Mode stability for massless scalars in five-dimensional black hole backgrounds

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The mode stability of the Kerr black hole in four dimensions was demonstrated by Whiting in 1989, by separating the Teukolsky equation that describes gravitational perturbations and then transforming the radial and angular equations in such a way that the problem can be reformulated as a wave equation in an auxiliary spacetime in which the proof of stability is greatly simplified, owing to the absence of an ergoregion. As a preliminary step towards extending these ideas to higher-dimensional black holes, we study the mode stability of the massless scalar wave equation in the five-dimensional black hole solutions of Einstein gravity and supergravity. We show how the wave equation can again be mapped into one in an auxiliary spacetime in which there is no ergoregion, allowing us to give a proof of the mode stability of the solutions of the scalar wave equation.

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I. INTRODUCTION

Establishing results on the stability of black hole solutions has been a major activity in the general relativity community for many years. This is especially subtle, and also important, in the case of stationary rotating black holes. Many different approaches have been followed, but one of the most fruitful has involved finding an integral transformation that maps the difficult analysis in the original variables into a considerably simpler analysis in the transformed variables. The idea was first developed in [1], where it was employed to establish the mode stability of the Kerr black hole [1]. The technique has been developed further in recent years, and found application in studies such as the global stability of black holes (see, for example, [2,3]), the stability of extremal black holes [4], and mode stability on the real frequency axis [5,6].

Establishing the mode stability of the Kerr black hole involved studying the properties of the mode functions in the separation of variables for the Teukolsky equation, which provides a gauge-invariant description of the perturbations around the Kerr background. In [1] the generalized Teukolsky equation with a spin parameter s was studied, with $s = \pm 2$ corresponding to the actual case of interest in which the equation describes the gravitational perturbations themselves. The case $s = 0$ corresponds to the massless scalar wave equation (the massless Klein-Gordon equation), while the $s = \pm 1$ case governs gauge-invariant components of the Maxwell field.

The techniques for analyzing the Teukolsky equation for spin s that were developed in [1] were broadly similar for all s, and in fact the essential features associated with the stability of the solutions could already be seen in the $s = 0$ case. This is a useful observation because if one looks at more complicated situations than black holes in pure Einstein gravity, such as black holes in Einstein-Maxwell theory or in supergravity, the analog of the

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Teukolsky analysis has not been implemented. In the absence of a gauge-invariant treatment of the perturbations of the black holes in these more complicated situations, one can at least study the analog of $s = 0$ Teukolsky equation, that is, the massless scalar wave equation in the black hole background. One may hope that this can provide a "proxy" for the full analysis, and that establishing stability results for the solutions of the massless wave equation may be indicative of what one might find in a more elaborate and complete perturbative analysis. This approach was adopted in [7], where the techniques of [1] were applied to the study of the mode stability for solutions of the massless scalar wave equation in the background of a class of four-dimensional supergravity black holes carrying four independent electric charges [8,9].

In this paper, we extend some of the four-dimensional techniques for mode-stability analysis that were developed in [1] to the case of five dimensions. Even in the pure Einstein case, the analog of the four-dimensional Teukolsky analysis is unknown. The stumbling block is that the four-dimensional analysis depended heavily upon the use of the Newman-Penrose formalism, and no particularly useful extension of this to five or higher dimensions has been constructed.[1] Thus for now, our approach will mirror the one that can be followed for more complicated theories in four dimensions, namely, we shall focus attention on establishing stability results for solutions of the five-dimensional massless scalar wave equation. This already allows us to develop a rather nontrivial generalization of the techniques that were employed in four dimensions, and it reveals ways in which the integral transformation that allows us to establish mode-stability results is substantially different from the one in four dimensions.

We shall describe in the subsequent sections how one can establish mode-stability results for solutions of the massless wave equation in the background of the five-dimensional Myers-Perry rotating black hole [11], and also in the background of general 3-charge rotating black holes in the five-dimensional STU supergravity theory [12].

II. MASSLESS SCALAR WAVE EQUATION IN FIVE-DIMENSIONAL BLACK HOLE BACKGROUND

A. Five-dimensional Myers-Perry black hole

The natural generalization of the four-dimensional rotating Kerr [13] black hole to higher spacetime dimensions is provided by the Myers-Perry black hole solutions [11]. These vacuum solutions of the D-dimensional Einstein equations are characterized by their mass M and by $[(D-1)/2]$-independent angular momenta, reflecting the fact that independent rotations can occur in each orthogonal spatial 2-plane. In this paper we shall be concerned specifically with the example of the five-dimensional rotating black hole. Its metric is given by [11]

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}[adt - (r^2+a^2)d\phi]^2 + \frac{\cos^2\theta}{\rho^2}[bdt - (r^2+b^2)d\psi]^2 + \frac{1}{r^2\rho^2}[abdt - b(r^2+a^2)\sin^2\theta d\phi - a(r^2+b^2)\cos^2\theta d\psi]^2, \quad (2.1)$$

where

$$\Delta = \frac{(r^2+a^2)(r^2+b^2)}{r^2} - 2M, \qquad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \quad (2.2)$$

Here, a and b are the two independent rotation parameters, with ϕ and ψ being the two associated azimuthal angles (each with period 2π). The latitude coordinate θ ranges over $0 \le \theta \le \frac{1}{2}\pi$.

B. Massless scalar wave equation

Our focus will be the investigation of solutions of the massless scalar wave equation $\Box\Psi = 0$ in the Myers-Perry background, with the goal of establishing that modes with time dependence $e^{-i\omega t}$ that are ingoing on the future horizon and outgoing at future null infinity cannot have a frequency ω with a positive imaginary part. In other words, we seek to show that there cannot exist spatially regular modes that would give rise to instabilities growing exponentially in time.

