, which we are looking for, enters into Eq. A2(1) through the quantity A_1 which appears both in the integrands of integrals (19), (20) and (22) and also in the limits of these integrals. Besides A_1 these integrals depend also on initial values of $\phi = \phi_0$ and $p = p_0$. Taking p_0 as a parameter by solving Eq. A2(1) for different values of ϕ_0 we obtain A_1 as a function ϕ_0.

To solve the Eq. A2(1) the CERN Library subroutine Gauss was used to calculate the above-mentoned integrals and CERN Library subroutine RZERO was applied to find the zeros of this equation.

The subroutine RZERO requires the lower and upper limits A and B between which the zero of the considered equations should be found. In our case the lower limit is given by $A = A_{1T} = p_0/(1 + \sin\phi_0)$. The upper limit B is a little more complicated if we try to define it with some precision. It can be taken to be equal:

$$B = A_{1To} = \frac{p_0}{\sin\phi_0} \text{ for } 0^\circ < \phi_0 \leq 150^\circ$$

B = a few times lower limit A for $150^\circ < \phi < 270^\circ$. There are no reflections for $-90^\circ < \phi_0 < 0^\circ$.

Having found the solution of Eq. A2(1) we have the value of A_1 and we can calculate now other interesiting quantities such as the electric field intensity E_R, minimum value of $p = p_{min}$, the phases ϕ_{R1} and ϕ_{R2} corresponding to the first and the second turning points, the distance $\Delta_{Z1} = \lambda_{\Delta S1}$ of the first turning points. The program also draws two curves $E_T = f_T(\phi_0)$ and $E_R = f_R(\phi_0)$, which divides the space (E, ϕ_0) into 3 regions:

1) $E < E_T$ - transmission without oscillations
2) $E_T < E < E_R$ - transmission with oscillations
3) $E > E_R$ - reflections

T A B L E I

FIELD INTENSITY LIMITS FOR TRANSITION, OSCILLATION AND REFLECTION OF ELECTRONS IN THE PREBUNCHER AS A FONCTIION OF GUN VOLTAGE Vg

Vg kV	$E_T(\phi_0 = 90^\circ)$ MV/m	$V_T = E_T{*}L$ kV	$E_{R\ MIN}$ MV/m	$V_{R\ MIN}$ kV	$\phi_{0R\ MIN}$ degrees	$\phi_{1R\ MIN}$ degrees	$Z_{1R\ MIN}$ mm	$E_{TO\ MIN}$ ($\phi_0 = \pi/2$ MV/m	$V_{TO\ MIN}$
50	7.27	107.96	11.46	170.18	107.75	198.51	6.15	14.54	215.9
60	8.01	118.95	12.59	186.96	107.70	198.59	6.71	16.02	237.9
70	8.69	129.05	13.64	202.55	107.65	198.68	7.22	17.38	258.1
80	9.33	138.55	14.63	217.26	107.60	198.79	7.68	18.66	27
90	9.94	147.61	15.57	231.21	107.55	198.90	8.11	19.88	29
100	10.53	156.37	16.46	244.43	107.50	198.98	8.52	21.05	31

T A B L E I

FIELD INTENSITY LIMITS FOR TRANSITION, OSCILLATION AND REFLECTION OF ELECTRONS

IN THE PREBUNCHER AS A FONCTIION OF GUN VOLTAGE Vg

Vg kV	$E_T(\phi_0 = 90^\circ)$ MV/m	$V_T = E_T$*L kV	E_R MIN MV/m	V_R MIN kV	ϕ_{OR} MIN degrees	ϕ_{1R} MIN degrees	Z_{1R} MIN mm	E_{TO} MIN ($\phi_0 = \pi/2$ MV/m	V_{TO} MIN
50	7.27	107.96	11.46	170.18	107.75	198.51	6.15	14.54	215.9
60	8.01	118.95	12.59	186.96	107.70	198.59	6.71	16.02	237.9
70	8.69	129.05	13.64	202.55	107.65	198.68	7.22	17.38	258.1
80	9.33	138.55	14.63	217.26	107.60	198.79	7.68	18.66	27
90	9.94	147.61	15.57	231.21	107.55	198.90	8.11	19.88	29
100	10.53	156.37	16.46	244.43	107.50	198.98	8.52	21.05	31

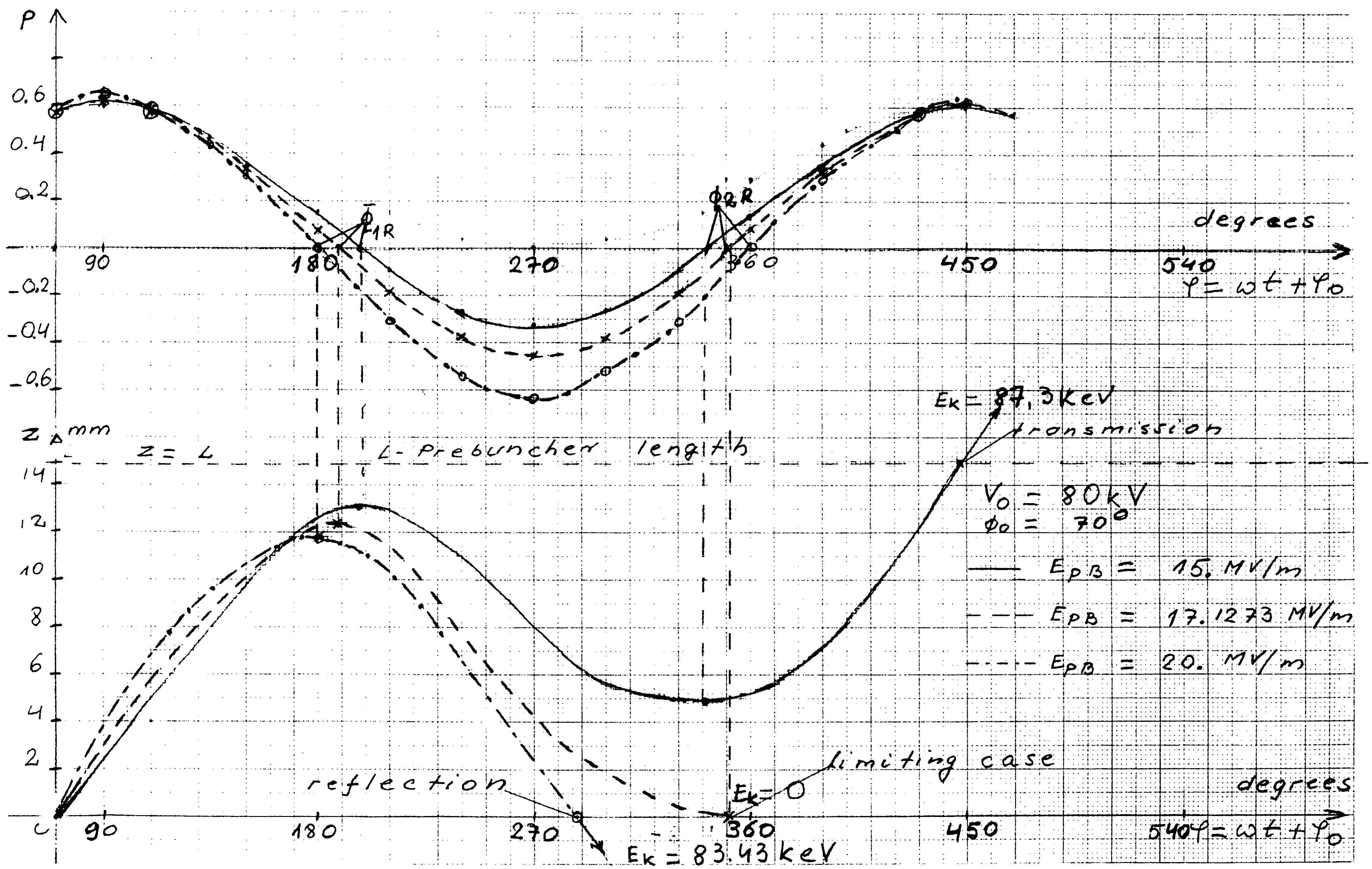
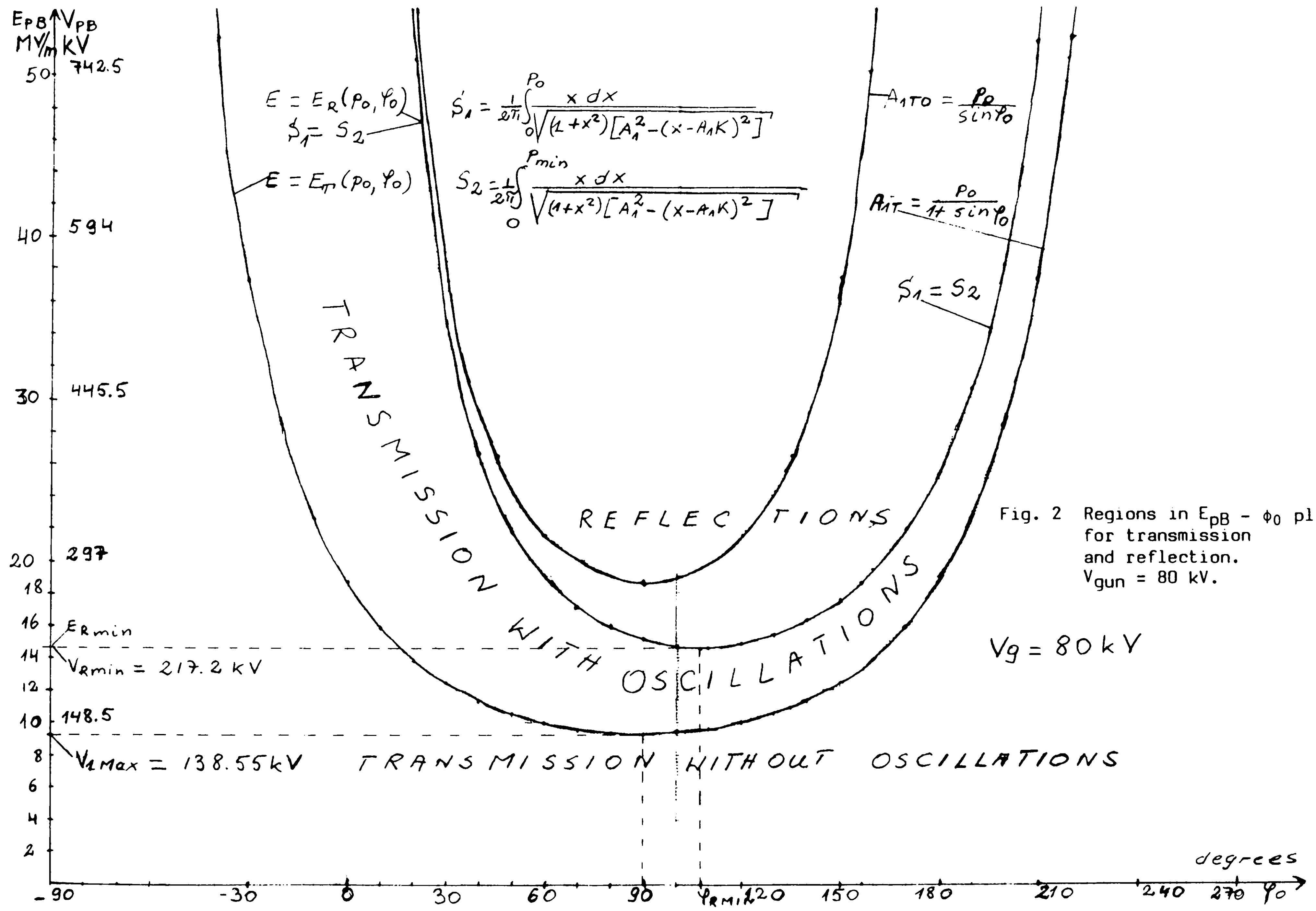


Fig.1 Momentum p and axial distance z as a function of $\phi = \phi_0 + \omega t$ for different values of field intensity E_{pB} in the prebuncher. $\phi_0 = 70°$.



$$S_1 = \frac{1}{2\pi}\int_0^{\rho_o} \frac{x\,dx}{\sqrt{(1+x^2)\left[A_1^2-(x-A_1K)^2\right]}}$$

$$S_2 = \frac{1}{2\pi}\int_0^{\rho_{min}} \frac{x\,dx}{\sqrt{(1+x^2)\left[A_1^2-(x-A_1K)^2\right]}}$$

Fig. 2 Regions in E_{pB} - ϕ_0 plane for transmission and reflection. V_{gun} = 80 kV.

