Cours/Lecture Series

1992 - 1993 ACADEMIC TRAINING PROGRAMME

LECTURE SERIES

SPEAKER : Michael BERRY / University of Bristol
TITLE : Geometric Phases
TIME : 8, 9, 10 February, 11.00 to 12.00 hrs
PLACE : Auditorium

ABSTRACT

An elementary account will be given of the mathematical phenomenon of "global change without local change", i.e. anholonomy, applied to phases in quantum mechanics and angles in classical mechanics. Phases and angles in photons, electrons, and molecules will be discussed, with a historical emphasis. More advanced topics may include a detailed illustration of the vector and scalar gauge forces associated with the geometric reaction of a light system or a heavy one (modern Born-Oppenheimer theory) and high-order adiabatic corrections to the geometric phase and its relation to recent developments in asymptotics.

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Some geometric phases

transparencies for three lectures at CERN

February 1993

Michael Berry

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

Some references

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Lecture 1

(Some geometric phases, Michael Berry Geneva 1993)

SOME GEOMETRIC PHASES

Three central ideas

1) Adiabaticity

Mechanical phenomena on the border between statics and dynamics

study of things, i.e. states that do not change; in quantumspeak: eigenstates of Hamiltonian operators

study of happenings: evolving states, changing external fields

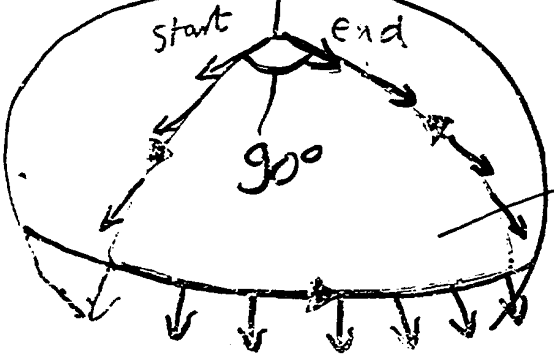
Border (the adiabatic regime): study of $\boxed{\text{slow changes}}$

2. Anholonomy

global change without local change

In geometry: when some quantities are varied round a cycle, others, dependent on them*, fail to return. e.g. parallel transport of vectors on a sphere

*and have no local rate of change



solid angle

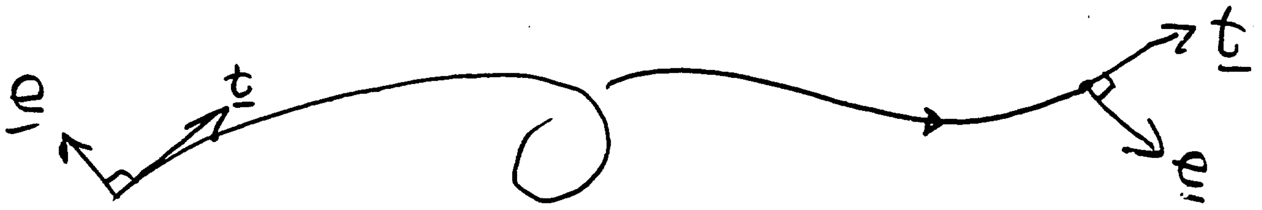
$\Omega = \frac{1}{8}$ sphere $= \frac{1}{8} 4\pi$

$= \pi/2$

Anholonomy in optics

Vladimirsky 1941

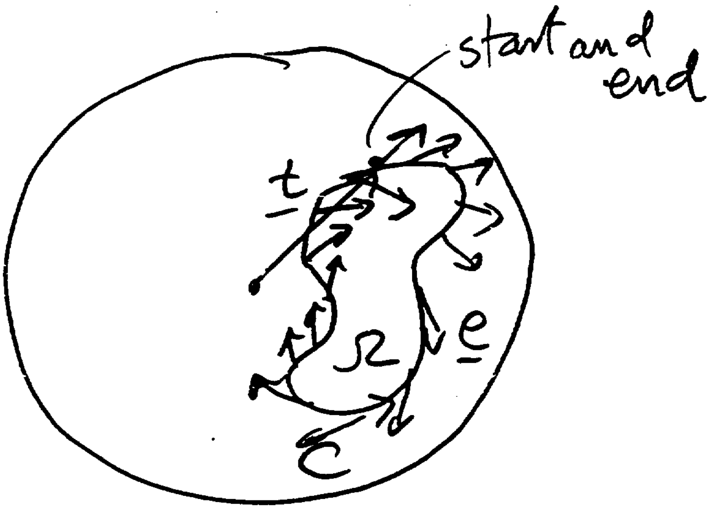
Light rays curved in space:



Curving of $\underline{t}$ governed by Snell's law (1611)

What about polarisation $\underline{e}$?

This is parallel-transported: no rotation of $\underline{e}$ about $\underline{t}$. So $\underline{e}$ need not return with $\underline{t}$: it rotates by the solid angle Ω that the $\underline{t}$ circuit subtends on its sphere:

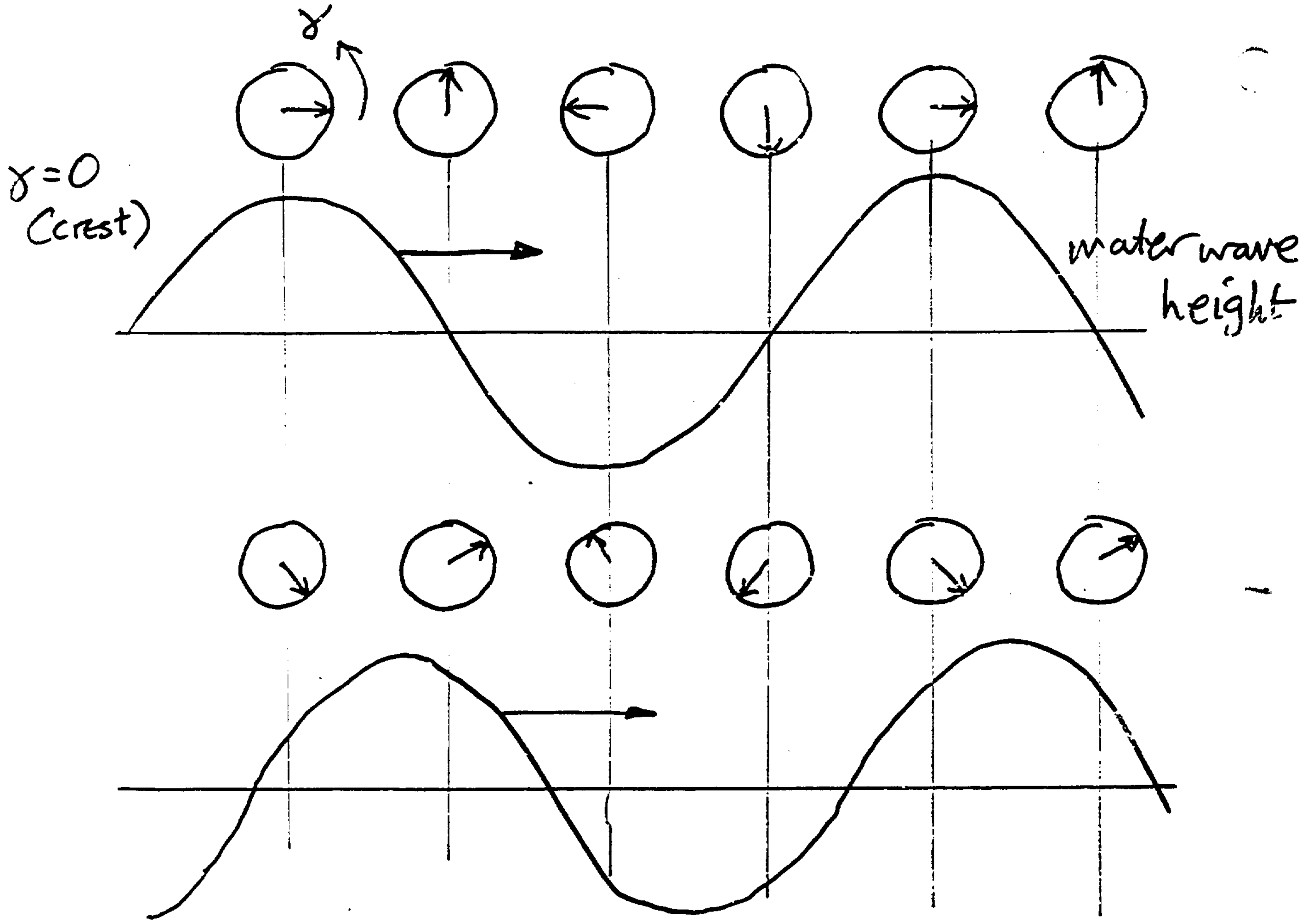


3. Phase

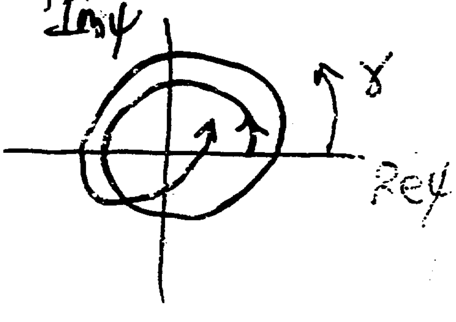
Oxford English Dictionary:

"A particular stage or point in a recurring sequence of movements or changes, e.g. a vibration or undulation, considered in relation to a reference position or time" (1861)

Naturally measured by an angle, γ.



Representation in terms of complex functions of time:



$$\psi(t) = \rho(t)\cos\gamma(t) + i\rho(t)\sin\gamma(t)$$

$$= \rho(t)\, e^{i\gamma(t)}$$

Solid-angle formula for parallel transport

Triad $\mathbf{r}, \mathbf{e}_1, \mathbf{e}_2$ $\mathbf{r}(t)$ radial, driving $\mathbf{e}_1, \mathbf{e}_2$

Let $\mathbf{e}$ be any vector $\perp \mathbf{r}$ (e.g. $\mathbf{e}_1$ or $\mathbf{e}_2$). Then

$$\dot{\mathbf{e}} = \mathbf{\Omega} \wedge \mathbf{e} \qquad \mathbf{\Omega} = \text{angular velocity}$$

Most general

$$\mathbf{\Omega} = a\mathbf{r} + b\dot{\mathbf{r}} + c\,\mathbf{r}\wedge\dot{\mathbf{r}}$$

$\boxed{\text{Parallel transport}}$ (no twist about $\mathbf{r}$) $\rightarrow$ $a = 0$

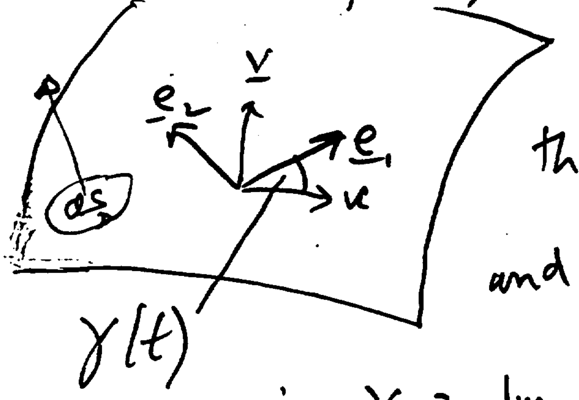
then $\mathbf{e}\cdot\mathbf{r} = 0$ $\longrightarrow$ $b = 0$, $c = 1$

$$\therefore \quad \mathbf{\Omega} = \mathbf{r}\wedge\dot{\mathbf{r}} \quad : \quad \boxed{\dot{\mathbf{e}} = (\mathbf{r}\wedge\dot{\mathbf{r}})\wedge\mathbf{e} = -\,\mathbf{e}\cdot\dot{\mathbf{r}}\;\mathbf{r}}$$

Useful expression for equation of motion:

let $\boldsymbol{\phi} \equiv \mathbf{e}_1 + i\mathbf{e}_2$. Then $\boxed{\boldsymbol{\phi}^*\cdot\dot{\boldsymbol{\phi}} = 0}$

Chart passage of $\mathbf{e}_1, \mathbf{e}_2$ relative to given orthogonal coordinates on sphere, with unit directions $\mathbf{u}, \mathbf{v}$, angle γ.



let $\mathbf{n} \equiv \mathbf{u} + i\mathbf{v}$

then $\boldsymbol{\phi} = \mathbf{n}\,e^{-i\gamma}$

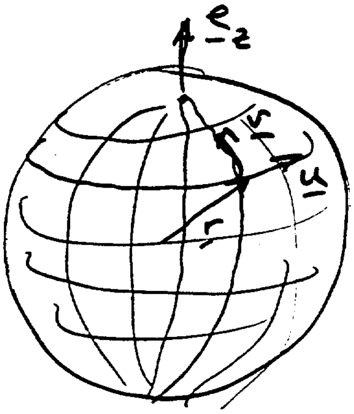
and $\boldsymbol{\phi}^*\cdot\dot{\boldsymbol{\phi}} = 0 \longrightarrow \dot{\gamma} = \mathrm{Im}\,\mathbf{n}^*\cdot\dot{\mathbf{n}}$

$$\therefore \quad \gamma = \mathrm{Im}\oint \mathbf{n}^*\cdot\dot{\mathbf{n}}\,dt = \mathrm{Im}\iint_{\partial S = C} \nabla\mathbf{n}^*\cdot\wedge\nabla\mathbf{n}\cdot d\mathbf{S}$$

$\wedge$, d on sphere
$\cdot$ scalar product

$$\boxed{\gamma(C) \equiv \mathrm{Im}\iint_{\partial S = C} d\mathbf{n}^*\cdot\wedge\, d\mathbf{n}}$$

For $\underline{u}$, $\underline{v}$ choose e.g. polar coordinate directions



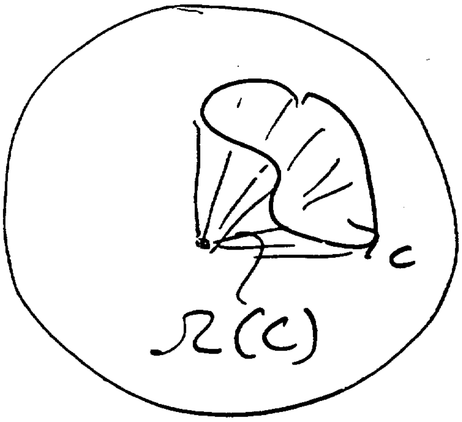
$$\underline{u} = \frac{\underline{r}\wedge\underline{e}_z}{|\underline{r}\wedge\underline{e}_z|}, \quad \underline{V} = \frac{\underline{r}\wedge\underline{u}}{r}$$

Then $\boxed{\mathrm{Im}\,\nabla\underline{n}^*\wedge\cdot\nabla\underline{n} = \frac{\underline{r}}{r^3}}$

= monopole field

$$\therefore \boxed{\gamma(C) = \iint_{\partial S = C} \frac{\underline{r}\cdot d\underline{S}}{r^3} = \Omega(C)}$$

solid angle subtended by C at $\underline{r}=0$

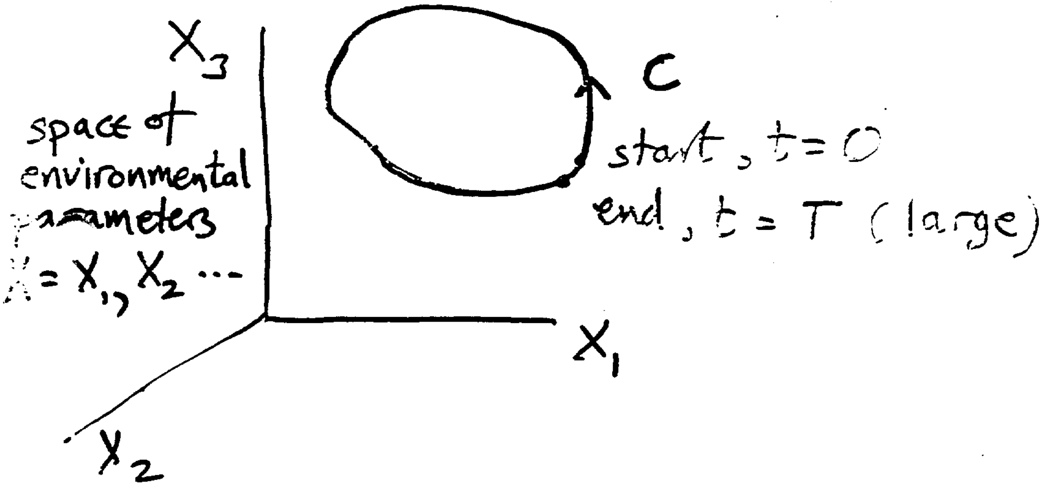


2-form notation: if X_1, X_2 are coordinates on sphere

$$\boxed{d\underline{n}^*\cdot\wedge d\underline{n} \equiv \left(\frac{\partial\underline{n}^*}{\partial X_1}\cdot\frac{\partial\underline{n}}{\partial X_2} - \frac{\partial\underline{n}^*}{\partial X_1}\cdot\frac{\partial\underline{n}}{\partial X_2}\right) dX_1\, dX_2}$$

The geometric phase

Take any quantum system (atom, electron, molecule in one of its discrete energy states (labelled n) and slowly alter its environment (e.g. electric or magnetic forces) round a cycle.



$X = X(t)$

Afterwards, environment has returned, and system is still in state n (quantum adiabatic theorem). But its phase has changed, by $\gamma = \gamma(T) - \gamma(0)$.

First guess at γ:

Imagine the environment doesn't change. Then

$$\gamma = -\omega(X)T = -\frac{E_n(X)}{\hbar}T \qquad (\text{Planck})$$

energy of state n

When X changes, generalize to

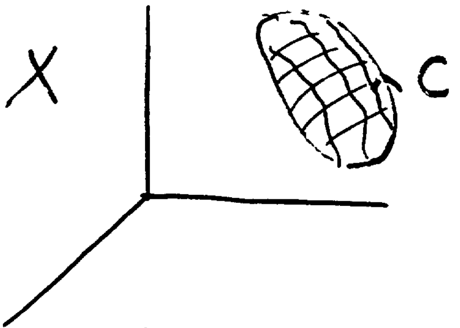
$$\boxed{\gamma = (?) = -\frac{1}{\hbar}\int_0^T dt\, E_n(X(t))}$$

But this is wrong. The true phase is . .

$$\boxed{\gamma = \underbrace{-\frac{1}{\hbar}\int_0^T dt\, E_n(X(t))}_{\text{"dynamical phase"}} + \underbrace{\gamma_n(C)}_{\text{"geometric phase"}}}$$

dynamical phase: increases with T; system's answer to:
"How long did your trip take?"

geometric phase: independent of T, dependent only on n and the geometry of C; system's answer to:



"where have you been?"

$\gamma_n(C)$ is phase anholonomy

As with other circuit-dependent quantities in physics,
$\gamma_n(C)$ = flux of something through C
(e.g. emf = flux of rate of change of magnetic field)

Here (from Schrödinger equation)

$$\gamma_n(C) = -\iint_{\partial S = C} \mathrm{Im}\,\langle dn| \wedge |dn\rangle$$

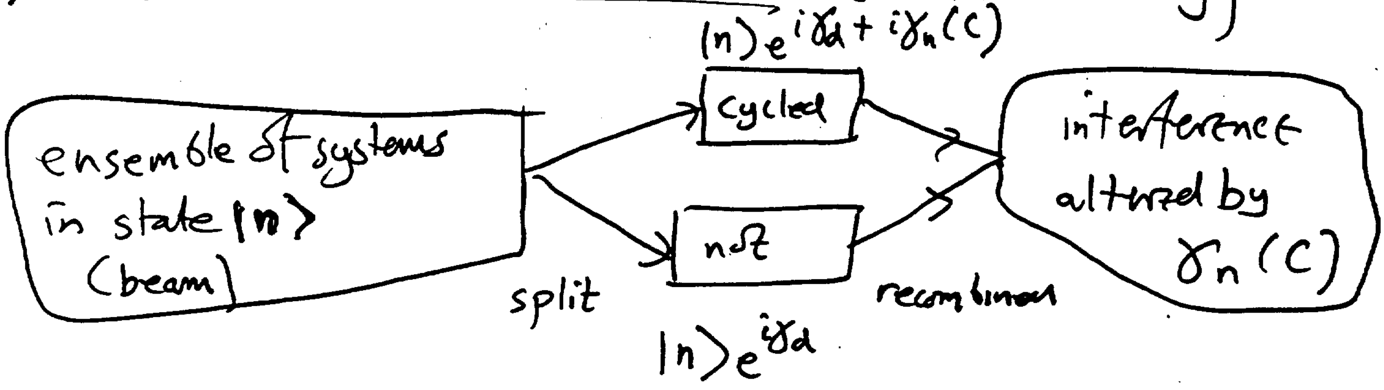
abstractly (2 form)

$$= -\iint_{\partial S=C} \underbrace{dX_1 dX_2}_{\text{coordinates spanning } C} \iiint d\underset{\sim}{r}\, \mathrm{Im}\left[\frac{\partial \psi_n^*(\underset{\sim}{r};X)}{\partial X_1}\frac{\partial \psi_n(\underset{\sim}{r};X)}{\partial X_2} - \frac{\partial \psi_n^*(\underset{\sim}{r};X)}{\partial X_2}\frac{\partial \psi_n(\underset{\sim}{r};X)}{\partial X_1}\right]$$

concretely

3 types of phase experiment

1) Two Hamiltonians, one state (interferometry)



2) One Hamiltonian, two states (superposition)

$$|\psi(0)\rangle = a_m|m\rangle + a_n|n\rangle$$

$$|\psi(T)\rangle = a_m|m\rangle e^{i\gamma_{dm} + i\gamma_m(C)} + a_n|n\rangle e^{i\gamma_{dn} + i\gamma_n(C)}$$

measure any operator $\hat{A}$ not commuting with $\hat{H}$:

$$\boxed{\begin{aligned}\langle\psi(T)|\hat{A}|\psi(T)\rangle &= |a_n|^2\langle n|\hat{A}|n\rangle + |a_m|^2\langle m|\hat{A}m\rangle \\ &+ 2\,\mathrm{Re}\, a_n^* a_m \langle n\hat{A}m\rangle e^{i\{\gamma_{dn} - \gamma_{dm} + \gamma_n(C) - \gamma_m(C)\}}\end{aligned}}$$

depends on $\gamma_n(C)$

3) Repeated cycling

$$|\psi(nT)\rangle = e^{-i\frac{n}{\hbar}\int_0^T E_n(t)dt}\, e^{in\gamma_n(C)}|\psi(0)\rangle$$

$$= e^{-i\left(\bar{\omega} - \frac{\gamma_n(C)}{T}\right)nT}|\psi(0)\rangle \qquad \left(\bar{\omega} = \frac{1}{\hbar T}\int_0^T dt\, E_n(t)\right)$$

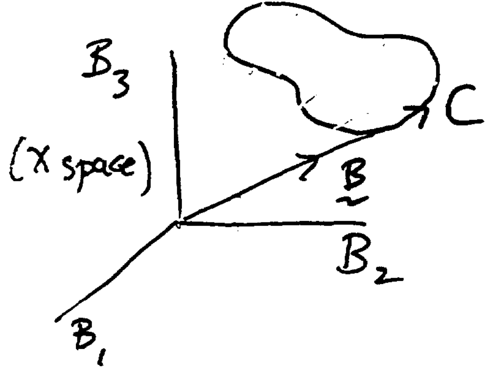
$$\boxed{|\psi(nT)\rangle \equiv e^{-i\omega' nT}|\psi(0)\rangle \rightarrow \text{frequency shift}}$$

Example TURNING SPINS

System : quantum spins

environment : whatever can change spin, e.g. a magnetic field $\underset{\sim}{B}$ for particles with magnetic moment.

cycle : slow circuit in $\underset{\sim}{B}$ space



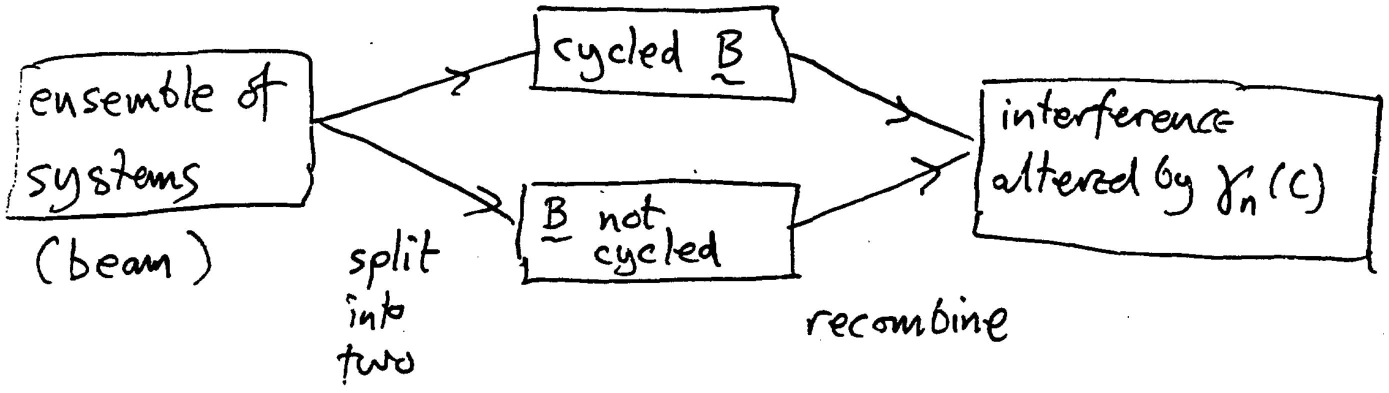
Quantum states labelled by component $n\hbar$ of angular momentum along $\underset{\sim}{B}$.

