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Lecture on Orbits in the Strong Focussing Synchrotron

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J. B. Adams   F. K. Goward

CERN/PS/JBA-FKG 1.

1. Introduction

The highest energy which has so far been achieved by a proton accelerator is about $2\frac{1}{2}$ GeV (2.5×10^9 eV) from the Brookhaven Cosmotron. In this machine particles are accelerated along an orbit of radius about 10 metres inside a vacuum chamber whose available aperture is about 70 cm. x 15 cm.. The cost of such a conventional synchrotron is 6 to 10 million dollars and using the same kind of techniques it would become very costly to extend the energy to 10 GeV. Some thought was given by C.E.R.N. to such an extension in early 1952 and a figure of about 15 million dollars was then estimated, provisionally, as the likely cost of a 10 GeV 'conventional' machine.

Some fundamental change in technique is desirable for higher energy machines and this lecture will describe such a change, first suggested by Courant, Livingston and Snyder* and since then extended and modified by work in both Europe and U.S.A. The C.E.R.N. laboratory is planning a 'strong-focussing synchrotron' of this type to give energies of 20-30 GeV.

In this lecture the conventional and strong-focussing methods will be compared and the developments in the strong-focussing method described, recalling first the essential principles of conventional focussing and then proceeding to the later developments. In this lecture only the motion of particles circulating at constant energy in static magnetic fields is considered. No problems associated with oscillations of the particles in energy or phase due to acceleration are included.

2. The Conventional Synchrotron

The focussing principle of a conventional synchrotron relies on two fundamental ideas:

is necessary in order to produce the outward 'bowing' of the field lines required to return a particle to the median plane (See fig. 1a).

(b) To give radial stability the rate of decrease of field with radius must be limited. The limit is shown by considering a particle of energy E_0 injected tangentially into a field at two different radii, r and r_0. The limit of zero focussing occurs when the particles simply maintain their radii i.e. when $H_z r = H_0 r_0$ or $H_z = H_0(r/r_0)^{-1}$. To obtain positive radial focussing of the particles towards a stable orbit at r_0 the axial field at r must be greater (for $r < r_0$) than that given by the above condition i.e.

$$H_z = H_0(r/r_0)^{-n} \text{ with } n < 1.$$

The condition for both axial and radial stability to coexist, therefore, is that

$$H_z = H_0(r/r_0)^{-n} \text{ with } 0 < n < 1 \quad \ldots \quad (1)$$

To obtain more quantitative information about the orbits the axial field, H_z, is approximated, putting $r - r_0 = \rho$, by the relationship

$$H_z = H_0\ (1 - n\rho/r_0) \quad \ldots \quad (2)$$

from which the radial field component is approximately

$$H_r = - H_0 . n . z/r_0 \quad \ldots \quad (3)$$

For such a 'linear' field motion is obtained which is, to a first approximation, independent in the axial and radial directions. Some coupling occurs in the second approximation because the machine is circular and there are consequent centrifugal terms in the equations of motion; in addition strictly linear fields radially and axially cannot coexist in a three dimensional machine. These factors will, however, be neglected and the equations of motion written:

$$\ddot{\rho} = - w_0^2(1-n)\rho, \quad \ldots \quad (4a)$$

$$z = - w_0^2\, nz \quad \ldots \quad (4b)$$

with solutions:

$$= a \cos w_0(1-n)^{\frac{1}{2}}t + b \sin w_0(1-n)^{\frac{1}{2}}t, \quad \ldots \quad (5a)$$

$$z = c \cos w_0(n)^{\frac{1}{2}}t + d \sin w_0(n)^{\frac{1}{2}}t \quad \ldots \quad (5b)$$

In these equations the angular frequency of rotation, dθ/dt, of the particle is

The condition for simultaneous stability in both ρ and z directions, $0 < n < 1$, follows from equation (5). Typical orbits for the radial motion are shown in Fig. 2. For equal focussing action in both directions n = 0.5 and particles originating in a point focus refocus after about 0.7 revolutions, the focussing or betatron wavelength being $\sqrt{2}$ circumferential lengths.

The first consideration in designing a synchrotron is to fix on the minimum aperture required. Such consideration will first be restricted to a machine whose magnetic field has no irregularities such as those which might arise from mechanical and electrical defects. Only the aperture required to accommodate the focussing oscillations, and not synchrotron oscillations will be studied. The aperture, A, required is then proportional to the angular spread, ϕ, of the source, or to some effective angular spread resulting from collisions with residual gas, and to the focussing wavelength, , that is

$$A \propto \phi \lambda, \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (6)$$

assuming that the source occupies a definite small fraction of the aprt Starting from a given injector and using focussing devices to bring the source size to the required constant fraction of the donut aperture, kA, the angular spread of the source is, with optimum adjustment, inversly proportion to the size of the aperture, that is:

$$\phi \propto A^{-1} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (7)$$

This follows from the Helmholz Lagrange condition in an optical system. From the equations (6) and (7)

$$A \propto \lambda^{\frac{1}{2}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (8)$$

It is desirable, therefore, to reduce the focussing wavelength, λ, to a small value as shown in fig. 3. Such a reduction is possible in, say, the radial direction by making n take large negative values, when the radial aperture, A_ρ, varies with n according to the relationship

$$A_\rho \propto n^{-\frac{1}{4}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (9)$$

since the wavelength is inversely proportional to $n^{\frac{1}{2}}$ (see equations 5). These n values would, however, give defocussing in the axial direction, the behaviour corresponding to the orbit marked '+' in figure 2, with z replacing quite clear but no practical way of achieving a reduction in both the axial and radial apertures simultaneously was advanced for some time. The suggestion made by Courant Livingston and Snyder is important because it does offer this possibility. It ensures a variation of aperture with field gradient of the form

$$A \propto n^{-\frac{1}{4}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (10)$$

where A now refers to both radial and axial apertures.