One can define new radial and latitude coordinates $\tilde{x}$ and y by writing

$$\tilde{x} = r^2, \qquad y = \cos^2\theta. \quad (2.3)$$

Using these, we have

$$\sqrt{-g} = \frac{\tilde{x} + a^2 y + b^2(1-y)}{4}, \quad (2.4)$$

and defining the quantity $G^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$, we have

[1] However, see [10].

$$G^{00} = -\frac{(\tilde{x}+2M+b^2)}{4} - \frac{M^2\tilde{x}}{D} + \frac{(b^2-a^2)y}{4},$$
$$G^{01} = G^{02} = 0, \qquad G^{03} = -\frac{aM(\tilde{x}+b^2)}{2D},$$
$$G^{04} = -\frac{bM(\tilde{x}+a^2)}{2D},$$
$$G^{11} = \tilde{x}^2 + (a^2+b^2-2M)\tilde{x} + a^2b^2,$$
$$G^{12} = G^{13} = G^{14} = 0,$$
$$G^{22} = y(1-y), \qquad G^{23} = G^{24} = 0,$$
$$G^{33} = \frac{(b^2-a^2)\tilde{x} + b^2(b^2-a^2-2M)}{4D} + \frac{1}{4(1-y)},$$
$$G^{34} = -\frac{abM}{2D},$$
$$G^{44} = \frac{(a^2-b^2)\tilde{x} + a^2(a^2-b^2-2M)}{4D} + \frac{1}{4y}, \tag{2.5}$$

where

$$D = (\tilde{x}+a^2)(\tilde{x}+b^2) - 2M\tilde{x}. \tag{2.6}$$

It can be shown [14] that the massless scalar wave equation $\Box\Psi = 0$, which may be written as $\partial_\mu(G^{\mu\nu}\partial_\nu\Psi) = 0$, is separable.

We can obtain the separated $\tilde{x}$ and y equations in Schrödinger form by writing[2]

$$\Psi = e^{-\mathrm{i}\omega t + \mathrm{i}m\phi + \mathrm{i}n\psi} R(\tilde{x}) S(y), \tag{2.7}$$

and then defining

$$R(\tilde{x}) = \frac{X(\tilde{x})}{\sqrt{D}}, \qquad S(y) = \frac{Y(y)}{\sqrt{y(1-y)}}. \tag{2.8}$$

This gives the separated equations

$$\frac{X''}{X} + U_{\tilde{x}} - \frac{\sigma}{D} = 0, \qquad \frac{Y''}{Y} + U_y + \frac{\sigma}{y(1-y)} = 0. \tag{2.9}$$

It is convenient now to make a further change of the radial variable. Defining ϵ_+ and ϵ_- by

$$2M-(a+b)^2 = 2M\epsilon_+^2, \quad 2M-(a-b)^2 = 2M\epsilon_-^2, \tag{2.10}$$

a further coordinate transformation from $\tilde{x}$ to x given by

$$\tilde{x} = 2M\epsilon_+\epsilon_- x + \frac{1}{2}M(\epsilon_+ - \epsilon_-)^2, \tag{2.11}$$

implies that D, defined in Eq. (2.6), becomes

$$D = 4M^2\epsilon_+^2\epsilon_-^2 x(x-1). \tag{2.12}$$

The outer horizon $r = r_+$ is located at $x = 1$, and the inner horizon $r = r_-$ at $x = 0$. The region where r goes to infinity corresponds to x goes to infinity.

In terms of the variable x, we can now write the radial equation in the form

$$\frac{X''}{X} + \frac{\kappa+\Lambda}{x} + \frac{\kappa-\Lambda}{x-1} + \frac{\frac{1}{4}-\beta^2}{x^2} + \frac{\frac{1}{4}-\gamma^2}{(x-1)^2} = 0, \tag{2.13}$$

finding

$$\kappa = \frac{1}{4}M\epsilon_+\epsilon_-\omega^2,$$
$$\beta = \frac{\mathrm{i}}{4\sqrt{2M}}\left[\frac{2M\omega-(a+b)(m+n)}{\epsilon_+} - \frac{2M\omega-(a-b)(m-n)}{\epsilon_-}\right],$$
$$\gamma = \frac{\mathrm{i}}{4\sqrt{2M}}\left[\frac{2M\omega-(a+b)(m+n)}{\epsilon_+} + \frac{2M\omega-(a-b)(m-n)}{\epsilon_-}\right],$$
$$\Lambda = \sigma + \frac{1}{2} - \frac{3}{4}M\omega^2 + \frac{1}{8}(a^2-b^2)\omega^2 + \frac{[2M\omega-(a+b)(m+n)]^2}{16M\epsilon_+^2} + \frac{[2M\omega-(a-b)(m-n)]^2}{16M\epsilon_-^2}. \tag{2.14}$$

In the angular direction, the equation for Y in (2.9) is of the form

$$\frac{Y''}{Y} + \frac{\hat{\kappa}+\hat{\Lambda}}{y} + \frac{\hat{\kappa}-\hat{\Lambda}}{y-1} + \frac{\frac{1}{4}-\hat{\beta}^2}{y^2} + \frac{\frac{1}{4}-\hat{\gamma}^2}{(y-1)^2} = 0, \tag{2.15}$$

[2]For the separated form of the equations before casting them in the Schrödinger form, see [14]. There the separation was already performed for the 5d STU black holes [12].

with

$$\hat{\kappa} = -\frac{1}{8}(a^2 - b^2)\omega^2, \qquad \hat{\beta} = \frac{n}{2}, \qquad \hat{\gamma} = \frac{m}{2}$$
$$\hat{\Lambda} = \frac{1}{2} + \sigma - \frac{(m^2 + n^2)}{4} + \frac{(a^2 - b^2)\omega^2}{8}. \tag{2.16}$$

