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BUNCHES WITH LOCAL ELLIPTIC ENERGY DISTRIBUTION

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Introduction and Summary

This distribution fits well with distributions observed in proton synchrotrons and makes several analytical calculations for bunched beams in longitudinal phase space possible. For any shape of the focusing force the line density becomes proportional to the potential well. Self-forces caused by space-charge and inductive wall impedances are thus proportional to the external force, making calculation of bucket area reduction and bunch lengthening easy. The microwave instability threshold, as given by the Keil-Schnell criterion with local values for current and energy spread, is independent of the azimuthal position along the bunch, and again analytical formulae are possible even for strongly non-linear focusing forces. The relative magnitude of the self-force and the microwave threshold turn out to be closely related, as the self-force is always 40% of the external force when the microwave threshold is reached. The classical longitudinal space-charge limit can therefore only be reached within a factor of 0.4. Other calculations with this "natural" distribution include analytical formulae for the rigid dipole mode threshold, and creation of flat-topped bunches with reduced peak line density resulting in a higher transverse space-charge limit.

Synchrotron equation with an arbitrary waveshape

The synchrotron equations

$$\frac{dW}{dt} = -\frac{\partial H}{\partial \phi} = \frac{e}{2\pi}\left[V(\phi) - V_0\right] \qquad (1)$$

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} = \frac{h\omega_0^2\eta}{\beta^2 E_s} W \qquad (2)$$

can be derived from the Hamiltonian

$$H = \frac{h\omega_0^2\eta}{2\beta^2 E_s} W^2 - \frac{e}{2\pi} U(\phi) , \qquad (3)$$

where $U(\phi)$ is the potential

$$U(\phi) = \int_{\phi_s}^{\phi} \left[V(\phi) - V_0\right] d\phi = \int_{\phi_s}^{\phi} V(\phi) d\phi - V_0(\phi - \phi_s) , \qquad (4)$$

where $W = \Delta E/\omega_0 = (E - E_s)/\omega_0$; $\eta = 1/\gamma_t^2 - 1/\gamma^2 = -(df/f)/(dp/p)$; eV_0 is the energy gain per turn of the synchronous particle; E_s, ω_0, and ϕ_s are energy, revolution frequency, and phase of the synchronous particle; $V(\phi)$ is the accelerating waveshape, which has zero mean and periodicity 2π; h is the harmonic number. $U(\phi)$ has been chosen so that the Hamiltonian of the synchronous particle is zero. There is area conservation in the (W,ϕ) phase plane.

The local elliptic energy distribution

The Hamiltonian being a constant of motion, a necessary and sufficient condition for a stationary (=time invariant) particle distribution is that the phase-space density $g(W,\phi)$ can be written as a function of the Hamiltonian. If we choose

$$g(W,\phi) = \frac{d^2N}{dWd\phi} = g(H) = c_1\sqrt{H_b - H} , \qquad (5)$$

where H_b is the Hamiltonian of the extreme (= boundary) particle, we get as function of energy,

$$g(W,\phi) = c_2\sqrt{W_b^2(\phi) - W^2} = \frac{c_2}{\omega_0}\sqrt{\Delta E_b^2(\phi) - \Delta E^2} , \qquad (6)$$

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where $W_b(\phi)$ or $E_b(\phi)$ is the bunch boundary in phase space. For any value of ϕ, the *density is an elliptic function of energy.* The line density:

$$\lambda(\phi) = \frac{dN}{d\phi} = \int g(W,\phi)dW = c_3\left[U(\phi) - U(\phi_2)\right] \qquad (7)$$

has the *same shape as the potential,* Fig. 1; $U(\phi_2)$ is the potential at one end of the bunch, and c_2 through c_3 are constants.

