EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH
ORGANISATION EUROPEENNE POUR LA RECHERCHE NUCLEAIRE

CERN - PS DIVISION

PS/ AE/ Note 2000-010
CLIC Note 459

AN ISOCHRONICITY-TUNABLE ACHROMAT MODULE

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This note provides closed form expressions (in the thin lens approximation) for the quadrupole strengths required to tune the R_{56} parameter of the isochronous module based on three identical dipoles (2) and lists the eight sets of possible ranges for the three lengths of the drift spaces separating the mangetic components. Thus it permits the design of an insertion satisfying precise geometric constraints as in the future CLIC test facility CTF3 and which is able to compress or stretch longitudinally the beam according to the settings of the quadrupoles. An application to a CTF3 transfer line is also shown.

Geneva, Switzerland
23 August 2000

1 Introduction

Among the many parameters which are essential in the CLIC study, the length of the bunch is very critical. It should be 30 μm inside the main linac and carefully controlled in the bends of the injector complex. The isochronous rings and transfer lines of the RF power source also require that the bunch length of the drive beam be modified, either by stretching, in order to limit the coherent synchrotron radiation effects, or by compression, in order to optimise the power transfer to the main beam. In the first order approximation the bunch length is proportional to the R_{56} parameter which is defined by the following integral :

$$R_{56} = \int_{s_1}^{s_2} \frac{D_x}{\rho(s)} ds \tag{1}$$

where D_x is the horizontal dispersion, $\rho(s)$ the radius of curvature, and s_1,s_2 are the longitudinal coordinates of the beginning and end of the beamline considered. The R_{56} parameter is positive if high momentum particles of the bunch travel longer paths. Of course the values of the R_{56} parameter of the various insertions can be fixed at the design stage, but the operation of both the accelerator and the decelerator are much easier if some flexibility is given to modify it in a given range. This flexibility becomes a feature in a test facility such as CTF3 [1], whose purpose is to validate most of the RF Power Source design and at the same time to study the behaviour of coherent synchrotron radiation for which the theory and the simulations remain to be confronted with experimental data. Thus a study was started to find an ensemble of several magnetic components (dipoles and quadrupoles) called an "insertion", which would be able to generate both a negative or a positive R_{56} parameter by only modifying the strength of the quadrupoles. Quite naturally the isochronous insertion developed five years ago [2] was chosen as a promising candidate. It turned out that it was possible to obtain the expression for the absolute values of the focal lengths as a function of the R_{56} parameter in the thin lens approximation. This will be shown in the next section. It demanded much more algebra to derive the conditions on the minimum and maximum values of the R_{56} parameter and on the lengths of the drift spaces, such that the absolute values of the focal lengths remain positive. Actually eight different sets of conditions can be found to cover all the physically valid configurations. They are derived in appendix A. It is impossible to decide analytically which one is best to optimise a given design. This depends upon the geometry and the constraints imposed on the Twiss parameters at the entrance and exit of the insertion. A simple interactive Excel program guides the user towards the best choice. The last section shows an application to the transfer line between the Delay Loop and the Isochronous Ring of CTF3.

2 Quasi-isochronous module based on three identical dipoles ($R_{56} \neq 0$)

Let us consider a module consisting of three bending magnets geometrically and magnetically symmetric around the median plane of the second magnet. To simplify the algebra, these magnets are treated as sector magnets of the same length l_m but of different deflection angles ϕ_1 and ϕ_2 for the first and second dipole respectively. The space between the first two magnets is filled by a space drift of length L_1, by a focalising quadrupole of length l_q and normalised gradient k_1, by a second space drift of length L_2, by a defocalising quadrupole of length l_q and normalised gradient k_2 and finally by a third space drift of length L_3. On the assumption that the quadrupoles are perfectly centered, the parameter R_{56} is given by (1). Assuming also that the dispersion and its derivative are zero at the entrance of the first dipole, the contributions of the first dipole and of half the second dipole to this integral are, respectively [2] :

$$\rho_1 (\phi_1 - \sin\phi_1)$$
$$D_j \sin(\phi_2/2) - \rho_2 D'_j [\cos(\phi_2/2) - 1] + \rho_2 [\phi_2/2 - \sin(\phi_2/2)]$$

where ρ_1 and ρ_2 are the curvature radii of the first and of the second dipole respectively and D_j and D'_j are the dispersion and its derivative at the entrance of the second dipole. Thus the following equation is obtained :

$$\frac{R_{56}}{2} = \rho_1 (\phi_1 - \sin\phi_1) + D_j \sin(\phi_2/2) - \rho_2 D'_j [\cos(\phi_2/2) - 1] + \rho_2 [\phi_2/2 - \sin(\phi_2/2)] \quad (2)$$