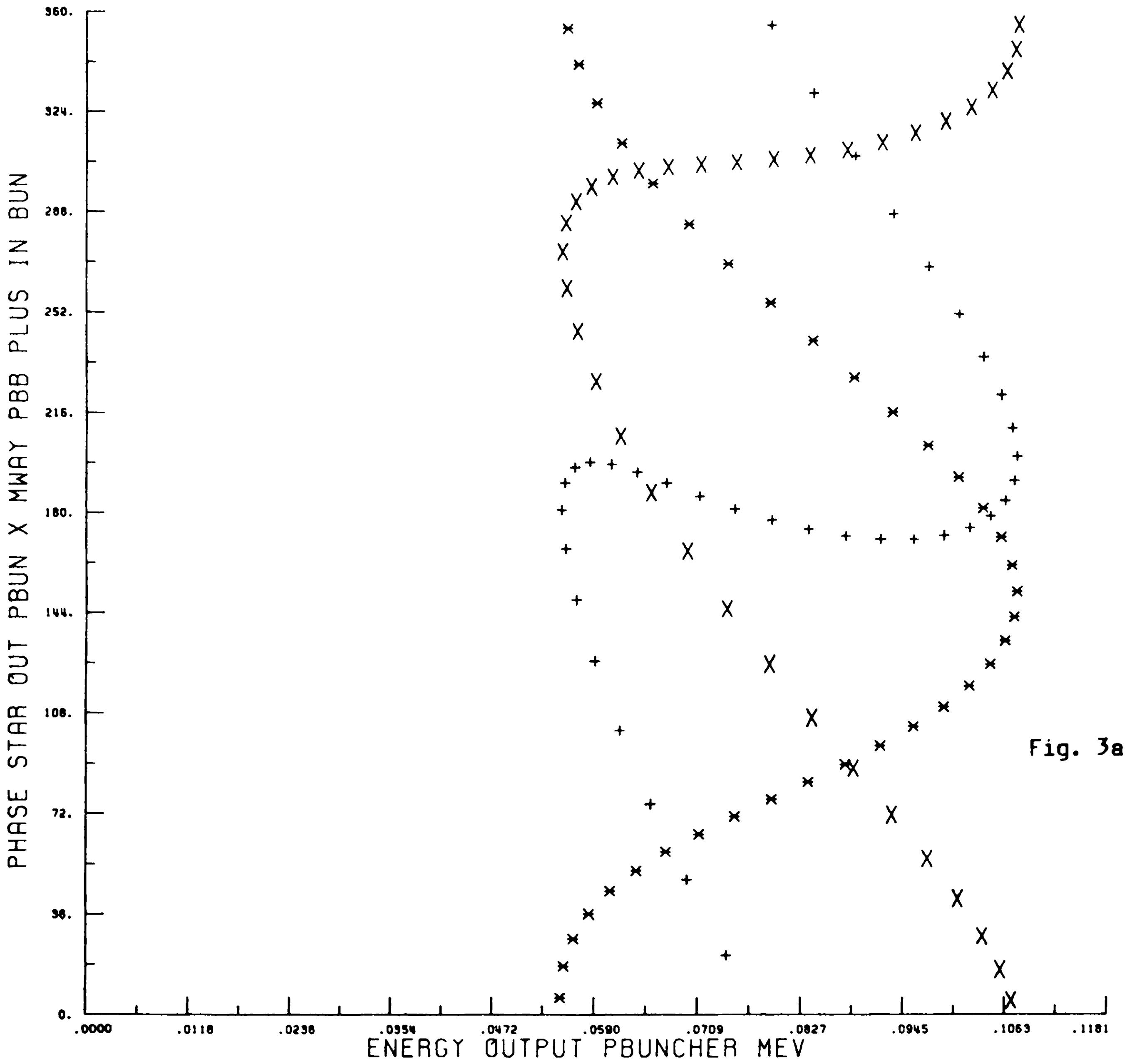


Fig. 3a Energy - Phase distribution of particles after prebuncher cavity with Gaussian field distribution E_{pBmax} = 2.2 MV/m.

* - at cavity exit

x - in the middle between prebuncher and buncher

\+ - at buncher entrance

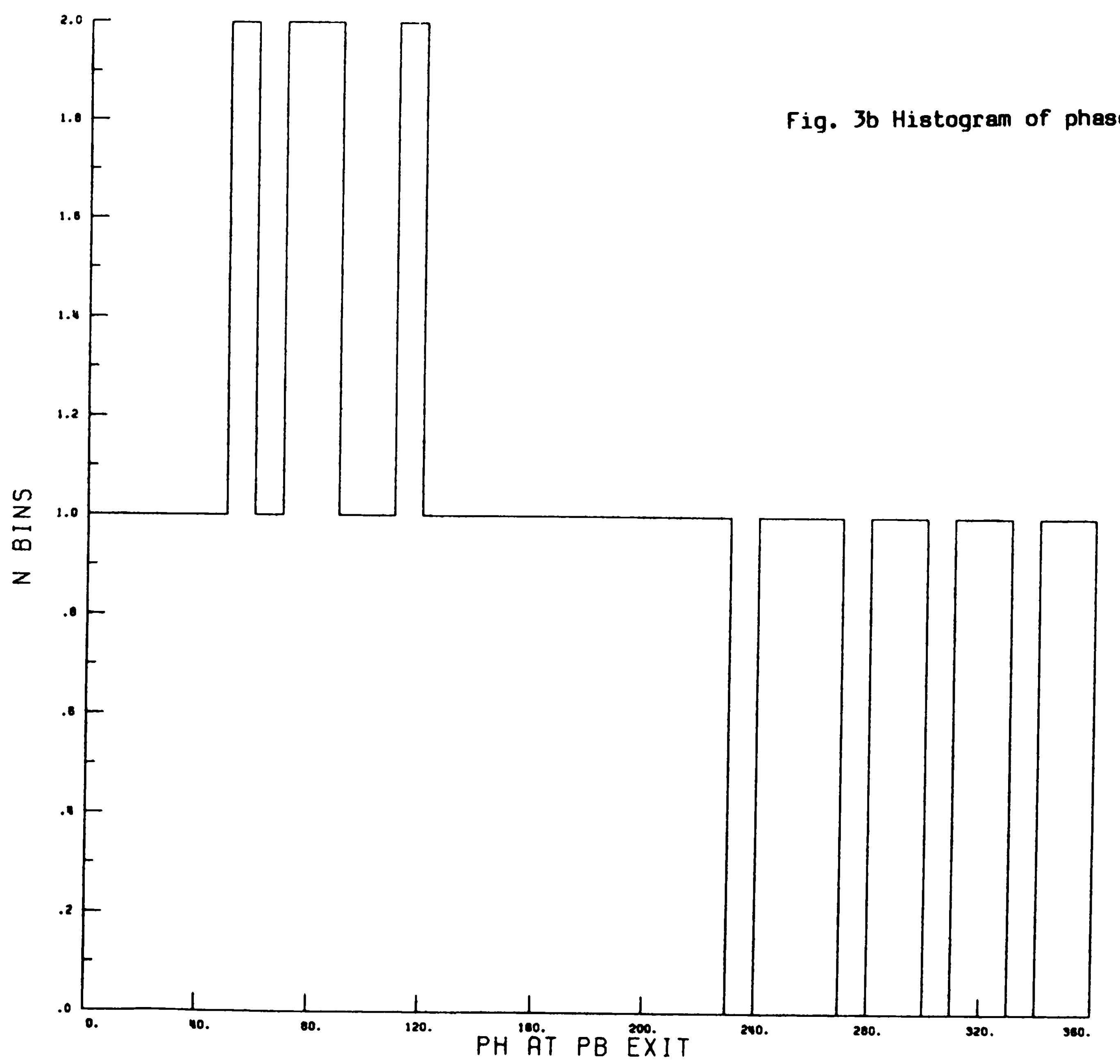


Fig. 3b Histogram of phase distribution at the cavity exit. E_{pB} = 2.2 MV/m

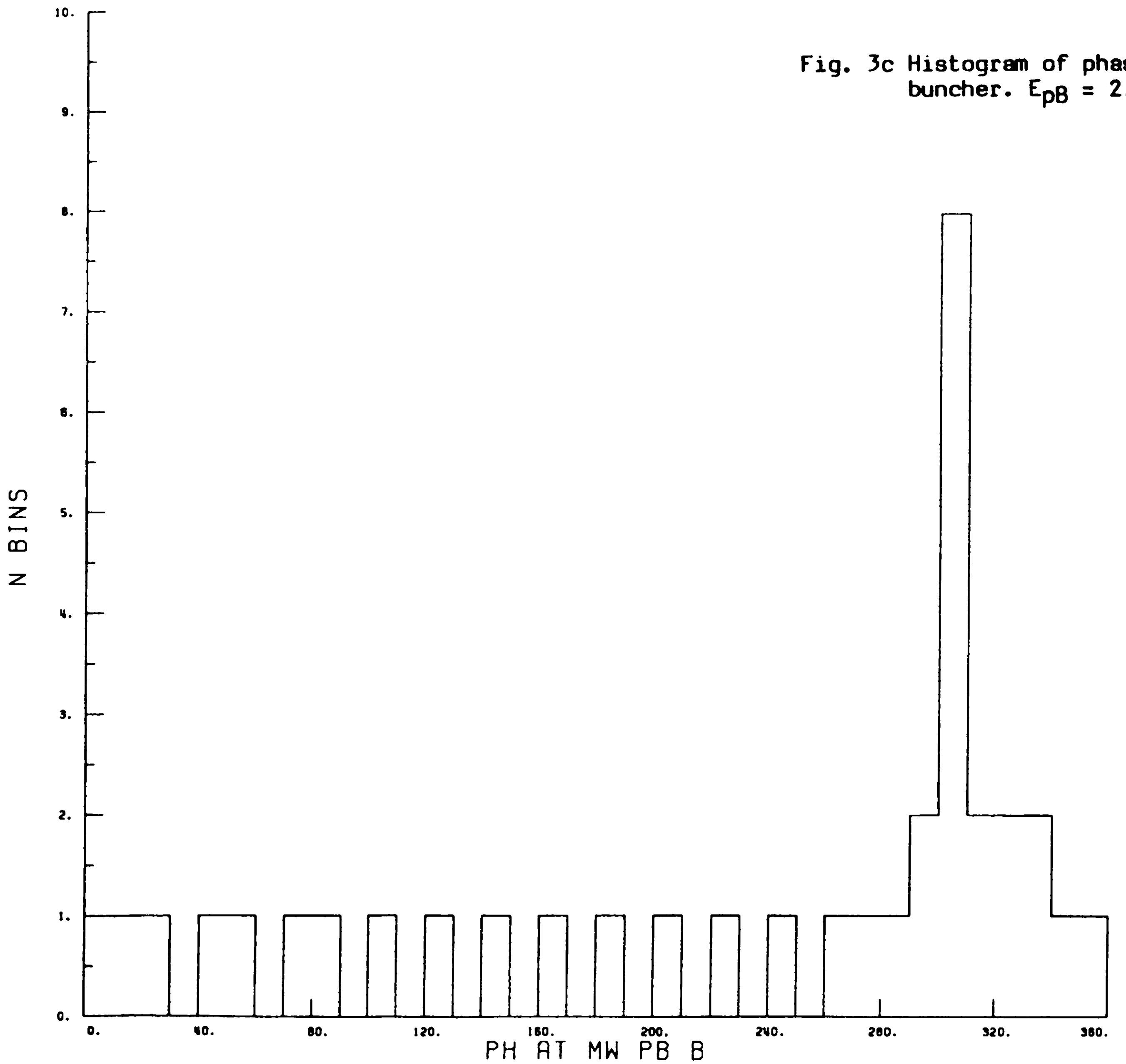


Fig. 3c Histogram of phase distribution the middle between prebuncher and buncher. E_{pB} = 2.2 MV/m