(n = integer for Bosons (α particles, photons)
= half-integer for Fermions (electrons, neutrons))

Geometric phase is

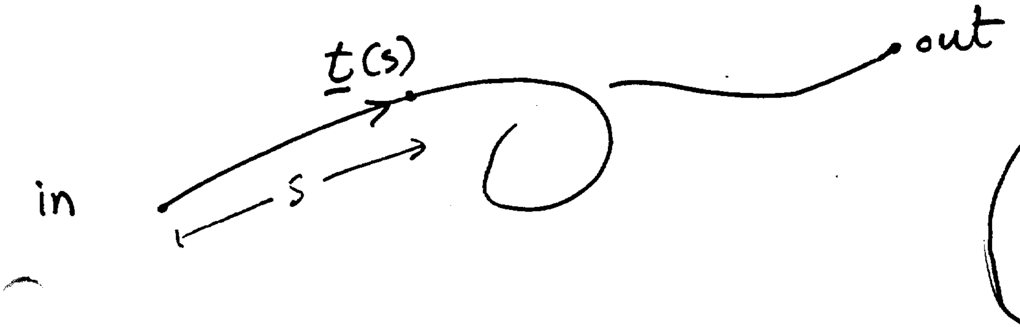
$$\boxed{\gamma_n(C) = -n\underbrace{\Omega}_{\text{solid angle subtended by } C \text{ at } \underset{\sim}{B}=0}(C)}$$

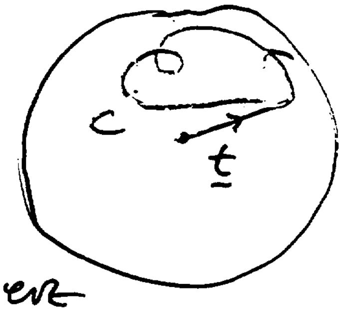
Schematic experiment



PHOTONS spin $n = \pm 1$ (Chiao-Wu-Tomita)

Photons don't interact with electric or magnetic fields, but their spins are parallel to their direction of propagation $\underline{t}$, and so can be turned by changing $\underline{t}$, e.g. by coiling an optical fibre along which the light is travelling.





sphere of unit tangents $\underline{t}$

phase shifts

$$\boxed{\begin{aligned} \gamma &= -\Omega(C) \quad (\text{spin } 1 : \text{right circular polarization}) \\ &\quad +\Omega(C) \quad (\text{spin } -1 : \text{left circular polarization}) \end{aligned}}$$

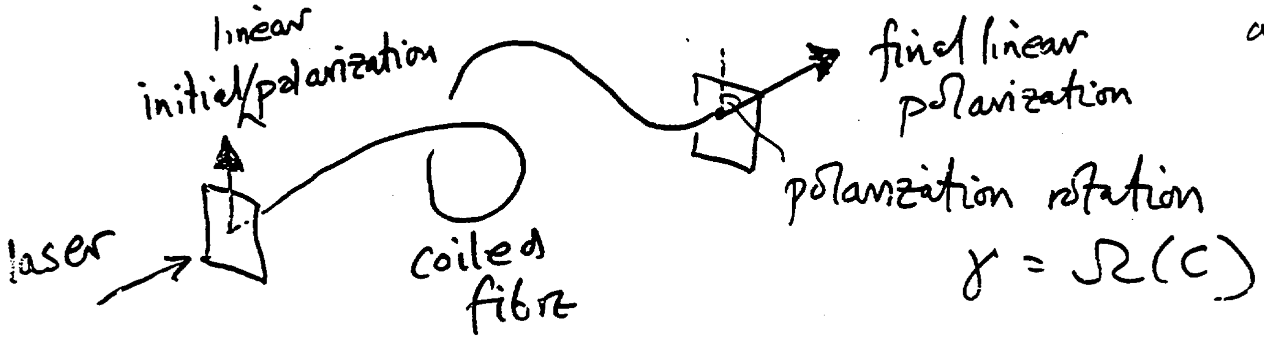
Instead of an interference experiment, it is easier to send in linearly polarized light:

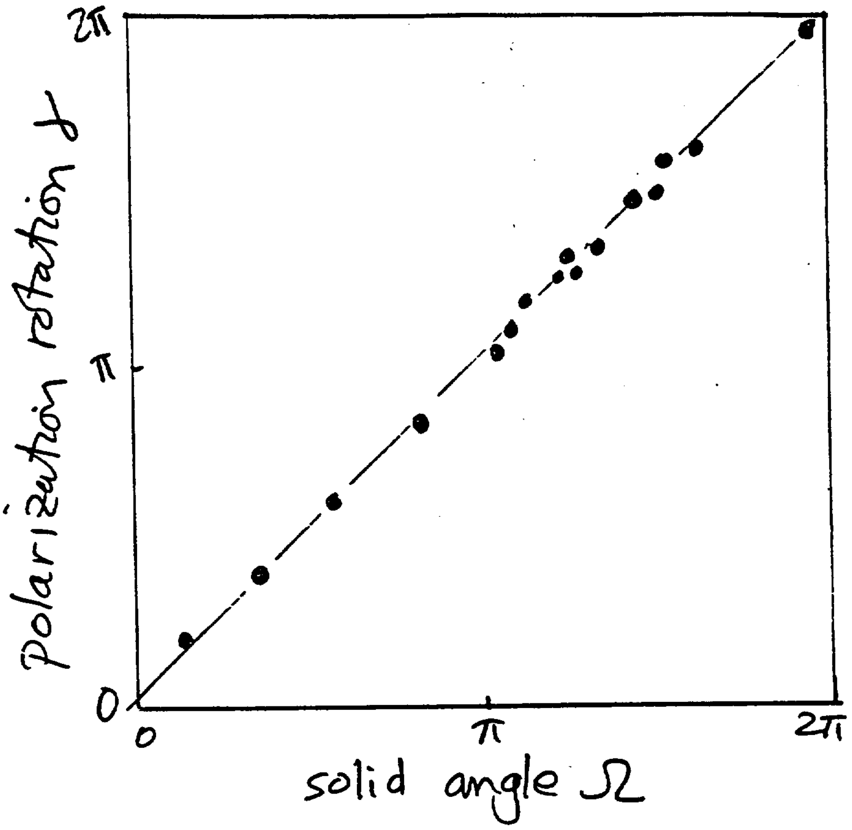
linear polarization = superposition of R circular + L circular

afterwards:

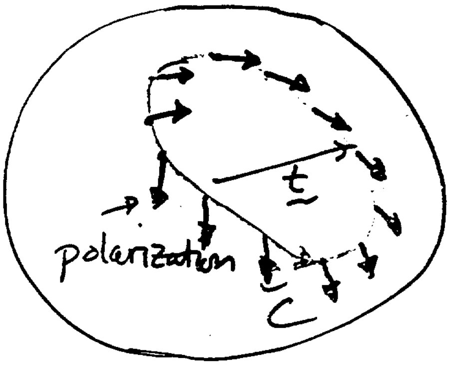
$$e^{-i\Omega} \times R + e^{+i\Omega} \times L = \text{linear polarization } \underline{\text{rotated by } \Omega}$$

Vladimirski again!





This is 'geometric optical activity' the solid-angle dependence means that the polarization direction is parallel-transported over the $\underset{\sim}{t}$ sphere



(Ross 1984)

A natural question Is the effect quantum or classical?

Chiao and Wu:

> "... would rather think of these effects as topological features of classical Maxwell theory which originate at the quantum level, but survive the correspondence-principle limit ($\hbar \to 0$) into the classical level"

(They also survive the short-wave limit ($\lambda \to 0$) into geometrical optics – Rytov 1938, Vladimirsky 1941)

Feynman "The photon equation ... is just the same as Maxwell's eq ray arguments strictly not applicable in monomode fibres; need full Maxwell equations.

Quantum or classical?

Pseudoproblem because (Feynman)

'The photon equation is just the same as maxwell's equations ...'

Maxwell

$$\boxed{\underset{\times i\hbar}{\partial_t\,(\text{Fields})} = \underset{\times i\hbar}{(\text{matrix linear in } \nabla)\,(\text{fields})}}$$

$$\boxed{i\hbar\,\partial_t\,(\text{Fields}) = (\text{matrix linear in } \underline{p} = -i\hbar\nabla)\,(\text{fields})}$$

→ Schrödinger equation if matrix is Hermitian

Connection with spin:

$$-i\hbar\,\nabla\wedge\underline{E} = \underline{p}\wedge\underline{E} = -i(\underline{p}\cdot\underline{\underline{\sigma}})\underline{E}$$

$$-i(\underline{p}\cdot\underline{\underline{\sigma}})\underline{E} = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$\boxed{\underline{\underline{\sigma}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

spin 1 matrices.

Explicitly:

$$\boxed{\partial_t \underline{D} = \nabla\wedge\underline{H}\ ;\ \partial_t \underline{B} = -\nabla\wedge\underline{E}\ ;\ \underline{B} = \mu(\underline{r})\underline{H}\ ;\ \underline{D} = \varepsilon(\underline{r})\underline{E}}$$

fibre refractive index

$$\longrightarrow$$

$$\boxed{i\hbar\,\partial_t|\underline{\Psi}\rangle = \hat{H}\,|\underline{\Psi}\rangle}$$

$$\boxed{|\underline{\Psi}\rangle = \begin{pmatrix} \varepsilon^{1/2}(\underline{r})\,\underline{E}(\underline{r},t) + i\mu^{1/2}(\underline{r})\,\underline{H}(\underline{r},t) \\ \varepsilon^{1/2}(\underline{r})\,\underline{E}(\underline{r},t) - i\mu^{1/2}(\underline{r})\,\underline{H}(\underline{r},t) \end{pmatrix}}$$

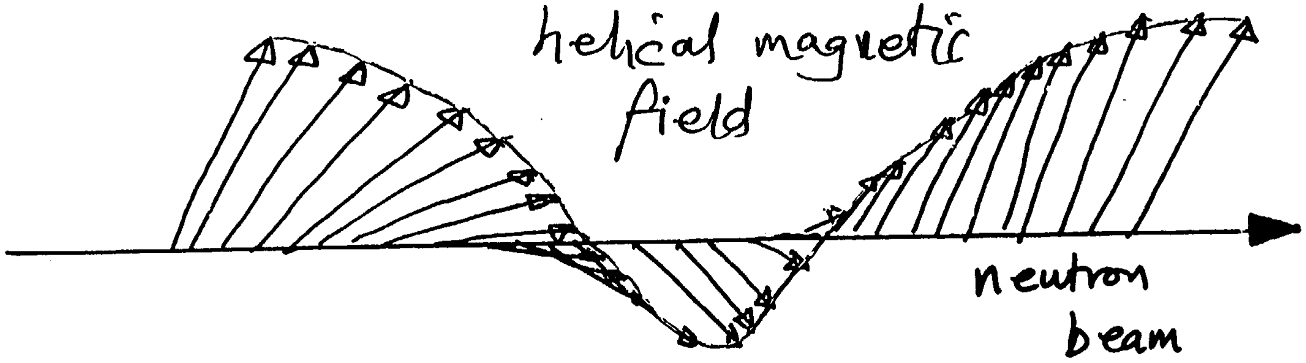
$$\boxed{\hat{H} = c\begin{pmatrix} \underline{\Pi}\cdot\underline{\sigma} & i\hbar\underline{\xi}\cdot\underline{\sigma} \\ -i\hbar\underline{\xi}\cdot\underline{\sigma} & -\underline{\Pi}\cdot\underline{\sigma} \end{pmatrix}}$$

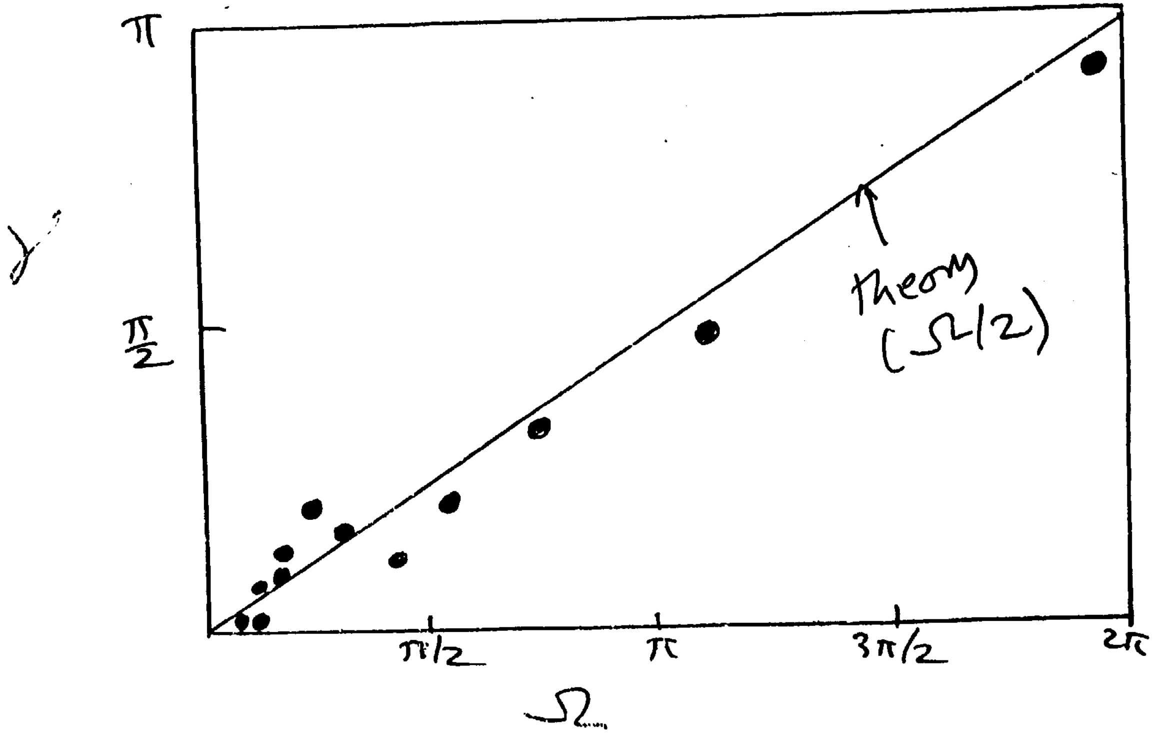
$$\boxed{\underline{\Pi} = n^{+1/2}(\underline{r})\,\underline{p}\,n^{1/2}(\underline{r})} \quad \boxed{n(\underline{r}) = \sqrt{\mu(\underline{r})\varepsilon(\underline{r})}}$$

$$\boxed{\underline{\xi}(\underline{r}) = \frac{1}{4n(\underline{r})}\nabla\log\frac{\varepsilon(\underline{r})}{\mu(\underline{r})}}$$

So, Maxwell's equations (1865?), not only relativistic but also quantum mechanical!

NEUTRONS, $n = 1/2$



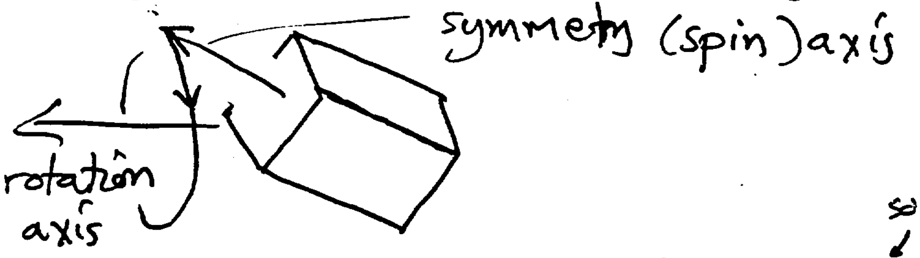


T. Bitter and
D. Dubbers

ship

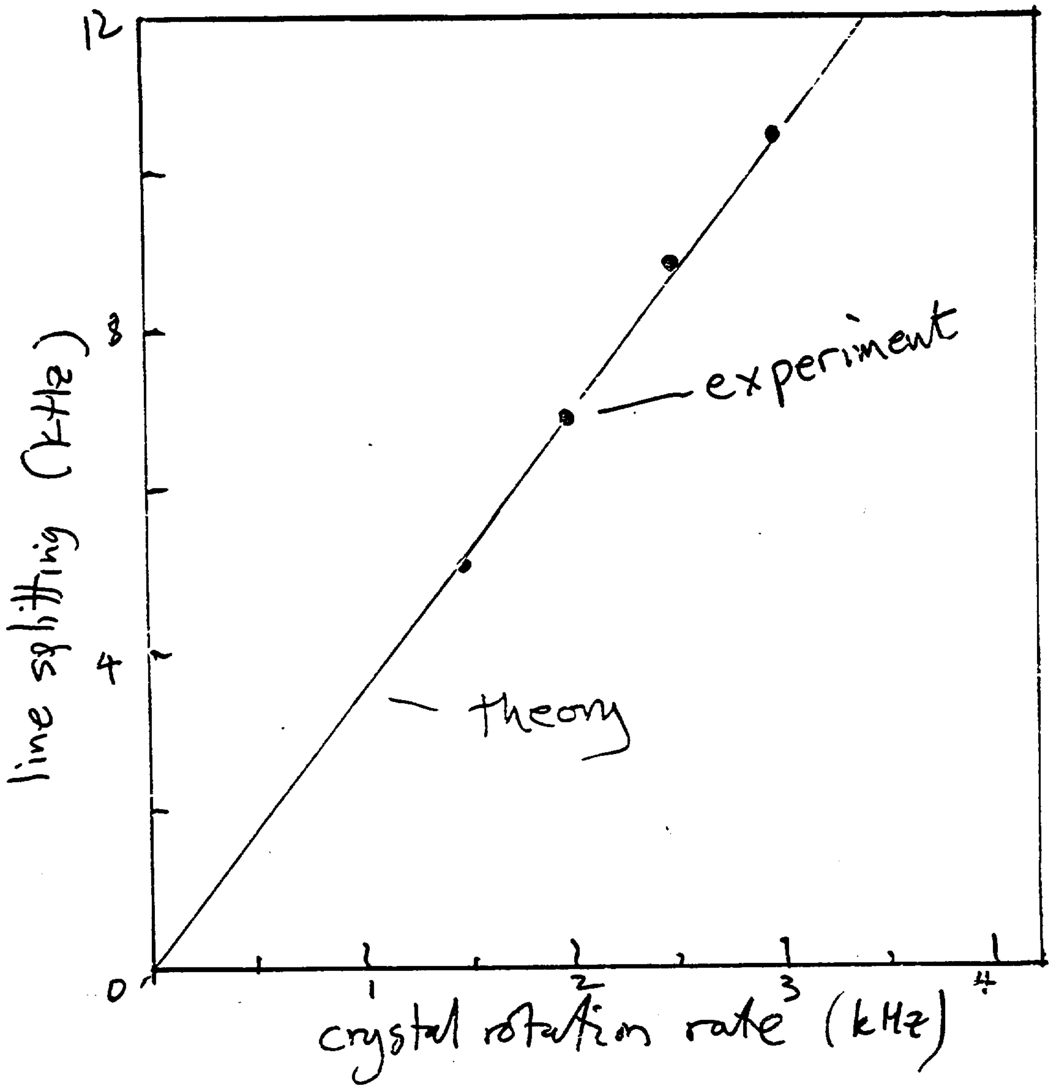
NUCLEI spin $n = \pm 3/2$

^{35}Cl nuclei in crystalline $NaClO_3$
Spin has electric (quadrupole) interaction with crystal electrons, and so can be turned by rotating a symmetry axis of the crystal



Phase of state increases by $\pm \frac{3}{2} \overset{\text{solid angle}}{\Omega}$ per rotation.

Ever-changing phase $\longrightarrow$ frequency shift $\left(\frac{d\,\text{phase}}{dt}\right)$
measurable as line splitting in nuclear magnetic resonance



R. Tycko

Geometric contribution to adiabatic transition probability

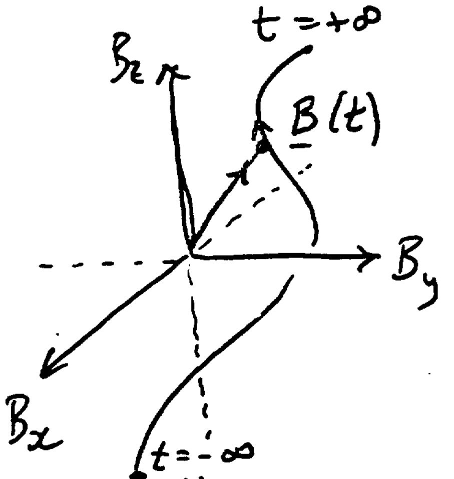
'Adiabatic transitions' are the exponentially weak transitions between states, governed by

$$\boxed{i\hbar \frac{\partial|\Psi\rangle}{\partial t} = \hat{H}(\delta t)|\Psi\rangle}$$

when $\hat{H}(\delta t)$ is non-degenerate and δ is small

Example: Spin-½ in a $\underline{B}$ field

$$\boxed{i\hbar \frac{\partial|\Psi\rangle}{\partial t} = \mu\,\underline{B}(\delta t)\cdot\underbrace{\underline{\sigma}}_{\text{Pauli}}|\Psi\rangle}$$



Hamiltonian curve

Separation of adiabatic (instantaneous) eigenstates is $2\mu|\underline{B}(\delta t)|$ – never zero for real t, so transitions arise from complex degeneracies

ie $B_x^2(\tau_c)+B_y^2(\tau_c)+B_z^2(\tau_c)=0$

Old theory: Transition probability

$$P_{+\to-} \sim e^{-\frac{4}{\hbar\delta}\mathrm{Im}\int_{\text{any real time}}^{\tau_c} d\tau\, B(\tau)} = e^{-\Gamma_d}$$

Old Theory wrong!

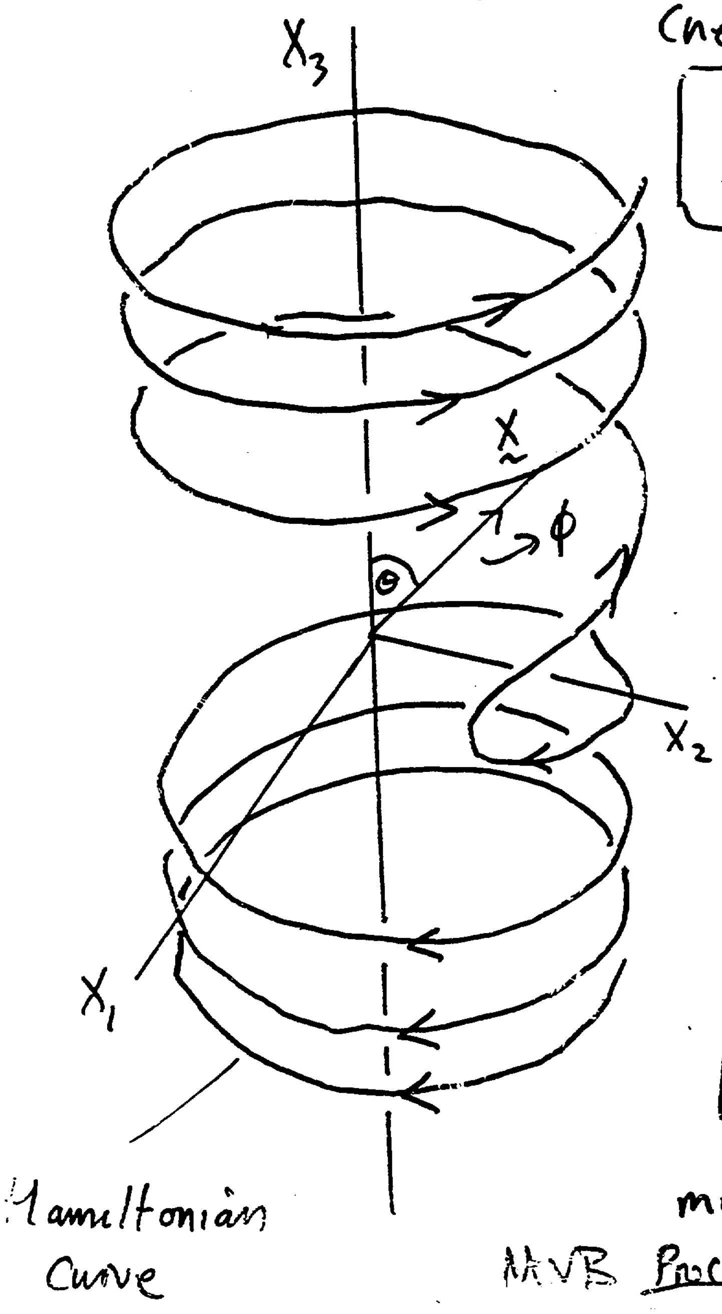
$$P_{+\to-}(\infty) = e^{-\Gamma_d}\, e^{\Gamma_g}$$

dynamical exponent (old)

$$\Gamma_d = \frac{4}{\hbar\delta}\,\mathrm{Im}\int_{\tau_0}^{\tau_c} d\tau\, B(\tau)$$

geometric exponent (new) – from ∥ transport of adiabatic states

$$\boxed{\Gamma_g = -2\,\mathrm{Im}\int_{\tau_0}^{\tau_c} d\tau\, \dot\phi \cos\theta}$$



This is purely geometric – independent of δ and $\hbar$.

Γ_g is a complex solid angle.