3. The Strong Focussing Synchrotron

Courant, Livingston and Snyder showed that focussing may be obtained in both radial and axial directions simultaneously if the magnet circumference is broken up into a number of sectors, N, with alternating large positive and negative values of n. The scheme may be illustrated by plots of some typical orbits, in both directions, as shown in fig. 4. This figure is drawn for the case in which the field gradient (or n value) and sector length are so related that a phase change of $\pi/2$ occurs in the normal betatron oscillation occurring in one focussing sector. These orbits are special cases of focussing oscillations in which repetition occurs in an integral number of sectors, but for a machine which is perfectly constructed there is a continuous range of values of n for which stable operation in both directions is possible. This stable region is shown in fig. 5., where the n values for the ~~positive~~ negative gradient sectors, n_1, are not necessarily equal to the n values for the positive gradient sectors, n_2. The limits of the stable region are found by a simple application of Floquet's theorem which states that, in problems of this type, solutions can be found which are merely multiplied by an exponential factor, $e^{\pm i\mu}$, in passing from one unit to the next, the unit consists of two sectors in this particular problem. Using this theorem, stable operation is possible provided that μ is real, when the factor $e^{\pm i\mu}$ is oscillatory. μ then corresponds to the phase shift between each pair of sectors, the phase being measured in terms of the focussing wavelength in the new composite system. Quantitatively the condition for stability is,

$$|\cos\mu| > 1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (11)$$

$$\text{where } \cos\mu = \cos\gamma_1 \cosh\gamma_2 - \tfrac{1}{2}\left(\frac{\gamma_1}{\gamma_2} - \frac{\gamma_2}{\gamma_1}\right)\sin\gamma_1 \sinh\gamma_2 \quad .. \qquad (11)$$

Some typical orbits (in the median plane) both inside and outside the stable region are shown in fig. 6, the machine being normally operated at some point on the diagonal of figure 5 to give equal stability radially and axially.

The orbits inside the stable area consist, approximately of long period sinusoidal oscillations with other oscillations superposed of wavelength equal to two sectors lengths (figs. 6a and 6b). The relative magnitude of these superposed oscillations decreases as we operate at points increasingly close to 0 in fig. 5 and there is thus some advantage to be gained, in reduction of aperture for given injection conditions, by operating the synchrotron rather nearer to 0 than is the case in the '$\pi/2$ mode' of fig. 4. The mode number, μ, is defined as the phase change in a pair of sectors. This phase change is defined in terms of the focussing oscillation in the composite system and not in terms of the betatron oscillation in a single focussing sector. The mode used will depend on a compromise between the focussing and synchrotron oscillation amplitudes, but operation with a mode number as low as the $\pi/5$ mode is quite likely. (In this mode the oscillation will repeat every 20 sectors).

At the limits of the stable region the oscillation amplitude builds up linearly (fig. 6c); this situation will clearly arise since a linear rise must form the boundary between focussing behaviour (sinusoidal envelope) and defocussing (exponential envelope) behaviour. A special orbit could have been drawn on fig. 6(c), which maintained constant amplitude, by making the orbit originate at a radial position r_0 in the centre of a defocussing sector. Finally fig. 6(d) shows an orbit outside the stable region.

The ideal geometrical arrangement so far studied will certainly not be used in practice for various reasons. First it is necessary to have some separation between sectors having positive and negative n values, for both mechanical and magnetic reasons. Next it is necessary to put breaks in the magnet every so often to accommodate the R.F. accelerators, the injection mechanism, and other devices. Then again, the sectors can only be constructed and aligned within certain dimensional tolerances and variations are sure to exist in the magnetic properties of the iron used at different points of the circumference. All these factors

4. Azimuthal Inhomogeneities

The operation of the strong-focussing synchrotron is very sensitive to irregularities in the geometrical alignment of the sectors, so this type of azimuthal inhomogeneity will be studied as a first example. The strong-focussing synchrotron is much more sensitive to alignment errors than a conventional machine, as is illustrated by the orbits plotted in fig. 7, where each type of machine is assumed to be manufactured accurately except for the displacement of δ, of a given small element of the circumference of length d. For convenience it has been assumed that the focussing oscillations are sinusoidal of wavelength λ although, as explained in Section 3, other smaller-wavelength oscillations are superposed. The displaced element, which is assumed short compared with λ, deflects the particle through an angle that depends on the error in magnetic field on the particles' orbit. The error in magnetic field depends on the displacement of the element, δ, and on the field index, n. As has been shown previously λ is inversely proportioned to $n^{\frac{1}{2}}$ so finally the angle of deflection, for a given displacement of the element, is inversely proportioned to λ^2. The corresponding amplitude of oscillation induced by the displaced element is proportional to λ and to the angle of deflection, and is therefore ultimately proportional to $1/\lambda$, or to $n^{\frac{1}{2}}$. For element lengths of $\lambda/4$ the oscillation amplitude reaches a value in excess of 2δ, the value depending on the mode number and on whether the displaced element is focussing or defocussing.

If all individual elements in the circumferential length are displaced from the true circle in a random manner then the R.M.S. oscillation induced per revolution is $\sqrt{S}$ times the R.M.S. displacement of a single element, where S is the number of elements. It is permissible to treat the irregularities as one 'effective' discontinuity (consisting approximately of the addition of a definite slope to the orbit) located at some point on the circumference.

In the strong focussing synchrotron the oscillation, already large, induced in one revolution can be further increased by resonant build-up of oscillations. This possibility exists as soon as the focussing wave-length becomes less than one circumferential length and is therefore not present in the conventional machine