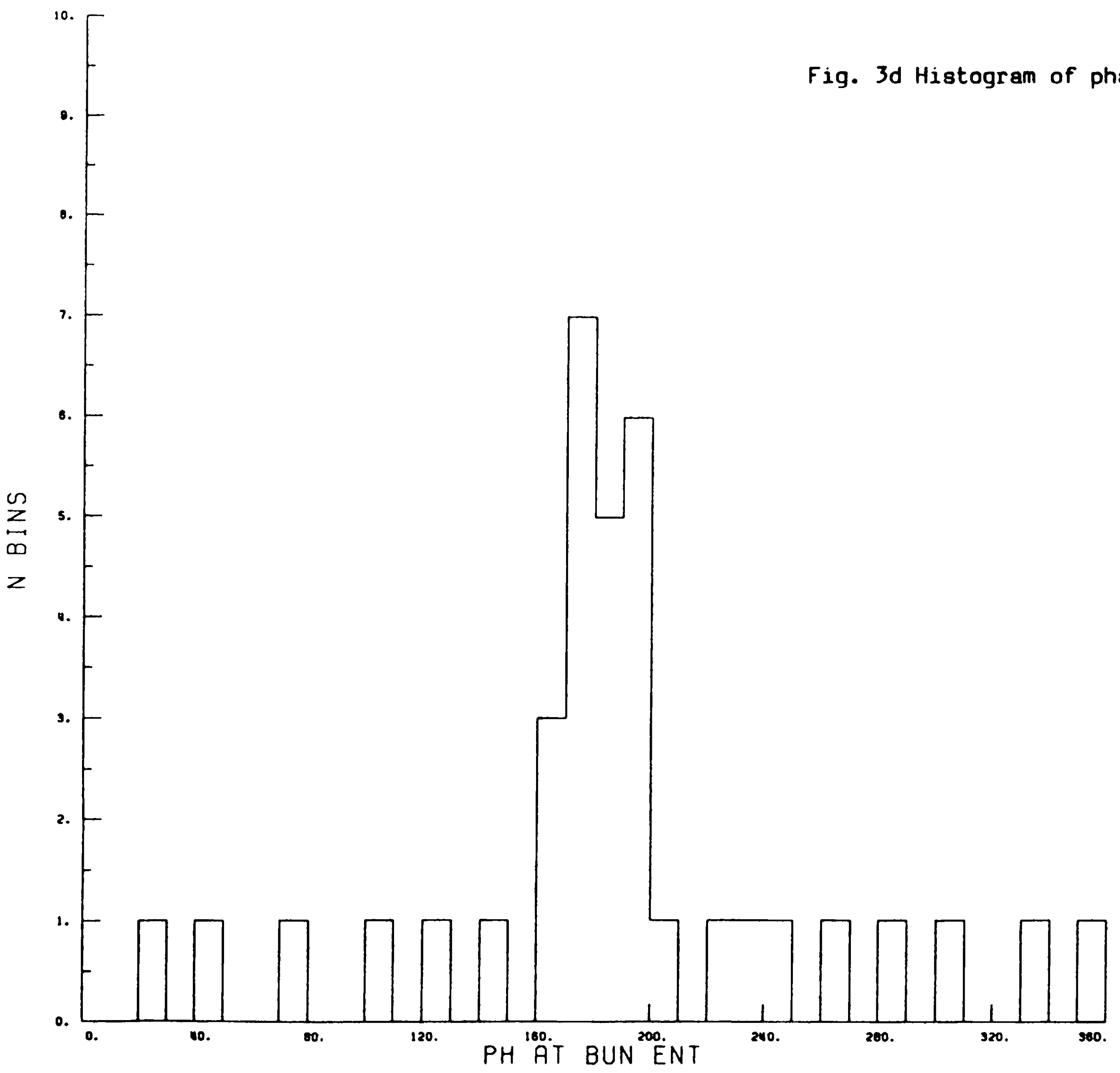


Fig. 3d Histogram of phase distribution at buncher entrance. E_{pB} = 2.2 MV/m

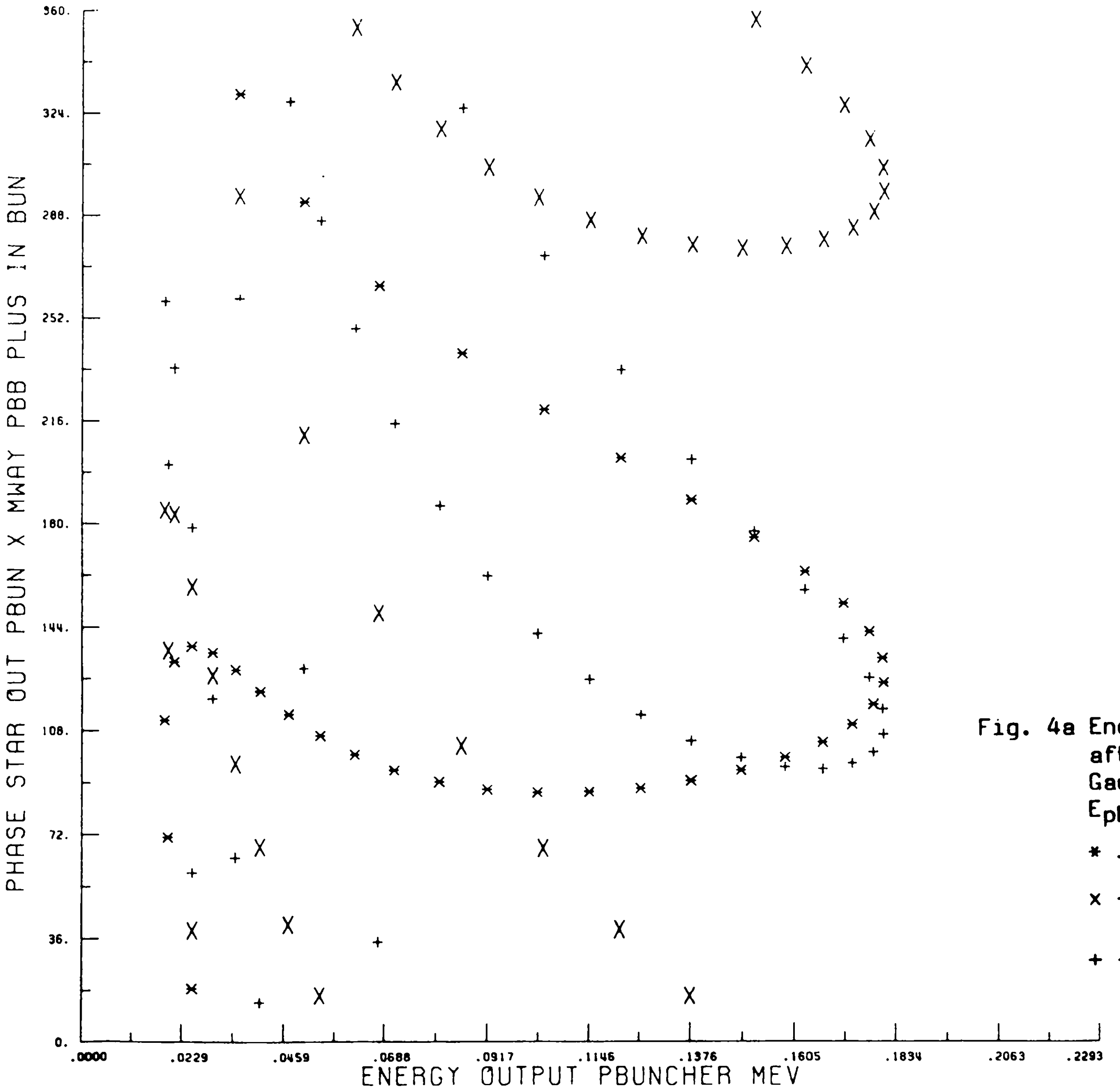


Fig. 4a Energy - Phase distribution of particles after prebuncher cavity with Gaussian field distribution E_{pBmax} = 7.5 MV/m.