In order to obtain a nondispersive module, the derivative of the dispersion at the point of symmetry should be zero providing a second equation :

$$-\frac{\sin(\phi_2/2)}{\rho_2} D_j + D'_j \cos(\phi_2/2) + \sin(\phi_2/2) = 0 \quad (3)$$

From these two equations it is easy to obtain :

$$\begin{aligned} D'_j &= \frac{1}{\rho_2}\left[\frac{R_{56}}{2} - l_m\left(\frac{3}{2} - \frac{\sin\phi_1}{\phi_1}\right)\right] \\ D_j &= \rho_2 \left[1 + D'_j \cot(\phi_2/2)\right] \end{aligned} \quad (4)$$

The first expression can be written more compactly

$$D'_j = \frac{x}{\rho_2} \quad (5)$$

where

$$x = \frac{R_{56}}{2} - l_m\left(\frac{3}{2} - \frac{\sin\phi_1}{\phi_1}\right) \quad (6)$$

It is possible to obtain in the same way as in reference [2] the expressions of the lengths of the first two drift spaces as functions of k_1,k_2 and of L_3 :

$$
\begin{aligned}
L_1 &= a\frac{C_2 q_1}{C_1 q_2}\left(L_3 - \frac{D_j}{D'_j} + q_2\right) - l + q_1 \\
L_2 &= q_1 - q_2 + \frac{b}{L_3 - \frac{D_j}{D'_j} + q_2}
\end{aligned}
\tag{7}
$$

where

$$
l = \rho_1 \tan(\phi_1/2) \qquad a = -\frac{D'_j}{\sin\phi_1} = -\frac{x}{\rho_2 \sin\phi_1} \tag{8}
$$

$$
b = \frac{q_2}{C_2}\left(\frac{q_2}{C_2} + \frac{q_1}{aC_1}\right) \qquad q_i = \frac{C_i}{S_i\sqrt{k_i}} \tag{9}
$$

$$
C_1 = \cos\left(l_q\sqrt{k_1}\right) \qquad S_1 = \sin\left(l_q\sqrt{k_1}\right) \tag{10}
$$

$$
C_2 = \cosh\left(l_q\sqrt{k_2}\right) \qquad S_2 = \sinh\left(l_q\sqrt{k_2}\right) \tag{11}
$$

The lengths L_1 and L_2 depend on the parameter R_{56} through the quantities D_j and D'_j. The aim of the study is to achieve R_{56} tuning that is to be able to vary this parameter between a minimum value (negative) $R_{56,min}$ and a maximum value (positive) $R_{56,max}$ without of course displacing the quadrupoles. Thus L_1 and L_2 are fixed and the normalized strengths k_1 and k_2 should be expressed as functions of R_{56} which implies to invert the system of the two equations (7). Unfortunately these are trascendental equations and no close form may be obtained for k_1 and k_2. However it can be shown that this is possible in the thin lens approximation that is for such a small l_q that the assumptions :

$$
\begin{aligned}
C_1 &= C_2 = 1 \\
S_1 &= l_q\sqrt{k_1} \\
S_2 &= l_q\sqrt{k_2}
\end{aligned}
$$

hold to a very good accuracy. Then the absolute values of the focal lengths $f_1 = l_q k_1$ and $f_2 = l_q k_2$ replace q_1 and q_2 respectively and the system (7) becomes :

$$
\begin{aligned}
L_1 &= a\frac{f_1}{f_2}\left(L_3 - \frac{D_j}{D'_j} + f_2\right) - l + f_1 \\
L_2 &= f_1 - f_2 + \frac{f_2\left(f_2 + \frac{f_1}{a}\right)}{L_3 - \frac{D_j}{D'_j} + f_2}
\end{aligned}
\tag{12}
$$

which can also be expanded in the form :

$$\begin{aligned}
&\frac{a+1}{a} f_1 f_2 + \left(L_3 - \frac{D_j}{D'_j}\right) f_1 - \frac{L_1+l}{a} f_2 = 0 \\
&\frac{a+1}{a} f_1 f_2 + \left(L_3 - \frac{D_j}{D'_j}\right) f_1 - \left(L_3 - \frac{D_j}{D'_j} + L_2\right) f_2 - L_2 \left(L_3 - \frac{D_j}{D'_j}\right) = 0
\end{aligned} \tag{13}$$