C. Transforming the radial equation

We start with the radial equation given by Eq. (2.13), with the constants given in (2.14). Next, introduce a new function $f(x)$, related to $X(x)$ by

$$X(x) = x^{\frac{1}{2} - \epsilon_1 \beta}(x-1)^{\frac{1}{2} - \epsilon_2 \gamma} f(x), \tag{2.17}$$

where $\epsilon_1^2 = \epsilon_2^2 = 1$. The function f therefore satisfies

$$[x(x-1)\partial_x^2 + (bx + c)\partial_x + dx + e]f(x) = 0, \tag{2.18}$$

with

$$b = 2(1 - \epsilon_1\beta - \epsilon_2\gamma), \quad c = -1 + 2\epsilon_1\beta,$$
$$d = 2\kappa, \quad e = \frac{1}{2} - \kappa - \epsilon_1\beta - \epsilon_2\gamma + 2\epsilon_1\epsilon_2\beta\gamma - \Lambda. \tag{2.19}$$

Next, we make an integral transform to a new radial variable $h(z)$, by defining

$$h(z) = e^{-\tilde{\alpha} z} e^{\tilde{\nu}/z} z^{1+\tilde{\gamma}} \int_1^\infty e^{2\tilde{\alpha} x z} f(x) dx. \tag{2.20}$$

Now, multiplying Eq. (2.18) by $e^{2\tilde{\alpha}xz}$ and integrating gives, after integration by parts,

$$0 = B + \int_1^\infty [4\tilde{\alpha}^2 x(x-1)z^2 + 2\tilde{\alpha}(4x - bx - 2 - c)z$$
$$+ dx + 2 - b + e] f(x) dx, \tag{2.21}$$

where B is the boundary term:

$$B = [e^{2\tilde{\alpha}xz} x(x-1)\partial_x f - \partial_x(e^{2\tilde{\alpha}xz} x(x-1)) f$$
$$+ e^{2\tilde{\alpha}xz}(bx + c) f]_1^\infty. \tag{2.22}$$

Note that Eq. (2.21) can be written as

$$0 = B + \int_1^\infty f(x)\mathcal{O}_z(e^{2\tilde{\alpha}xz})dx, \tag{2.23}$$

where

$$\mathcal{O}_z = z^2\partial_z^2 + \left(4z - 2\tilde{\alpha}z^2 - bz + \frac{d}{2\tilde{\alpha}}\right)\partial_z$$
$$+ 2 - 2\tilde{\alpha}cz - 4\tilde{\alpha}z - b + e. \tag{2.24}$$

It can then be seen that provided the boundary term B vanishes, a question to which we shall return later, then $h(z)$ defined in Eq. (2.20) satisfies $\mathcal{O}_z(e^{\tilde{\alpha}z} e^{-\tilde{\nu}/z} z^{-1-\tilde{\gamma}} h(z)) = 0$, and thus

$$\left(\partial_z^2 - \tilde{\alpha}^2 + \frac{2\tilde{\alpha}\tilde{\kappa}}{z} + \frac{\tilde{\Lambda}}{z^2} + \frac{2\tilde{\gamma}\tilde{\nu}}{z^3} - \frac{\tilde{\nu}^2}{z^4}\right)h(z) = 0, \tag{2.25}$$

where the constants are chosen so that

$$\tilde{\kappa} = -\epsilon_1\beta + \epsilon_2\gamma, \qquad \tilde{\Lambda} = \frac{1}{2} - \beta^2 - \gamma^2 - \Lambda,$$
$$\tilde{\gamma} = \epsilon_1\beta + \epsilon_2\gamma, \qquad \tilde{\nu} = -\frac{\kappa}{2\tilde{\alpha}}. \tag{2.26}$$

The constant $\tilde{\alpha}$ is arbitrary at this stage [it really just sets the scale of the new radial variable z that is introduced in Eq. (2.20)]. We shall find it convenient to define it to be

$$\tilde{\alpha} = \frac{\mathrm{i}\sqrt{M}\epsilon_-\omega}{2\sqrt{2}}. \tag{2.27}$$

D. Behavior of unstable modes

If they existed, unstable modes would be solutions that were purely outgoing at $\mathcal{I}$ and purely ingoing at the horizon $\mathcal{H}$, that is, asymptotically, they would have support only at $\mathcal{I}^+$ and $\mathcal{H}^+$. We are interested in establishing the nonexistence of unstable modes arising from frequencies in the upper half of the complex ω plane, that is for $\omega = \omega_0 + i\omega_1$, with $\omega_1 > 0$, since these would grow exponentially in the future. Thus, near $\mathcal{I}^+$, we must have

$$\Psi \sim \exp -i\omega(t - r_*), \quad \text{as} \quad \{t, r_*\} \to \infty, \tag{2.28}$$

and, near $\mathcal{H}^+$, we must have

$$\Psi \sim \exp -i\omega(t + r_*), \quad \text{as} \quad \{-t, r_*\} \to -\infty. \tag{2.29}$$

From the definition of the r_* coordinate in Eq. (A1), it can be seen that in terms of the radial coordinate x introduced in Eq. (2.11), we shall have asymptotically

$$\text{Near } \mathcal{I}: \quad r_* = \sqrt{2M\epsilon_+\epsilon_-} x^{\frac{1}{2}} + \mathcal{O}(x^{-1/2}), \tag{2.30}$$

$$\text{Near } \mathcal{H}: \quad r_* = \frac{\sqrt{M}(\epsilon_+ + \epsilon_-)}{2\sqrt{2}\epsilon_+\epsilon_-}\log(x-1) + \mathcal{O}(x-1). \tag{2.31}$$