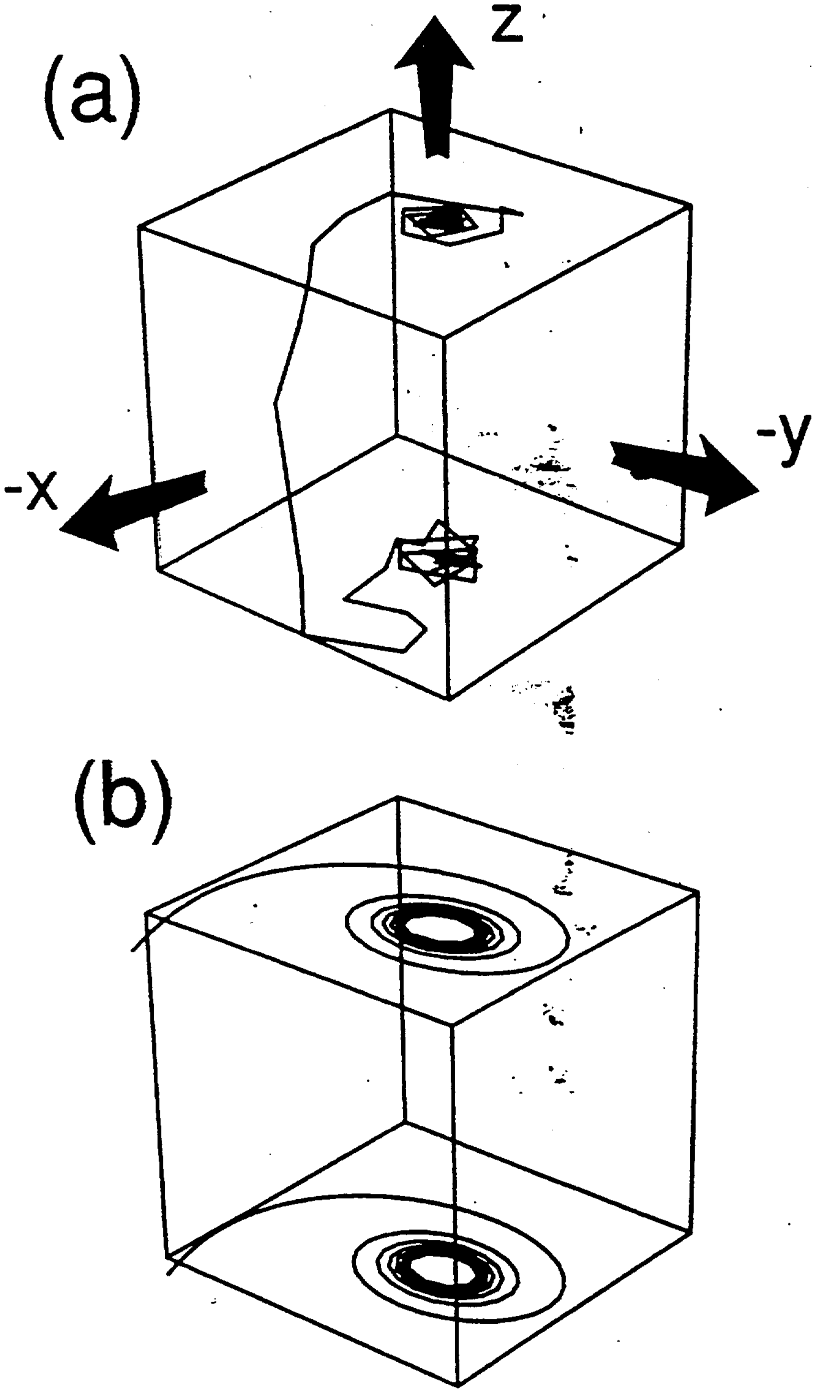
Example: Twisted Landau – Zener:

$$X_3 = A\tau$$
$$X_1 = \Delta\cos B\tau^2$$
$$X_2 = \Delta\sin B\tau^2$$
$$\Gamma_g = -\frac{\Delta^2}{A^2} B\,\mathrm{sgn}\,A$$

measurable in a spin experiment

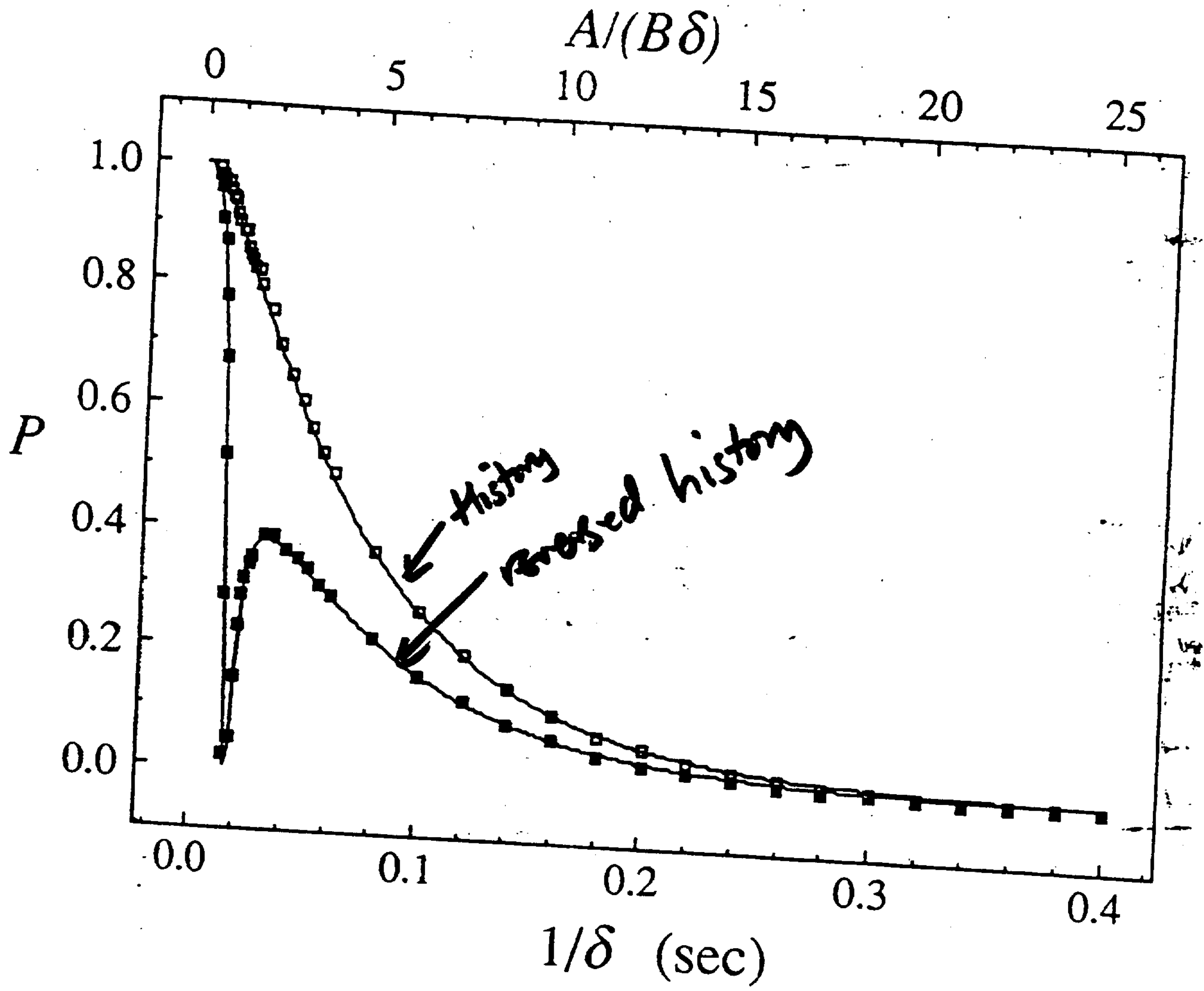
MVB *Proc. Roy. Soc.* 1990 A430 405–411

Histories in Hamiltonian space



J.W. Zwanziger, S.P. Rucker and G.C. Chingas Phys Rev A 43 (1991) 3232-9 figure 3

NMR in ^{13}C



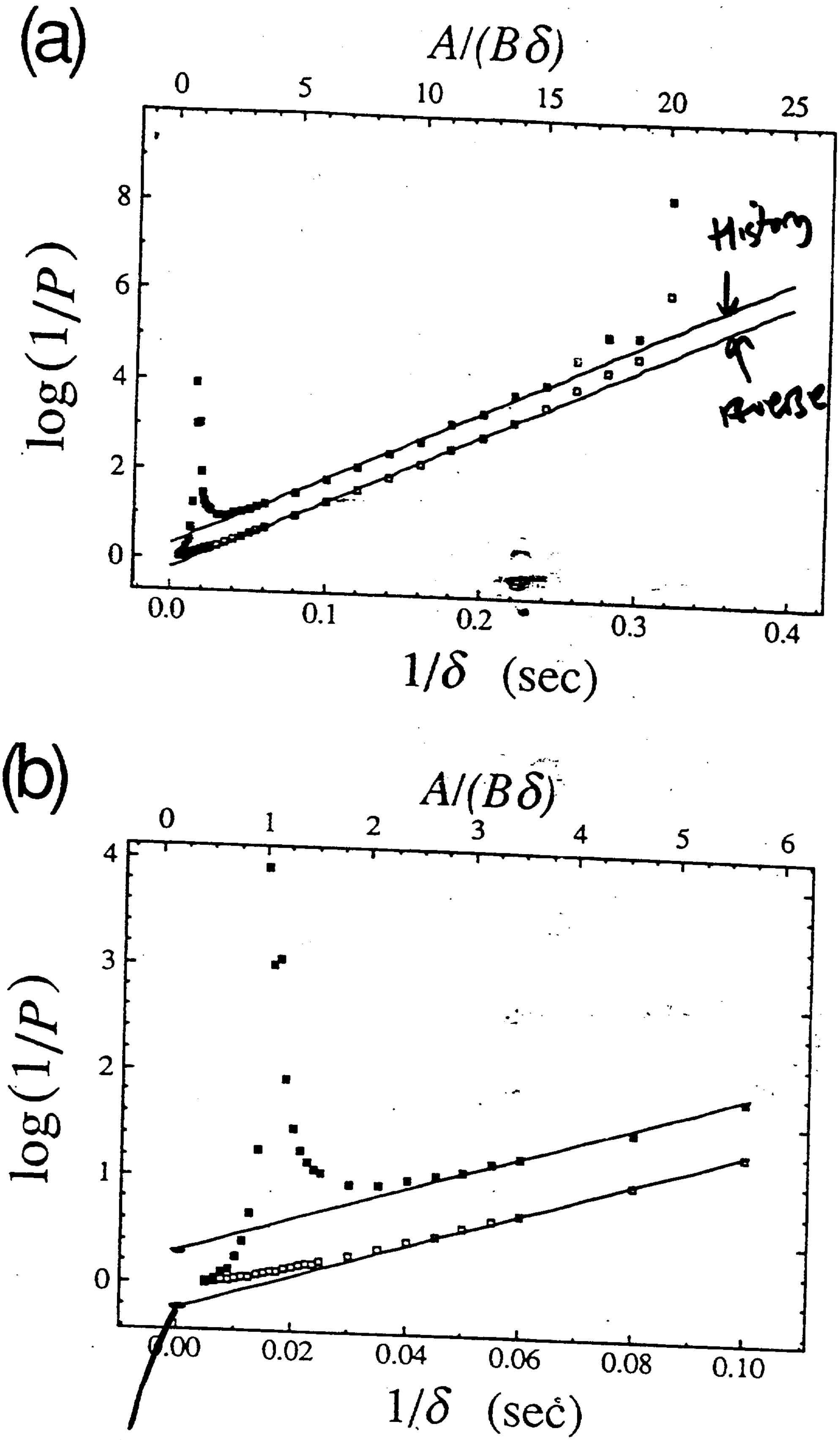
Predict

$$P_{+\to -} = e^{\underbrace{\Gamma_s}_{O(\delta)} - \underbrace{\Gamma_d}_{\frac{\mathrm{Im}}{\delta\hbar}\int (E_1 - E_2)\,dE}}$$

under reversed, Γ_s reverses

Γ_d does not

Figure 4

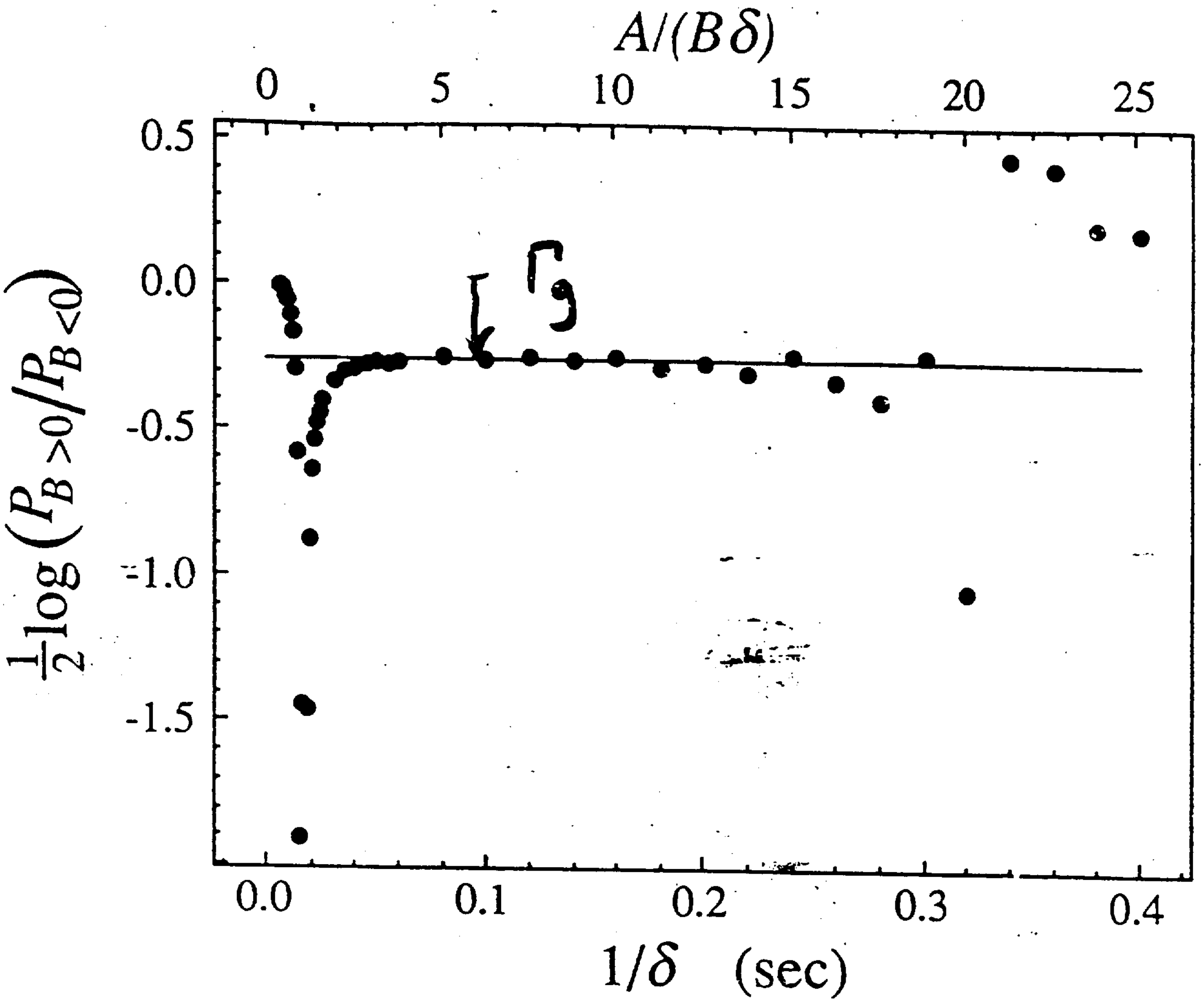


History

Reverse

intercepts Γ_g

Figure 5



Theory Γ_g = -0.243

Expt Γ_g = -0.26 ± ·01

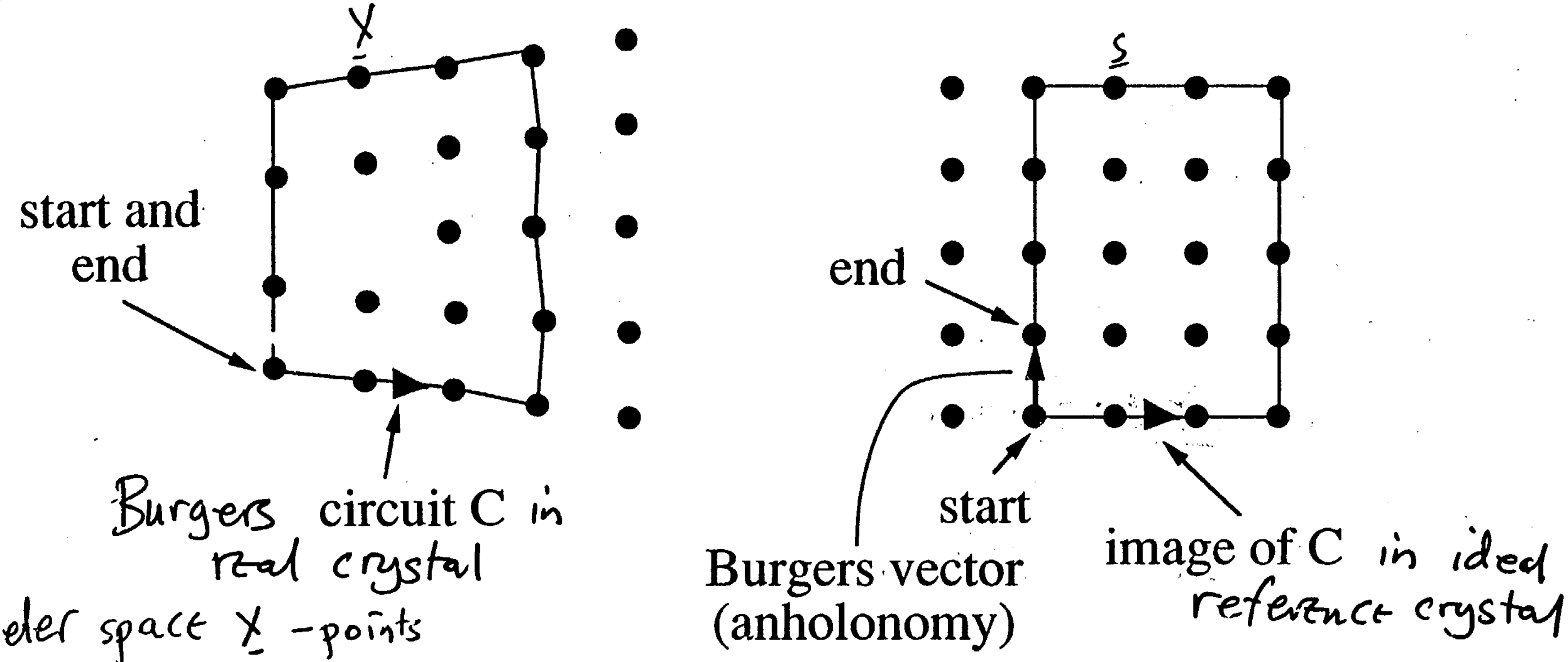
Figure 6

Lecture 2

(Some geometric phases, Michael Berry Geneva 1993)

BRISTOL ANHOLONOMY CALENDAR

1. Crystal dislocations as anholonomy (Frank 1951)



Parameter space $\underline{X}$ -points in real crystal

Figure 2. Crystal dislocation as anholonomy

Slaved variables $\underline{s}$ - points in ideal crystal

2 Dislocations in waves (Nye and Berry, 1974)

Wave $\boxed{\psi(x) = \rho(x)\, e^{i\chi(x)}}$

wavefronts

$\chi(x) = \text{constant mod } 2\pi$

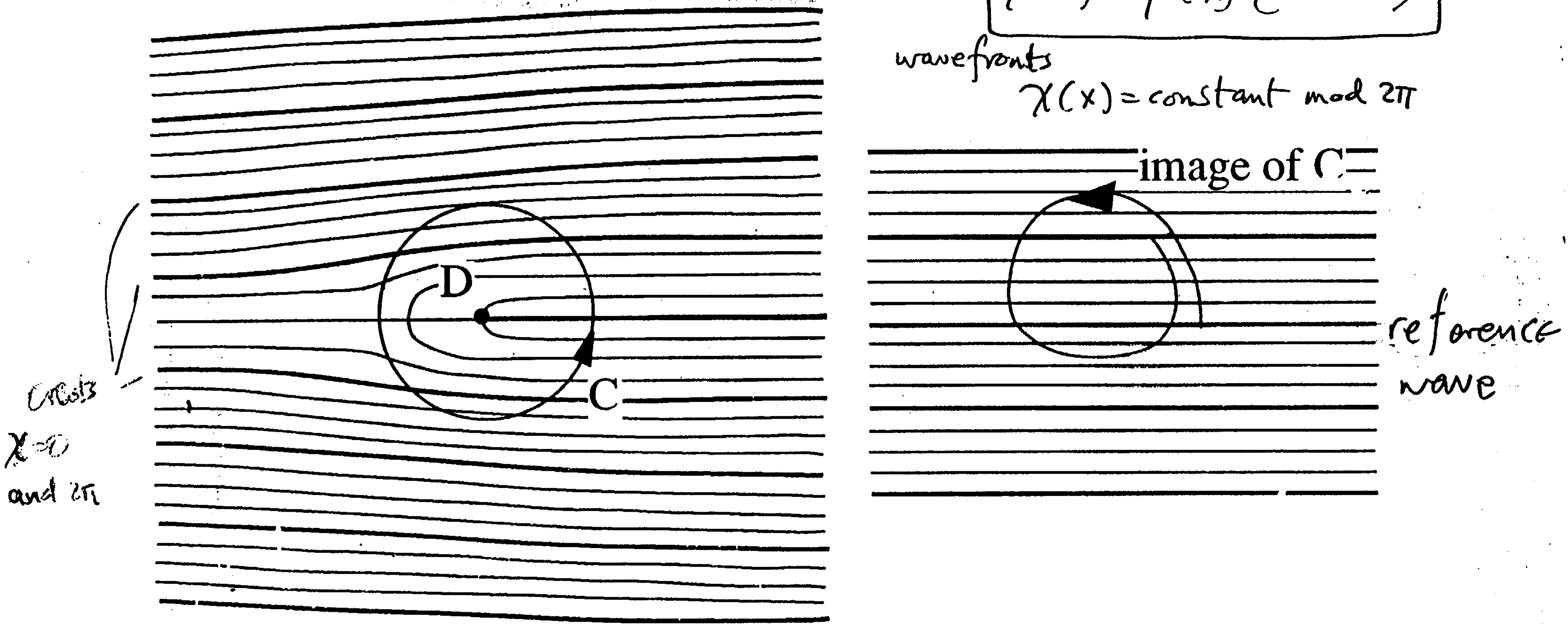


Figure 5. Wavefronts (at intervals of $\pi/4$ with crests bold) of a dislocated wave with a Burgers circuit surrounding the dislocation D, and its image in a plane wave

At dislocations, χ has a 2π anholonomy, and ρ is zero

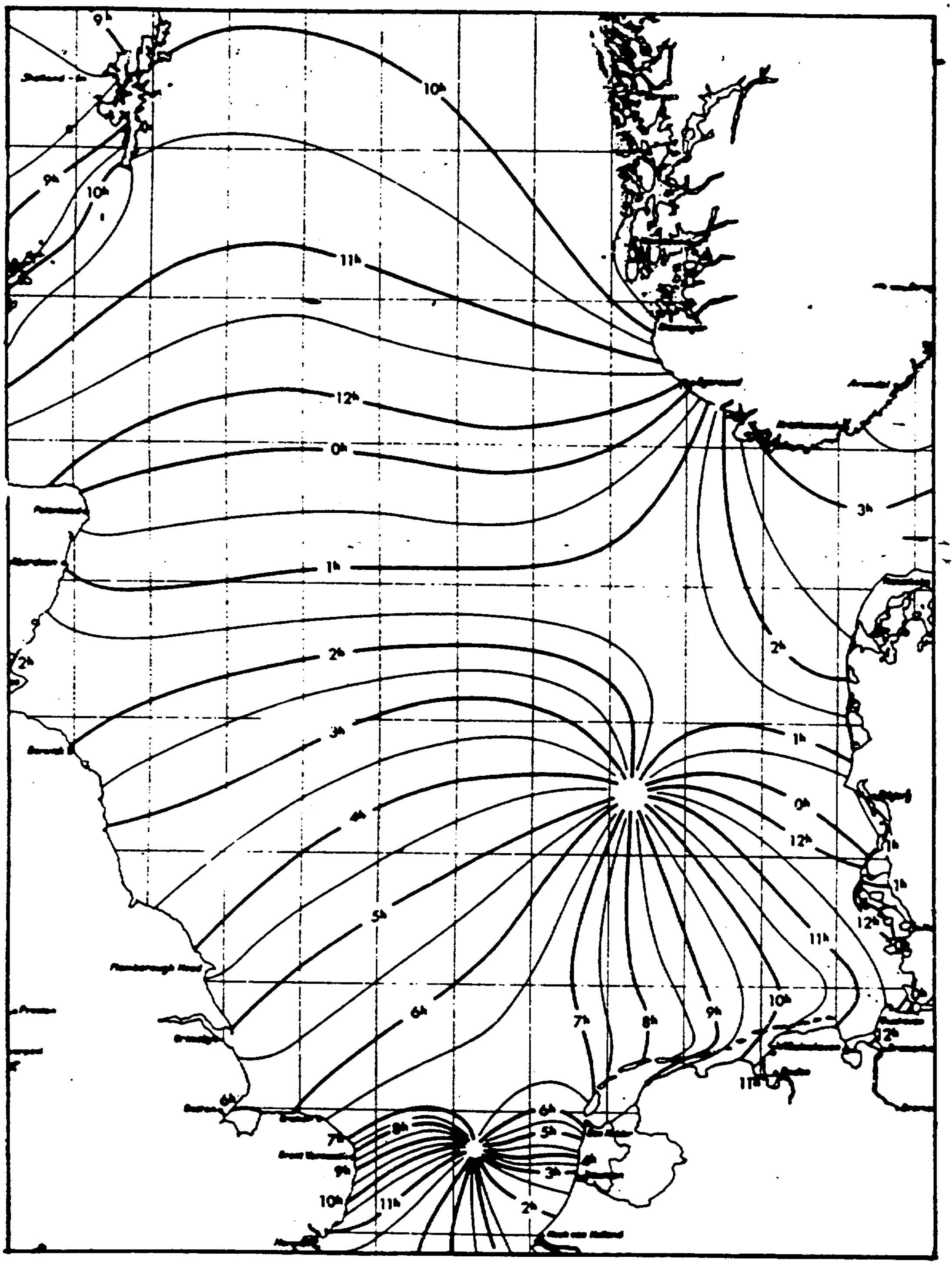
$$\boxed{\frac{1}{2\pi}\oint_C d\chi = \text{integer}}$$