Substracting the two equations, f_2 can be obtained :

$$f_2 = \frac{a L_2 \left(L_3 - \frac{D_j}{D'_j}\right)}{L_1 + l - a\left(L_2 + L_3 - \frac{D_j}{D'_j}\right)} \tag{14}$$

Replacing this value in the first equation, f_1 is given by :

$$f_1 = \frac{L_2 (L_1 + l)}{L_2 + L_1 + l - a\left(L_3 - \frac{D_j}{D'_j}\right)} \tag{15}$$

By using the expressions (4),(5) and (9) the quantity $a\frac{D_j}{D'_j}$ becomes :

$$a\frac{D_j}{D'_j} = -\frac{1}{\sin\phi_1}\left[x \cot(\phi_2/2) + \rho_2\right] \tag{16}$$

Using this expression and the definition of a, the absolute values of the focal lengths can be written in the following compact form :

$$\begin{aligned}
f_1 &= \frac{\rho_2 L_2 (L_1 + l)\sin\phi_1}{x\mathcal{L}_3 - \rho_2^2 + \rho_2 (L_1 + L_2 + l)\sin\phi_1} \\
f_2 &= -L_2 \frac{x\mathcal{L}_3 - \rho_2^2}{x L_2 + x\mathcal{L}_3 - \rho_2^2 + \rho_2 (L_1 + l)\sin\phi_1}
\end{aligned} \tag{17}$$

where :

$$\mathcal{L}_3 = L_3 - \rho_2 \cot(\phi_2/2) \tag{18}$$

In order to design a R_{56} tunable module it is necessary to find the intervals of L_1, L_2, L_3 such that f_1 and f_2 remain positive when R_{56} varies in the interval $R_{56,min} < R_{56} < R_{56,max}$ with $R_{56,min} < 0$ and $R_{56,max} > 0$.
Let x_{min} and x_{max} be defined by :

$$\begin{aligned}
x_{min} &= \frac{R_{56,min}}{2} - l_m \left(\frac{3}{2} - \frac{\sin\phi_1}{\phi_1}\right) < 0 \\
x_{max} &= \frac{R_{56,max}}{2} - l_m \left(\frac{3}{2} - \frac{\sin\phi_1}{\phi_1}\right)
\end{aligned} \tag{19}$$

The valid ranges of the lengths L_1, L_2 and L_3 can be determined by eight sets of conditions. The algebra to obtain them is tedious and can be found in appendix A. Hereafter are summarised the results :

First set

$$\frac{\sin\phi_1}{\phi_1} + \frac{\tan(\phi_2/2)}{\phi_2} < \frac{3}{2}$$
$$x_{max} < -\rho_2 \tan(\phi_2/2)$$
$$L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$
$$-\mathcal{L}_3 - \frac{d\rho_2 \sin\phi_1}{x_{min}} < L_2 < -\mathcal{L}_3$$
$$L_1 < -d - \frac{x_{min}(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Second set

$$\frac{\sin\phi_1}{\phi_1} + \frac{\tan(\phi_2/2)}{\phi_2} < \frac{3}{2}$$
$$x_{max} < -\rho_2 \tan(\phi_2/2)$$
$$L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$
$$L_2 > -\mathcal{L}_3$$
$$L_1 < -d - \frac{x_{max}(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Third set

$$x_{max} < \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3$$
$$L_1 > -d - \frac{x_{max}(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Fourth set

$$x_{max} > \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1 - x_{min}}$$
$$L_1 > -d - L_2 - \mathcal{L}_3 \frac{x_{max}}{\rho_2 \sin\phi_1}$$

fifth set

$$x_{max} < \rho_2 \sin\phi_1$$

$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$

$$L_2 > -\mathcal{L}_3$$

$$L_1 > -d - (L_2 + \mathcal{L}_3)\,\frac{x_{min}}{\rho_2 \sin\phi_1}$$

sixth set

$$x_{max} > \rho_2 \sin\phi_1$$

$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$

$$L_2 > -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1 - x_{min}}$$

$$L_1 > -d - (L_2 + \mathcal{L}_3)\,\frac{x_{min}}{\rho_2 \sin\phi_1}$$

seventh set

$$x_{max} < 0$$

$$L_3 > \rho_2 \cot(\phi_2/2)$$

$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\, x_{min}}{\rho_2 \sin\phi_1}$$

eigth set

$$x_{max} > 0$$

$$\rho_2 \cot(\phi_2/2) < L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$

$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\, x_{min}}{\rho_2 \sin\phi_1}$$