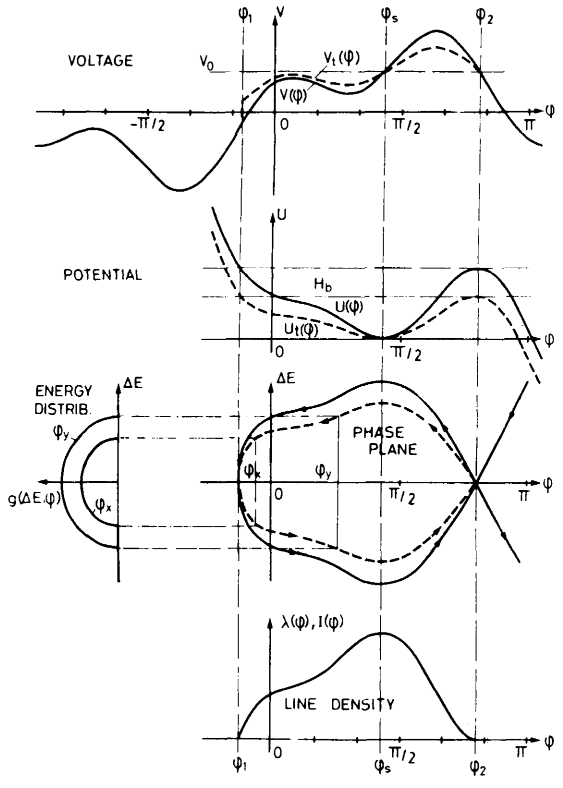

Fig. 1 Bunch with elliptic energy distribution

Space-charge and inductive wall effects

At low frequencies (long bunches) the effective coupling impedance including space-charge forces is mostly reactive[1]

$$\frac{Z_e}{n} = j\left(\omega_0 L - \frac{g_0 Z_0}{2\beta\gamma^2}\right) = j\omega_0 L_e , \qquad (8)$$

where $n = \omega/\omega_0$ and L_e is the effective inductance. The space-charge force is thus equivalent to a negative, energy-dependent wall inductance. The induced voltage is therefore proportional to the derivative of the local current I, which for a bunch extending from ϕ_1 to ϕ_2 with N_b particles per bunch is

$$\lambda(\phi) = \frac{dN}{d\phi} = N_b \frac{U(\phi) - U(\phi_2)}{u(\phi_1,\phi_2)} \qquad (9)$$

$$u(\phi_1,\phi_2) = \int_{\phi_1}^{\phi_2} \left[U(\phi) - U(\phi_2)\right] d\phi , \qquad \phi_1 < \phi_2 \qquad (10)$$

$$I(\phi) = e\frac{dN}{d\phi}\frac{d\phi}{dt} = 2\pi h I_b \frac{U(\phi) - U(\phi_2)}{u(\phi_1,\phi_2)} \qquad (11)$$

where $I_b = eN_b\omega_0/2\pi$ is the mean current per bunch. The total voltage $V_t(\phi)$ becomes

$$V_t(\phi) = \begin{cases} V(\phi) - 2\pi h^2 I_b \,\mathrm{Im}\{Z_e/n\} \dfrac{V(\phi) - V_0}{u(\phi_1,\phi_2)}, & \phi_1 < \phi < \phi_2, \\ V(\phi), \text{ elsewhere} & \end{cases} \qquad (12)$$

The induced voltage has thus the same shape as the applied voltage.

Below transition $\eta < 0$ and $u(\phi_1,\phi_2) < 0$, so a dominating space-charge force ($\mathrm{Im}\{Z_e/n\} < 0$) implies a reduced focusing voltage, while above transition $u(\phi_1,\phi_2) > 0$, so a dominating inductive wall impedance will result in reduced focusing.

The area of a full bucket is changed relative to its low intensity value A_0 by the square root of the relative change k_t in total focusing voltage:

$$A = A_0\sqrt{k_t} = A_0\sqrt{\frac{V_t - V_0}{V - V_0}} = A_0\sqrt{1 - \frac{2\pi h^2 I_b \,\mathrm{Im}\{Z_e/n\}}{u(\phi_1,\phi_2)}} \qquad (13)$$

where the bunch boundaries in this case are the bucket boundaries. For the limiting intensity[1,2]:

$$I_{b,max} = \frac{u(\phi_1,\phi_2)}{2\pi h^2 \,\mathrm{Im}\{Z_e/n\}}, \qquad (14)$$

the induced voltage cancels the applied voltage, the bucket area is reduced to zero, and the required phase-space density is infinite. As we will see in the following section, this limit can only be reached within a certain factor for stability reasons.