Dislocations are lines threads of silence

sound, microwaves, light, tides,

Dislocations in the tide wave (Whewell 1833)

Wavefronts : loci of instantaneous high tides at given times
Dislocations : 'amphidromic points' of no tide



The tide wave (12h period) is a forced vibration of the whole earth. Rotating source → no time-reversal symmetry → complex waves (stationary but not standing) $\boxed{h(\theta,\phi;t) = \mathrm{Re}\left[e^{-i\omega t}\underbrace{\psi(\theta,\phi)}_{\text{complex}}\right]}$

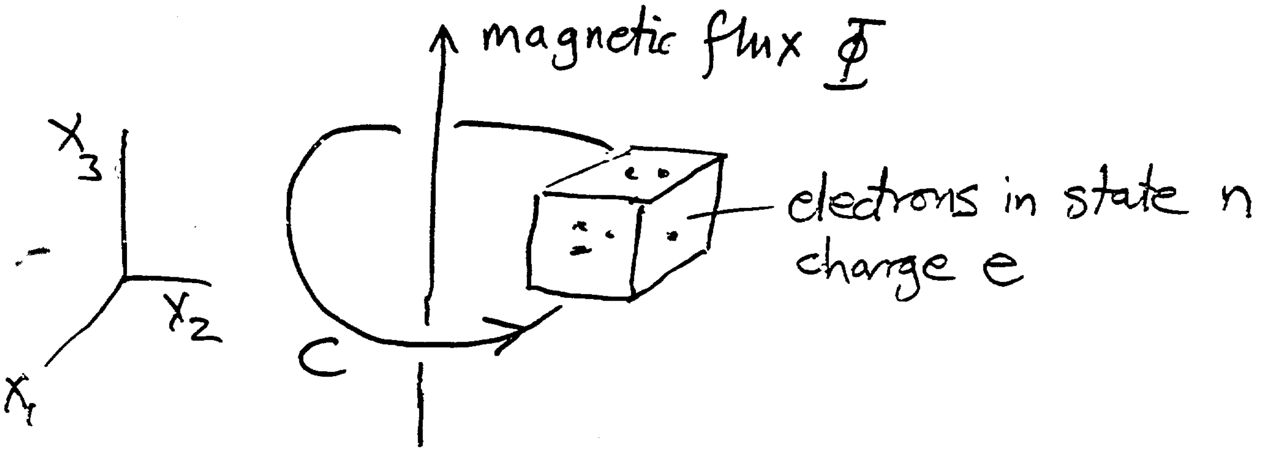
3. AHARONOV-BOHM EFFECT

(1959)

System : electrons

environment: 3-D space of positions, containing a line of magnetic flux (e.g. long solenoid)

cycle : circuit around the flux line

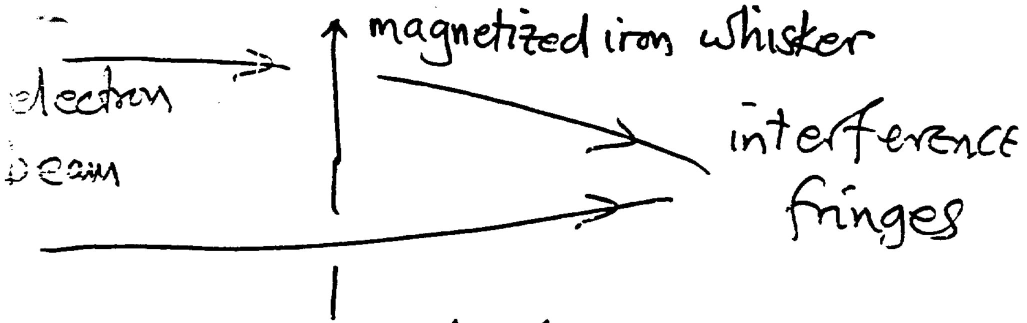


After the circuit, geometric phase is

$$\boxed{\gamma = e\Phi/\hbar}$$

(electrons influenced quantall by field elsewhere

Chambers' experiment 1959



The flux Φ shifts the fringes, via an extra phase:

$$\text{phase}(\rightarrow) - \text{phase}(\curvearrowright)$$
$$= \text{phase}(\rightarrow) + \text{phase}(\curvearrowleft)$$
$$= \text{phase}(\circlearrowleft) \quad = \text{geometric phase}$$

FOUR CLASSES OF AB EFFECT

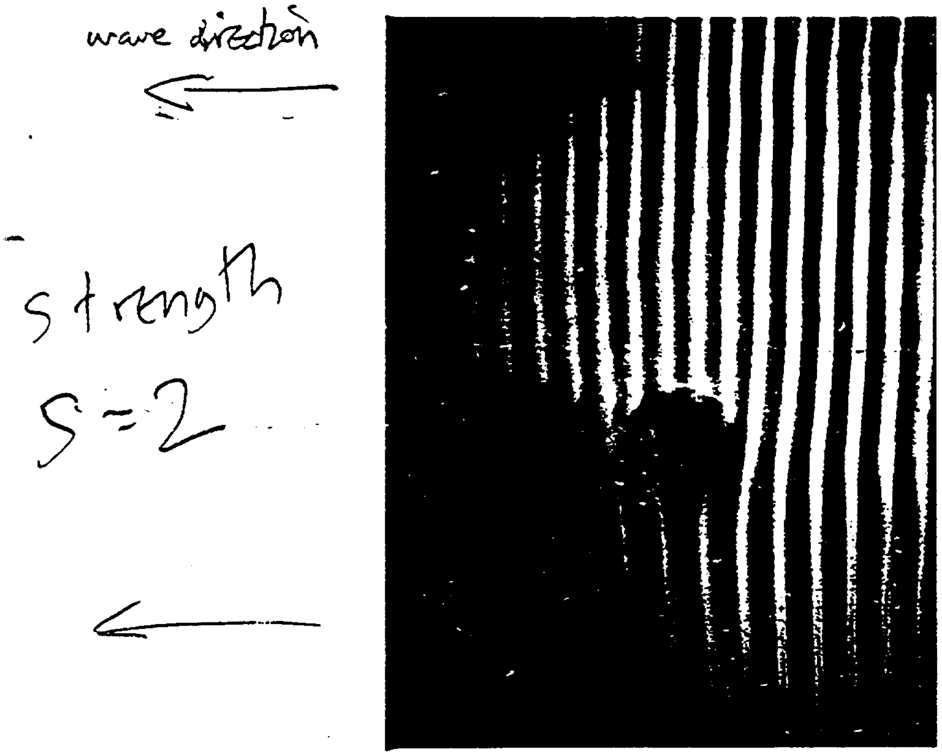
Effect	wave	flux	
AB	quantum	classical	electrons, mag flux
Tonomura	quantum	quantum	electrons on flux in super-conduct
Steinberg	classical	quantum	sound waves on quantized vortex in He
Bathtub	classical	classical	ripples on bathtub vortex

But, can observe water-wave analogue of AB effect

For waves on a medium moving with speed $\underset{\sim}{v}(\underset{\sim}{r})$ ($\ll c$), $\underset{\sim}{v}(\underset{\sim}{r})$ acts like a vector potential, so for AB need

$\nabla \wedge \underset{\sim}{v} = A\,\delta(\underset{\sim}{r})$, ie. a vortex line in irrotational flow.

This can be realized with surface ripples scattered from a bathtub vortex



dimensionless flux

$$\boxed{\alpha = \frac{\oint \underset{\sim}{v}\cdot d\underset{\sim}{l}}{\lambda\, c_{group}}}$$

wavelength
group vel.
circulation

M.V.B., R.G. Chambers et. al, Eur. J. Phys 1 (1980) 154-162

Dislocations can be
1) unattached to any singularity of the medium (generic case)
2) coincident with line singularities (AB)
3) attached to a point singularity and otherwise movable (but not removable) by gauge transformation (Dirac string)

4. Liquid crystal disclinations as anholonomy

(Frank 1959)

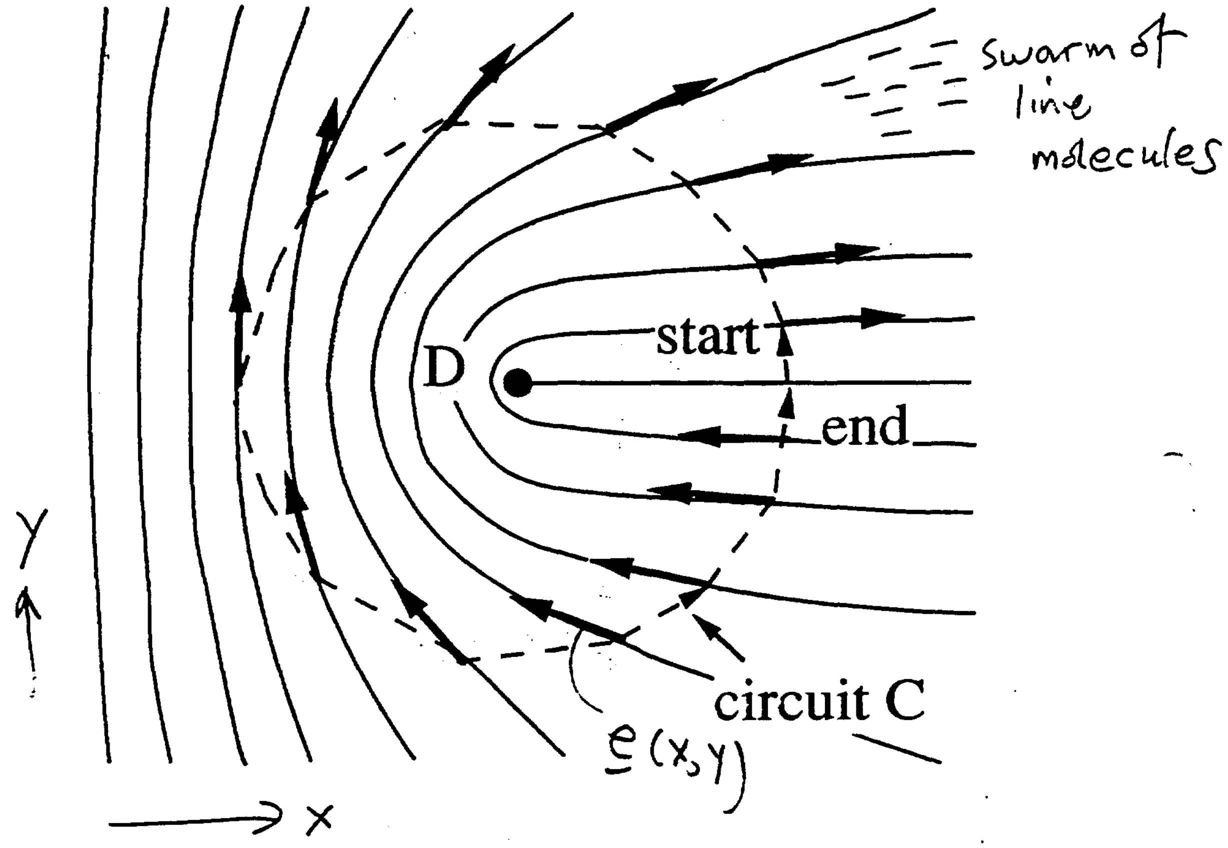


Figure 3. Reversal (π anholonomy) of liquid crystal direction around a disclination D

Can regard line field as field of eigendirections of a 2×2 real symmetric matrix

$$\boxed{M(X,Y) = \begin{pmatrix} M_{11}(X,Y) & M_{12}(X,Y) \\ M_{12}(X,Y) & M_{22}(X,Y) \end{pmatrix}}$$

$$\boxed{\underline{\underline{M}}\,\underline{e} = \lambda\,\underline{e}}$$

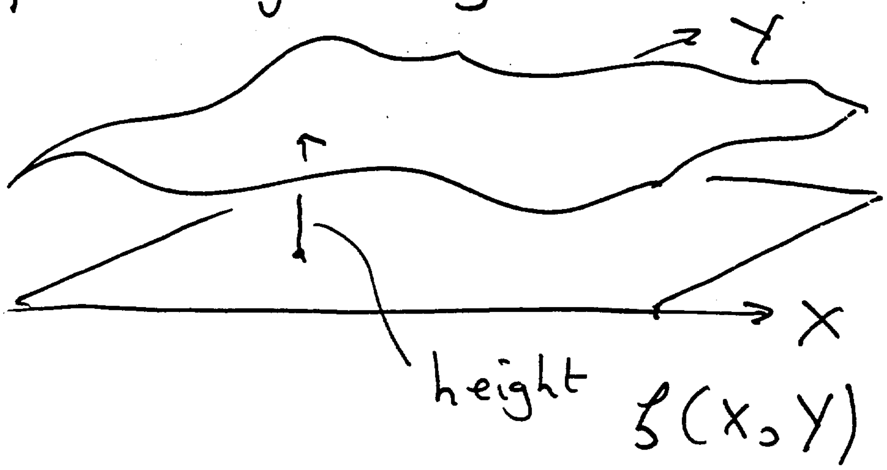
Disclination D is a degeneracy $M_{11} = M_{22}$, $M_{12} = 0$

If C encloses D, $\underline{e}$'s change sign -

$\boxed{\text{Geometric phase of } \pi}$

Darboux (1896)

Differential geometry of curved surfaces.



Curvature matrix

$$H(X,Y) = \begin{pmatrix} \partial^2\zeta/\partial X^2 & \partial^2\zeta/\partial X\partial Y \\ \partial^2\zeta/\partial Y\partial X & \partial^2\zeta/\partial Y^2 \end{pmatrix}$$ real, symmetric

parameters

Eigenvalues: principal curvatures at X, Y

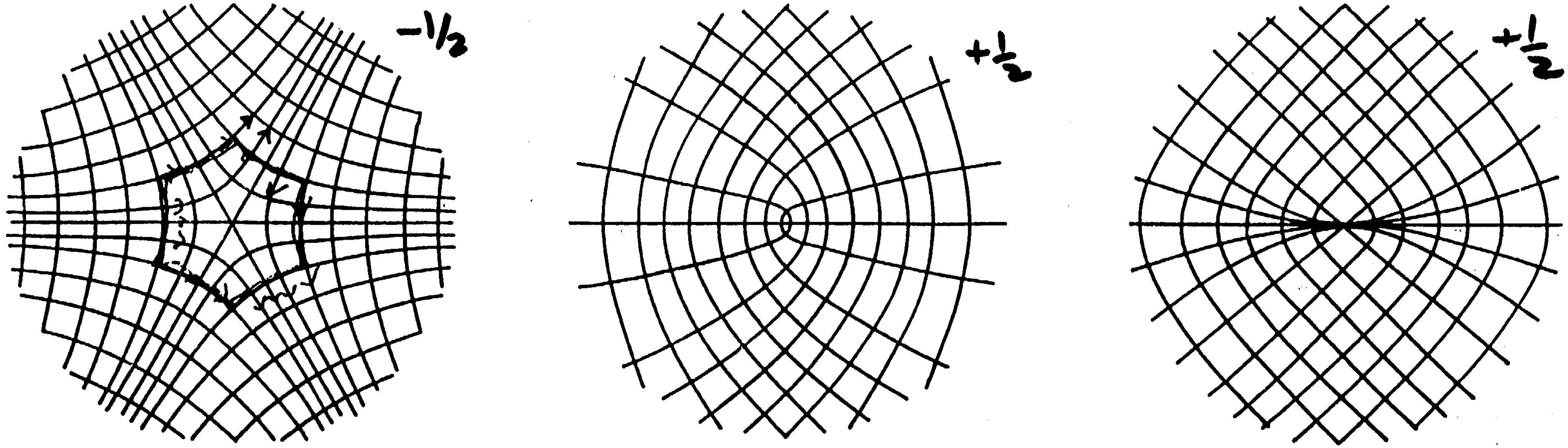
Eigenvectors: (orthogonal) directions of principal curvatures at X, Y.

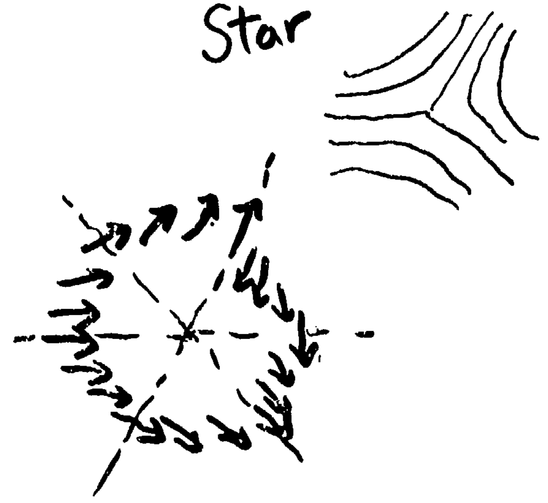
degeneracy: umbilic point on surface (locally spherical)

Sign change → a line of curvature turns through π (reverses) in a circuit of an umbilic point

Lines of curvature near an umbilic point of a surface

(surface : $z(x,y)$; curvature matrix: $\begin{pmatrix} \partial^2 z/\partial x^2 & \partial^2 z/\partial x \partial y \\ \partial^2 z/\partial y \partial x & \partial^2 z/\partial y^2 \end{pmatrix}$; eigenvalues: principal curvatures ;
eigenvectors: directions of principal curvature ; degeneracy = umbilic point (locally spherical)





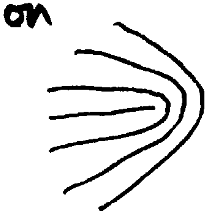
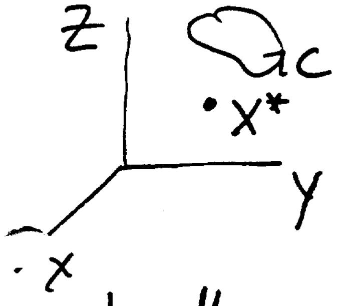


fig.44
(disclinations of directors of liquid crystals)

(Berry and Hannay (1977)

Degeneracies These are singularities of the 2-form generating the phase

$$\boxed{\gamma_n(C) = \iint_{\partial S=C} \mathrm{Im}\langle dn|\wedge|dn\rangle \equiv \iint_{\partial S=C} V_n(\underline{X})}$$



Singularities of V_n at X^*, where $|n\rangle$ (transported state) degenerates with $|n+1\rangle$ or $|n-1\rangle$

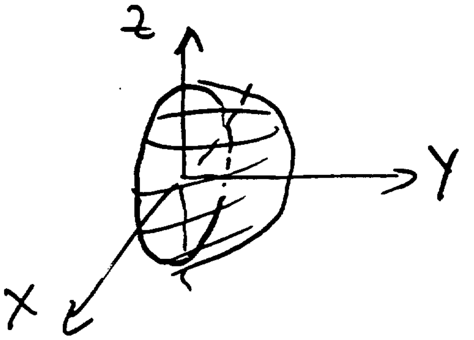
locally a 2-state system (like spin $-\frac{1}{2}$), so can choose parameters in which Hamiltonian is, locally

$$\boxed{H = \frac{1}{2}\begin{pmatrix} Z & X-iY \\ X+iY & -Z \end{pmatrix}} \rightarrow \boxed{V_n \approx \frac{\underline{R}\cdot d\underline{S}}{R^3}}$$

monopole

Then $\boxed{\gamma_n(C) \approx \pm\frac{1}{2}\Omega(C)}$ for C near X^*

Important special case: H real (e.g. time-reversal symmetry) i.e $Y=0$. Circuits in ZX plane



$\Omega = 2\pi$ (hemisphere)

$\therefore\ \gamma = \pm\pi$

$$\boxed{e^{i\gamma} = -1}$$

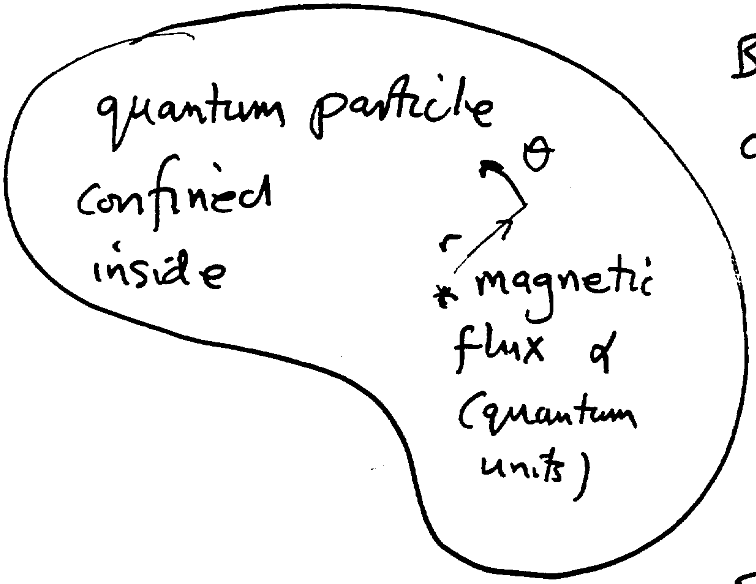
Sign change of real symmetric matrix round degeneracy

Exploring monopoles in parameter space

with Raul Mondragon

Need a 3-parameter family of quantum systems, uncomplicated by any symmetries (e.g. time reversal)

Such a family is Aharonov-Bohm chaotic billiards



Boundary is a cubic conformal transformation of unit disc, i.e

$$w(z) = z + Bz^2 + Ce^{i\chi}z^3$$

$$|z| = 1$$

Fix $\chi = \pi/3$

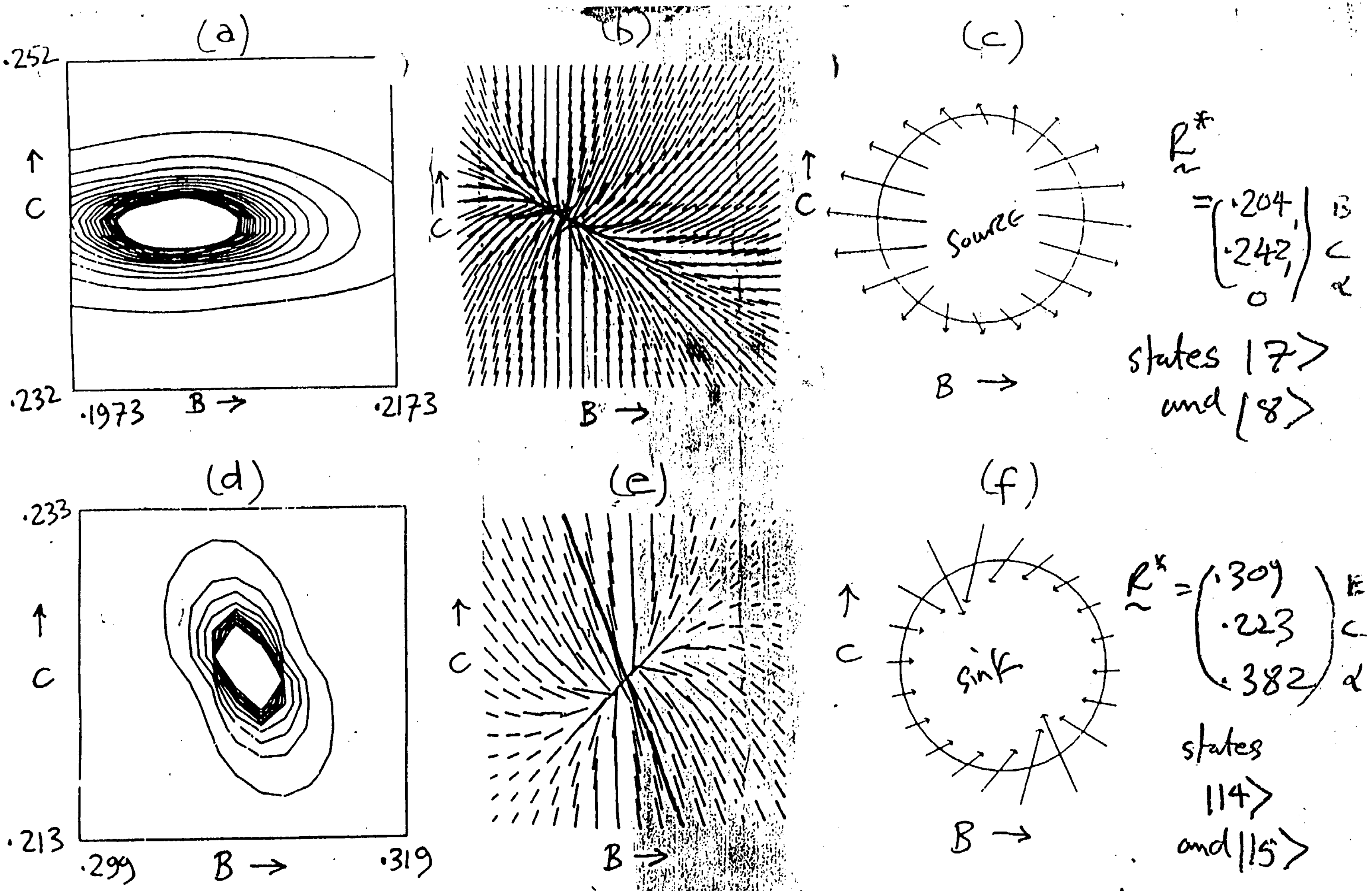
Hamiltonian

$$\hat{H} = -\left(\nabla - i\frac{\alpha\,\underline{e}_\theta}{r}\right)^2 \quad ; \quad \hat{H}|\psi\rangle = E|\psi\rangle$$