3 Application to a CTF3 transfer line

The CTF3 transfer line between the Delay Loop and the Combiner Ring should be able to increase or decrease the bunch length by 1.6 mm. Given the $\Delta p/p$ of the order of 1 %, the range of R_{56} is between -0.16 m and 0.16 m. To accomodate this transfer line in a 'S' shape inside the available space, it is made of two insertions, one bending the beam by 75° and the other bending it back by $-75°$. The analytical approach has permitted an identification of the ranges of possible solutions without using numerical searches which are very unstable in this specific problem. Thus the insertion could be optimised to find a compromise between the overall length imposed by the building dimensions, and the optics (Twiss parameters). The most useful set of conditions in the design of this CTF3 transfer line has been the third. The three dipoles of the selected insertion have the same length (0.4 m) and generate the same beam deflection (25°). The drift lengths are $L_1 = 1.2$ m, $L_2 = 0.6$ m and $L_3 = 1.55$ m. All the quadrupoles have the same length of 0.2 m.

The Figures 1, 2 and 3 show the optical functions of the full insertion when the R_{56} parameter of half one single insertion is -0.04 m, 0 m, 0.04 m respectively.

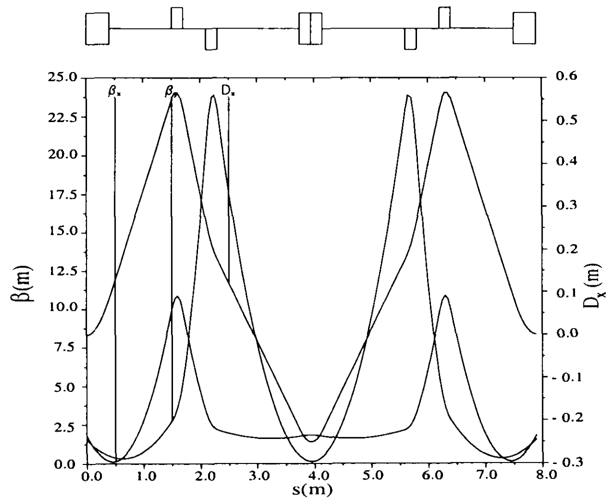

Figure 1: Optical functions for $R_{56} = -0.04$ m.

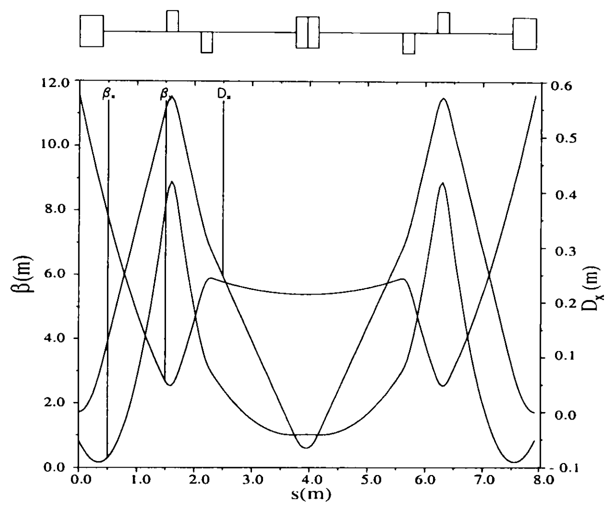

Figure 2: Optical functions for $R_{56} = 0$ m.

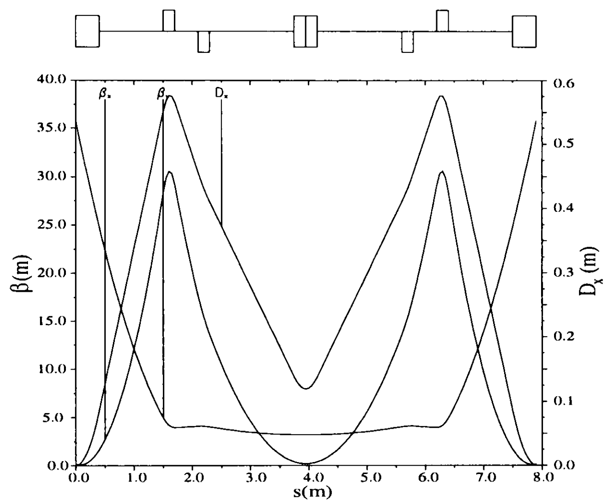

Figure 3: Optical functions for $R_{56} = 0.04$ m.

For a beam energy of 400 MeV, the gradients of the first and second quadrupoles vary between 12.04 T/m and 7.81 T/m, and between 12.13 T/m and 1.29 T/m respectively. They are shown in Figure 4.