* - at cavity exit

x - in the middle between prebuncher and buncher

\+ - at buncher entrance

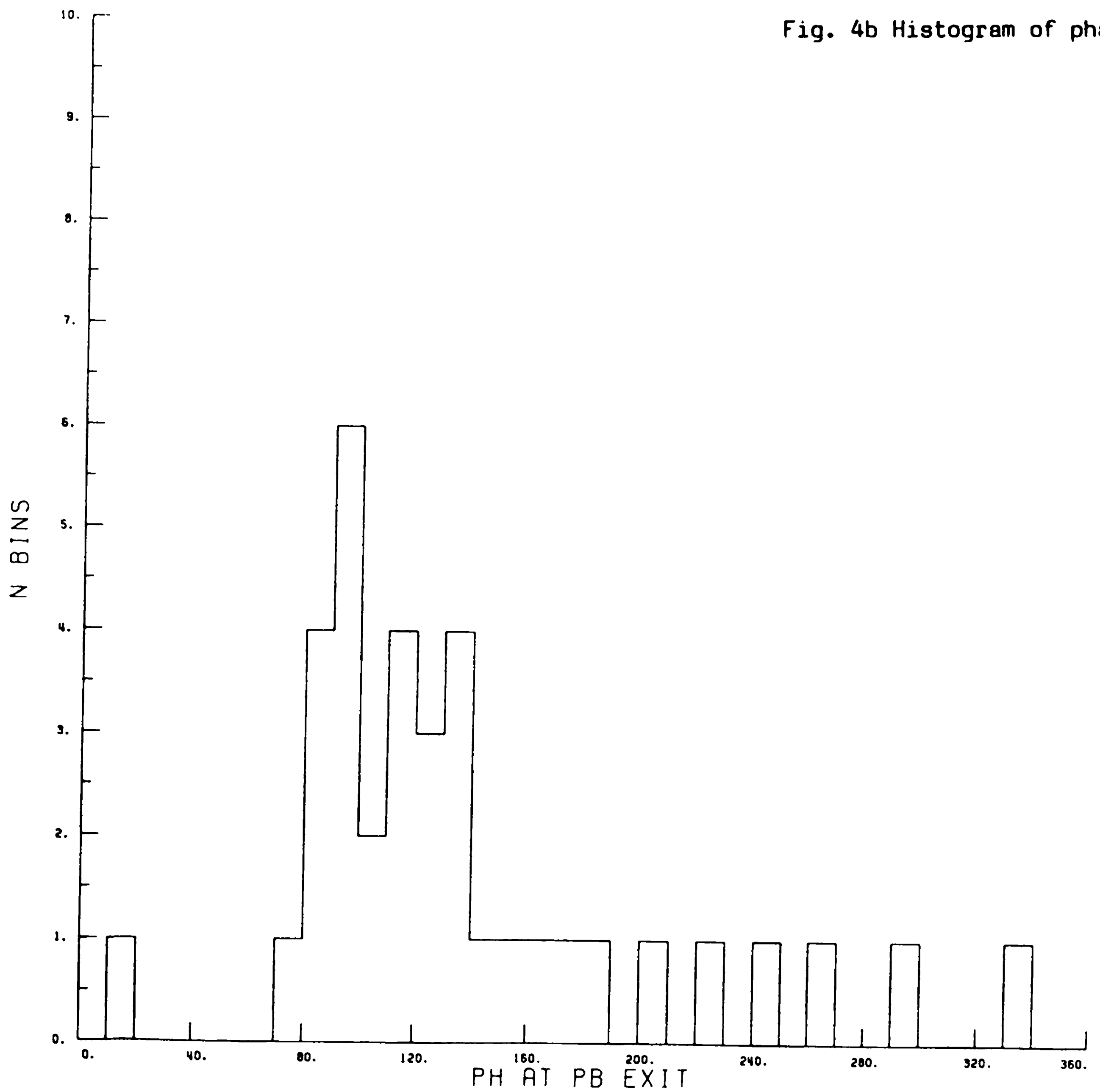
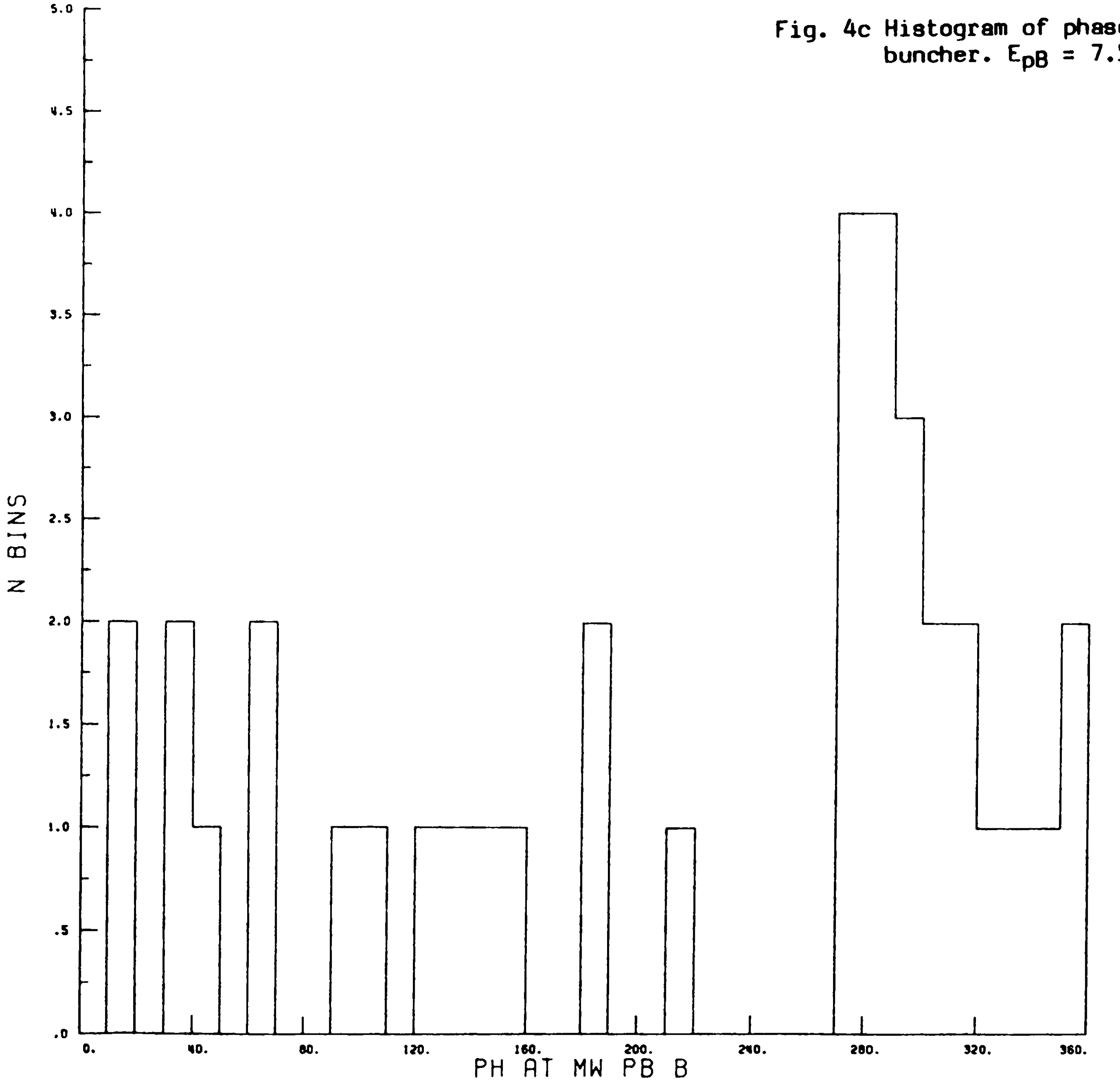


Fig. 4b Histogram of phase distribution at the cavity exit, E_{pB} = 7.5 MV/m

Fig. 4c Histogram of phase distribution in the middle between prebuncher and buncher. E_{pB} = 7.5 MV/m