$\psi(\underline{r}) = \langle \underline{r}|\psi\rangle = 0$ on Boundary

Parameters $(B, C, \alpha) \equiv \underline{R}$

2 Form $\underline{V} = \mathrm{Im}\langle \nabla_{\underline{R}}\psi_n| \wedge |\nabla_{\underline{R}}\psi_n\rangle$



Contours of $|\underline{V}|$

lines of $\underline{V}$

Directions of $\underline{V}$ on a circuit of $\underline{R}^*$

Figure 5

Monopole behaviour of $\underline{V}$:

$$|\underline{V}| \propto \frac{1}{r^2}$$

where $r = \sqrt{(B-B^*)^2 + (C-C^*)^2}$

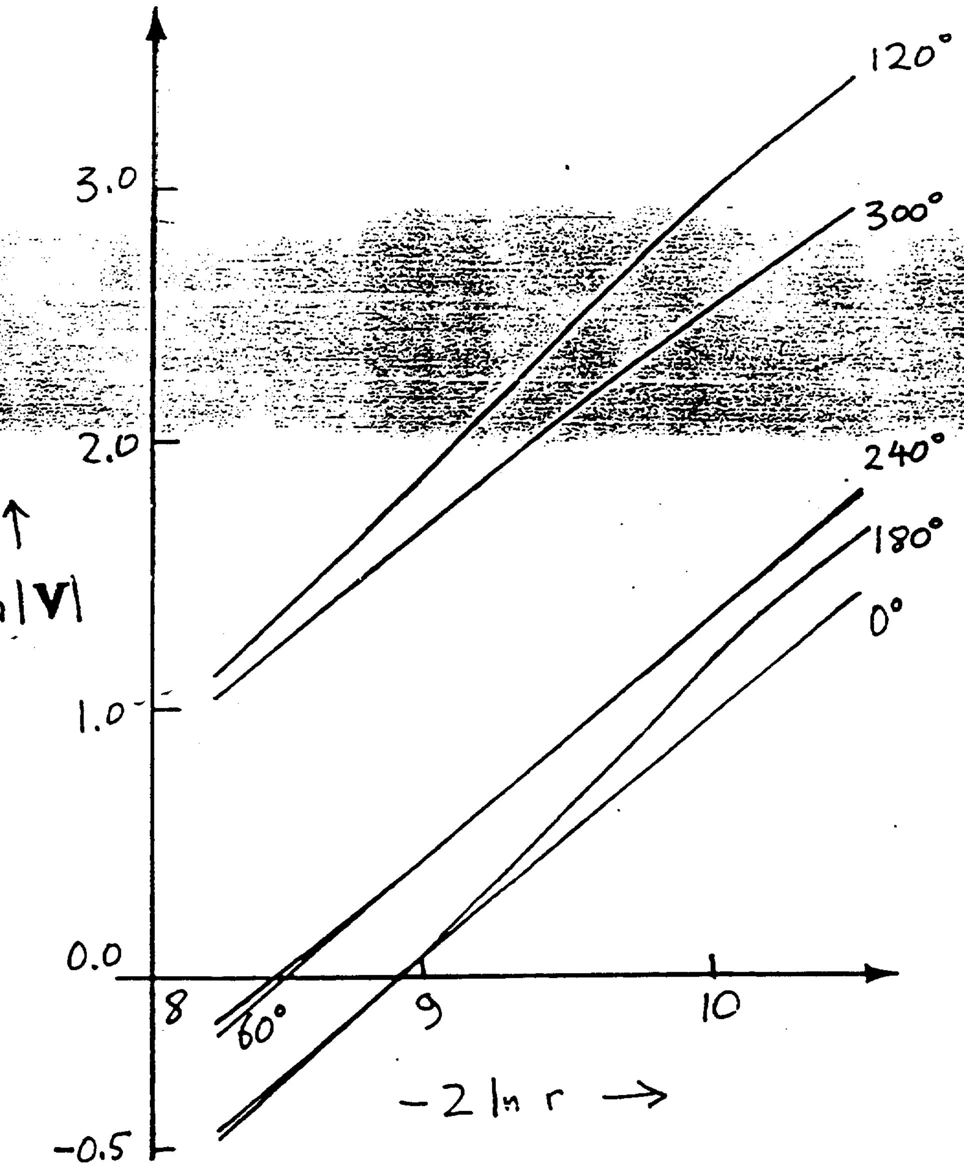
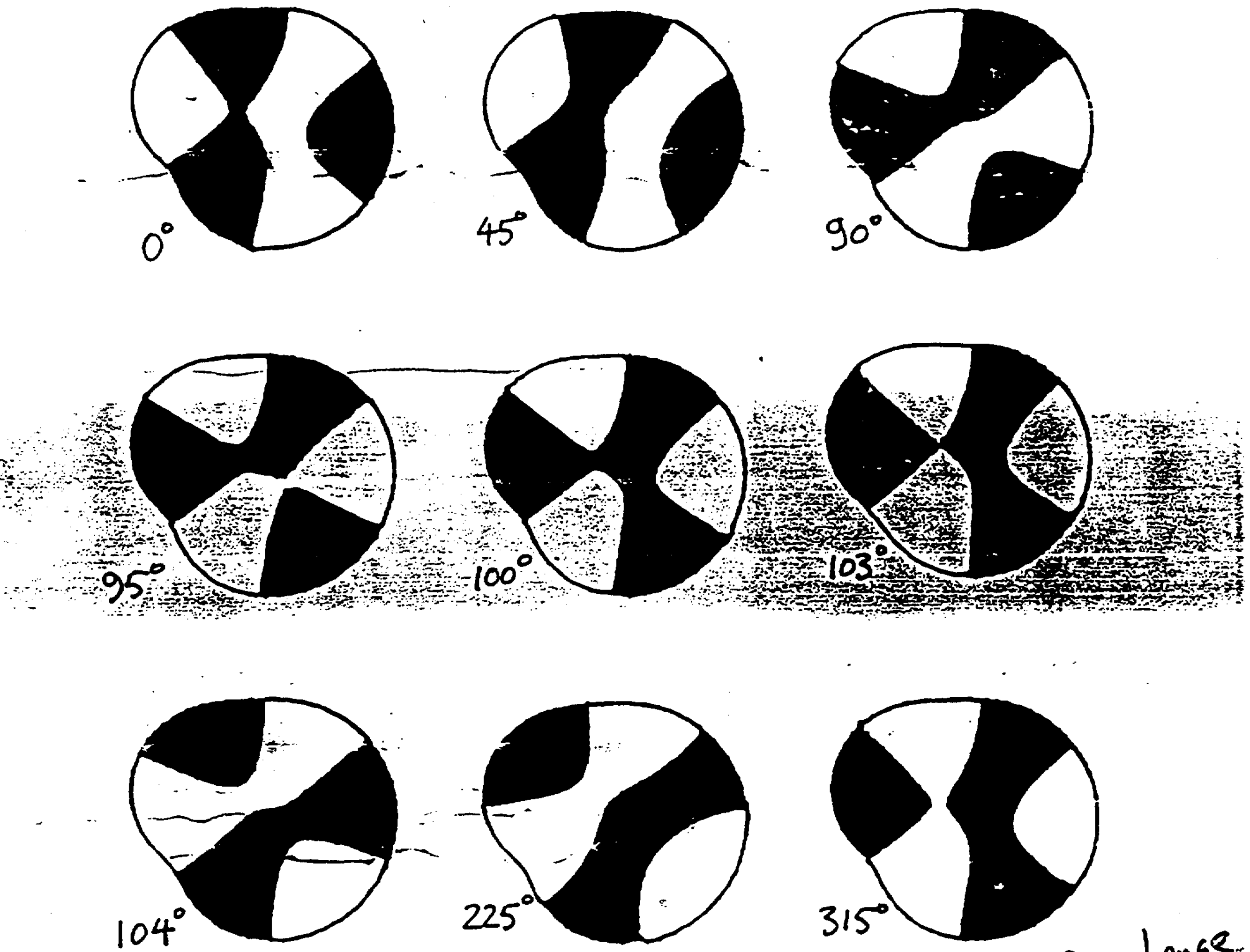


Figure 2

Rearrangement of nodal cells round degeneracy

at $\underset{\sim}{R}^* = (\underset{B}{.204} \quad \underset{C}{.242} \quad \underset{\alpha}{0}) \longrightarrow \therefore$ states real

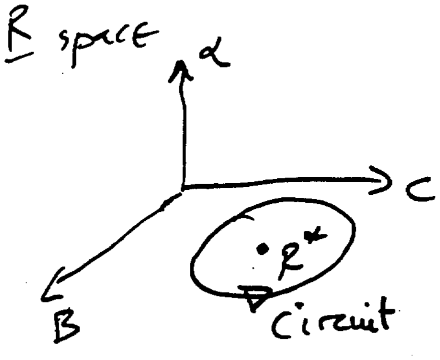


sign change

$$|\psi\rangle = \cos\frac{\theta}{2}\,|7\rangle + \sin\frac{\theta}{2}\,|8\rangle \qquad 0 \leq \theta \leq 2\pi$$

two states at $\underset{\sim}{R}^*$

Figure 7

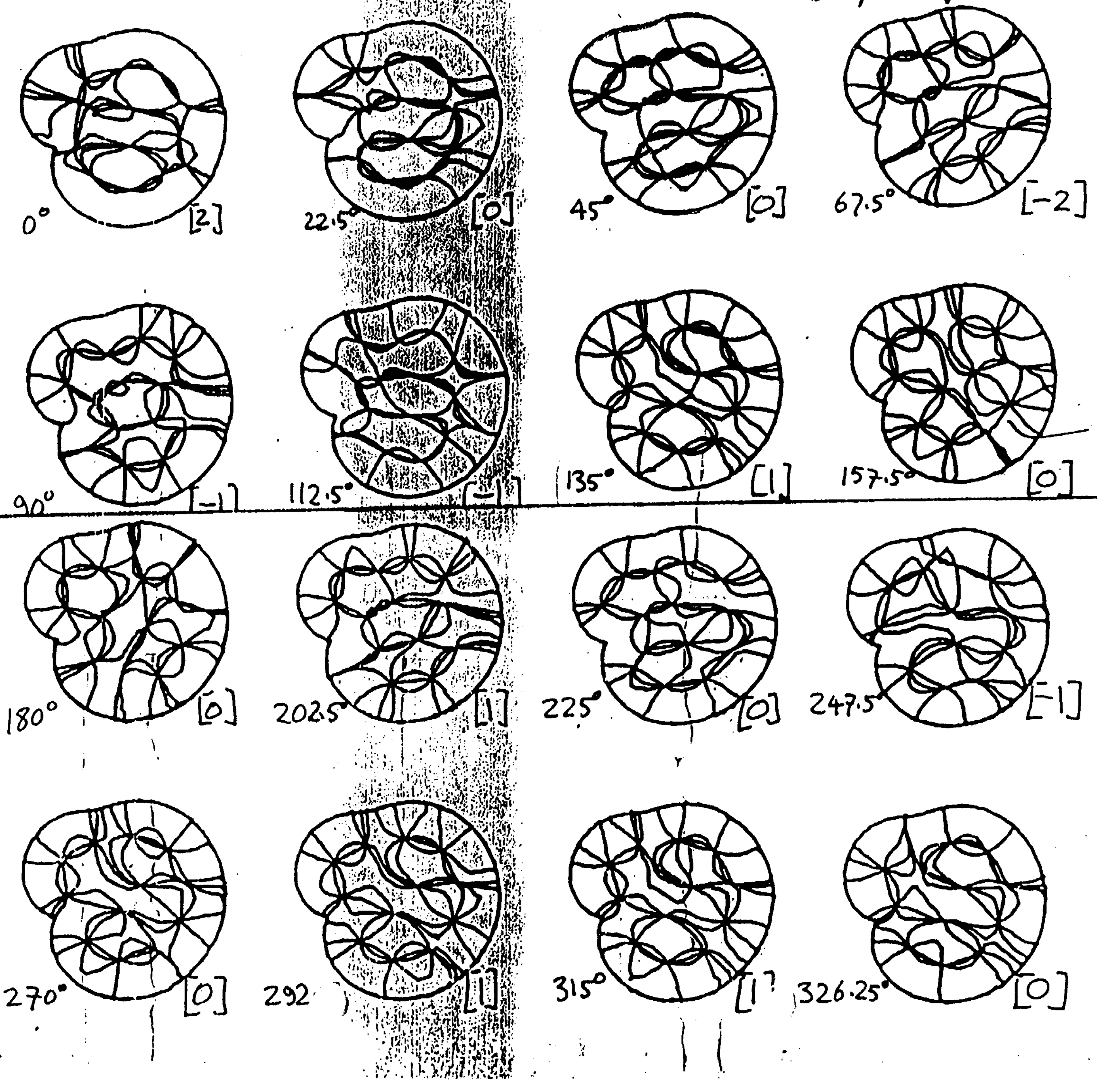


singularities within singularities — contours of phase χ of complex wavefunction

$$\psi(\underline{r}) = \rho(\underline{r}) \times e^{i\chi(\underline{r})}$$

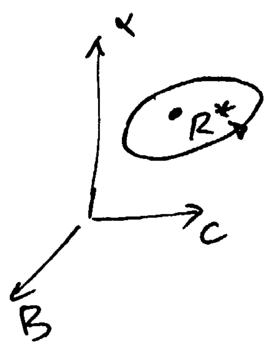
round a circuit of degeneracy at $\underset{\sim}{R}^* = \begin{pmatrix} .309 \\ .223 \\ .382 \end{pmatrix}$

states $|14\rangle$ and $|15\rangle$



wavefront dislocations

Circuit



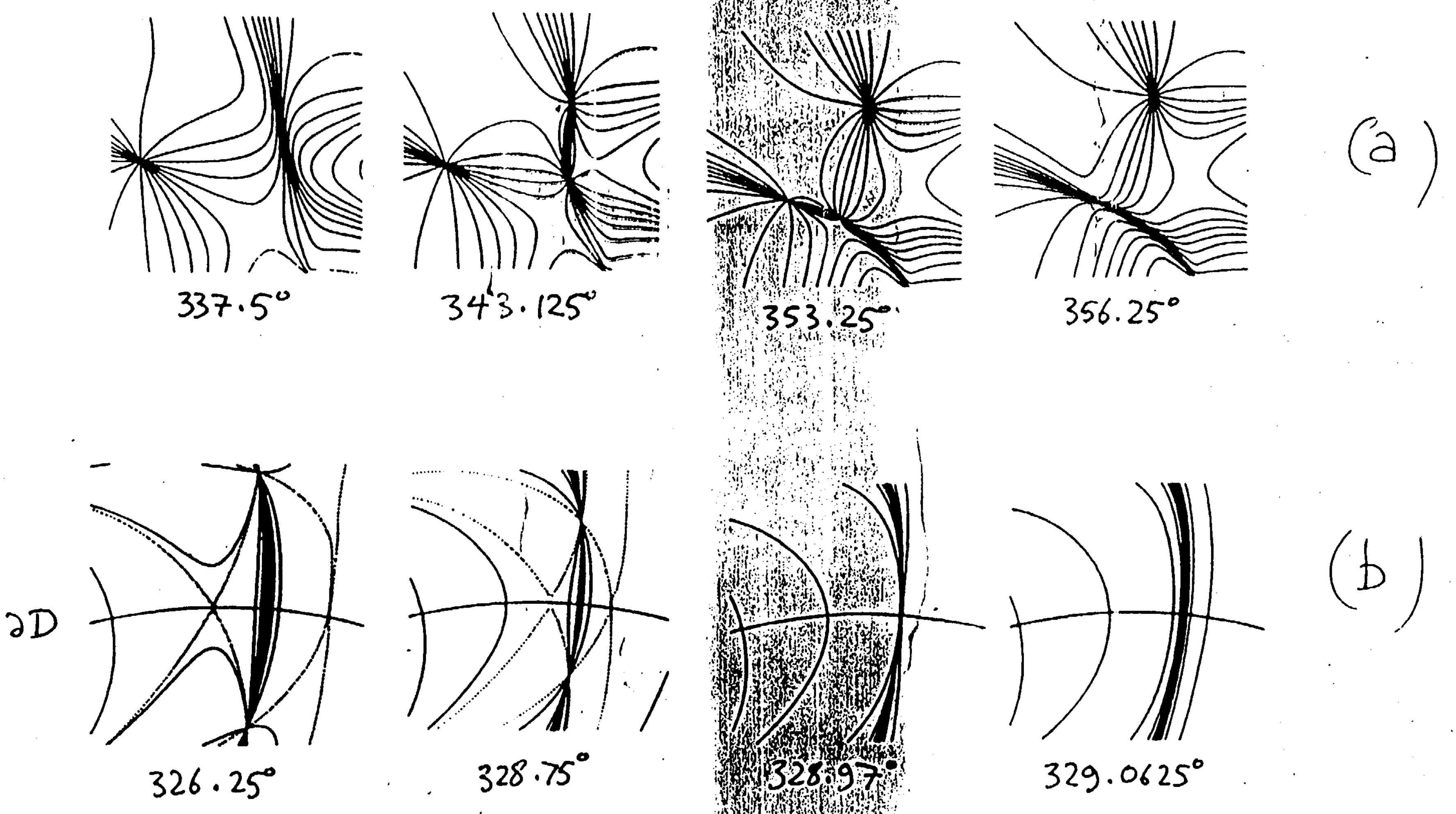


Figure 10

Singularities within singularities within singularities

collisions dislocation monopole

5. Disclinations in waves (Nye 1983)

In general 3-D spatially varying vector waves, there are singular lines where $\underline{E}$ is linearly polarized or circularly polarized.

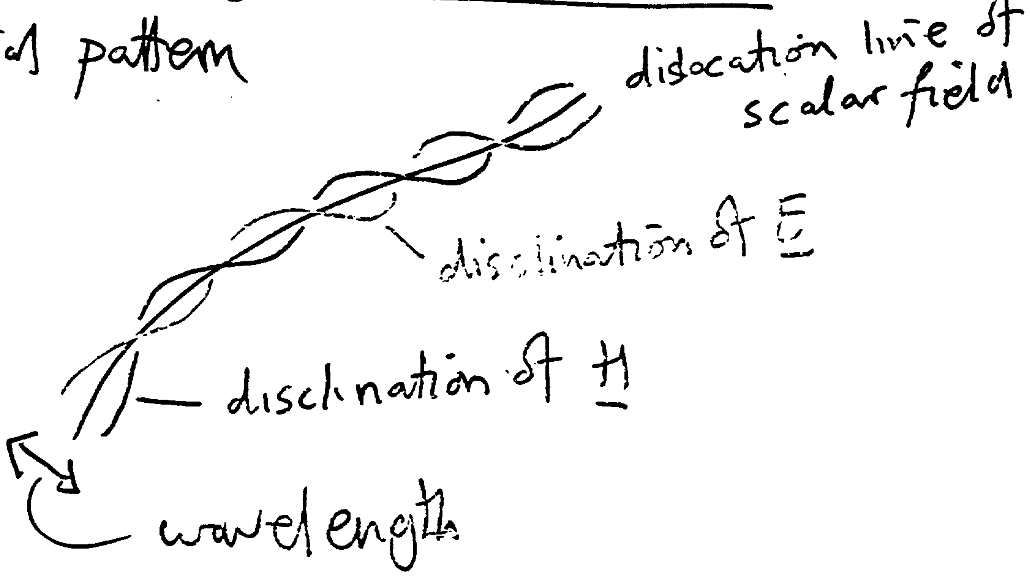
2π anholonomy — π anholonomy

These line singularities are 'a useful way of describing the spatial and temporal geometrical features of electromagnetic fields that would otherwise be far too complicated to visualize. The disclination carries with it a certain local field structure; therefore, a description of the arrangement and motion of the disclinations contains much of the essential geometrical information about the field itself. They constitute elements of structure in the field.'

NPL

Electromagnetic double helix :

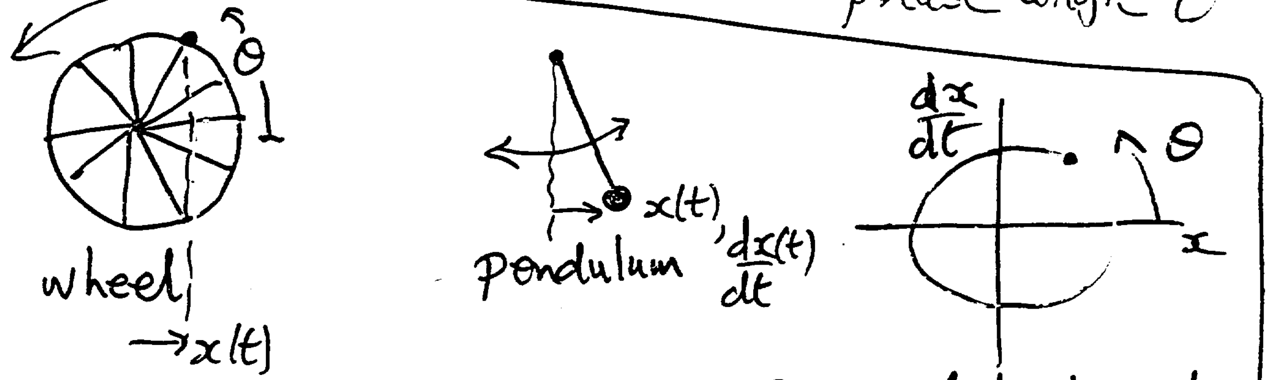
typical pattern



NEWTONIAN ANHOLONOMY

A quantum state with a fixed environment is just a complicated oscillator, and shows phase anholonomy as the environment is altered. In classical mechanics, oscillators abound — wheels, pendulums, in fact any periodic or multiply periodic motion — so these should show anholonomy too.

$$\boxed{\underbrace{x}_{\text{oscillator coordinate}}(\underbrace{t}_{\text{time}};\underbrace{X}_{\text{parameters}}) = \underbrace{A(X)}_{\text{amplitude}}\ {}^{\sin}_{\text{or}\cos}\Big\{\underbrace{\underbrace{\omega(X)}_{\text{frequency}}t}_{\text{generalized phase angle }\theta}\Big\}}$$



θ is an angle in space

θ is an abstract angle in phase(!) space

over time T, θ changes by ω(X)T for fixed environment X. When X is slowly cycled, then, as before, θ acquires not only the expected dynamical change but a geometric one:

$$\omega = -\frac{\partial\mathcal{E}}{\partial n}$$

$$\boxed{\theta_{\text{end}} - \theta_{\text{start}} = \underbrace{\int_0^T dt\,\omega(X(t))}_{\text{dynamical}} + \underbrace{\Delta\theta(C)}_{\text{Hannay's angle}}}$$

Chaos?

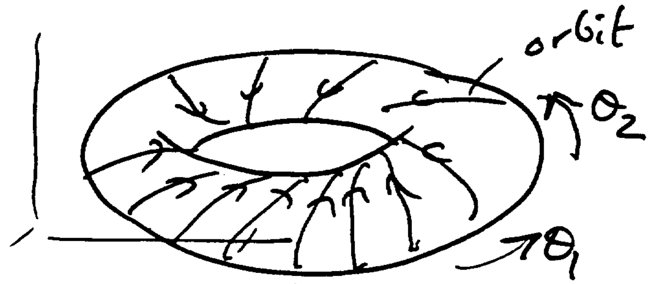
Hannay's angles in classical multiply periodic systems.