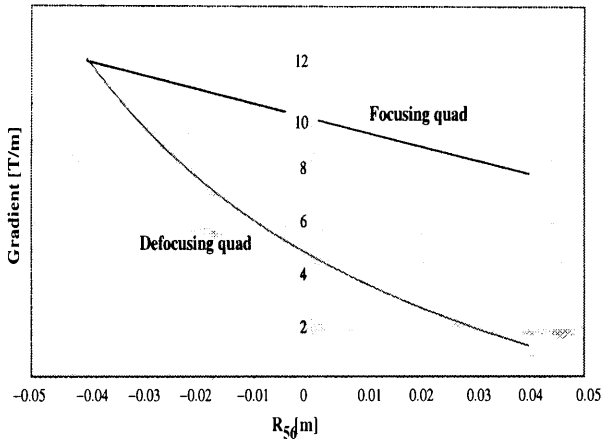

Figure 4: Quadrupole gradients at 400 MeV

4 Concluding remarks

The method described above is a very powerful tool to design a transfer line which is able either to compress or stretch longitudinally the beam in a given range. The drift lengths can be adjusted to fit a given geometry and optimised in order to obtain the best Twiss parameters at both ends of the insertion. Presently this optimisation is done interactively in one of the eight Excel programs corresponding to each set of conditions. In the future it is intended to automate the processing to speed it up. The matching triplets have also to be treated because the changes in R_{56} induce changes in the Twiss parameters and the matching should be modified accordingly, but this does not seem to be a difficult problem. The experience gathered until now has shown that the passage to thick lenses can be handled easily by a standard accelerator program like MAD. Here again the procedure should be automated. Finally it should be stressed that this method is also very valuable to obtain and control isochronicity and to avoid marginal configurations.

References

[1] CLIC Study Team, Proposal for Future CLIC Studies and a New CLIC Test Facility (CTF3), CLIC Note 402, Geneva (1999).

[2] T.E. d'Amico, G. Guignard, 'First-order design of a new type of isochronous arc', CERN SL/95-120, 1995.

A Derivation of the permitted ranges of the lengths L_1, L_2 and L_3

The aim of this appendix is to find all the valid ranges of the lengths L_1, L_2 and L_3 for which the absolute values of the focal lengths f_1 and f_2 become positive in a given range of R_{56}.
Let us start from the expressions of f_1 and f_2 (17) :

$$\begin{aligned} f_1 &= \frac{L_2\,(L_1+l)\,\rho_2 \sin\phi_1}{x\mathcal{L}_3 - \rho_2^2 + \rho_2\,(L_1+L_2+l)\sin\phi_1} \\ f_2 &= -L_2 \frac{x\mathcal{L}_3 - \rho_2^2}{xL_2 + x\mathcal{L}_3 - \rho_2^2 + \rho_2\,(L_1+l)\sin\phi_1} \end{aligned} \tag{20}$$

or in a more compact form :

$$\begin{aligned} f_1 &= \frac{L_2\,(L_1+l)\,\rho_2 \sin\phi_1}{\mathcal{L}_3\,(x-x_1)} \\ f_2 &= -L_2 \frac{x\mathcal{L}_3 - \rho_2^2}{(L_2+\mathcal{L}_3)\,(x-x_2)} \end{aligned} \tag{21}$$

where :

$$\begin{aligned} x_1 &= -\rho_2 \sin\phi_1 \frac{L_1+L_2+d}{\mathcal{L}_3} \\ x_2 &= -\rho_2 \sin\phi_1 \frac{L_1+d}{L_2+\mathcal{L}_3} \\ d &= l - \frac{\rho_2}{\sin\phi_1} \end{aligned}$$

It is shown in Appendix B that d is always negative for a total deflection angle of the insertion $\phi < 4$ rad.
For the sake of simplicity ϕ is assumed to be $\phi < \pi$ covering most of the practical insertion designs.
Let us distinguish the two cases $x\mathcal{L}_3 > \rho_2^2$ and $x\mathcal{L}_3 < \rho_2^2$.

1. $x\mathcal{L}_3 > \rho_2^2$

In this case the expressions (20) show that f_1 is always positive and that f_2 is positive if the denominator of its expression is negative. This is possible only if $x < 0$ implying $x_{max} < 0$. Thus :

$$\mathcal{L}_3 < \frac{\rho_2^2}{x_{max}}$$

By definition L_3 should be positive, giving :

$$x_{max} < -\rho_2 \tan(\phi_2/2)$$

But $R_{56,max}$ is also assumed to be positive which implies :

$$\frac{3}{2} - \frac{\sin\phi_1}{\phi_1} - \frac{\tan(\phi_2/2)}{\phi_2} > 0$$

Let us now find the conditions which ensure that the denominator of f_2 is negative.