From the radial equation (2.13), it can be seen that near the horizon $x = 1$, the solutions will have the form

$$X(x) = (x-1)^{\frac{1}{2}\pm\gamma} F_{\pm}^{(1)}(x-1), \tag{2.32}$$

where the functions $F_{\pm}^{(1)}(x-1)$ are analytic around $x=1$, with $F_{\pm}^{(1)}(0)$ a nonzero constant. As noted above, an unstable mode would have ω dependence of the form $e^{-i\omega r_*}$ near H, and hence from (2.31) and (2.14) it would correspond to the minus-sign choice in Eq. (2.32). Thus, for an unstable mode X_{um}, we must have the near-horizon behavior:

$$\text{Near}\,\mathcal{H}: \quad X_{um}(x) = (x-1)^{\frac{1}{2}-\gamma} F_{-}^{(1)}(x-1). \tag{2.33}$$

In the asymptotic region near $x=\infty$, it can be seen from the radial equation (2.13) that the asymptotic form of the solutions will be

$$X(x) = x^{\frac{1}{4}} e^{\pm 2i\sqrt{2\kappa}\sqrt{x}} F_{\pm}^{(\infty)}\left(\frac{1}{\sqrt{x}}\right), \tag{2.34}$$

with the functions $F_{\pm}^{(\infty)}(\frac{1}{\sqrt{x}})$ being asymptotic series with $F_{\pm}^{(\infty)}(0)$ a nonzero constant. Now, as we saw, an unstable mode would correspond to having a ω dependence of the form $e^{i\omega r_*}$ near infinity, and so from (2.30) and (2.14) it would correspond to the plus-sign choice in Eq. (2.34). Thus, for an unstable mode $X_{um}(x)$, we must have the asymptotic behavior

$$\text{Near}\,\mathcal{I}: \quad X_{um}(x) = x^{\frac{1}{4}} e^{2i\sqrt{2\kappa}\sqrt{x}} F_{+}^{(\infty)}\left(\frac{1}{\sqrt{x}}\right). \tag{2.35}$$

Before going on, we point out that infinitely many modes with these analytic properties do exist with ω_1, the imaginary part of ω, being <0. These modes decay exponentially in time, have been extensively studied [15], and are known as the quasinormal modes which, taken together, uniquely characterize a black hole spacetime to which they correspond.

We turn now to the behavior of the transformed radial function $h(z)$, defined by Eq. (2.20). In particular, we shall be concerned with the behavior in the coordinate range $0 \le z \le \infty$.

Near $z=0$, it can be seen from Eq. (2.25) that the leading-order behavior of the solutions will be

$$h(z) = e^{\pm\tilde{\nu}/z} z^{1\pm\tilde{\gamma}} G_{\pm}^{(0)}(z), \tag{2.36}$$

where $G_{\pm}^{(0)}(z)$ are analytic functions with $G_{\pm}^{(0)}(0)$ being nonzero constants. It is evident from (2.20) that at $z=0$ the integrand for an unstable mode is just an analytic function of x, and so the leading-order behavior of $h(z)$ near $z=0$ will be given by the prefactor functions $e^{\tilde{\nu}/z} z^{1+\tilde{\gamma}}$. In other words, the unstable mode corresponds to the plus-sign choice in Eq. (2.36):

$$\text{Near}\, z=0: \quad h_{um}(z) = e^{\tilde{\nu}/z} z^{1+\tilde{\gamma}} G_{+}^{(0)}(z), \tag{2.37}$$

As a check, we see from Eqs. (2.14), (2.26) and (2.27) that

$$\tilde{\nu} = \frac{i\sqrt{M}\epsilon_+\omega}{2\sqrt{2}}, \tag{2.38}$$

whose real part is negative when ω has a positive imaginary part, thus implying that $h_{um}(z)$ in Eq. (2.37) is finite, and goes to zero, as z goes to zero. [This justifies the sign choice in the definition of $\tilde{\alpha}$ in Eq. (2.27).]

It can be seen from Eq. (2.25) that near $z=\infty$, the function $h(z)$ has the behavior

$$h(z) = e^{\pm\tilde{\alpha}z} z^{\mp\tilde{\kappa}} G_{\pm}^{(\infty)}\left(\frac{1}{z}\right), \tag{2.39}$$

where the functions $G_{\pm}^{(\infty)}(\frac{1}{z})$ are asymptotic in z^{-1} with $G_{\pm}^{(\infty)}(0)$ being nonvanishing constants. In the expression (2.20) the leading behavior of $h(z)$ near $z=\infty$ is governed by the behavior of $f(x)$ near $x=1$. Using the previously determined behavior of $X(x)$, and hence $f(x)$, for an unstable mode we see that the integrand in Eq. (2.20) has the behavior

$$\int_1^\infty e^{2\tilde{\alpha}xz} f(x)dx = e^{2\tilde{\alpha}z} \int_1^\infty e^{2\tilde{\alpha}(x-1)z} (x-1)^{(\epsilon_2-1)\gamma} (1+\mathcal{O}(x-1))dx. \tag{2.40}$$

Substituting $v = -2\tilde{\alpha}(x-1)z$, we then have

$$\int_1^\infty e^{2\tilde{\alpha}xz} f(x)dx \propto e^{2\tilde{\alpha}z} z^{-1+\gamma(1-\epsilon_2)} \times \int_0^\infty e^{-v} v^{(\epsilon_2-1)\gamma} \left(1+\mathcal{O}\left(\frac{v}{z}\right)\right) dv. \tag{2.41}$$

Combining with the prefactor we then see, provided

$$\epsilon_2 = -1, \tag{2.42}$$

that

$$h(z) \propto e^{+\tilde{\alpha}z} y^{\tilde{\gamma}+(1-\epsilon_2)\gamma} \left(1+\mathcal{O}\left(\frac{1}{z}\right)\right) \sim e^{\tilde{\alpha}z} z^{-\tilde{\kappa}}. \tag{2.43}$$

Thus, for an unstable mode, the plus-sign choice in Eq. (2.39) is selected:

$$\text{Near } z=\infty\text{:}\qquad h_{um}(z)=e^{\tilde{\alpha}z}z^{-\tilde{\kappa}}G_+^{(\infty)}\left(\frac{1}{z}\right). \tag{2.44}$$

As a check, we note that if ω has a positive imaginary part, $\tilde{\alpha}$, given in Eq. (2.27), will have a negative real part, and so $h_{um}(z)$ will be finite as z goes to infinity. [This motivated the sign choice in the definition of $\tilde{\alpha}$ in Eq. (2.27).]