Classical Hamiltonian

$$H\,(\underline{q}\,,\,\underline{p}\,,\,\underline{X}(t))$$ D freedoms

$\underline{q}$: $q_1 \cdots q_D$ $\quad$ $\underline{p}$: $p_1 \cdots p_D$ $\quad$ $\underline{X}(t)$: parameters $\{X_1, \cdots\}$

For fixed $\underline{X}$, H is integrable : D conserved quantities, motion on D - torus in 2D dim. phase space:



Torus labelled by

$$\underline{I} = \{I_1 \cdots I_D\}$$

$$I_j = \frac{1}{2\pi} \oint_{\gamma_j} \underline{p} \cdot d\underline{q}$$

$1 \leq j \leq D$: D $\boxed{\text{actions}}$ (conserved)

Points on torus labelled by

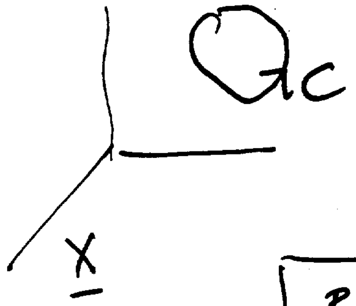
q's $\boxed{\text{angles}}$ $\underline{\theta} = \{\theta_1 \cdots \theta_D\}$

Can express H in terms of actions

$H = H(\underline{I})$; frequencies

$$\boxed{\omega_j = \frac{\partial H(\underline{I})}{\partial I_j} = \dot{\theta}_j}$$

Now let parameters $\underline{X}$ vary round a cycle C



At the end, the torus returns ($\underline{I}$ conserved: classical adiabatic theory)

$\boxed{\text{But } \underline{\theta} \text{ does not.}}$

Hannay:

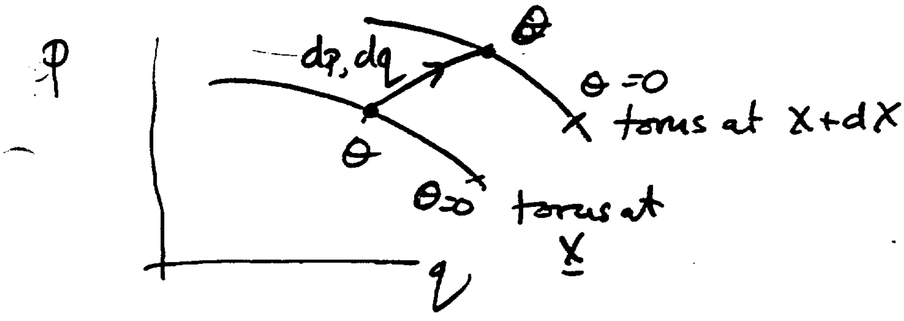
$$\theta_{j\,\mathrm{end}} - \theta_{j\,\mathrm{start}} = \underbrace{\int_0^T dt\, \omega_j(\underline{X}(t))}_{\text{dynamical}} + \underbrace{\Delta\theta_j(C)}_{\text{Hannay}}$$

$$\boxed{\Delta\theta_j = -\frac{\partial}{\partial I_j}\iint_{\partial S = C} \langle d\underline{p} \wedge \cdot\, d\underline{q} \rangle}$$

From Hamilton's equations

$$\langle\ \rangle = \frac{1}{(2\pi)^D}\int_0^{2\pi} d\theta_1 \cdots \int d\theta_D$$

= torus average



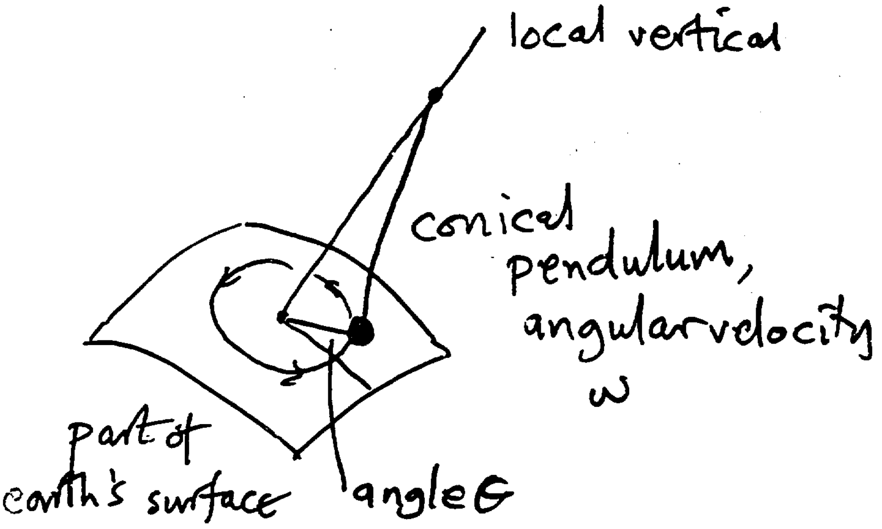
Semiclassical theory : from WKB association torus ⟷ eigenstate

$$\gamma_n(C) \approx \frac{1}{\hbar}\iint_{\partial S = C} \langle d\underline{p} \wedge \cdot\, d\underline{q} \rangle$$

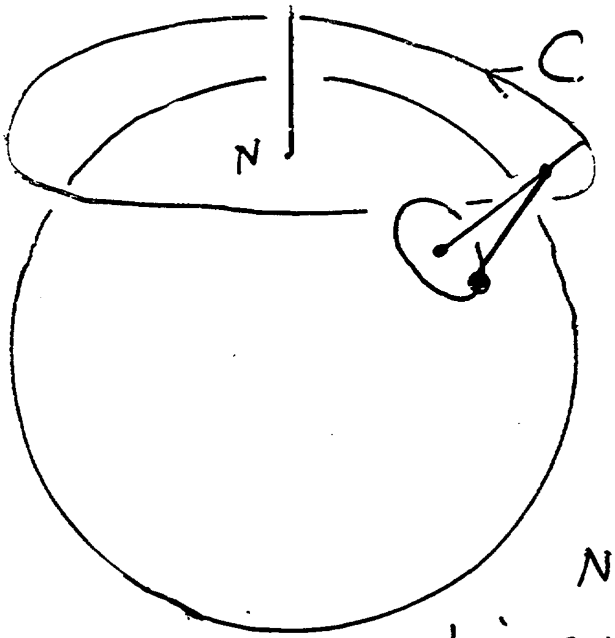
i.e $\boxed{\Delta\theta_j = -\hbar\frac{\partial}{\partial I_j}\gamma_n(C) = -\frac{\partial}{\partial n_j}\gamma_{\underline{n}}(C)}$

e.g. spin $\boxed{\gamma_n(C) = -n\Omega}$, so $\boxed{\Delta\theta = \Omega}$

Example : Foucault pendulum



During 24 hours $(=T)$, the local vertical (environment) turns :



angle turned through by bob of pendulum is

$$\theta_{end} - \theta_{start} = \omega T + \Omega$$

solid angle of C
- Hannay

Now,

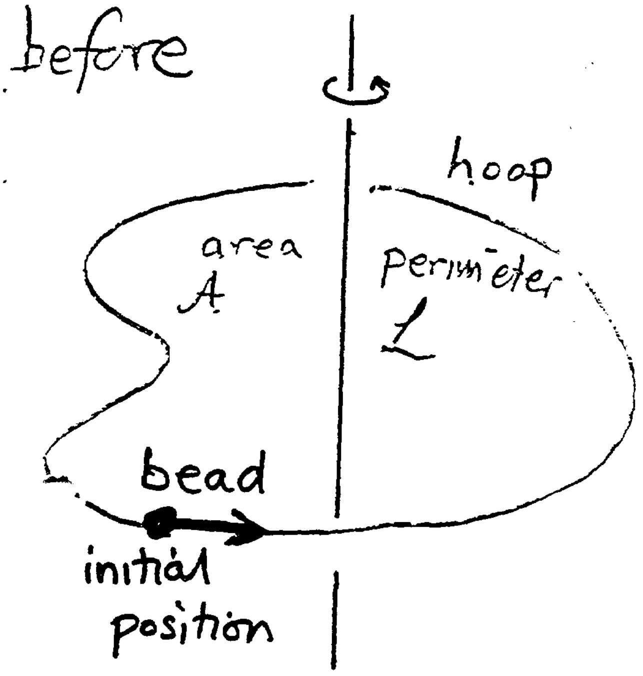
linear vibration = circular + circular
⤢ = ⟳ + ⟲

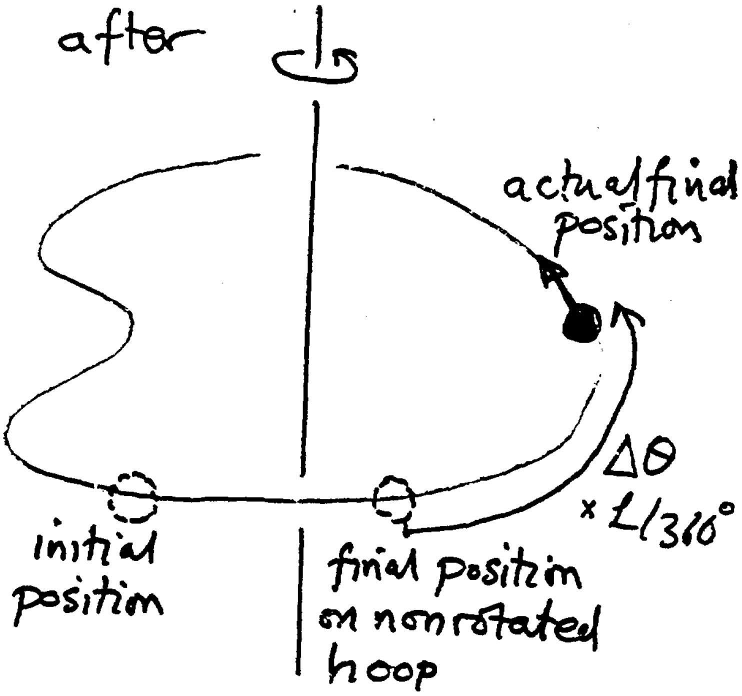
So, circular angle shift Ω $\longrightarrow$ rotation of linear vibration by $\Omega = 2\pi(1 - \sin(\text{latitude}))$

$\rightarrow$ angular vel. of direction relative to earth's surface
$= (\Omega - 2\pi)/T = \frac{2\pi}{T}\sin(\text{latitude})$ (11.7°/hr in Bristol)

= parallel transport of vibration (again!)

Example: Rotated rotator (Hannay's hoop)





Hannay angle:

$$\boxed{\Delta\theta = 2\pi\left(1 - \frac{4\pi\mathcal{A}}{\mathcal{L}^2}\right)}$$

- purely geometric; zero for circle ;
  Euler force.

For chaotic classical dynamics, no tori no actions, no Hannay angles.

Does

$$\boxed{V = \mathrm{Im}\langle dn \wedge dn\rangle} \qquad \left(\gamma_n(C) = \iint_{\partial S = C} V\right)$$

have a classical limit? Yes. Main steps of derivatio

1) $$\boxed{\sum_m |m\rangle\langle m| = 1}$$

2) $$\boxed{\langle m\, dn\rangle = \frac{\langle m|d\hat{H}|n\rangle}{E_n - E_m} \qquad (\text{twice})}$$

3) $$\boxed{\frac{1}{(E_n-E_m)^2} = -\frac{1}{\hbar^2}\int_0^\infty dt\, e^{\frac{it}{\hbar}(E_n-E_m)}}$$

4) quantum commutator classical Poisson bracket

$$\boxed{[\hat{a},\hat{b}] \rightarrow i\hbar\,\{a(z), b(z)\}} \qquad z = \{\underline{q}, \underline{p}\}$$

5) $$\boxed{\langle n|\hat{a}|n\rangle \rightarrow \langle a(z)\rangle} \quad \text{average over classical invariant manifolds corresponding to } |n\rangle$$

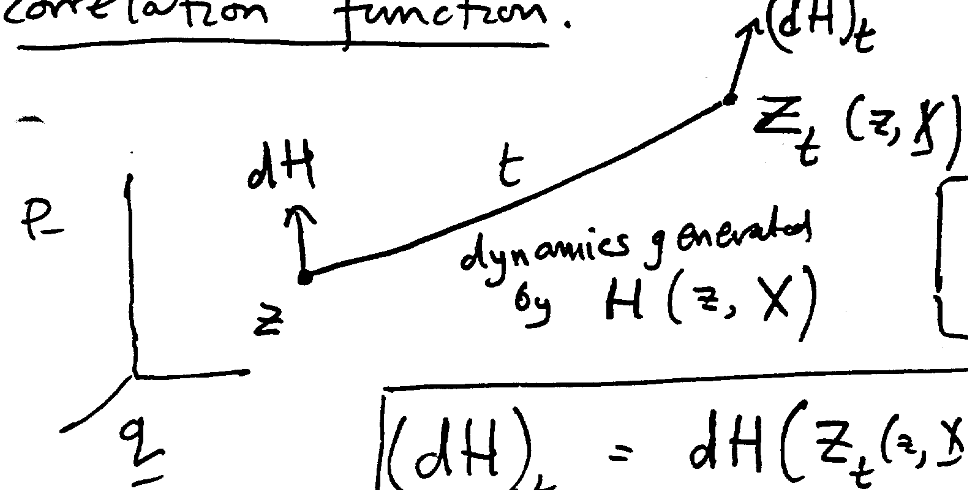
$$= \frac{1}{\partial_E \Omega}\int dz\, \delta(E - H(z,\underline{X}))\,a(z) \quad \text{in ergodic case}$$

$\partial_E \Omega$: phase volume of energy surface $H=E$

Result

$$\boxed{V(\underline{X}) \xrightarrow{\hbar\to 0} -\frac{1}{2\hbar\,\partial_E\Omega}\,\partial_E\left[\partial_E\Omega\int_0^\infty dt\,\left\langle (dH)_t \wedge dH\right\rangle\right]}$$

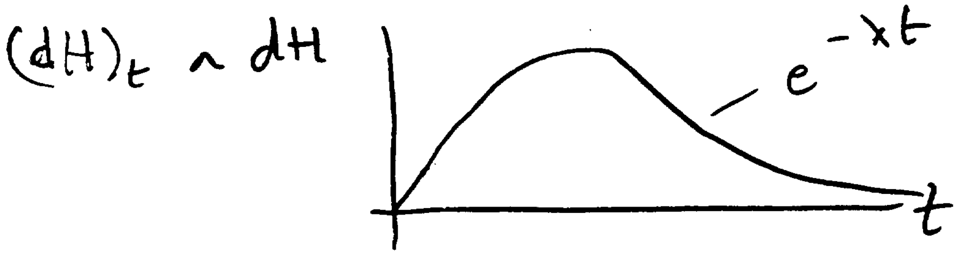
This is the time integral of a classical correlation function.



$$\boxed{dH = H(\underline{X}+d\underline{X}) - H(\underline{X})}$$

$$\boxed{(dH)_t = dH\left(Z_t(z,\underline{X}),\,\underline{X}\right)}$$

V involves in an essential way the history of the chaos. Estimate:



λ = entropy (rate of exponential separation)

Thus

$$\boxed{V(x) \approx -\frac{1}{2\hbar\,\partial_E\Omega}\,\partial_E\left[\frac{\partial_E\Omega}{\lambda^2}\left\langle \{H, dH\}\wedge dH\right\rangle\right]}$$

particles in $\underline{B}$ fields : $\langle \underline{L}_t \wedge \underline{L}\rangle$

Lecture 3

(Some geometric phases, Michael Berry Geneva 1993)

DYNAMICS OF THE ENVIRONMENT

So far, have divided the world into

$$\boxed{\text{World} = \text{system} + \text{environment}}$$

acting through parameters $X(t)$

e.g. spin, electron

But there is no unilateral action in physics, so the system must react back on the dynamics of the parameters $X(t)$, which are themselves subject to quantum mechanics. One reaction effect is on the phase of the quantum state of the environment: roughly, the world's state is single-valued, so the environment must acquire a phase $-\gamma_n(C)$ to compensate the system's $+\gamma_n(C)$.

This affects $\boxed{\text{molecules}}$ when divided into:

molecule = electrons + nuclei

"system" - light, swift

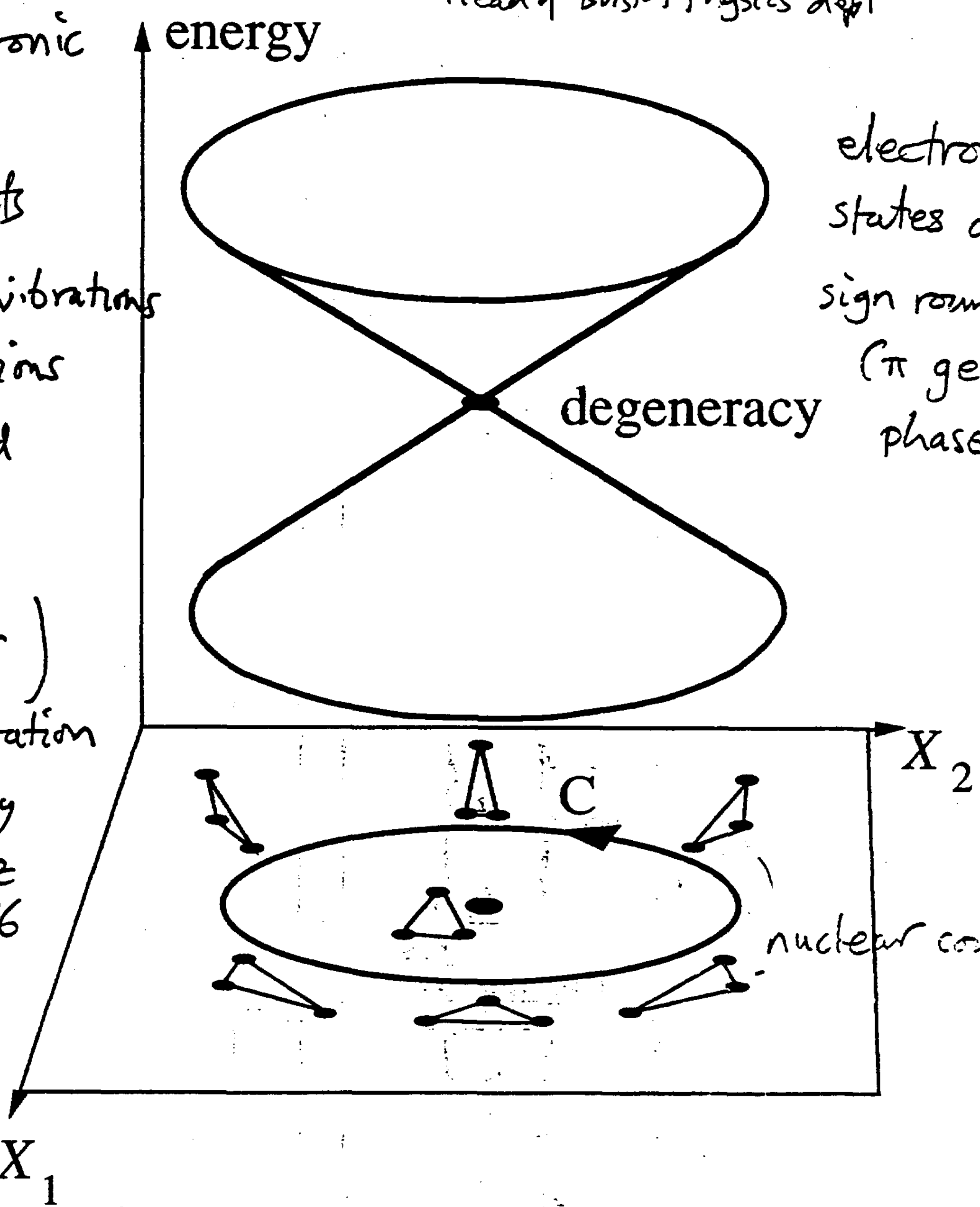
"environment" - ponderous, slow

Environmental parameters are nuclear position coordinates X, whose quantization gives the vibration-rotation spectrum of the molecule.

6. Molecular electronic degeneracies

(Longuet-Higgins, Öpik, Pryce, Sack 1958)

Head of Bristol Physics dept



electron π affects nuclear vibrations and rotations (half-odd vibronic quantum number)

- pseudorotation observed by Delacrétaz et al 1986

electronic states change sign round C (π geometric phase)

nuclear configurations

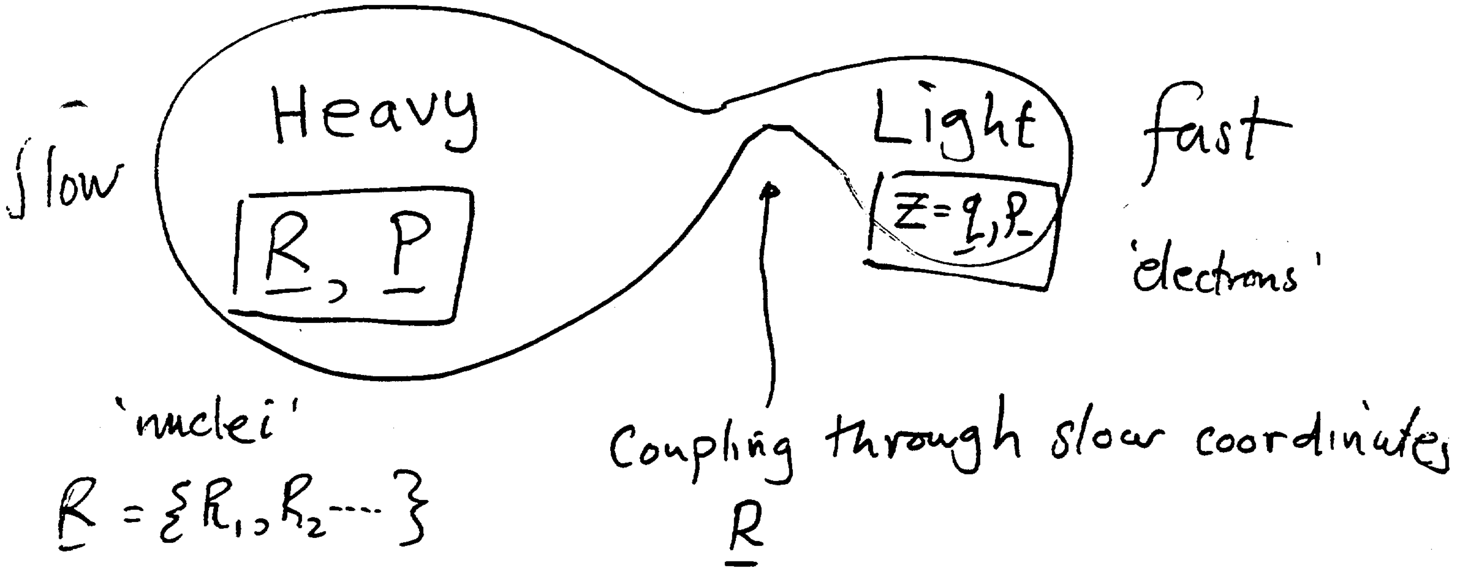
Figure 4. Circuit in shape space X of nuclei surrounding the equilateral molecule for which there is an electron energy degeneracy (conical intersection)

'This half oddness is at first sight strange, but may be understood by noting that [around a circuit] the electronic factor in the wave function will be multiplied by −1, so that the angular part of the nuclear factor must do likewise if the total wavefunction is to be single-valued'
(wheel balancing)

Generalization :

CLASSICAL GEOMETRIC REACTION FORCES

Coupled systems (classical or quantum)



$\underline{R} = \{R_1, R_2 \cdots\}$

Here, concentrate on slow dynamics.

Simplest approximation (Born–Oppenheimer, Adiabatic averaging) :

Averaged fast energy $E_{BO}(\underline{R})$ depends on slow coordinates, and its gradient acts as a reaction force on the slow system

$$\boxed{-\partial_i E_{BO}(\underline{R})} \qquad \partial_i \equiv \frac{\partial}{\partial R_i}$$

Born-Oppenheimer approximation depends on an adiabatic separation of time scales:

Fast motion can be calculated with slow coordinates $\underline{R}$ acting as parameters, i.e. frozen.

