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ON THE MEASUREMENT OF Q

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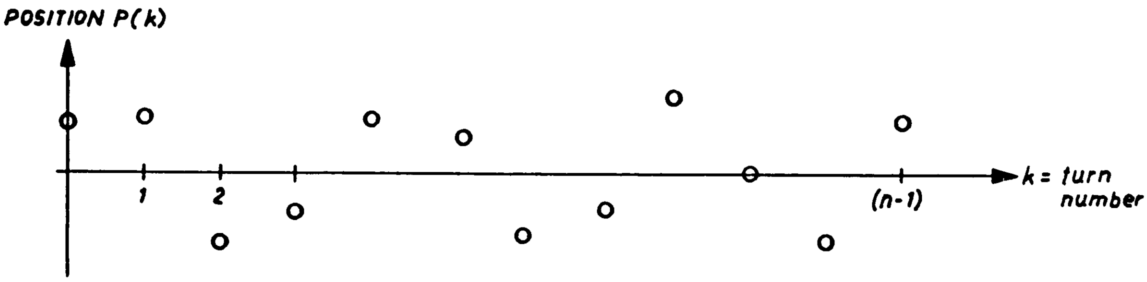
This note treats some generalities concerning Q measurement, but is primarily an elaboration of MPS/CO Note 69-17 on 'Q measurement using computer frequency analysis'.

One method of measuring Q is to record the beam position at a given point in the ring over a number n of sequential turns. The position can then be plotted on a piece of graph paper to give a picture something like that shown below.



A sine wave can then be fitted to these points and Q obtained from the frequency of this sine wave.

It is shown here that the non-integer part of Q can be obtained from these measurements with an accuracy $\frac{1}{n}$ and a resolution $\frac{1}{N}$ where N can be chosen arbitrarily.

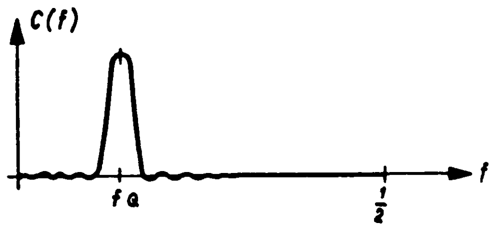
Two sources of error are present: firstly the accuracy of the position measurement; secondly the accuracy with which one can fit the sine wave to the points. The inaccuracy of the measurement can be considered as a 'noise' obscuring the real position measurement so that the best we can do is to determine Q from a 'best fit sine wave'.

The beam position can be obtained either optically from photographs of a pick-up signal (e.g. spiralling beam at injection) or by computer sampling of a pick-up output. The sine wave fitting can be done by eye (procedure used at present for photographs) or by some computer algorithm. Here is considered the accuracy which can be expected from a limited number n of position measurements.

For the computer method some best fit algorithm must be adopted. A common algorithm (very powerful for separating a known signal from noise) is that of correlation. Consider the function

$$C(f) = \sqrt{a^2(f) + b^2(f)} \quad \text{where} \quad a(f) = \sum_{k=0}^{n-1} p_k \cos 2\pi fk, \qquad b(f) = \sum_{k=0}^{n-1} p_k \sin 2\pi fk$$

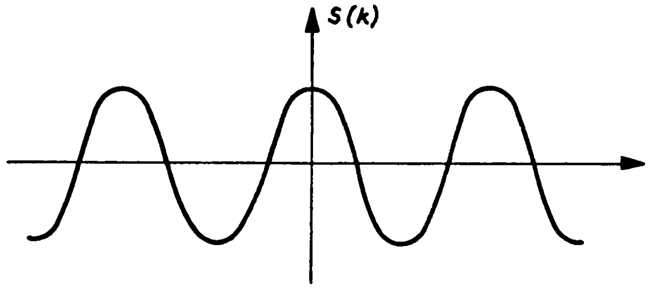
$C(f)$ is a measure of the correlation between the position and a sine wave of frequency f and 'best correlating' phase. If a graph of $C(f)$ against f is drawn a peak may be observed and the frequency of this peak f_Q gives the non-integer part of Q.

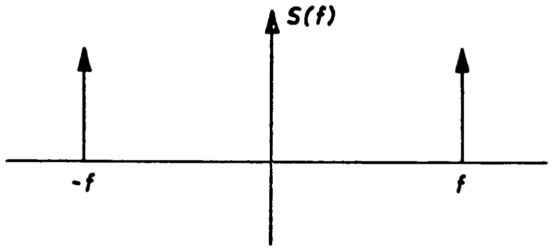


The definition of $C(f)$ above suggests a fourier transformation. It is shown in the appendix how the above graph of $C(f)$ against f can be approximated to any desired accuracy using currently available fast discrete fourier transform techniques.

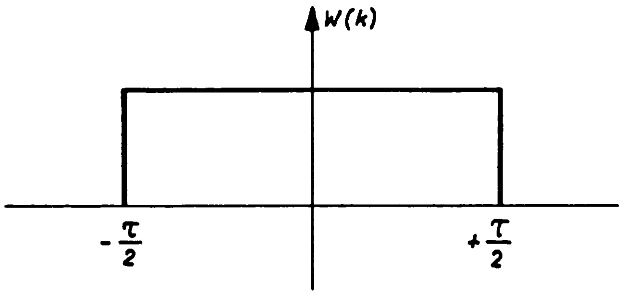
The discrete fourier transform is very closely related to the continuous fourier transform and for simplicity continuous time functions will be used for accuracy studies.