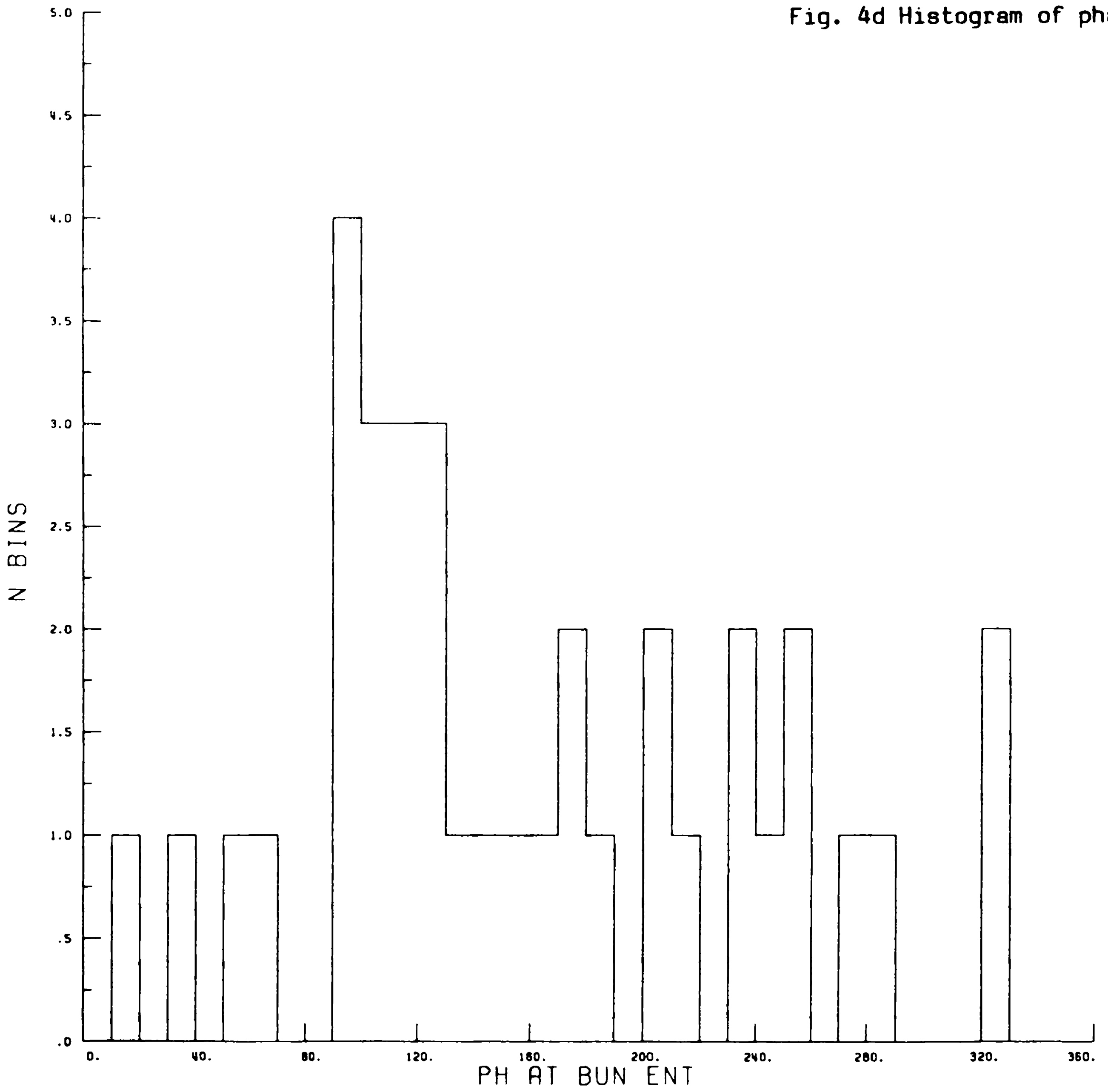


Fig. 4d Histogram of phase distribution at buncher entrance. E_{pB} = 7.5 MV/m

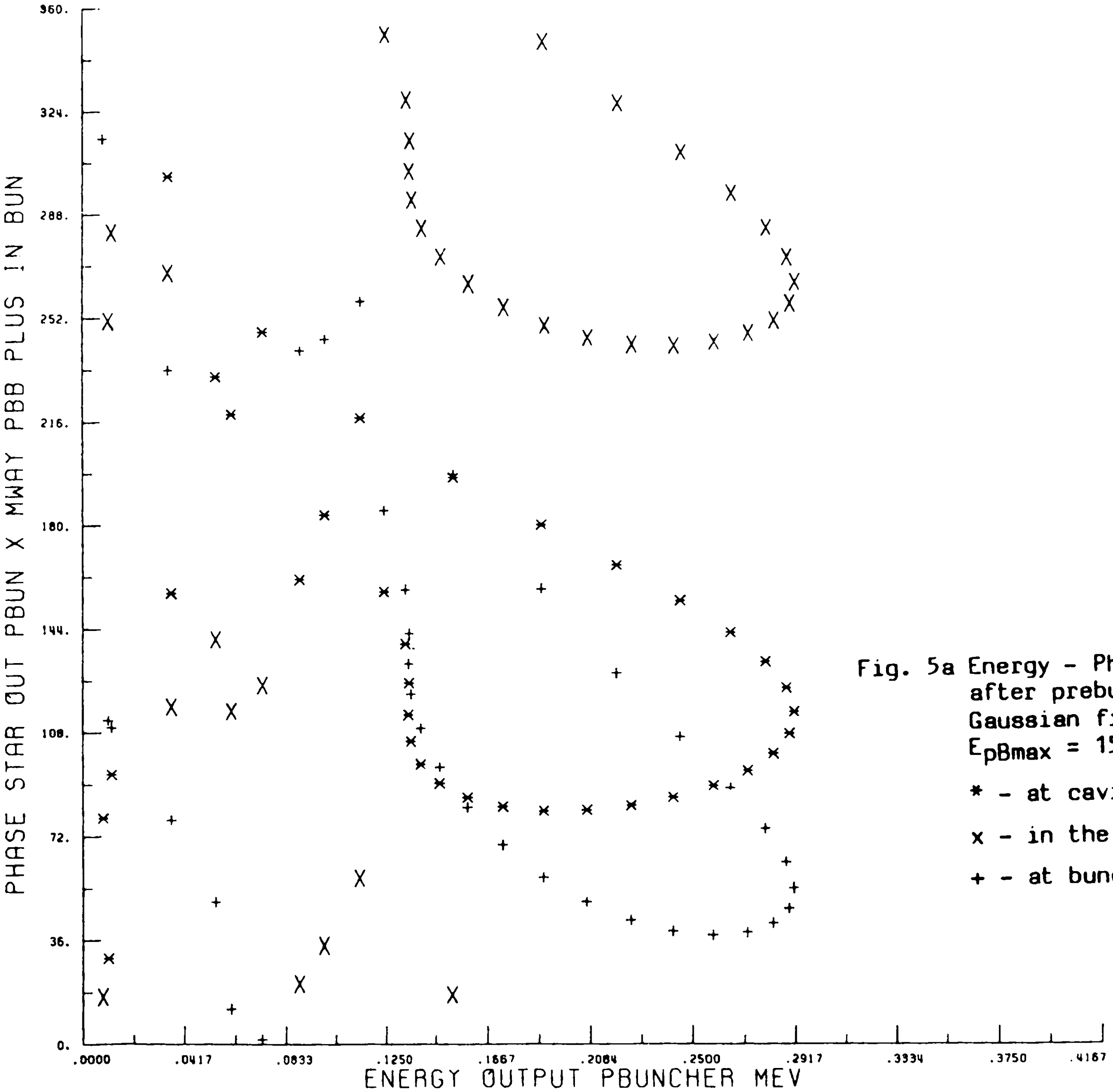


Fig. 5a Energy - Phase distribution of particles after prebuncher cavity with Gaussian field distribution
E_{pBmax} = 15 MV/m.

* - at cavity exit

x - in the middle between prebuncher and buncher

+ - at buncher entrance

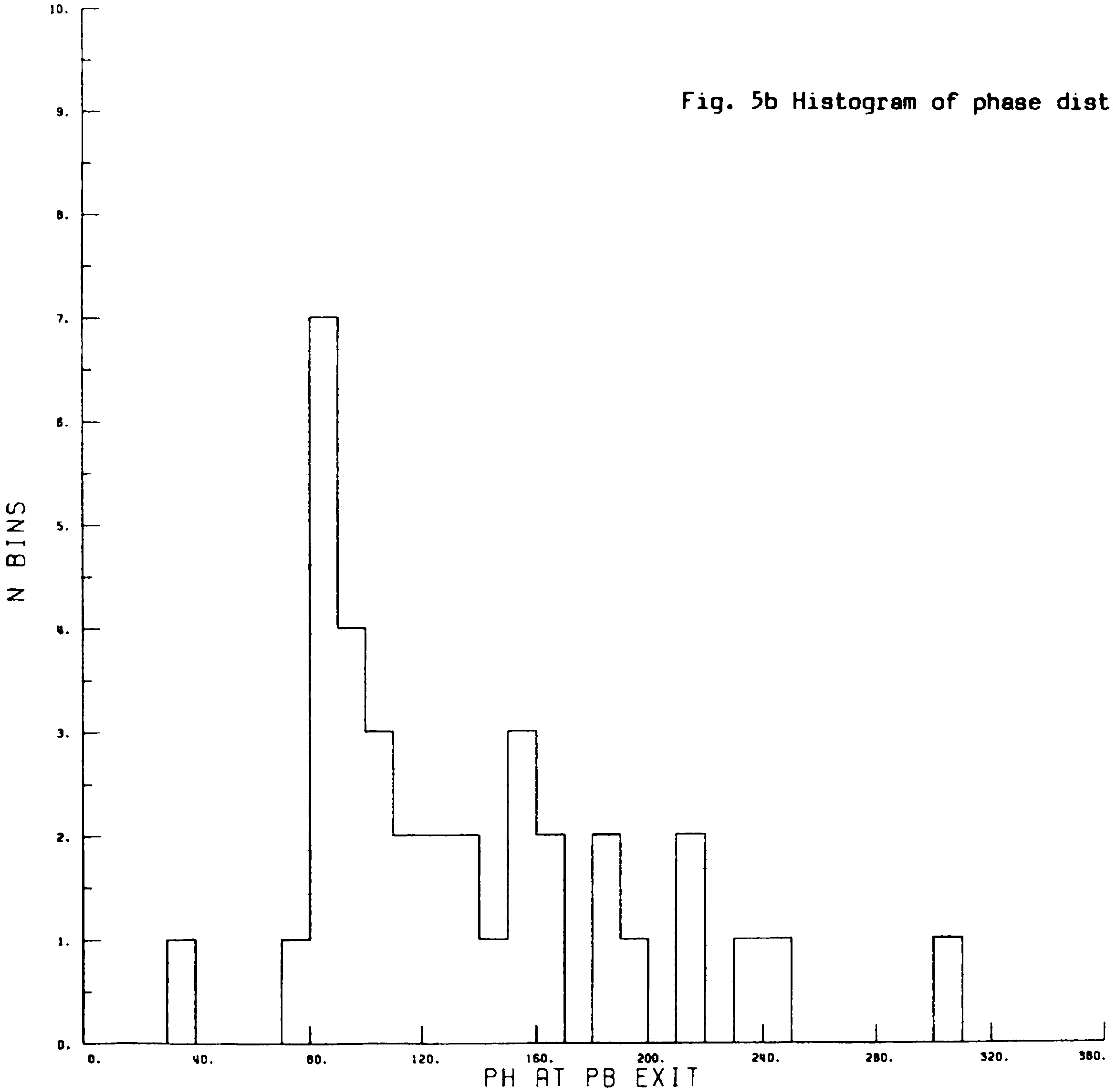


Fig. 5b Histogram of phase distribution at the cavity exit. E_{pB} = 15 MV/m

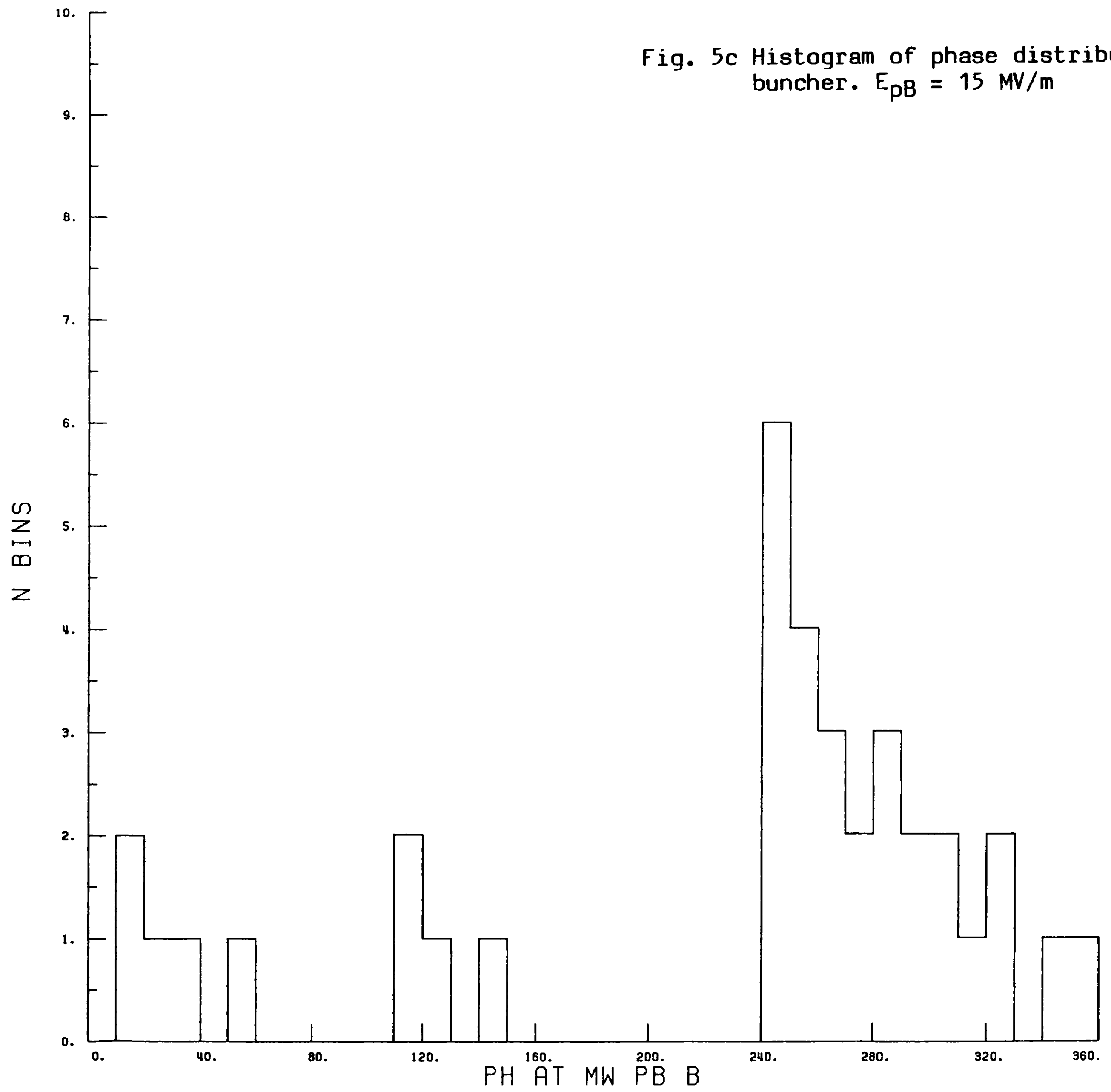


Fig. 5c Histogram of phase distribution in the middle between prebuncher and buncher. E_{pB} = 15 MV/m