An improved approximation, more consistent with the adiabatic assumption, includes two extra reaction forces on the slow motion: 'magnetic' and 'electric'

These first appeared in quantum mechanics (modifying vibration-rotation ('slow') energy levels in molecules) but have classical counterparts.

$$\boxed{H(\underline{R},\underline{P},z) = \overset{\text{slow}}{\frac{1}{2}\sum\sum \underset{\text{inverse mass matrix (small)}}{C_{ij}} P_i P_j} + \overset{\text{fast}}{h(z,\underline{R})}}$$

Gauge approximation to slow motion based on effective Hamiltonian

$$\boxed{\begin{aligned} H_g(\underline{R},\underline{P}) = \tfrac{1}{2}\sum\sum Q_{ij}\,(P_i - A_i(\underline{R}))(P_j - A_j(\underline{R})) \\ + E_{BO}(\underline{R}) + \Phi(\underline{R}) \end{aligned}}$$

So 3 reaction forces act

Born - Oppenheimer	$-\partial_i E_{BO}(\underline{R})$
Magnetic gauge	$B_{ij} = \partial_i A_j(\underline{R}) - \partial_j A_i(\underline{R})$
Electric gauge	$-\partial_i \Phi(\underline{R})$

<u>Quantum formulae for geometric forces</u>: let fast h have eigenstates $|n(\underline{R})\rangle$ and eigenvalues $E_n(\underline{R})$. Adiabatic assumption: no transitions, i.e. fast state remains fixed in one of the $|n\rangle$'s. Then effective slow Hamiltonian is

$$H_g = \langle n|H|n\rangle = \tfrac{1}{2}\sum\sum Q_{ij}\langle n(\underline{R})|\,P_i P_j\,|n(\underline{R})\rangle + E_n(\underline{R})$$

But $P_i = -i\hbar\dfrac{\partial}{\partial R_i}$, so

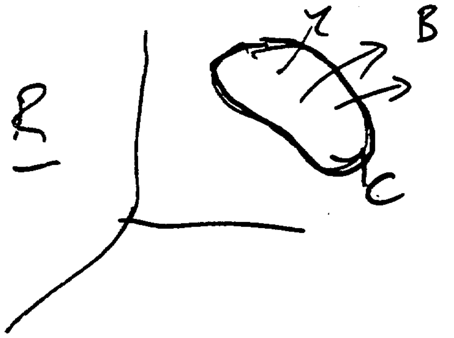
$$P_j|n\rangle = -i\hbar|\partial_i n\rangle + |n\rangle P_j\,,$$

leading to

$$\boxed{\begin{aligned} & E_{Bo}(\underline{R}) = E_n(\underline{R}) \\ \hline & B_{ij}(\underline{R}) = i\hbar\left(\langle \partial_i n | \partial_j n\rangle - \langle \partial_j n | \partial_i n\rangle\right) \\ & \quad 2\text{form}: \quad \langle dn | \wedge | dn\rangle \\ \hline & \Phi(\underline{R}) = \sum\sum Q_{ij} g_{ij} \\ & g_{ij} = \frac{\hbar^2}{2} \langle \partial_i n | (1 - |n\rangle\langle n|) | \partial_j n\rangle \end{aligned}}$$

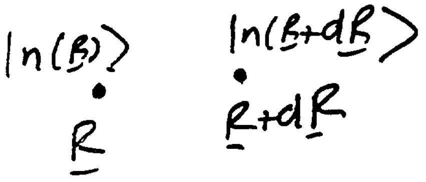
NB: B_{ij} and Φ reflect (in the slow system) geometric aspects of the fast motion:

Geometric phase: in a cycle of $\underline{R}$, phase of fast state changes by



$$\boxed{\gamma_n(C) = i \iint_{\partial S = C} \langle dn | \wedge | dn\rangle}$$

Fast metric



$$\boxed{\begin{aligned} (\text{distance})^2 &\equiv 1 - |\langle n | n+dn\rangle|^2 \\ &= g_{ij}\, dR_i\, dR_j \end{aligned}}$$

(Provost & Vallee)

Remarkably, $\hbar$ drops out of the classical limit of these formulae, leaving geometric reaction forces in purely classical dynamics.

Main steps of classical limit: (with J.M. Robbins)

1) $$\sum_m |m\rangle\langle m| = 1$$

2) $$\langle m | \partial_i n\rangle = \frac{\langle m|\partial_i h|n\rangle}{E_n - E_m}$$ (twice)

3) $$\frac{1}{(E_n-E_m)^2} = -\frac{1}{\hbar^2}\int_0^\infty dt\, t\, e^{i\frac{t}{\hbar}(E_n-E_m)}$$

4) $$\overset{\text{quantum commutator}}{[a,b]} \mapsto \overset{\text{classical Poisson bracket}}{i\hbar\{a(z), b(z)\}}$$

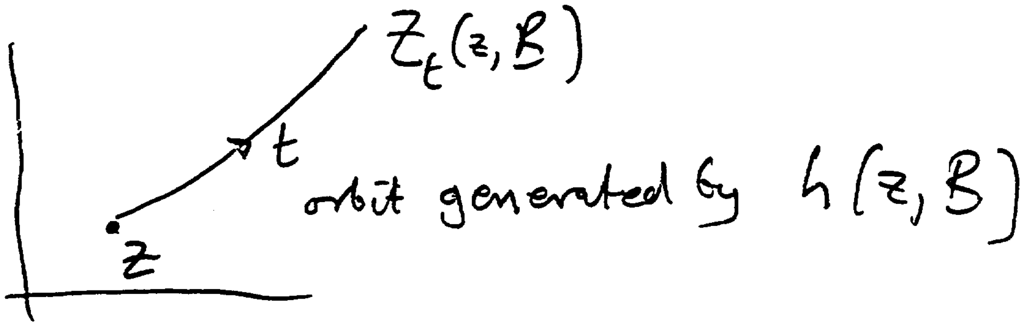
5) $$\langle n| a |n\rangle \to \langle a(z)\rangle$$ average over classical invariant manifold corresponding to $|n\rangle$

$$= \frac{1}{\underbrace{\partial_E\Omega}_{\text{phase volume}}}\int dz\, \delta(E-h(z,R))\, a(z)$$ in ergodic case

These lead to classical formulae that apply also to chaotic fast systems:

$$\boxed{\begin{aligned}&E_{B_0}(\underline{B}) = E(\underline{B}) \quad \text{, determined by} \\ &\text{constancy of adiabatic invariant : phase volume} \\ &\Omega(E,\underline{B}) = \int dz\, \underset{\substack{\uparrow \\ \text{unit step}}}{\Theta}\{E - h(z,\underline{B})\}\end{aligned}}$$

$$\boxed{\begin{aligned}&B_{ij}(\underline{B}) = -\frac{1}{2\partial_E\Omega}\,\partial_E\left[\partial_E\Omega \int_0^\infty dt \left\langle (\partial_i h)_t\, \partial_j h - (\partial_j h)_t\, \partial_i h \right\rangle_E\right] \\ &\text{where } (\partial_i h)_t \equiv \partial_i h\left(Z_t(z,\underline{B}), \underline{B}\right)\end{aligned}}$$



$$\boxed{g_{ij}(\underline{B}) = -\frac{1}{2}\int_0^\infty dt\, t \left\langle \underbrace{\left((\partial_i h)_t - \partial_i E\right)}_{\text{fluctuation in } \partial_i h \text{ at } t}\left(\partial_j h - \partial_j E\right)\right\rangle_E}$$

These involve correlations

$$\boxed{C_{ij}(t) \equiv \left\langle (\partial_i h - \partial_i E)_t\, (\partial_j h - \partial_j E)\right\rangle_E}$$

Connection with linear response theory :

Can derive geometric forces from classical mechanics, using shift of distribution of fast variables away from microcanonical.

In the chaotic case, get, in addition,

FRICTION – dissipation of slow motion by energy absorption by fast motion. Velocity dependent forces

$$\boxed{\ddot{R}_i = -k_{ij}\dot{R}_i}$$

antisymetric part = magnetic force

symmetric part is friction

$$\boxed{\frac{1}{2\partial_E\Omega}\partial_E\left[\partial_E\Omega\int_0^\infty dt\,\{C_{ij}(t)+C_{ji}(t)\}\right]}$$

N.B. Rate of energy absorption by fast chaos is

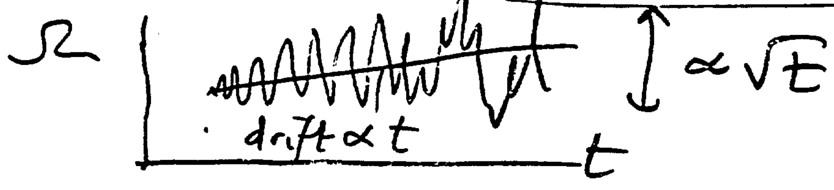
$$\boxed{\dot{E}_a = \dot{R}_i k_{ij}\dot{R}_j}$$

Drift in adiabatic invariant

$$\boxed{\dot{\Omega} = \partial_E\Omega\;\dot{R}_i k_{ij}\dot{R}_j}$$

– superimposed on fluctuations

$$\boxed{(\Delta\Omega)^2_t \underset{t\to\infty}{\Rightarrow} \frac{(\partial_E\Omega)^2}{2}\dot{R}_i\dot{R}_j\left\{t\int_0^\infty d\tau\,C_{ij}(\tau) - \int_0^\infty d\tau\,\tau\,C_{ij}(\tau)\right\}}$$

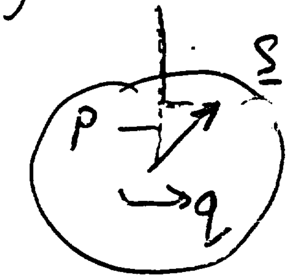


Spin model (with J. M. Robbins)

$$\boxed{H = \frac{P^2}{2} + \underline{R}\cdot\underline{S}}$$

Heavy particle in 3D, coupled with spin $\underline{S}$ via position $\underline{R}$ (sphere of monopolium)

$\underline{S}$ is fast system (1 freedom)



Adiabatic regime

$$\boxed{R = |\underline{R}| \text{ large}}$$

(from eliminating mass)

Exact equations of motion (4 freedoms, 8D phase space)

$$\boxed{\text{fast :} \quad \dot{\underline{S}} = \underline{R}\wedge\underline{S}}$$

$\underline{S}$ precesses about instantaneous $\underline{R}$

$$\boxed{\text{slow} \quad \dot{\underline{R}} = \underline{V} = \underline{P} , \ddot{\underline{R}} = \dot{\underline{P}} = -\underline{S}}$$

$\underline{S}$ forces $\underline{R}$

Motion probably nonintegrable, but can eliminate fast motion (unusual!) exactly, using conservation of total angular momentum:

$$\boxed{\underline{J} = \underline{R}\wedge\underline{V} + \underline{S} \quad = \text{constant}}$$

Thus $$\boxed{\dot{\underline{V}} = \underline{R} \wedge \underline{V} - \underline{J}}$$ (measure-preserving but not Hamiltonian $\nabla \cdot \underline{B} \neq 0$)

(charged particle in monopolium, plus 'gravity')

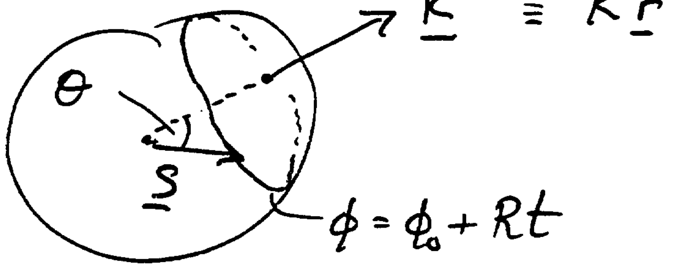
conserved: $$\boxed{\begin{array}{l} E = \frac{1}{2}V^2 + \underline{J} \cdot \underline{R} \\ \hline |S| = |\underline{J} - \underline{R} \wedge \underline{V}| \end{array}}$$ (energy) (magnitude of acceleration)

Geometric forces

Fast motion generated by

$$h = \underline{R} \cdot \underline{S} \quad \text{, i.e. } \dot{\underline{S}} = \underline{R} \wedge S \text{ , } \underline{R} \text{ fixed}$$

need $\nabla_{\underline{R}} h = \underline{S}$



motion integrable and ergodic

Invariant manifolds : latitude circle θ

Adiabatic invariant

$$\boxed{I = \underline{S} \cdot \underline{r} = S\cos\theta} \left(= \frac{1}{2\pi}\Omega = \frac{1}{2\pi}\oint p\,dq \right)$$

Thus $E_{B0} = IR$

$$\underline{B} = -\frac{1}{2}\partial_E \int_0^\infty dt \left\langle \underline{S}_t \wedge \underline{S} \right\rangle_E$$

$$\underline{\Phi} = -\frac{1}{2}\int_0^\infty dt\, t \left\langle (\underline{S}_t - I\underline{r}) \cdot (\underline{S} - I\underline{r}) \right\rangle_E$$

Magnetic $\underline{B} = -I\,\frac{\underline{r}}{R^2}$ monopole

electric $\Phi = \frac{S^2 - I^2}{2R^2}$ inverse-cube repulsive force

Thus adiabatic theory predicts acceleration

$$\boxed{\dot{\underline{V}} = \underbrace{-I\,\underline{r}}_{\text{Born Oppenheimer}} \underbrace{- I\,\frac{\underline{V}\wedge\underline{r}}{R^2}}_{\text{magnetic}} + \underbrace{\frac{(S^2-I^2)\,\underline{r}}{R^3}}_{\text{electric}}}$$

Aharonov & Stern

Study the geometric forces in isolation.

Magnetic only

Eliminate Born-Oppenheimer and electric (both radial) by confining motion to a sphere R = constant. Implement this constraint with

$$\boxed{P^2 \longrightarrow P^2 - (\underline{P}\cdot\underline{r})^2}$$

Then

$$\boxed{\underline{V} = \underline{P} - \underline{P}\cdot\underline{r}\,\underline{r} \text{ i.e. } \underline{V}\cdot\underline{r} = 0}$$

Exact equations of motion for slow variables

$$\boxed{\dot{\underline{V}} = \underline{R}\wedge\underline{V} - \underline{J} + \underline{J}\cdot\underline{r}\,\underline{r} - \frac{V^2}{R}\underline{r}}$$

(conserves R, E, $S = |\underline{J} - \underline{R}\wedge\underline{V}|$)   seek $\underline{r} = (\theta, \phi)$ about $\underline{J} = J\underline{z}$

This motion is integrable :

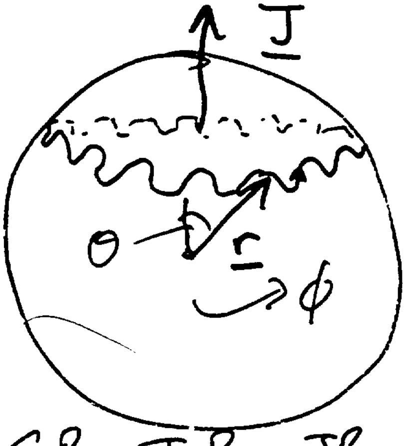
$$K \equiv E + \frac{1}{2R^2}(J^2 - S^2) = \underline{R}\cdot\underline{J} + \frac{(\underline{r}\wedge\underline{V})\cdot\underline{J}}{R}$$

conserved

$$\therefore\quad 2(E-K) + \frac{|\underline{J}\wedge\underline{r}|^2}{R^2} = \left(\underline{V} - \frac{\underline{J}\wedge\underline{r}}{R}\right)^2$$

$$\therefore\quad \boxed{\begin{aligned} &\frac{2(E-K)}{R^2} + \frac{J^2\sin^2\theta}{R^4} = \dot{\theta}^2 + \sin^2\theta\left(\dot{\phi} - \frac{J}{R^2}\right)^2 \\ &K = J(\sin^2\theta\,\dot{\phi} + R\cos\theta) \end{aligned}}$$

Motion is precession about $\underline{J}$ ($\dot{\phi}$)
with nutation (θ oscillations)

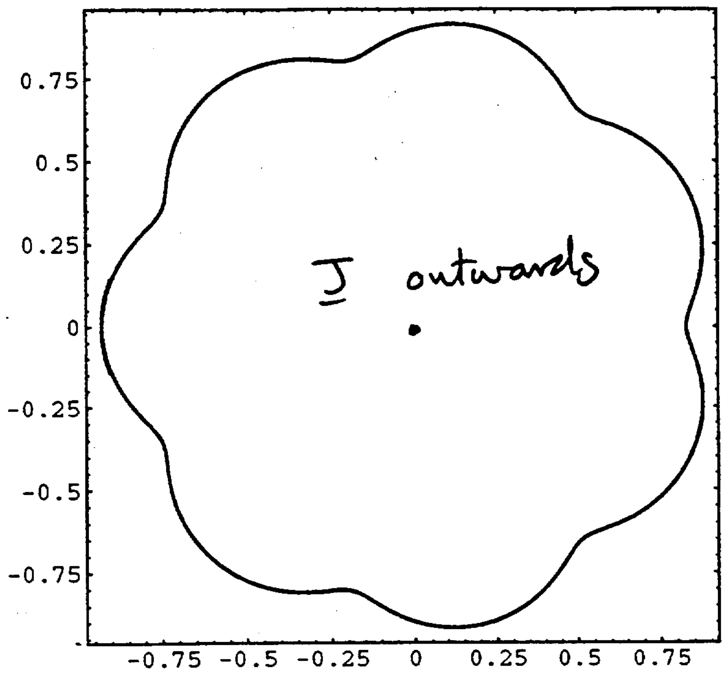


Adiabatic limit
R large : $\dot{\theta} \to 0$
(no nutation)

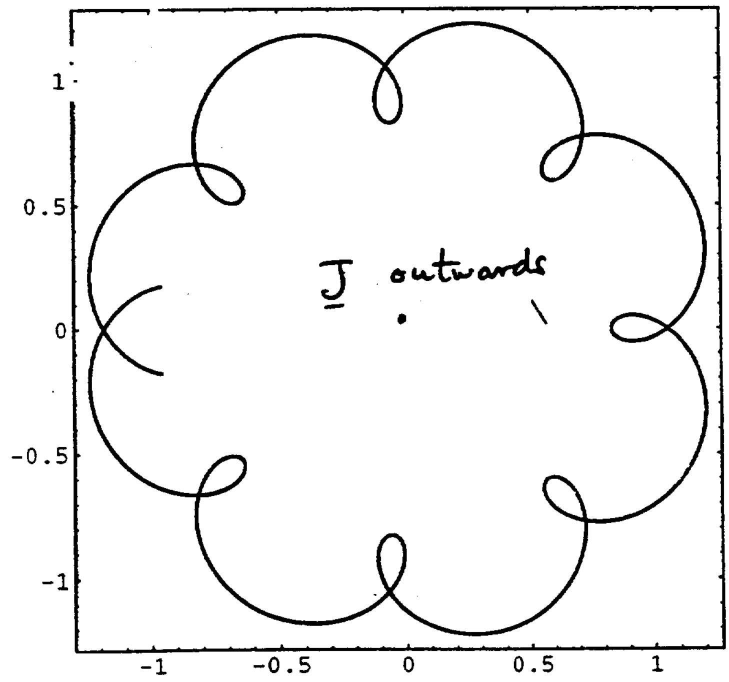
$$\dot{\phi} \to \frac{J}{R^2} \to \boxed{V = R\sin\theta\,\dot{\phi} = \frac{I\tan}{R}}$$

$(I = \underline{S}\cdot\underline{R} = \underline{J}\cdot\underline{R} = JR\cos\theta)$

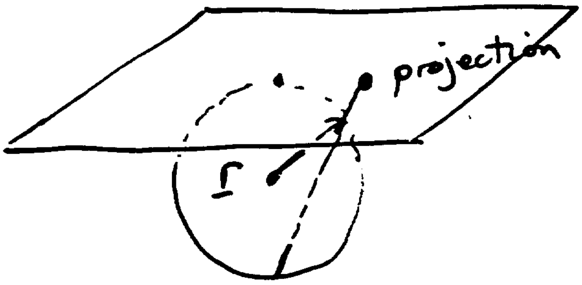
Stereographic projections from S pole at $\underline{r}$ sphere

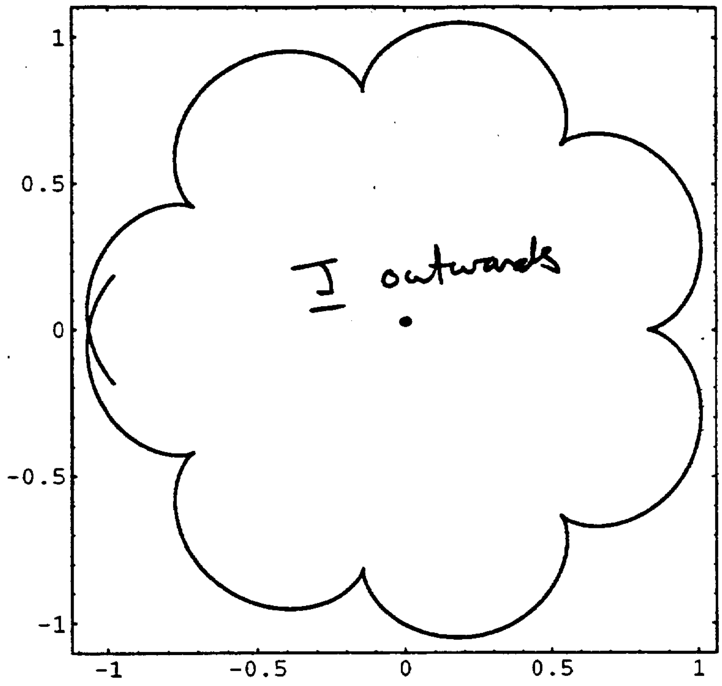


$R = 3 \quad J = 3$

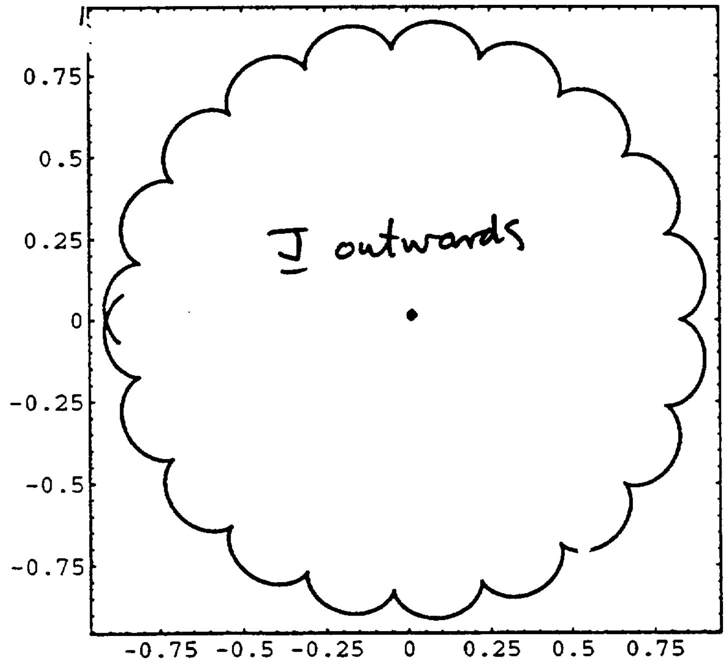


$R = 3 \quad J = 3$





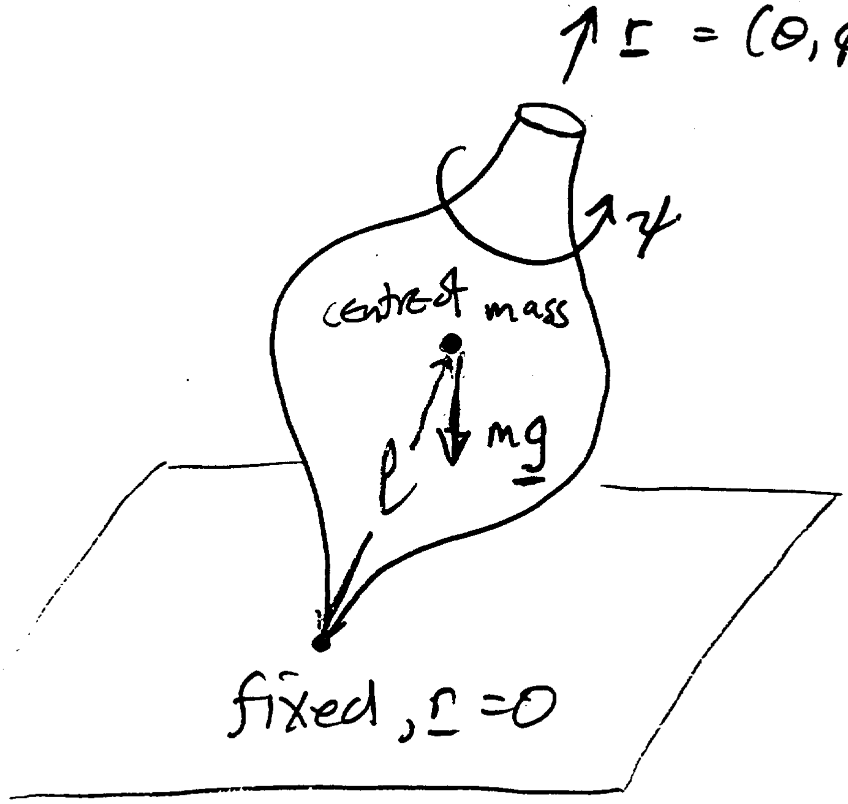
R=3 J=3



R=4 J=3

Adiabatically, amplitude of nutation diminishes

Analogy with heavy symmetrical (Lagrange) top



Moments of inertia
A, B, B

Angular velocity

$$\boxed{\underline{\Omega} = (\dot{\psi} + \dot{\phi}\cos\theta)\,\underline{r} + \underline{r}\wedge\dot{\underline{r}}}$$

momentum $$\boxed{\underline{L} = A(\dot{\psi} + \dot{\phi}\cos\theta)\,\underline{r} + B\,\underline{r}\wedge\dot{\underline{r}}}$$

$L_\psi = \underline{L}\cdot\underline{r} = A(\dot{\psi} + \dot{\phi}\cos\theta)$ conserved

$L_\phi = \underline{L}\cdot\underline{z} = L_\psi\,\underline{r}\cdot\underline{e}_z + B\,\underline{r}\wedge\dot{\underline{r}}\cdot\underline{e}_z$ conserved

torque equation $\dot{\underline{L}} = -mg\ell\,\underline{r}\wedge\underline{z}$

gives

$$\boxed{\ddot{\underline{r}} = \frac{L_\psi}{B}\,\underline{r}\wedge\dot{\underline{r}} - \frac{mg\ell}{B}(\underline{z} - \underline{z}\cdot\underline{r}\,\underline{r}) - |\dot{\underline{r}}|^2\,\underline{r}}$$

Identical to equations of motion of spin model (constrained to a sphere) if

$$\boxed{R = \frac{L_\psi}{B} \qquad \underline{J} = \frac{mg\ell L_\psi}{B^2}\underline{z}}$$

∴ Adiabatic regime $\boxed{R \text{ large} \longrightarrow L_\psi \text{ large} = \text{fast top}}$

$\left(K = \frac{J L_\phi}{B}\right)$

Puzzle: no obvious top analogy for spin

$$\boxed{\underline{S} = \frac{L_\psi}{B^2}\left[mg\ell\underline{z} - L_\psi \underline{r}\wedge(\underline{B}\wedge\underline{r})\right]}$$

Geometric force approximation:

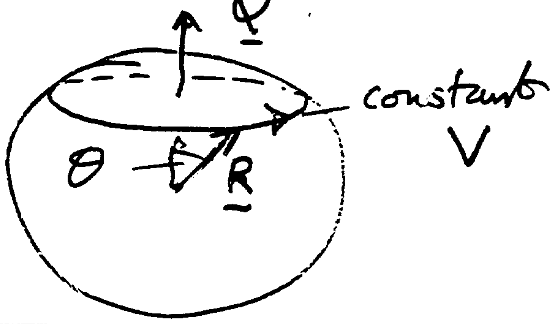
$$\boxed{\dot{\underline{V}} = -I\frac{\underline{V}\wedge\underline{r}}{R^2} - \frac{V^2}{R}\underline{r}}$$

monopole

conserves R, $E = \frac{1}{2}V^2$ and

$$\boxed{\underline{Q} = \underline{R}\wedge\underline{V} + I\underline{r}}$$

Hence $\underline{r}\cdot\underline{Q}$ constant →



speed given by

$\tan\theta = \frac{|\underline{R}\wedge\underline{V}|}{I}$ → $\boxed{V = \frac{I\tan\theta}{R}}$

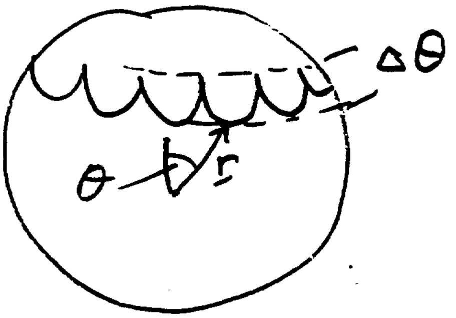
✱ Thus, magnetic gauge force describes exactly the adiabatic precession. ✱

Born – Oppenheimer fails completely, predicting $\underline{R}$ at rest.