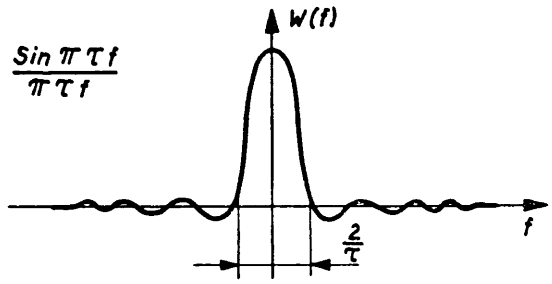
We suppose the samples were from a continuous perfect sinusoïd of frequency f_Q (below left) which has the continuous fourier transform S(f) (below right).



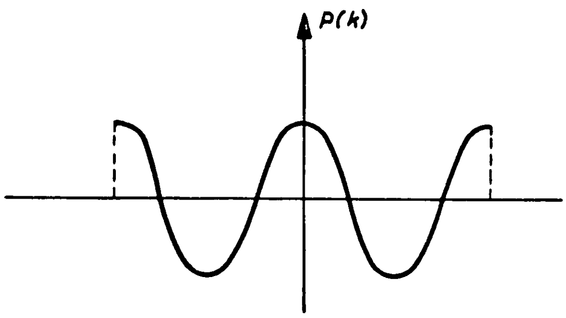


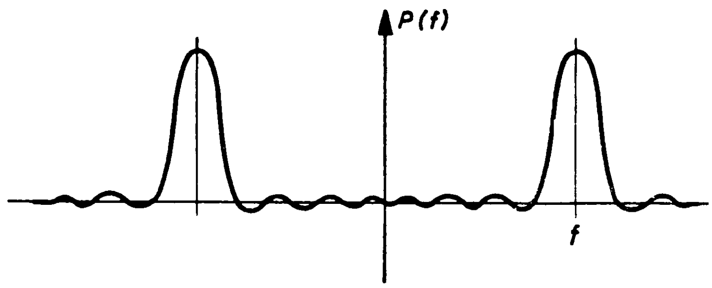
In practice we follow the sinusoïd for only a limited time, the equivalent of multiplying by the rectangular window (below left) of fourier transform (below right).





The resultant spectrum from the product of the two time funçtions is the convolution of the two frequency functions giving the actual signal and its spectrum as below.



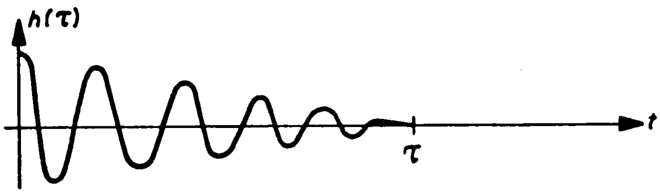


The modulus of the positive half of this spectrum (above right) gives the function C(f) defined above. Thus even if the position measurements are samples of a perfect sine wave, the peak in C(f) will have a half width of $\frac{1}{T}$. In the ideal case we can always find the peak, and hence Q, precisely. In the practical case with noise and distortion a reasonable measure of the accuracy might be $\pm \frac{1}{T}$, i.e. 100 turns for $f \pm 0.01$. Under good clean signal conditions, however, better estimates of the peak position can perhaps be made with a consequent reduction of the number of turns required.

Notes

1. The above results are fundamental - no amount of ingenuity can extract information which is not there.

2. The apparent conflict with the claim (G. Schneider, MPS/SR/69-10) that Q can be measured accurately in one betatron period, i.e. ~ 4 turns, can be resolved as follows; in Schneider's method the measurement is made at the output of a tuned filter. The impulse response h(t) of this filter is a damped sinusoid. The output of the tuned filter is the convolution of the input p(t) and the impulse response



i.e. $$O(t) = \int_0^\infty h(s)\, p(t - s)\, ds$$

If this impulse response is considered zero after time T we have

$$O(t) = \int_0^T h(s)\, p(t - s)\, ds$$

$$= \int_{-T}^{0} h(-u)\, p(u + t)\, du$$

which is the correlation between the input over the preceding time τ and the time inverted impulse response. Thus with the filter method correlation with something like a sinusoid is involved, as in the computer method; and a finite measuring time, determined by the bandwidth of the tuned filter, is taken.

3. Is there anything to be gained by following the beam round the machine? In this case the peak in C(f) would be at Q x rotation frequency but its width would still be determined by the length of record. Thus for $f_Q = 6.25 \pm 0.01$ requires the same length of record as for $f_Q = 0.25 \pm 0.01$. The sampling acts as a sort of perfect mixing which decreases the percentage resolution required.

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APPENDIX

The required function is

$$C(f) = \sqrt{a^2(f) + b^2(f)} \qquad \text{with}$$

$$a(f) = \sum_{k=0}^{n-1} p_k \cos 2\pi fk; \quad b(f) = \sum_{k=0}^{n-1} p_k \sin 2\pi fk$$

For digital calculation the function $C(f)$ must be sampled, say at intervals f_0 over the range $f = 0 - \frac{1}{2}$. We can then write

$$C_r = C(rf_0) \qquad r = 0, 1 \ldots N/2$$

$$f_0 = \frac{1}{N}$$

By making N large we can have f_0 small to give good resolution of $C(f)$. This gives

$$C_r = \sqrt{a_r^2 + b_r^2} \qquad r = 0,1 \ldots N/2$$

$$a_r = \sum_{k=0}^{n-1} p_k \cos (2\pi\ kr/N)$$

$$b_r = \sum_{k=0}^{n-1} p_k \sin (2\pi\ kr/N)$$

If now we define $p_k = 0 \quad n-1 < k \leq N-1$ we have

$$C_r = |A_r| \qquad r = 0.1 \ldots N/2$$

where

$$A_r = \sum_{k=0}^{N-1} p_k \exp(-2\pi rk/N)$$

A_r is the discrete fourier transform of the real series p_k made up of n samples of position and $N - n$ zeros.