If $L_2 + \mathcal{L}_3 < 0$, x should be larger than x_2 implying :

$$x_{min} > -\rho_2 \sin\phi_1 \frac{L_1 + d}{L_2 + \mathcal{L}_3}$$

which provides an upper bound for L_1 :

$$L_1 < -d - \frac{x_{min}\,(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

including the inequality $L_1 < -d$ which is required to ensure that $x_{min} < 0$. By definition L_1 should be positive providing a lower bound for L_2 :

$$L_2 > -\mathcal{L}_3 - \frac{d\rho_2 \sin\phi_1}{x_{min}}$$

Summarising, a first set of conditions is obtained :

First set

$$\frac{\sin\phi_1}{\phi_1} + \frac{\tan(\phi_2/2)}{\phi_2} < \frac{3}{2}$$

$$x_{max} < -\rho_2 \tan(\phi_2/2)$$

$$L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$

$$-\mathcal{L}_3 - \frac{d\rho_2 \sin\phi_1}{x_{min}} < L_2 < -\mathcal{L}_3$$

$$L_1 < -d - \frac{x_{min}\,(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

If $L_2 + \mathcal{L}_3 > 0$, x should be smaller than x_2 implying :

$$x_{max} < -\rho_2 \sin\phi_1 \frac{L_1 + d}{L_2 + \mathcal{L}_3}$$

which provides an upper bound for L_1 :

$$L_1 < -d - \frac{x_{max}\,(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Summarising, another set of conditions is obtained :

Second set

$$\frac{\sin\phi_1}{\phi_1} + \frac{\tan(\phi_2/2)}{\phi_2} < \frac{3}{2}$$

$$x_{max} < -\rho_2 \tan(\phi_2/2)$$

$$L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$

$$L_2 > -\mathcal{L}_3$$

$$L_1 < -d - \frac{x_{max}\,(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

2. $x\mathcal{L}_3 < \rho_2^2$

In this case the expressions for f_1 and f_2 are positive if their denominators are also positive.

Let us compute the difference $x_2 - x_1$:

$$x_2 - x_1 = \frac{\rho_2 L_2 \sin\phi_1}{\mathcal{L}_3 (L_2 + \mathcal{L}_3)} (d + L_1 + L_2 + \mathcal{L}_3)$$

Let us treat in turn the three cases depending upon the signs of $\mathcal{L}_3$ and $L_2 + \mathcal{L}_3$.

First case : $\mathcal{L}_3 < 0$ and $L_2 + \mathcal{L}_3 < 0$

This case implies the following inequalities :

$$\frac{\rho_2^2}{x_{min}} < \mathcal{L}_3 < 0$$
$$x < x_1$$
$$x < x_2$$

Let us observe that :

$$\begin{aligned} x_2 < x_1 \quad &\text{if} \quad L_1 < -d - L_2 - \mathcal{L}_3 \\ x_1 < x_2 \quad &\text{if} \quad L_1 > -d - L_2 - \mathcal{L}_3 \end{aligned} \tag{22}$$

Let us study the first inequality. It is evident that L_1 is positive only if :

$$L_2 < -\mathcal{L}_3 - d$$

which is included in the inequality $L_2 < -\mathcal{L}_3$. Thus $x_{max} < x_2$ implying :

$$L_1 > -d - \frac{x_{max} (L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

The existence of L_1 provides the following inequality :

$$-\frac{x_{max} (L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1} < -L_2 - \mathcal{L}_3$$

and by dividing both terms by the positive quantity $-L_2 - \mathcal{L}_3$, it gives :

$$x_{max} < \rho_2 \sin\phi_1$$

Thus another set of conditions is obtained :

Third set

$$x_{max} < \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3$$
$$-d - \frac{x_{max} (L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1} < L_1 < -d - L_2 - \mathcal{L}_3$$

Let us study the second inequality of (22). Thus $x_{max} < x_1$ implying :

$$L_1 > -d - L_2 - \mathcal{L}_3 \frac{x_{max}}{\rho_2 \sin \phi_1}$$

This lower bound of L_1 is smaller than that given by (22) if $x_{max} < \rho_2 \sin \phi_1$. Thus two new sets of conditions are obtained :