Accuracy of conservation of adiabatic invariant

$$I = J\cos\theta$$

$$\boxed{\frac{\Delta I}{I} = \frac{\Delta\cos\theta}{\cos\theta} \to \frac{+2J\sin^2\theta_0}{R^3\cos\theta_0}} \quad \text{as } R \to \infty$$



So, precession of a top, with the axis (slow) forced by reaction from the spin (fast), is 'caused' by the monopole at the point that is held fixed.

Electric only

Eliminate Born-Oppenheimer and magnetic by choosing $I=0$

In particular, choose

$$\boxed{\underline{J}=0 \quad \rightarrow \quad I=\underline{S}\cdot\underline{r}=\underline{J}\cdot\underline{r}=0 \text{ always}}$$

Exact motion is

$$\boxed{\dot{\underline{V}}=\underline{R}\wedge\underline{V}}$$

('spin' $\underline{V}$, driven by $\underline{R}$, and coupled by $\dot{\underline{R}}=\underline{V}$, or charged particle in uniform sphere of monopolium)

This can be solved exactly.

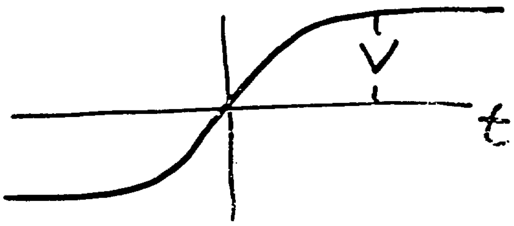
Speed $\boxed{V=\text{constant}}$

$$\frac{1}{2}(\ddot{R^2})=\dot{(\underline{R}\cdot\underline{V})}=V^2 \rightarrow \boxed{R^2(t)=R_0^2+V^2t^2}$$

radial repulsion.

'Adiabatic invariant' for 'spin' $\underline{V}$ is not constant:

$$\boxed{\underline{r}\cdot\underline{V}=\frac{\underline{R}\cdot\underline{V}}{R}=\frac{Vt}{\sqrt{t^2+R_0^2/V^2}}}$$



So, asymptotically, orbits recede along radii

Can set $V = 1$, and parameterize orbit shapes just with R_0

Count of variables : 6 quantities to describe an orbit

eliminate : 1 by choosing $t=0$ at closest approach

3 by rigid rotation of orbit about $\underline{R} = 0$

1 by scaling : $\alpha \underline{R}(\alpha t)$ is an orbit if $\underline{R}(t)$ is

$5 \longrightarrow 1$ left $= R_0$

Thus time t = arc length along orbit, so $\underline{V}(t)$ is unit tangent. Orbit as space curve:

curvature

$$\boxed{K = \sqrt{\dot{\underline{V}} \cdot \dot{\underline{V}}} = \text{constant} = R_0}$$

Normal and binormal

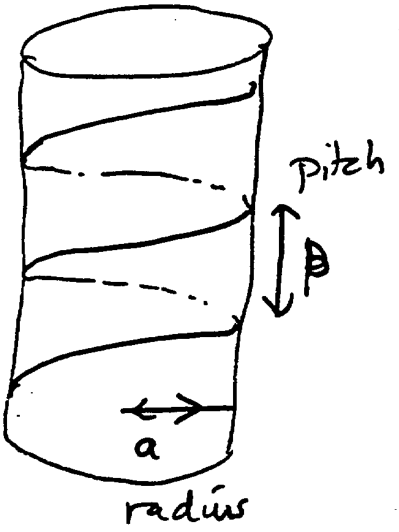
$$\underline{n} = \frac{\underline{R} \wedge \underline{V}}{R_0} \qquad \underline{b} = \frac{\underline{R} - t\underline{V}}{R_0}$$

torsion

$$\boxed{\tau = -\dot{\underline{b}} \cdot \underline{n} = t}$$

What curves have constant curvature R_0 and changing torsion t ?

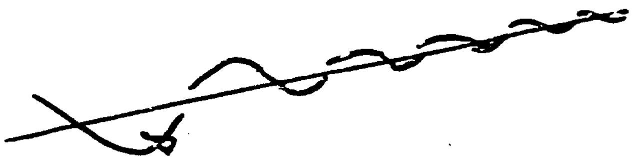
Preliminary : uniform helix



$$K = \frac{4\pi^2 a}{p^2 + 4\pi^2 a^2} \qquad \tau = \frac{2\pi p}{p^2 + 4\pi^2 a^2}$$

$$\boxed{\begin{aligned} a &= \frac{R_0}{R_0^2 + t^2} \longrightarrow \frac{R_0}{t^2} \\ p &= \frac{2\pi t}{R_0^2 + t^2} \longrightarrow \frac{2\pi}{t} \end{aligned}}$$

So, orbits coil round asymptotes with shrinking radius and pitch



Winding reverses at $t=0$ (zero torsion), so predict $\boxed{\text{curly antelope horns}}$ for $\underline{R}(t)$

and for $\underline{V}$ on its unit sphere predict a curve whose speed is the curvature R_0 of $\underline{R}(t$ and whose curvature is the torsion t of $\underline{R}(t)$

i.e. $\boxed{\text{an asymptotically infinitely coiling } S}$

Antelope horns: numerical solution

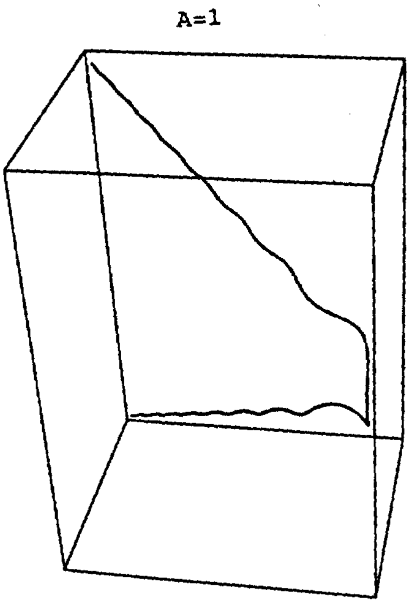
of $\underline{R}'' = \underline{R} \wedge \underline{R}''$

$V = |\underline{R}'| = 1$

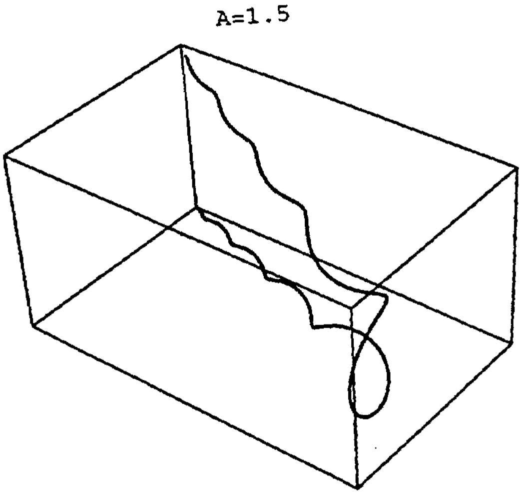
closest approach

$R = R_0$

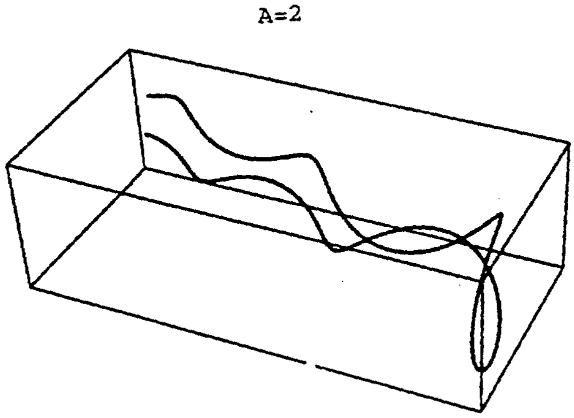
$R_0 = 1$



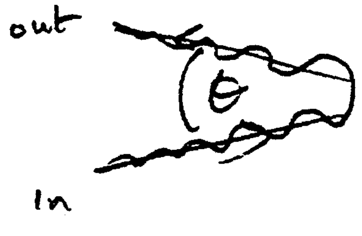
$R_0 = 1.5$



$R_0 = 2$

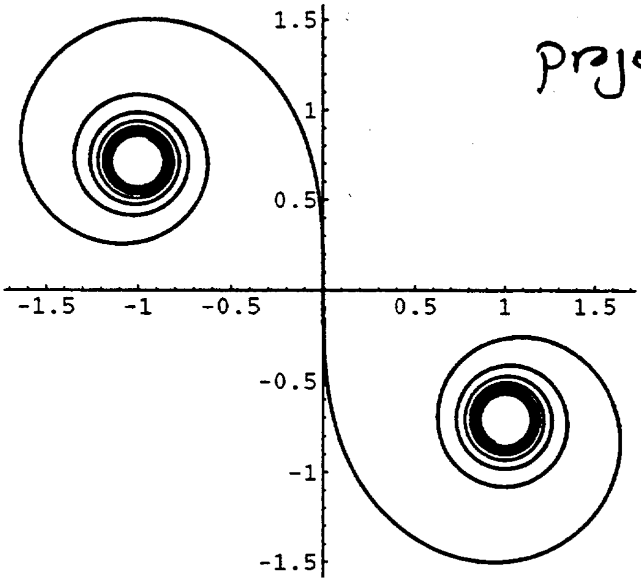


Horns close up as R_0 increases

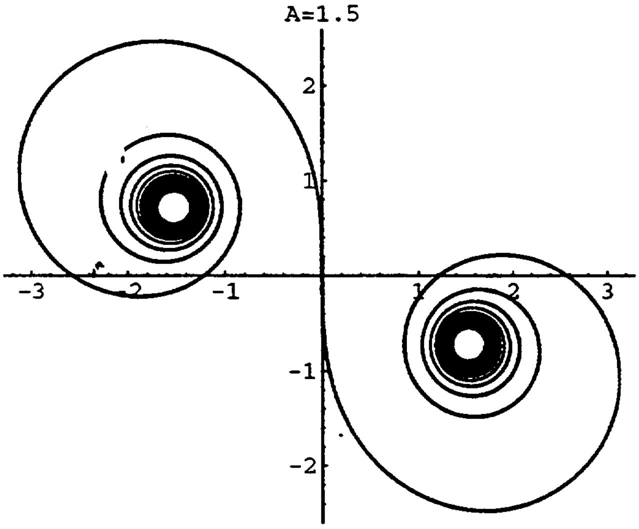


seek opening angle $\Theta(R_0)$

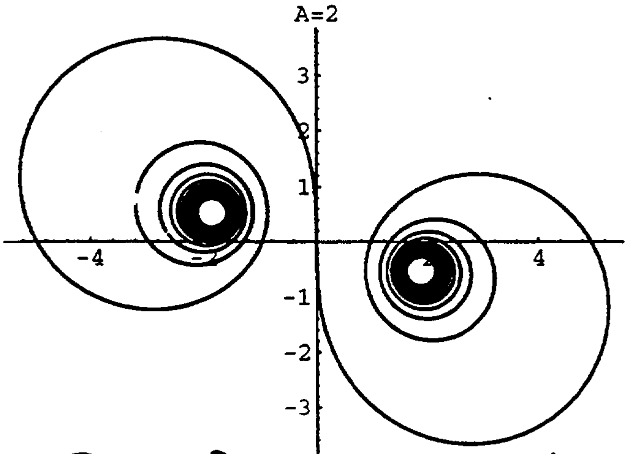
Track of velocity $\underline{V}$ on ~~its~~ ~~unit~~ sphere
(stereographic S pole
projection)



$R_0 = 1$



$R_0 = 1.5$



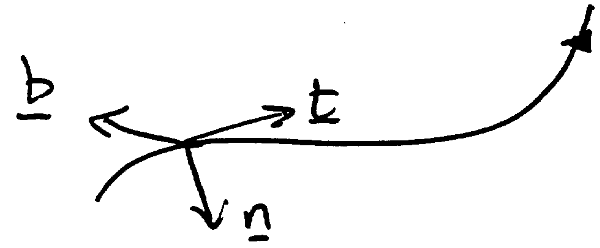
$R_0 = 2$

As R_0 increases, velocity asymptotes
become antipodal

Intermezzo Constructing the curve

with given curvature $\kappa(s)$ | functions of
torsion $\tau(s)$ | arc length s

Frenet equations



$$\boxed{\begin{aligned}\underline{t}' &= \kappa\underline{n}\\ \underline{n}' &= -\kappa\underline{t} + \tau\underline{b}\\ \underline{b}' &= -\tau\underline{n}\end{aligned}}$$

9 linear equations. Reduce to 2 by mapping onto evolution of a quantum spin-$\frac{1}{2}$ particle in a 'magnetic field'

$$\boxed{\underline{B}(s) = \left(\kappa(s), 0, -\tau(s)\right)}$$

Schrödinger equation

$$\boxed{i|\psi\rangle' = \underline{\sigma}\cdot\underline{B}(s)|\psi\rangle} \qquad \boxed{|\psi\rangle = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}}$$

$$\boxed{\underline{\sigma} = \frac{1}{2}\underbrace{\left[\underset{x}{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}, \underset{y}{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}, \underset{z}{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}\right]}} = \text{Pauli matrices}$$

take any 3 solutions $|\psi_x\rangle, |\psi_y\rangle, |\psi_z\rangle$

with $\langle\psi_i|\underline{\sigma}|\psi_i\rangle$ orthogonal at $t=0$.

Then

$$\boxed{\underline{t} = 2\left[\langle\psi_x|\sigma_z|\psi_x\rangle, \langle\psi_y|\sigma_z|\psi_y\rangle, \langle\psi_z|\sigma_z|\psi_z\rangle\right]}$$

given $\underline{t}(s)$, the curve is $\underline{R}(s) = \int_0^s dt\, \underline{t}(s)$

In the spin model $\dot{\underline{V}} = \underline{B}\wedge\underline{V}$

the 'quantum Hamiltonian' is

$$\boxed{\underline{\sigma}\cdot\underline{B}(t) = \frac{1}{2}\begin{pmatrix} -t & R_0 \\ R_0 & t \end{pmatrix}}$$

defining the Landau-Zener problem of quantum transit theory

Thus the velocity (unit tangent) is

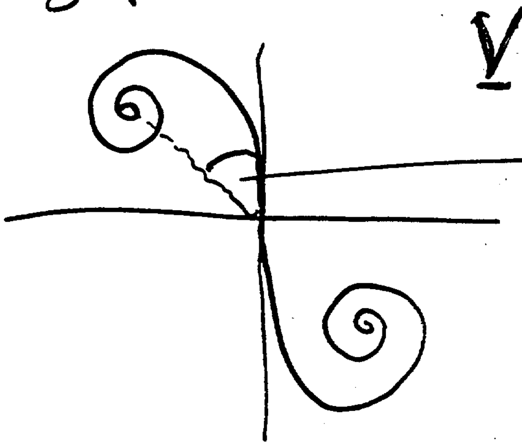
$$\boxed{\underline{V}(t) = \left\{R_0 \operatorname{Im} y_1 y_2^*,\ R_0 \operatorname{Re} y_1 y_2^*,\ |y_1|^2 - \frac{1}{4}R_0^2|y_2|^2\right\}}$$

where y_1 y_2 are the even and odd parabolic cylinder (Kummer, confluent hypergeometric) functions

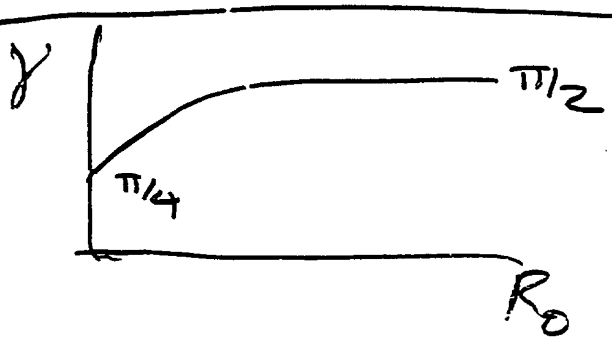
$$\boxed{y_1(t) = e^{-\frac{it^2}{4}} M\left(\frac{1}{8}iR_0^2, \frac{1}{2}, \frac{1}{2}it^2\right)}$$

$$\boxed{y_2(t) = t\, e^{-\frac{it^2}{4}} M\left(\frac{1}{8}iR_0^2 + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}it^2\right)}$$

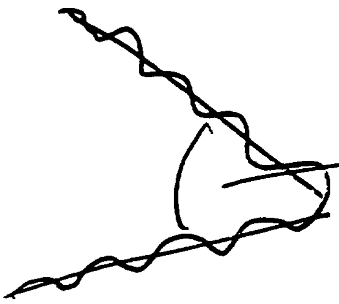
Asymptotics of these functions gives



$$\boxed{\gamma = \frac{\pi}{4} + \arg\left[\Gamma\left(\frac{1}{2} - i\frac{R_0^2}{8}\right)\Gamma\left(1+\frac{iR_0^2}{8}\right)\right.}$$



and



$$\boxed{\theta(R_0) = 2\arcsin e^{-\frac{\pi R_0^2}{4}}}$$

Geometric force approximation

For large R_0, particle spirals in and back out along same direction, so 'guiding centre' motion is purely radial.

Gauge theory (electric force only) $I = 0$ so

$$\boxed{\dot{\underline{V}} = \frac{S^2}{R^3}\underline{r} \quad , \quad S = |\underline{R}(0)\wedge\underline{V}(0)| = R_0}$$

solution $\boxed{\underline{R}(t) = \sqrt{R_0^2+t^2}\;\underline{x}}$

* So electric gauge force describes exactly the repulsion of the guiding centre *

Born-Oppenheimer fails completely.

Conclusions

1. Geometric reaction forces of electric and magnetic type give a clear picture of average slow motion beyond Born-Oppenheimer, and can capture the main features of this motion with little calculation.

2. New general formulae, giving B_{ij} and Φ in terms of fast orbits with fixed $\underline{R}$, are easy to use and apply also to chaotic fast motion.

3. The particular model $\frac{P^2}{2} + \underline{S}\cdot\underline{R}$ much richer than the two special cases described, and includes chaos in exact slow motion, but geometric forces still produce distinctive effects (e.g. swerving out of plane) observed in numerical computations.

4. For chaotic fast motion, the magnetic force is the neglected antisymmetric cousin of friction.

MORE CLASSICAL ANHOLONOMY

SWIMMING BUGS (~~Shaper~~ Shapere and Wilczek)

Swimming = cyclic alteration of a body's shape producing net translation through a fluid. 

The shape variables return; the position variables do not. Therefore $\boxed{\text{swimming is anholonomy}}$.

For small creatures (insects, bacteria) the translation vector depends only on the sequence of body shapes, and not on how fast the movements are made, because the fluid reaction is dominated by viscous (rather than inertial) forces. So reciprocating motions get nowhere.

The body makes a circuit C in the abstract space of shapes. Wilczek et. al show:

anholonomy (= translation vector)

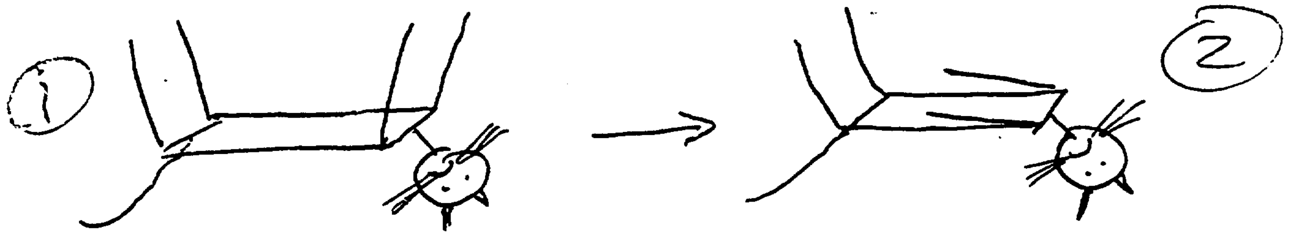
= flux of something (complicated) through C

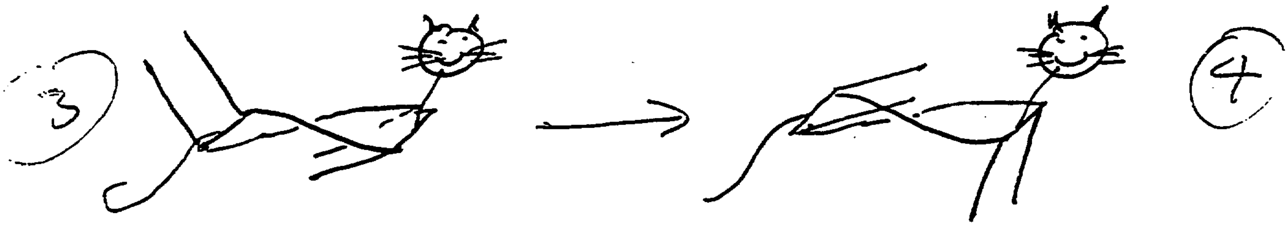
5.29

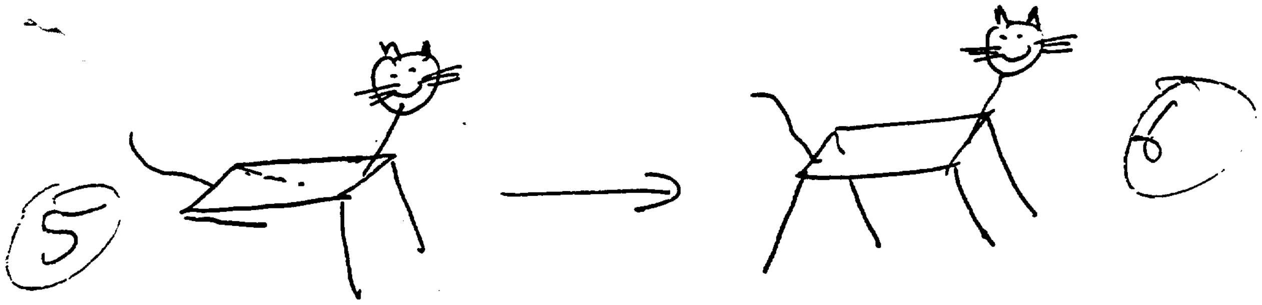
Nityananda

CATS IN SPACE (Shapere and Wilczek)

With nothing to push against, the cat's angular momentum remains zero. Nevertheless it can rotate, by changing its shape :







The shape variables return, but the orientation variables do not, so the cat's turn is anholonomy.

A formula for the net rotation of the cat:

R_{ij} = rotation matrix turning initial into final object (with the same shape).

$$= P \exp\left[\oint A^{\mu}_{ij}(S)\, dS_{\mu}\right]$$

path ordered product

cycle in space of shapes S

Infinitesimal rotation from infinitesimal μ deformation

This is geometry. For A, need dynamics, i.e. angular momentum zero:

$$A^{\mu}_{ij}\frac{dS_{\mu}}{dt} = -\varepsilon_{ijk}\left(\underline{I} - 1\,\mathrm{Tr}\, I\right)^{-1}_{kl} L_{l}$$

traceless inertia tensor of body in shape S in standard orientation

apparent angular momentum of deforming body when reduced to standard orientation

A^{μ}_{ij} is a vector in shape space (with axes μ). It is the (matrix-valued) potential of the gauge field

$$F^{\mu\nu} = \frac{\partial A^{\mu}}{\partial S_{\nu}} - \frac{\partial A^{\nu}}{\partial S_{\mu}} + \left[A^{\mu}, A^{\nu}\right]$$