Microwave instabilities

High-frequency instabilities within a bunch can occur if the coupling impedance exceeds the value given by the Boussard criterion[3,4]:

$$\frac{|Z_e|}{n} \le F\,\frac{E|\eta|}{e\beta^2}\,\frac{[\Delta E(\phi)/E]^2}{I(\phi)} \qquad (15)$$

which is the coasting beam criterion[5] with local values for energy spread and current. F is a form factor and $\Delta E(\phi)$ the full spread at half height, FWHH.

For the elliptic distribution, FWHH is $\sqrt{3}$ times the energy boundary $E_b(\phi)$, which can be found from the Hamiltonian with the total potential $U_t(\phi) = k_t U(\phi)$:

$$\left[\frac{\Delta E(\phi)}{E}\right]^2_{FWHH} = \frac{3\Delta E_b^2}{E^2} = \frac{3\beta^2 e}{\pi h\eta E}\, k_t\,[U(\phi) - U(\phi_2)], \qquad (16)$$

so the ratio between local current (11) and energy spread squared is *constant along the bunch* and we get the threshold current:

$$I_b \le \frac{3Fk_t|u(\phi_1,\phi_2)|}{2\pi^2 h^2 |Z_e|/n}. \qquad (17)$$

From (13) we then get the *voltage reduction at the microwave threshold:*

$$k_t = \frac{V_t - V_0}{V - V_0} = 1\Big/\left[1 \pm \frac{3F\,\mathrm{Im}\{Z_e/n\}}{\pi|Z_e|/n}\right] \quad \begin{matrix} + \text{ for } \gamma > \gamma_t \\ - \text{ for } \gamma < \gamma_t \end{matrix} \qquad (18)$$

$\mathrm{Im}\{Z_e/n\}$ is the reactive part at low frequencies while $|Z_e|/n$ is the magnitude of the impedance at high frequencies corresponding to many wavelength within the bunch. For not too short bunches and well-damped resonance these can be considered of equal magnitude, so for defocusing self-forces we get:

$$k_t = 1/(1 + \frac{3F}{\pi}) = 0.6 \qquad (19)$$

$$I_b \le \frac{|u(\phi_1,\phi_2)|}{5\pi h^2|Z_e|/n} = 0.4\, I_{b,max}, \qquad (20)$$

where $F \simeq 0.7$ is based on the stability diagram of a coasting beam with the same distribution[4,6].

As $|\mathrm{Im}\{Z_e/n\}| \le |Z_e|/n$, the induced voltage V_i will *never exceed 40%* of the applied voltage, as microwave instabilities would otherwise blow up the bunch area and thus reduce the induced voltage.

As for short bunches $\phi_\ell = \phi_2 - \phi_1 \propto 1/\sqrt[4]{V_t}$ for protons (A constant) and $\phi_\ell \propto 1/\sqrt{V_t}$ for electrons (ΔE constant), *bunch lengthening due to potential well reduction can never exceed 14% for protons and 29% for electrons,* as $1/\sqrt[4]{0.6} = 1.14$ and $1/\sqrt{0.6} = 1.29$. This is in good agreement with experimental results (fig. 1 of ref. 4).

The bucket area reduction by space-charge ($\gamma < \gamma_t$) or inductive wall ($\gamma > \gamma_t$) will never exceed 23%, as $\sqrt{0.6} = 0.77$. In other words, the *space-charge limit (14)* can only be reached *within a factor of 0.4* in reasonable agreement with experiments (fig. 7 of ref. 1).