Fourth set

$$x_{max} < \rho_2 \sin \phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3$$
$$L_1 > -d - L_2 - \mathcal{L}_3$$

Fifth set

$$x_{max} > \rho_2 \sin \phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3$$
$$L_1 > -d - L_2 - \mathcal{L}_3 \frac{x_{max}}{\rho_2 \sin \phi_1}$$

Second case :	$\mathcal{L}_3 < 0$	and	$L_2 + \mathcal{L}_3 > 0$

This case implies the following inequalities :

$$\begin{aligned} \frac{\rho_2^2}{x_{min}} &< \mathcal{L}_3 < 0 \\ x &< x_1 \\ x &> x_2 \end{aligned} \tag{23}$$

Thus x_1 must be larger than x_2 which is possible only if :

$$L_1 > -L_2 - \mathcal{L}_3 - d \tag{24}$$

The third inequality of (23) necessitates $x_{min} > x_2$ which ,expanded, gives :

$$L_1 > -d - (L_2 + \mathcal{L}_3) \frac{x_{min}}{\rho_2 \sin \phi_1} \tag{25}$$

x_2 must be negative to comply with the definition of x_{min} implying $L_1 > -d$, which is included in the inequality (25) together with the inequality (24). The second inequality of (23) necessitates $x_{max} < x_1$ which ,expanded, gives :

$$L_1 > -d - L_2 - \mathcal{L}_3 \frac{x_{max}}{\rho_2 \sin \phi_1} \tag{26}$$

This lower bound of L_1 is larger than the lower bound provided by (25) if :

$$-\mathcal{L}_3 < L_2 < -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1 - x_{min}} \tag{27}$$

which is possible only if $x_{max} > \rho_2 \sin\phi_1$. Thus three new sets of conditions are obtained :

sixth set

$$\begin{gathered} x_{max} < \rho_2 \sin\phi_1 \\ \frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2) \\ L_2 > -\mathcal{L}_3 \\ L_1 > -d - (L_2 + \mathcal{L}_3) \frac{x_{min}}{\rho_2 \sin\phi_1 - x_{min}} \end{gathered}$$

seventh set

$$\begin{gathered} x_{max} > \rho_2 \sin\phi_1 \\ \frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2) \\ -\mathcal{L}_3 < L_2 < -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1 - x_{min}} \\ L_1 > -d - L_2 - \mathcal{L}_3 \frac{x_{max}}{\rho_2 \sin\phi_1} \end{gathered}$$

eigth set

$$\begin{gathered} x_{max} > \rho_2 \sin\phi_1 \\ \frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2) \\ L_2 > -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1} \\ L_1 > -d - (L_2 + \mathcal{L}_3) \frac{x_{min}}{\rho_2 \sin\phi_1 - x_{min}} \end{gathered}$$

Third case : $\mathcal{L}_3 > 0$ and $L_2 + \mathcal{L}_3 > 0$

This case implies the following inequalities :

$$\begin{gathered} x > x_1 \\ x > x_2 \end{gathered} \tag{28}$$

Let us observe that :

$$\begin{aligned} x_2 < x_1 \quad &\text{if} \quad L_1 < -d - L_2 - \mathcal{L}_3 \\ x_1 < x_2 \quad &\text{if} \quad L_1 > -d - L_2 - \mathcal{L}_3 \end{aligned} \tag{29}$$

Let us study the first inequality where it is evident that L_1 is positive only if :

$$L_2 < -d - \mathcal{L}_3$$

Then the condition $x_{min} > x_1$ must hold implying $x_1 < 0$ to comply with the definition of x_{min}. This is possible only if :

$$L_1 > -d - L_2$$

This inequality contradicts the first inequality of (29) and thus no valid range exits for L_1.
Let us study the second inequality of (29). Thus $x_{min} > x_2$ implying :

$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\, x_{min}}{\rho_2 \sin \phi_1}$$

which includes both $L_1 > -d$ required because x_{min} must be negative and the second inequality of (29).
The inequality $x\mathcal{L}_3 < \rho_2^2$ generates two cases :

$$\begin{array}{ccc} \mathcal{L}_3 > 0 & \text{for} & x_{max} < 0 \\ \text{or} & & \\ 0 < \mathcal{L}_3 < \dfrac{\rho_2^2}{x_{max}} & \text{for} & x_{max} > 0 \end{array} \tag{30}$$