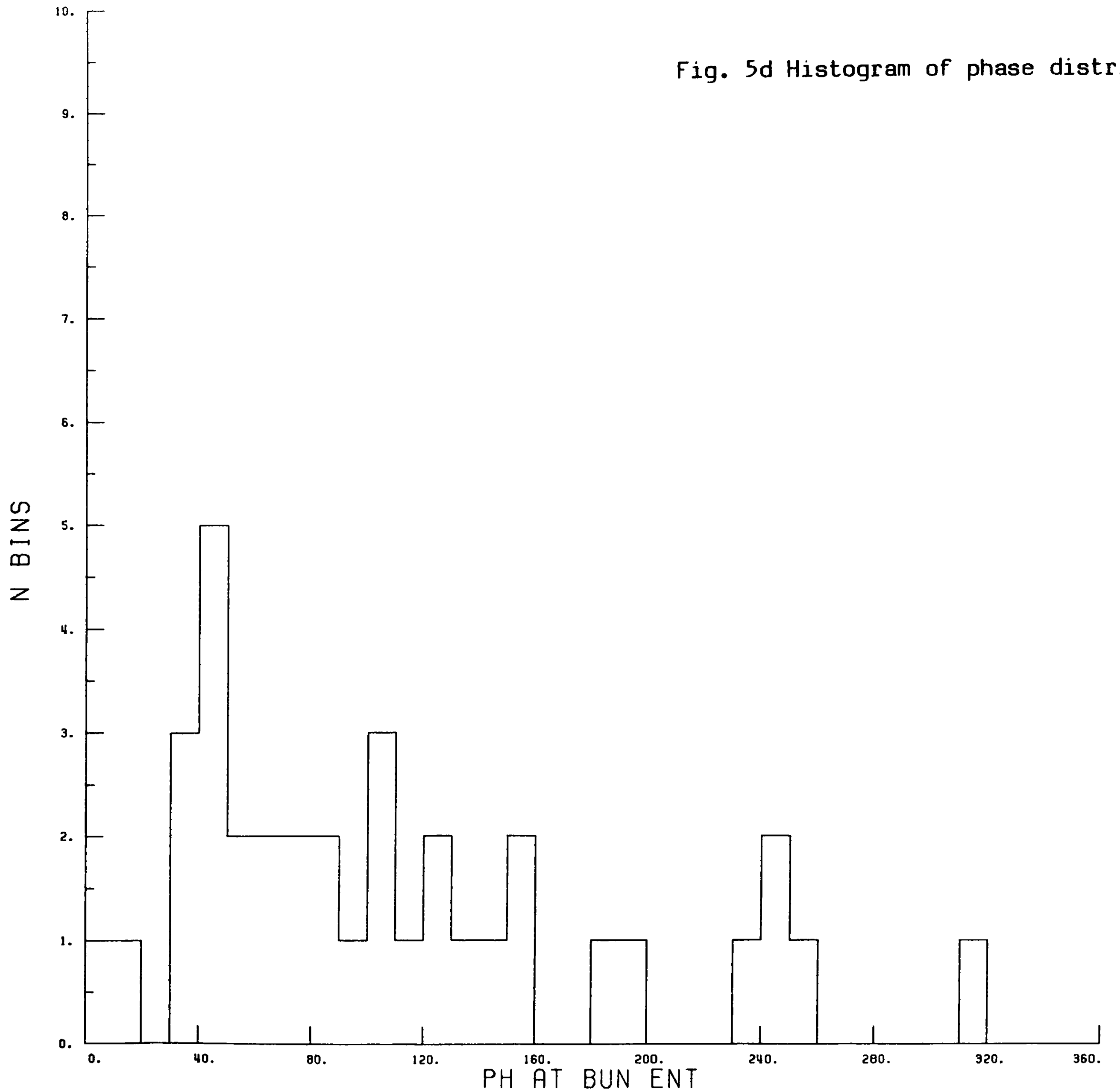


Fig. 5d Histogram of phase distribution at buncher entrance. E_{pB} = 15 MV/m

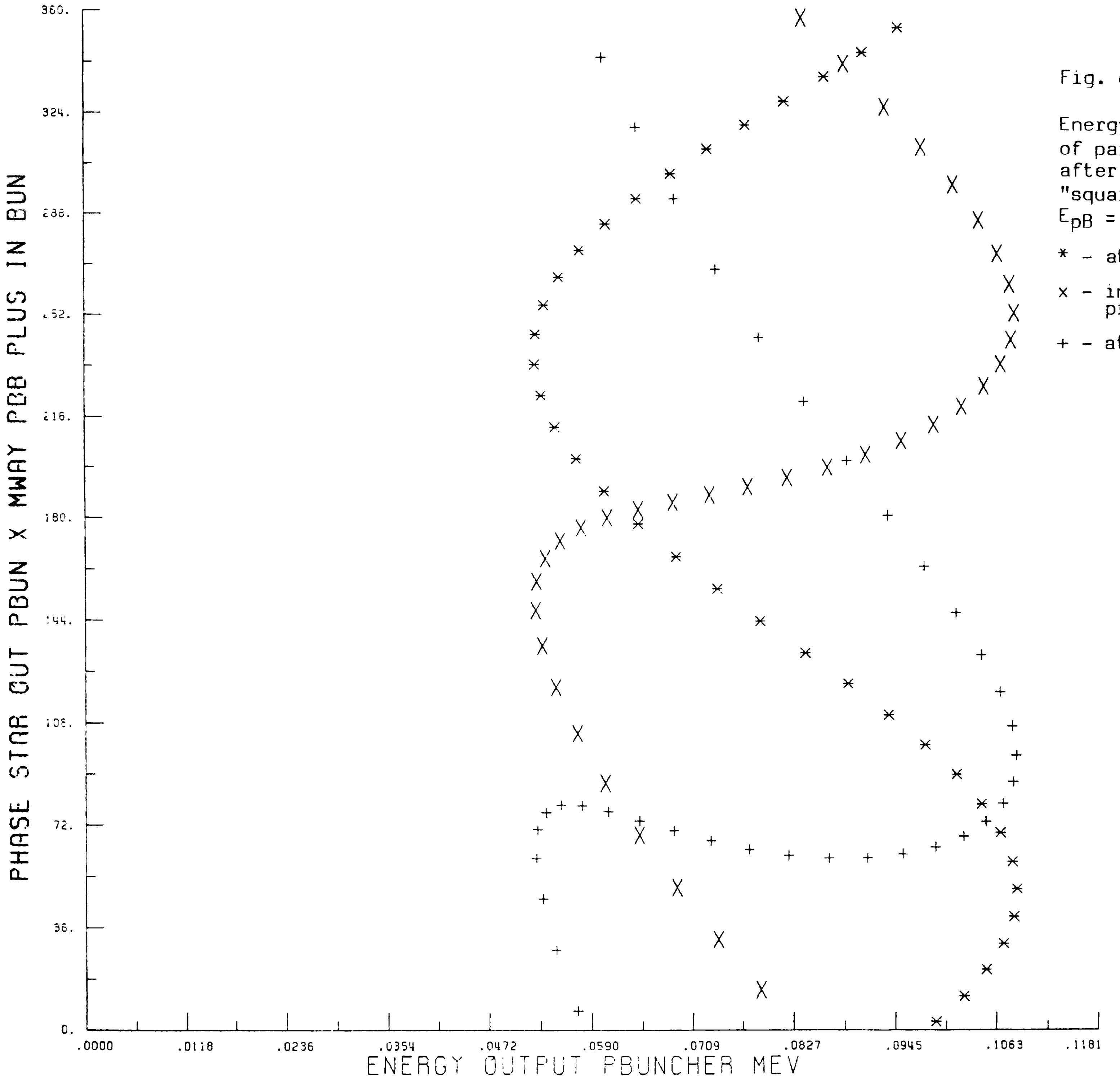


Fig. 6a

Energy - Phase distribution
of particles
after prebuncher with
"square" field distribution.
E_{pB} = 2.2 MV/m

* - at cavity exit

x - in the middle between prebuncher and buncher

+ - at buncher entrance

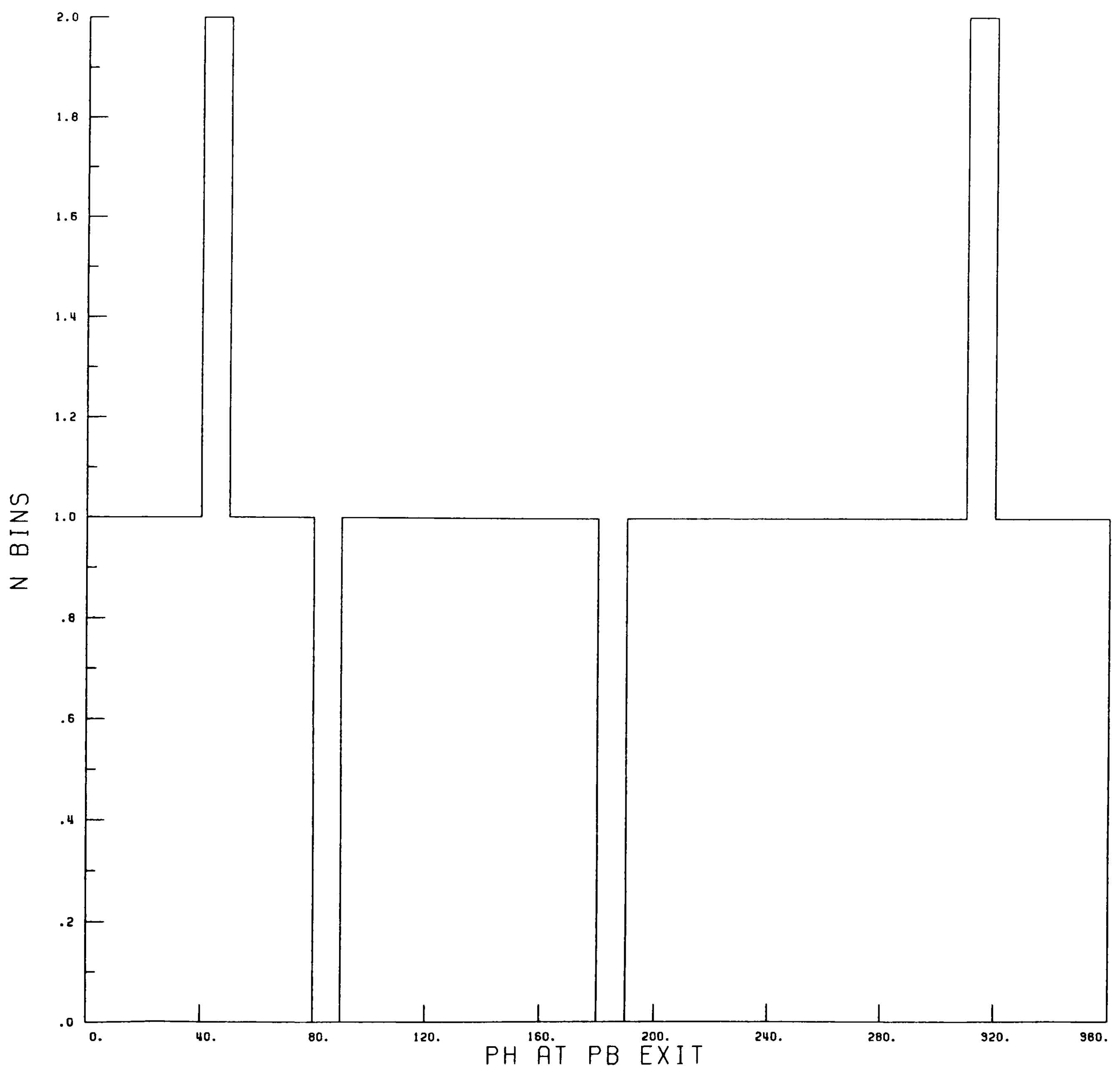


Fig. 6b Histogram of phase distribution at the cavity exit. E_{pB} = 2.2 MV/m "square" shape.