Sinusoidal voltage

For a sinusoidal voltage,

$$V(\phi) = V_1 \sin\phi, \quad V_0 = V_1 \sin\phi_s = V_1\Gamma, \qquad (21)$$

the normalizing integral $u(\phi_1,\phi_2)$ is

$$u(\phi_1,\phi_2) = \int_{\phi_1}^{\phi_2} [U(\phi) - U(\phi_2)]\, d\phi = -V_1 f(\phi_1,\phi_2) \qquad (22)$$

$$f(\phi_1,\phi_2) = \sin\phi_2 - \sin\phi_1 - \frac{1}{2}(\phi_2 - \phi_1)(\cos\phi_1 + \cos\phi_2), \qquad (23)$$

and the line density and relative focusing k_t:

$$\lambda(\phi) = N_b\,\frac{\cos\phi + \phi\sin\phi_s - \cos\phi_2 - \phi_2\sin\phi_s}{f(\phi_1,\phi_2)}$$

$$k_t = \frac{V_t - V_0}{V - V_0} = 1 + \frac{2\pi h^2 I_b\,\mathrm{Im}\{Z_e/n\}}{V_1 f(\phi_1,\phi_2)}, \qquad (24)$$

which with ϕ_1 and ϕ_2 being bucket limits gives the bucket area,

$$A = A_0\sqrt{1 + \frac{2\pi h^2 I_b\,\mathrm{Im}\{Z_e/n\}}{V_1 f(\phi_1,\phi_2)}}, \qquad (25)$$

where $f(\phi_1,\phi_2) > 0$ for $\gamma < \gamma_t$, and $f(\phi_1,\phi_2) < 0$ for $\gamma > \gamma_t$, resulting in a positive $\lambda(\phi)$ for $\phi_1 < \phi < \phi_2$ in both cases as required. The bucket area (25) agrees with Bigliani's[7] formula for the $\phi_s = 0$ case, as the distributions are identical ($\cos^2$), and agrees also rather well with numerical calculations assuming constant density in phase space[2,8].

For defocusing self-forces and $|\mathrm{Im}\{Z_e/n\}| \simeq |Z_e|/n$ the microwave threshold is

$$I_b \le F\frac{3k_t V_1|f(\phi_1,\phi_2)|}{2\pi^2 h^2|Z_e|/n} \simeq \frac{V_1|f(\phi_1,\phi_2)|}{5\pi h^2|Z_e|/n} = 0.4\, I_{b,max}. \qquad (26)$$

Short bunches

In the limit of short bunches, the focusing force and the self-force are linear; the potential and the line density are parabolic. The formulae can be obtained by expanding $f(\phi_1,\phi_2)$ in $\phi_\ell = \phi_2 - \phi_1$ (in radians):

$$f(\phi_1,\phi_2) = (\phi_\ell^3 \cos\phi_s)/12.$$

Relative voltage and microwave threshold:

$$k_t = \frac{V_t - V_0}{V - V_0} = 1 + \frac{24\pi h^2 I_b\,\mathrm{Im}\{Z_e/n\}}{\phi_\ell^3 V_1 \cos\phi_s} \qquad (28)$$

$$I_b \le \frac{\phi_\ell^3 V_1 \cos\phi_s}{60\pi h^2|Z_e|/n}. \qquad (29)$$

Rigid dipole mode threshold

The net force dF acting on a bunch rigidly displaced $d\phi$, is the integral of the product of the voltage $V(\phi) - V_0$ and the line density of the displaced bunch $\lambda(\phi + d\phi)$.

Relative to the focusing force for an infinite short bunch with the same charge, we get for sinusoidal voltage

$$k_c = \frac{(\phi_2 - \phi_1)\cos 2\phi_s - \frac{1}{2}(\sin 2\phi_2 - \sin 2\phi_1)}{2\, f(\phi_1,\phi_2)\cos\phi_s} . \qquad (30)$$

The coherent rigid dipole mode frequency is $\omega_c = \omega_{s0}\sqrt{k_c}$, where ω_{s0} is the small-amplitude, zero-intensity synchrotron frequency, while the small-amplitude incoherent frequency ω_i is changed as the square root of the relative change in total focusing voltage, $\omega_i = \omega_{s0}\sqrt{k_t}$, and is thus intensity dependent.