Thus the two last sets of conditions are obtained :

ninth set

$$x_{max} < 0$$
$$L_3 > \rho_2 \cot(\phi_2/2)$$
$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\, x_{min}}{\rho_2 \sin \phi_1}$$

tenth set

$$x_{max} > 0$$
$$\rho_2 \cot(\phi_2/2) < L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$
$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\, x_{min}}{\rho_2 \sin \phi_1}$$

Let us observe that the third and the fourth sets of conditions can be combined because they differ only by the fourth inequalities which are complementary. Similarly the fifth and the seventh sets can be combined because they differ only by the third inequalities also complementary. Thus the number of valid sets of conditions is reduced to eight and they are listed below :

First set

$$\frac{\sin\phi_1}{\phi_1} + \frac{\tan(\phi_2/2)}{\phi_2} < \frac{3}{2}$$
$$x_{max} < -\rho_2 \tan(\phi_2/2)$$
$$L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$
$$-\mathcal{L}_3 - \frac{d\rho_2 \sin\phi_1}{x_{min}} < L_2 < -\mathcal{L}_3$$
$$L_1 < -d - \frac{x_{min}(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Second set

$$\frac{\sin\phi_1}{\phi_1} + \frac{\tan(\phi_2/2)}{\phi_2} < \frac{3}{2}$$
$$x_{max} < -\rho_2 \tan(\phi_2/2)$$
$$L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$
$$L_2 > -\mathcal{L}_3$$
$$L_1 < -d - \frac{x_{max}(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Third set

$$x_{max} < \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3$$
$$L_1 > -d - \frac{x_{max}(L_2 + \mathcal{L}_3)}{\rho_2 \sin\phi_1}$$

Fourth set

$$x_{max} > \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 < -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1 - x_{min}}$$
$$L_1 > -d - L_2 - \mathcal{L}_3 \frac{x_{max}}{\rho_2 \sin\phi_1}$$

fifth set

$$x_{max} < \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 > -\mathcal{L}_3$$
$$L_1 > -d - (L_2 + \mathcal{L}_3)\frac{x_{min}}{\rho_2 \sin\phi_1}$$

sixth set

$$x_{max} > \rho_2 \sin\phi_1$$
$$\frac{\rho_2^2}{x_{min}} + \rho_2 \cot(\phi_2/2) < L_3 < \rho_2 \cot(\phi_2/2)$$
$$L_2 > -\mathcal{L}_3 \frac{x_{max} - x_{min}}{\rho_2 \sin\phi_1 - x_{min}}$$
$$L_1 > -d - (L_2 + \mathcal{L}_3)\frac{x_{min}}{\rho_2 \sin\phi_1}$$

seventh set

$$x_{max} < 0$$
$$L_3 > \rho_2 \cot(\phi_2/2)$$
$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\,x_{min}}{\rho_2 \sin\phi_1}$$

eigth set

$$x_{max} > 0$$
$$\rho_2 \cot(\phi_2/2) < L_3 < \frac{\rho_2^2}{x_{max}} + \rho_2 \cot(\phi_2/2)$$
$$L_1 > -d - \frac{(L_2 + \mathcal{L}_3)\,x_{min}}{\rho_2 \sin\phi_1}$$

B To prove that d is negative for a total deflection angle of the insertion less than π.

The expression of d can be written under the form :

$$d = l - \frac{\rho_2}{\sin\phi_1} = \frac{\rho_2}{\sin\phi_1}\left(\frac{l\sin\phi_1}{\rho_2} - 1\right)$$

Thus :

$$d < 0 \qquad \text{if} \qquad \frac{l\sin\phi_1}{\rho_2} < 1$$

Recalling the definition of l, this expression becomes :

$$\frac{l\sin\phi_1}{\rho_2} = 2\frac{\phi_2}{\phi_1}\sin^2\frac{\phi_1}{2} = 2\frac{\phi - 2\phi_1}{\phi_1}\sin^2\frac{\phi_1}{2} = \frac{\phi_1}{2}(\phi - 2\phi_1)\left[\frac{\sin\frac{\phi_1}{2}}{\frac{\phi_1}{2}}\right]^2$$

An upper bound of $\frac{l\sin\phi_1}{\rho_2}$ is given by :

$$\frac{l\sin\phi_1}{\rho_2} < \frac{\phi_1}{2}(\phi - 2\phi_1)$$

because $\frac{\sin\frac{\phi_1}{2}}{\frac{\phi_1}{2}}$ is always less than 1. The second-order polynomial in ϕ_1 reaches a maximum of $\phi^2/16$ for $\phi_1 = \phi/4$. Thus $\frac{l\sin\phi_1}{\rho_2} < 1$ for $\phi < 4$ and of course also for $\phi < \pi$.