It is now straightforward to check, using the asymptotic properties of the radial functions for unstable modes established in this section, that the boundary term given in Eq. (2.22) will vanish for any unstable mode. Thus, we have established that there is a one-to-one mapping between exponentially unstable modes in the original untransformed radial function $X(x)$ and exponentially unstable modes in the radial function $h(z)$ obtained by means of the integral transform (2.20). The final steps in the proof of the mode stability will be presented in the next section; this will entail establishing for the transformed radial equation that there cannot exist any exponentially unstable modes.

III. MODES IN THE TRANSFORMED SPACETIME

A. Combining angular and transformed radial equations

The constant σ that was introduced in the original process of separating variables is present in the transformed radial equation (2.25) through the quantity Λ [see Eqs. (2.14) and (2.26)] and in the angular equation (2.15) through the quantity $\hat{\Lambda}$ [see Eq. (2.16)]. It follows therefore that if we form the combination

$$\frac{z^2}{h(z)}\partial_z^2 h(z)+\frac{y(1-y)}{Y(y)}\partial_y^2 Y(y) \tag{3.1}$$

and make use of Eqs. (2.25) and (2.15), then we shall obtain an equation in which all the σ dependence has canceled. This equation can in fact be interpreted as the result of performing a separation of variables in which we write

$$\Psi(t,z,y,\phi,\psi)=h(z)Y(y)e^{-\mathrm{i}\omega t}e^{\mathrm{i}m\phi}e^{\mathrm{i}n\psi}. \tag{3.2}$$

We postpone writing the full "unseparated" equation for now, and just focus on the terms proportional to ω^2. (That is, the $-\partial_{tt}$ terms in the full five-dimensional wave equation.) Together with the terms involving the radial and angular derivatives, these are

$$z^2\partial_z^2+y(1-y)\partial_y^2+\frac{\epsilon_+^2M\omega^2}{8z^2}+\frac{\epsilon_-^2Mz^2\omega^2}{8}+\frac{[-\epsilon_1(\epsilon_--\epsilon_+)-\epsilon_2(\epsilon_-+\epsilon_+)]M\omega^2}{4\epsilon_-z}$$
$$+\frac{[\epsilon_1(\epsilon_--\epsilon_+)-\epsilon_2(\epsilon_-+\epsilon_+)]Mz\omega^2}{4\epsilon_+}+\frac{3}{4}M\omega^2-\frac{1}{8}(a^2-b^2)(1-2y)\omega^2+\text{rest}. \tag{3.3}$$

Since

$$\epsilon_+=\sqrt{1-\frac{(a+b)^2}{2M}},\qquad \epsilon_-=\sqrt{1-\frac{(a-b)^2}{2M}}, \tag{3.4}$$

[see Eqs. (2.10)], and we always assume a and b are non-negative, it follows that $\epsilon_-\geq\epsilon_+$. Consequently, all the ω^2 terms in Eq. (3.3) will be non-negative, provided that we choose the sign of ϵ_2 to be

$$\epsilon_2=-1. \tag{3.5}$$

Note that the necessity for this choice of sign was already seen in the previous section, in Eq. (2.42). The choice of sign for ϵ_1 is undetermined by these considerations. We shall, for definiteness, make the choice

$$\epsilon_1=-1. \tag{3.6}$$

For the remaining ω^2 terms, namely with coefficient

$$\frac{3}{4}M-\frac{1}{8}(a^2-b^2)(1-2y), \tag{3.7}$$

we note that the range of the angular coordinate is $0\leq y\leq 1$, and so $-1\leq 1\text{–}2y\leq 1$. We also have that

$$(a+b)^2\leq 2M, \tag{3.8}$$

[see Eq. (3.4)], and so we have $a^2-b^2=(a+b)(a-b)\leq(a+b)^2\leq 2M$. Thus, the remaining terms (3.7) contributing to ω^2 terms in (3.3) are always positive.

In summary, we have seen that the overall coefficient of ω^2 in Eq. (3.3) is always positive, implying that in the transformed metric $\tilde{g}_{\mu\nu}$ obtained by the process of unseparating variables, $\frac{\partial}{\partial t}$ is always timelike outside the horizon.

We now present the complete result for the combination of the radial and angular equations. With $\epsilon_1 = \epsilon_2 = -1$ as discussed above we obtain

$$\left\{z^2\partial_z^2 + y(1-y)\partial_y^2 - \left(\frac{(a+b)}{4z} + \frac{(a-b)z}{4}\right)m\omega - \left(\frac{(a+b)}{4z} - \frac{(a-b)z}{4}\right)n\omega + \left(\frac{\epsilon_+^2 M}{8z^2} + \frac{\epsilon_-^2 M z^2}{8} + \frac{M}{2z} + \frac{Mz}{2} + \frac{3}{4}M - \frac{1}{8}(a^2-b^2)(1-2y)\right)\omega^2 - \frac{m^2}{4(1-y)} - \frac{n^2}{4y} + \frac{1}{4y(1-y)}\right\}\Psi = 0. \quad (3.9)$$