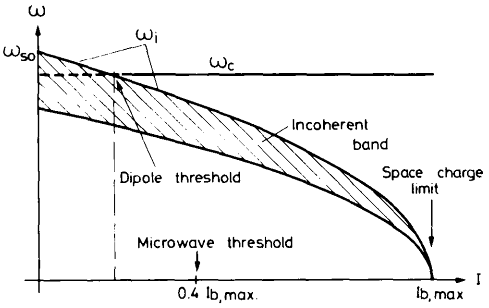

Fig. 2 Rigid dipole mode threshold

Landau damping is lost when the coherent frequency is outside the band of incoherent frequencies. The threshold is given by $k_c = k_t$, so for sinusoidal voltage we get

$$I_b = \frac{V_1}{2\pi h^2\, \mathrm{Im}\{Z_e/n\}} \left\{ \frac{1}{2\cos\phi_s}\left[(\phi_2 - \phi_1)\cos 2\phi_s - \frac{1}{2}\left(\sin 2\phi_2 - \sin 2\phi_1\right)\right] - \sin\phi_2 + \sin\phi_1 + \frac{1}{2}(\phi_2 - \phi_1)(\cos\phi_1 + \cos\phi_2)\right\} \qquad (31)$$

which for stationary buckets ($\phi_s = 0$ or π) reduces to

$$I_b \le \pm \frac{V_1}{2\pi h^2\, \mathrm{Im}\{Z_e/n\}} \left(\frac{1}{2}\sin\phi_\ell - \frac{\phi_\ell}{2} + 2\sin\frac{\phi_\ell}{2} - \phi_\ell\cos\frac{\phi_\ell}{2}\right). \qquad (32)$$

For short bunches k_c and k_t may be expanded in ϕ_ℓ:

$$k_c = 1 - \phi_\ell^2/(40\cos^2\phi_s) \qquad (33)$$

$$I_b \le -\frac{V_1}{2\pi h^2\, \mathrm{Im}\{Z_e/n\}} \times \frac{\phi_\ell^5}{480\cos\phi_s} . \qquad (34)$$

The crucial point in this derivation is the assumption of a rigid motion. The threshold obtained from a dispersion relation[9,10] for the same distribution is 20% lower than given by the short-bunch formula above. The long-bunch formulae (31) and (32) have been verified experimentally in the PS Booster[11], where the dominating space-charge impedance can be calculated with good accuracy.

Flat-topped bunches

By subtracting two elliptic distributions corresponding to different bunch lengths, flat-topped bunches can be created, the net phase space density being positive as required, fig. 3. The advantage of this line density is a higher average current for a given peak current and bunch length, and thus an increased transverse space charge limit in terms of average current.

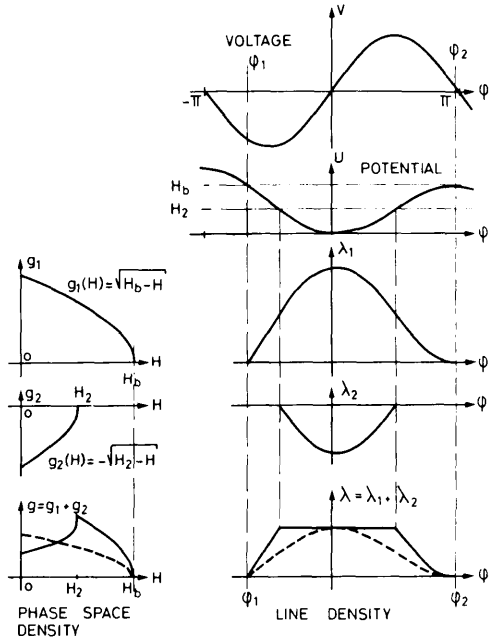

Fig. 3 Flat-topped bunch

To obtain this bunch shape, a double peaked energy distribution with a reduced central density has to be created prior to trapping.

Alternatively, flat-topped bunches can be created by a flat-bottomed potential well obtained by means of a higher harmonic cavity[4]. Both means are being pursued in the PS Booster. Theoretically each method could increase the intensity by 30-35% and combined 55-60%.

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The invariance of the Boussard criterion along a parabolic bunch has been known for some time[3,12]. Later this was shown to be the case also for the elliptic distribution in a full bucket of a sinusoidal waveshape[13].

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