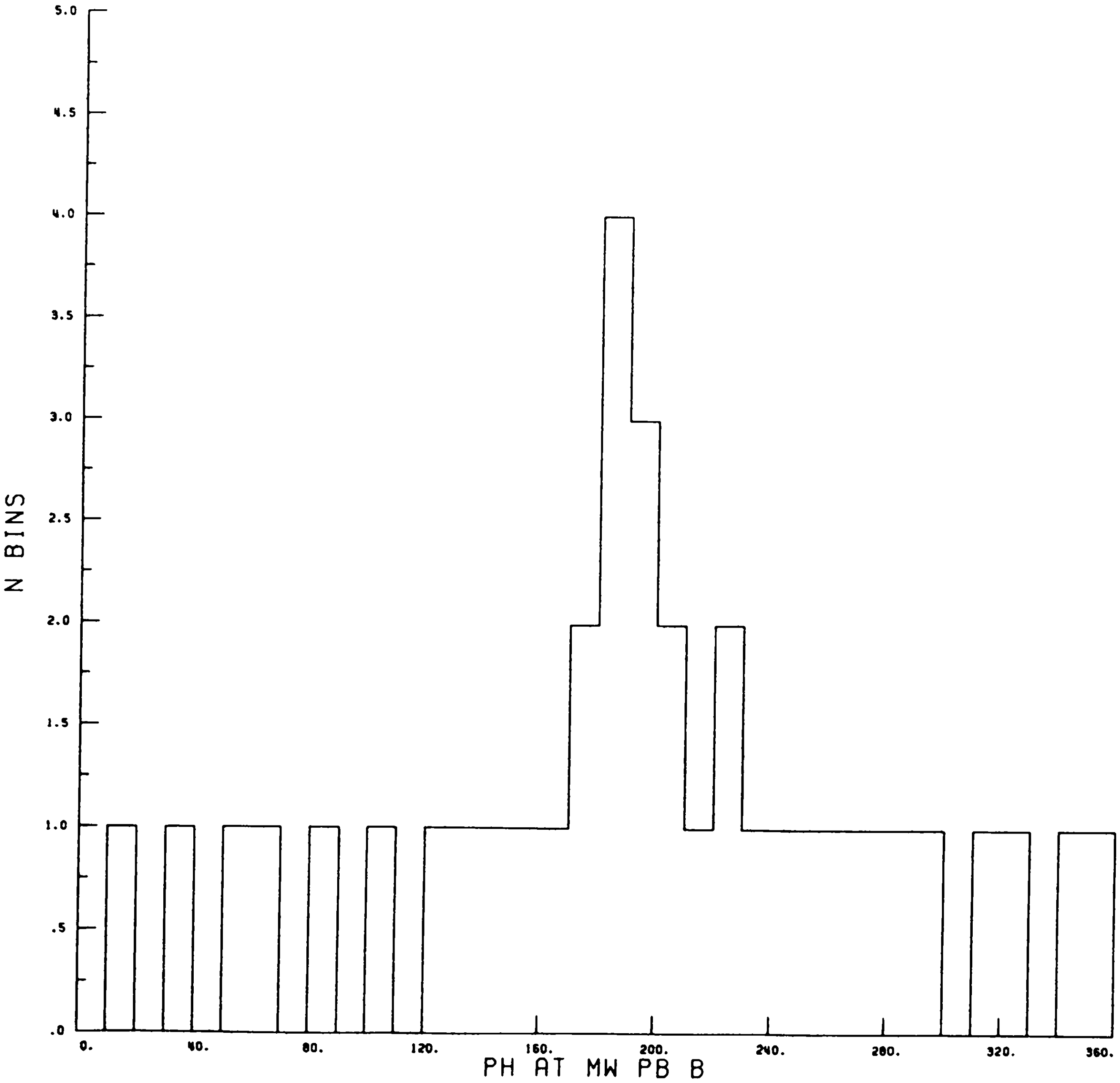


Fig. 6c Histogram of phase distribution in the middle between prebuncher and buncher. E_{pB} = 2.2 MV/m "square" shape.

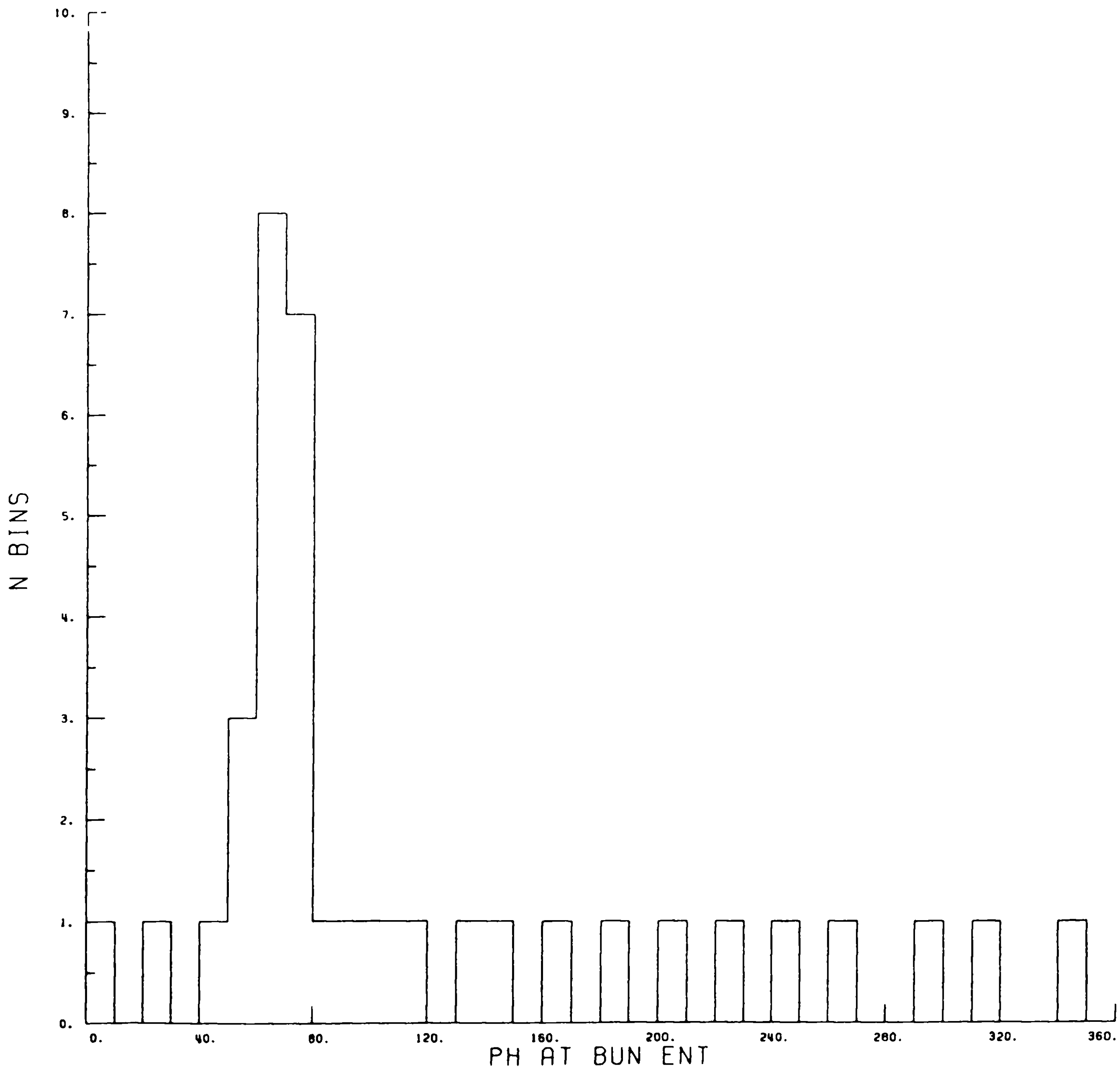


Fig. 6d Histogram of phase distribution at buncher entrance. E_{pB} = 2.2 MV/m "square" shape.

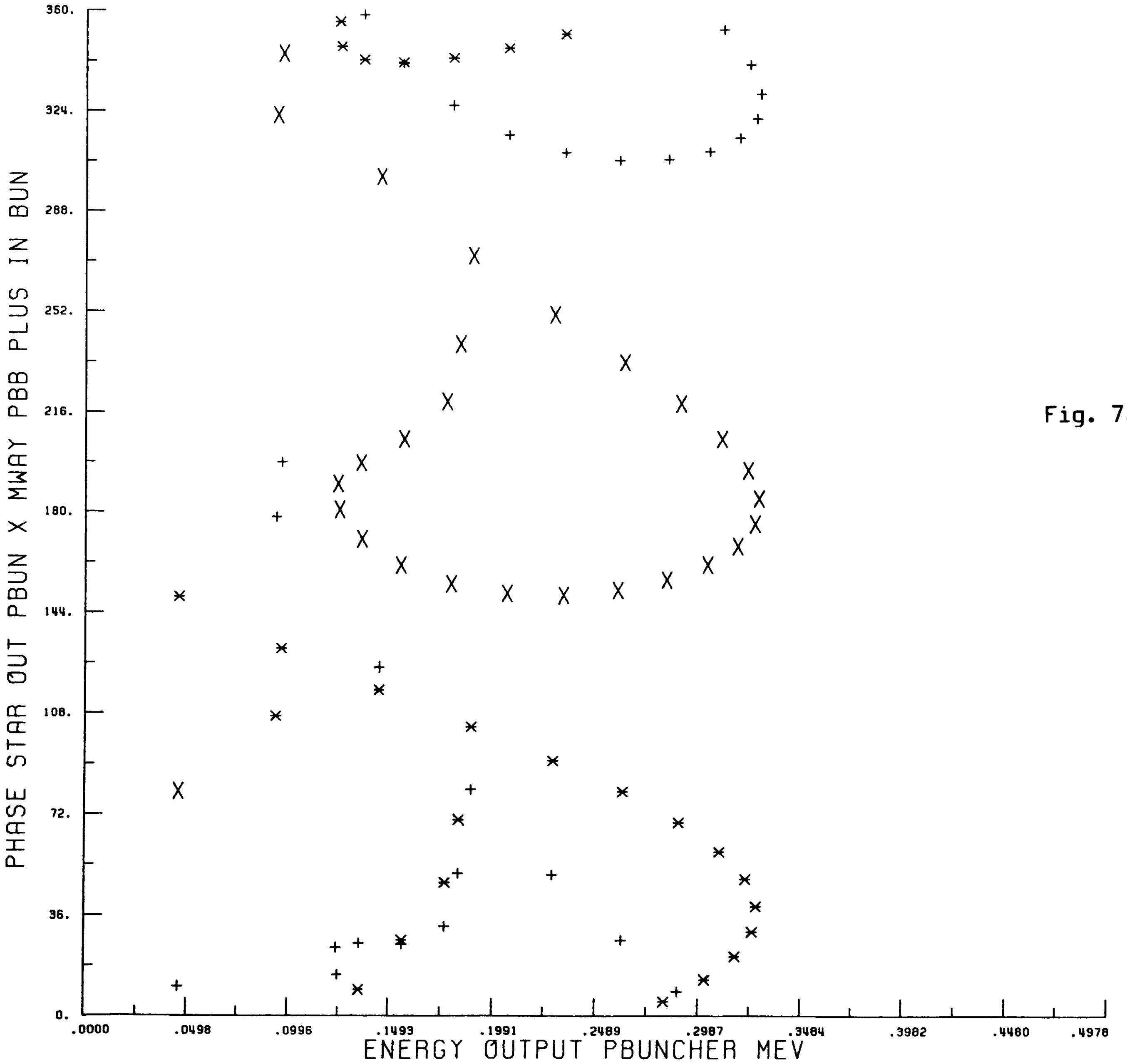


Fig. 7a Energy - Phase distribution of particles
after prebuncher cavity
with "square" field distribution.
E_{pB} = 15 MV/m