Using the replacements

$$\omega \to \mathrm{i}\partial_t, \qquad m \to -\mathrm{i}\partial_\phi, \qquad n \to -\mathrm{i}\partial_\psi \quad (3.10)$$

we can read off the components $\tilde{g}^{\mu\nu}$ of an inverse metric in a transformed spacetime, such that Eq. (3.9) can be written as

$$\tilde{g}^{\mu\nu}\partial_\mu\partial_\nu\Psi + \frac{1}{4y(1-y)}\Psi = 0, \quad (3.11)$$

with

$$\tilde{g}^{zz} = z^2, \qquad \tilde{g}^{yy} = y(1-y), \qquad \tilde{g}^{\phi\phi} = \frac{1}{4(1-y)}, \qquad \tilde{g}^{\psi\psi} = \frac{1}{4y},$$
$$\tilde{g}^{t\phi} = -\frac{(a+b)}{8z} - \frac{(a-b)z}{8}, \qquad \tilde{g}^{t\psi} = -\frac{(a+b)}{8z} + \frac{(a-b)z}{8},$$
$$\tilde{g}^{tt} = -\left(\frac{\epsilon_+^2 M}{8z^2} + \frac{\epsilon_-^2 M z^2}{8} + \frac{M}{2z} + \frac{Mz}{2} + \frac{3}{4}M - \frac{1}{8}(a^2-b^2)(1-2y)\right). \quad (3.12)$$

We may find a suitable conformal factor Ω^2 and a redefined wave function Φ such that the D'Alembertian of Φ in a rescaled metric $\hat{g}_{\mu\nu} = \Omega^2\tilde{g}_{\mu\nu}$ gives rise to Eq. (3.11). We define

$$\Phi(t,z,y,\phi,\psi) = \frac{1}{z\sqrt{y(1-y)}}\Psi(t,z,y,\phi,\psi), \quad (3.13)$$

and, noting from (3.12) that we have

$$\sqrt{-\hat{g}} = \frac{8\sqrt{2}\Omega^5}{\sqrt{M}(1+z)^2}, \quad (3.14)$$

it can be seen that if we choose Ω so that

$$\Omega^3 = \frac{(1+z)^2\sqrt{M}}{8\sqrt{2}}, \quad (3.15)$$

then the transformed Eq. (3.11) is equivalent to the following equation for Φ in the $\hat{g}_{\mu\nu}$ metric:

$$\partial_\mu(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu\Phi) = 0. \quad (3.16)$$

This can be derived from the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\mu\bar{\Phi}\partial_\nu\Phi. \quad (3.17)$$

From the resulting energy-momentum tensor

$$T_{\mu\nu} = \partial_{(\mu}\bar{\Phi}\partial_{\nu)}\Phi - \frac{1}{2}\hat{g}_{\mu\nu}\hat{g}^{\rho\sigma}\partial_\rho\bar{\Phi}\partial_\sigma\Phi, \quad (3.18)$$

we may construct a conserved current $J^\mu = -K^\nu T^\mu{}_\nu$, where $K = \frac{\partial}{\partial t}$ is the time-translation Killing vector. This gives rise to a conserved energy

$$\mathcal{E} = \int \sqrt{-\hat{g}}J^0 d^4x, \quad (3.19)$$

with

$$J^0 = -\hat{g}^{t\rho}\partial_{(\rho}\bar{\Phi}\partial_{t)}\Phi + \frac{1}{2}\hat{g}^{\rho\sigma}\partial_\rho\bar{\Phi}\partial_\sigma\Phi$$
$$= -\frac{1}{2}\hat{g}^{tt}|\partial_t\Phi|^2 + \frac{1}{2}\hat{g}^{zz}|\partial_z\Phi|^2 + \frac{1}{2}\hat{g}^{yy}|\partial_y\Phi|^2 + \frac{1}{2}\hat{g}^{\phi\phi}|\partial_\phi\Phi|^2 + \frac{1}{2}\hat{g}^{\psi\psi}|\partial_\psi\Phi|^2. \quad (3.20)$$

The integrand in the energy integral (3.19) is therefore given by

$$\sqrt{-\hat{g}}J^0 = \frac{1}{2z^2}\left\{P|\partial_t\Phi|^2 + z^2|\partial_z\Phi|^2 + y(1-y)|\partial_y\Phi|^2 + \frac{1}{4(1-y)}|\partial_\phi\Phi|^2 + \frac{1}{4y}|\partial_\psi\Phi|^2\right\}, \tag{3.21}$$

where

$$P = \frac{\epsilon_+^2 M}{8z^2} + \frac{\epsilon_-^2 M z^2}{8} + \frac{M}{2z} + \frac{Mz}{2} + \frac{3}{4}M - \frac{1}{8}(a^2-b^2)(1-2y). \tag{3.22}$$

The four-dimensional integration in Eq. (3.19) is over the coordinates of the 3-sphere (with the ranges $0 \le y \le 1$, $0 \le \phi < 2\pi$, $0 \le \psi < 2\pi$), and over the transformed radial variable z. This ranges over $0 \le z \le \infty$, and as we discussed in Sec. II. 4 the transformed radial function $h(z)$ for any putative unstable mode goes rapidly to zero at $z = 0$ [see Eq. (2.37)], and it goes rapidly to zero at $z = \infty$ [see Eq. (2.44)], ensuring the convergence of the integrals of all the terms in Eq. (3.19).

From Eqs. (3.4) we have $|a^2 - b^2| \le 2M$, and since y lies in the interval $0 \le y \le 1$, it follows that the quantity P satisfies $P \ge 0$. Since every term in the energy integral (3.19) is integrable for any putative unstable mode, and each contribution is non-negative, it follows in particular that the integral of the $\frac{1}{2z^2}P|\partial_t\Phi|^2$ term is bounded from above by the conserved energy $\mathcal{E}$. Thus, Φ cannot grow exponentially in time, and therefore there cannot in fact exist any exponentially unstable modes.

IV. 3-CHARGE FIVE-DIMENSIONAL STU SUPERGRAVITY BLACK HOLES

The 3-charge rotating black-hole solution in five-dimensional STU supergravity was obtained in [12], by using a solution-generating procedure. A convenient form of the solution was given in [16]. With minor change of notation, to achieve consistency with our present conventions, the metric is given by

$$ds^2 = (H_1H_2H_3)^{1/3}(\tilde{x}+\tilde{y})d\hat{s}^2, \tag{4.1}$$

where

$$d\hat{s}^2 = -\Phi(dt+\mathcal{A})^2 + ds_4^2, \tag{4.2}$$

with

$$ds_4^2 = \frac{d\tilde{x}^2}{4X} + \frac{d\tilde{y}^2}{4Y} + \frac{U}{G}\left(d\chi - \frac{Z}{U}d\sigma\right)^2 + \frac{XY}{U}d\sigma^2. \tag{4.3}$$

The various functions above are given by[3]

$$\begin{aligned} X &= (\tilde{x}+a^2)(\tilde{x}+b^2) - 2M\tilde{x}, \qquad Y = -(a^2-\tilde{y})(b^2-\tilde{y}),\\ G &= (\tilde{x}+\tilde{y})(\tilde{x}+\tilde{y}-2M), \qquad U = \tilde{y}X - \tilde{x}Y, \qquad Z = ab(X+Y),\\ \mathcal{A} &= \frac{2Mc_1c_2c_3(\tilde{x}+\tilde{y})}{G}[(a^2+b^2-\tilde{y})d\sigma - ab\,d\chi] - \frac{2Ms_1s_2s_3}{\tilde{x}+\tilde{y}}(ab\,d\sigma - \tilde{y}d\chi),\\ \Phi &= \frac{G}{(\tilde{x}+\tilde{y})^3H_1H_2H_3}, \qquad H_i = 1 + \frac{2Ms_i^2}{\tilde{x}+\tilde{y}}. \qquad i = 1,2,3. \end{aligned} \tag{4.4}$$

Here $s_i = \sinh\delta_i$ and $c_i = \cosh\delta_i$, where δ_i are the boost parameters that correspond to turning on the three electric charges. When $\delta_i = 0$, the metric reduces to the five-dimensional Myers-Perry black hole.

The coordinates σ and χ are related to the standard azimuthal angular coordinates ϕ and ψ (each with period 2π) by

$$\sigma = \frac{a\phi - b\psi}{a^2-b^2}, \qquad \chi = \frac{b\phi - a\psi}{a^2-b^2}, \tag{4.5}$$

as can be seen from Eq. (15) in [16] after turning off the gauge-coupling constant g. The standard radial and angular coordinates r and θ are related to $\tilde{x}$ and $\tilde{y}$ by

$$\tilde{x} = r^2, \qquad \tilde{y} = a^2\cos^2\theta + b^2\sin^2\theta. \tag{4.6}$$

Thus, $\tilde{x}$ here is the same as $\tilde{x}$ in Eq. (2.6) of the uncharged case. The coordinate $\tilde{y}$ is related to our coordinate $y = \cos^2\theta$ by

$$\tilde{y} = (a^2-b^2)y + b^2. \tag{4.7}$$

Proceeding as in the earlier uncharged case, we may separate variables and write the radial equation in the same form as Eq. (2.13), and the angular equation in the same form as Eq. (2.15). Only the expressions for the various κ, Λ, β and γ coefficients will change when the charges are turned on.

The coefficients κ, Λ, β and γ in the potential for the radial equation, generalizing those in (2.14) for Myers-Perry, are now given by

[3]There was one typo in [16]: a missing factor of $(\tilde{x}+\tilde{y})$ in the first of the two terms in the expression for the 1-form $\mathcal{A}$. This is corrected here.

$$\begin{aligned}
\kappa &= \frac{1}{4}M\epsilon_+\epsilon_-\omega^2,\\
\beta &= \frac{\mathrm{i}}{4\sqrt{2M}}\left[\frac{2(\Pi_c+\Pi_s)M\omega-(a+b)(m+n)}{\epsilon_+}-\frac{2(\Pi_c-\Pi_s)M\omega-(a-b)(m-n)}{\epsilon_-}\right],\\
\gamma &= \frac{\mathrm{i}}{4\sqrt{2M}}\left[\frac{2(\Pi_c+\Pi_s)M\omega-(a+b)(m+n)}{\epsilon_+}+\frac{2(\Pi_c-\Pi_s)M\omega-(a-b)(m-n)}{\epsilon_-}\right]\\
\Lambda &= \sigma+\frac{1}{2}-\frac{1}{4}M(3+2s_1^2+2s_2^2+2s_3^2)\omega^2+\frac{1}{8}(a^2-b^2)\omega^2\\
&\quad+\frac{[2(\Pi_c+\Pi_s)M\omega-(a+b)(m+n)]^2}{16M\epsilon_+^2}+\frac{[2(\Pi_c-\Pi_s)M\omega-(a-b)(m-n)]^2}{16M\epsilon_-^2}.
\end{aligned} \tag{4.8}$$

Crucial properties that held previously in the uncharged case continue to hold here. In particular, since

$$\Pi_c+\Pi_s \geq \Pi_c-\Pi_s \geq 1, \tag{4.9}$$

together with the usual inequalities $\epsilon_+ \leq \epsilon_- \leq 1$, it follows that when ω has a positive imaginary part, the real parts of β and γ will be negative.