* - at cavity exit

x - in the middle between prebuncher and buncher

+ - at buncher entrance

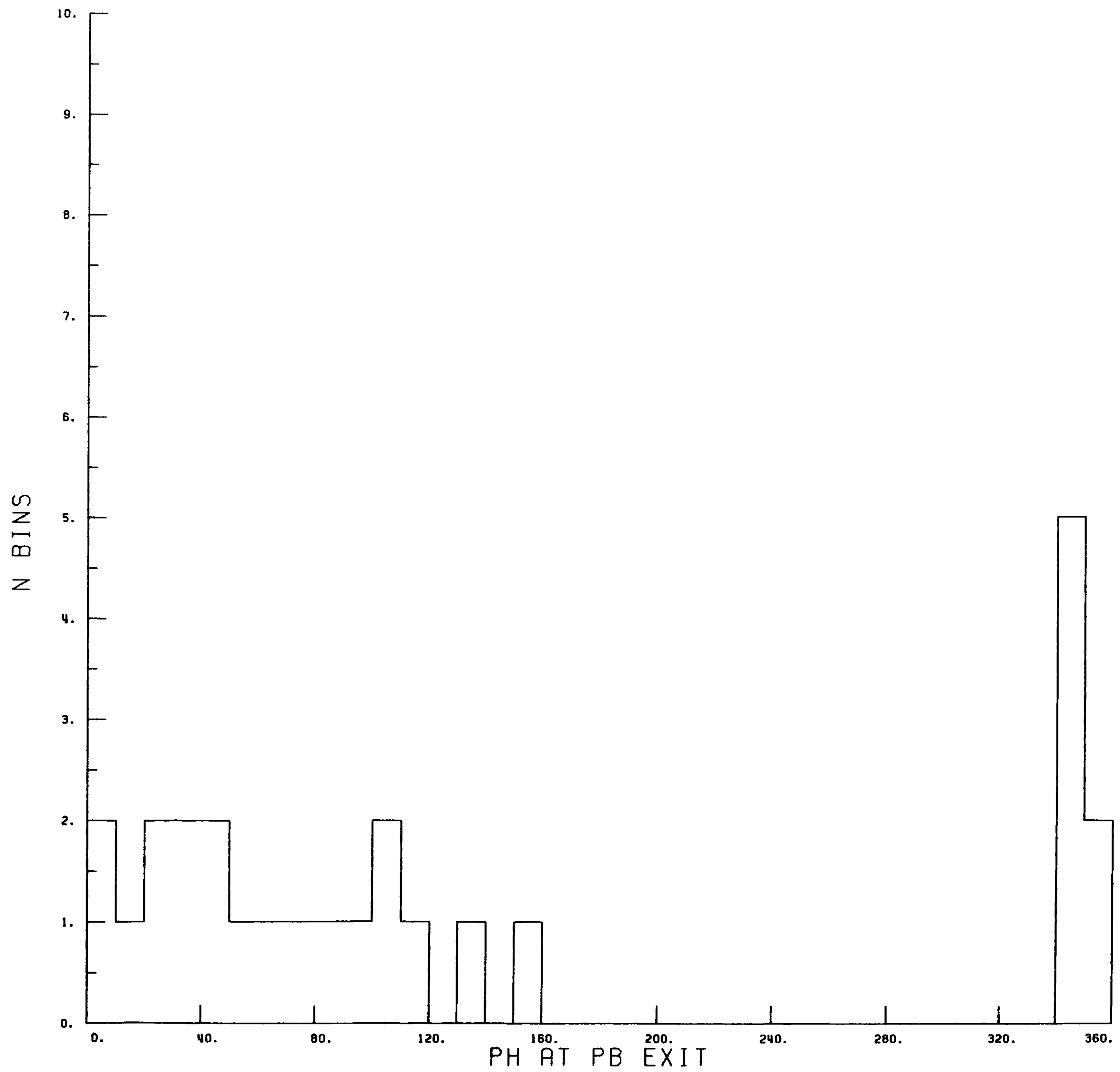


Fig. 7b Histogram of phase distribution at the cavity exit. E_{pB} = 15 MV/m "square" shape.

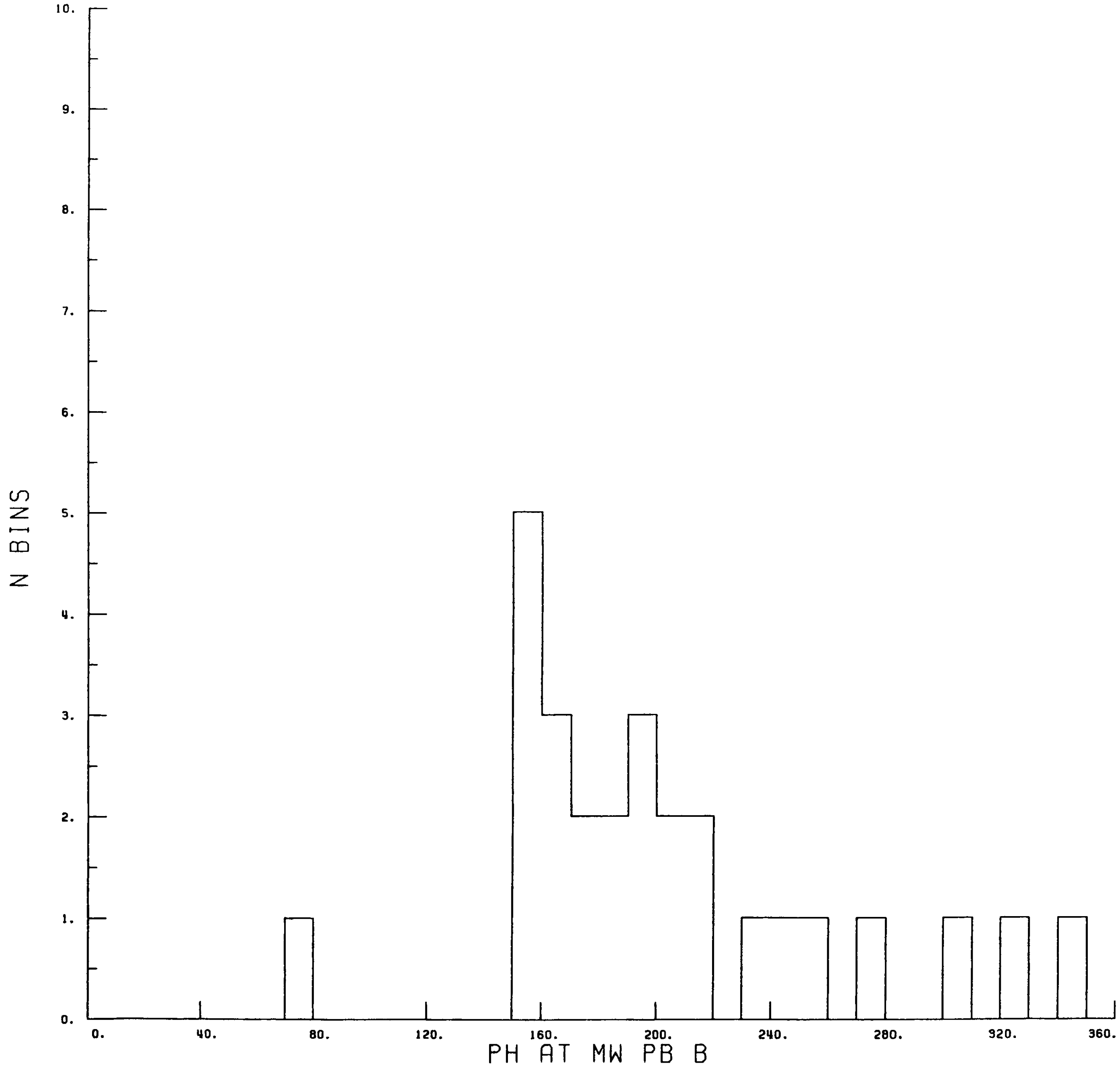


Fig. 7c Histogram of phase distribution in the middle between prebuncher and buncher. E_{pB} = 15 MV/m "square" shape.

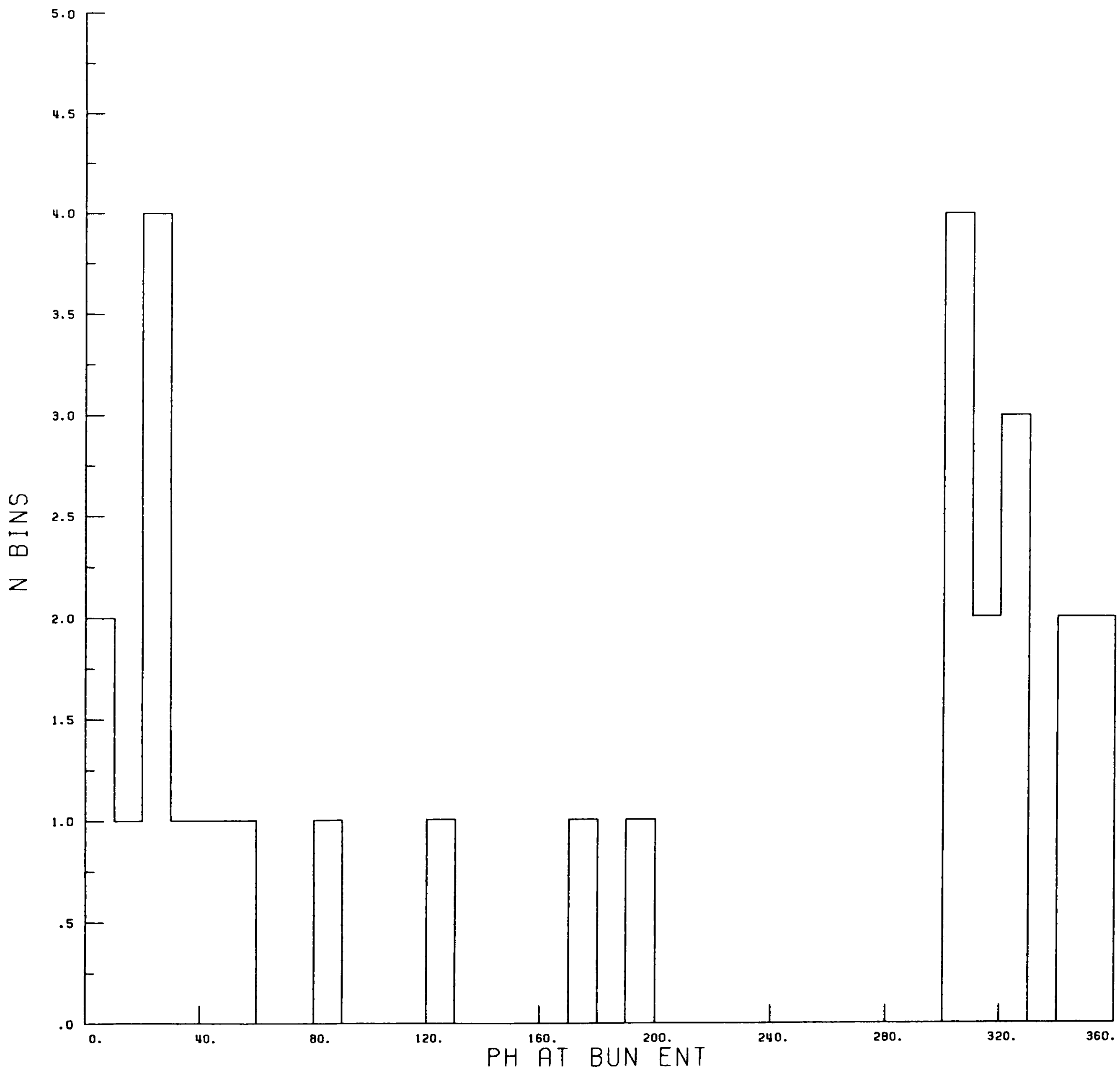


Fig. 7d Histogram of phase distribution at buncher entrance. E_{pB} = 15 MV/m "square" shape.