For the angular equation, the hatted quantities $\hat{\kappa}$, $\hat{\Lambda}$, $\hat{\beta}$ and $\hat{\gamma}$ are given by

$$\begin{aligned}
&\hat{\kappa} = -\frac{1}{8}(a^2-b^2)\omega^2, \qquad \hat{\beta}=\frac{n}{2}, \qquad \hat{\gamma}=\frac{m}{2}\\
&\hat{\Lambda} = \frac{1}{2}+\sigma-\frac{(m^2+n^2)}{4}+\frac{(a^2-b^2)\omega^2}{8},
\end{aligned} \tag{4.10}$$

unchanged from the results (2.16) for the uncharged black holes.

Following the same steps as we did previously for the uncharged Myers-Perry black hole, we find that after implementing the same integral transformation of the radial equation as before, we again arrive at an "unseparated" equation of the form (3.9), with the only difference being in the coefficient of ω^2:

$$\begin{aligned}
&\left\{z^2\partial_z^2+y(1-y)\partial_y^2-\left(\frac{(a+b)}{4z}+\frac{(a-b)z}{4}\right)m\omega-\left(\frac{(a+b)}{4z}-\frac{(a-b)z}{4}\right)n\omega\right.\\
&\quad+\left(\frac{\epsilon_+^2M}{8z^2}+\frac{\epsilon_-^2Mz^2}{8}+\frac{(\Pi_c+\Pi_s)M}{2z}+\frac{(\Pi_c-\Pi_s)Mz}{2}+\frac{1}{4}M\left(3+2\sum_i s_i^2\right)-\frac{1}{8}(a^2-b^2)(1-2y)\right)\omega^2\\
&\quad\left.-\frac{m^2}{4(1-y)}-\frac{n^2}{4y}+\frac{1}{4y(1-y)}\right\}\Psi=0.
\end{aligned} \tag{4.11}$$

This correspondingly implies that the components of the tilded inverse metric $\tilde{g}^{\mu\nu}$ are unchanged except for $\tilde{g}^{tt}$, which becomes

$$\tilde{g}^{tt} = -\left(\frac{\epsilon_+^2M}{8z^2}+\frac{\epsilon_-^2Mz^2}{8}+\frac{(\Pi_c+\Pi_s)M}{2z}+\frac{(\Pi_c-\Pi_s)Mz}{2}+\frac{1}{4}M\left(3+2\sum_i s_i^2\right)-\frac{1}{8}(a^2-b^2)(1-2y)\right). \tag{4.12}$$

Calculating the determinant, we now find that instead of Eq. (3.14) we have

$$\sqrt{-\hat{g}} = \frac{8\sqrt{2}e^{\delta_1+\delta_2+\delta_3}\Omega^5}{\sqrt{M}}[(e^{\delta_1+\delta_2}+e^{\delta_3}z)(e^{\delta_2+\delta_3}+e^{\delta_1}z)(e^{\delta_1+\delta_3}+e^{\delta_2}z)(1+e^{\delta_1+\delta_2+\delta_3}z)]^{-\frac{1}{2}}. \tag{4.13}$$

Following the remaining steps of the previous discussion for the uncharged case, we find that the conserved energy is given by integrating $\sqrt{-\hat{g}}J^0$ as in Eq. (3.21), with the function P now given not by Eq. (3.22) but instead

$$P = \frac{\epsilon_+^2M}{8z^2}+\frac{\epsilon_-^2Mz^2}{8}+\frac{(\Pi_c+\Pi_s)M}{2z}+\frac{(\Pi_c-\Pi_s)Mz}{2}+\frac{1}{4}M\left(3+\sum_i 2s_i^2\right)-\frac{1}{8}(a^2-b^2)(1-2y). \tag{4.14}$$

The same arguments that established that P was non-negative in the uncharged case show that $P \geq 0$ here also, and hence again there cannot exist any unstable exponentially growing modes.

V. DISCUSSION

It has become apparent that many of the equations governing massless fields on black hole spacetimes are of Heun type (in cases where the cosmological constant is nonzero), or one of its many confluent variants, such as for nonextreme Kerr [1], or the extreme case [4]. The differential equations we find for massless scalar fields in the five-dimensional Myers-Perry black hole spacetime [see (2.13) and (2.15)] are of yet another confluent Heun type. That observation has allowed us to extend to this (and the related STU) case the analysis originally applied to massless fields of all spin in the Kerr spacetime [1]. Before our present work, that analysis had also been extended: (i) by using a different integration contour, to rule out unstable modes on the real axis for the Kerr spacetime [6], (ii) by considering a modified integral transform, to deal with the extreme ($|a| = M$) Kerr black hole [4], and (iii) by looking carefully at more complicated examples, to establish the absence of unstable modes for massless scalar fields in STU spacetimes and all more specialized subcases [7]. Remarkably, the integral transform we have used here is, effectively, an inverse of that developed for the extreme Kerr spacetime [4]. In this context, it is also worth noting that quite different techniques, stemming from Seiberg-Witten theory (see, for example [17]), and based on the spectral properties of the operators involved, have been used to discuss both Kerr quasinormal modes [18] and Kerr–de Sitter stability [19]. The relevance of such an approach to the spacetimes we consider here is yet to be determined.

VI. CONCLUSION

We have shown that a massless scalar field has no exponentially unstable modes in the five-dimensional Myers-Perry black hole spacetime. We have also shown that the same is true in the five-dimensional supergravity-motivated STU spacetimes and, previously [7], that this holds, too, for the four-dimensional STU spacetimes. Together, these encompass a number of other special cases which arise from restricting the parameters in these more general examples. Although these results may serve as suggestive for the behavior for fields of higher spin—in particular, Maxwell fields and gravitational perturbations—it would be useful to have some more direct indication, perhaps by writing down (at least) the analog of the Teukolsky equation in these more general cases. That task currently remains for future work.

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APPENDIX: GLOBAL STRUCTURE OF THE MYERS-PERRY BLACK HOLE

Defining a coordinate r^* by the relation

$$dr_* = \frac{(r^2+a^2)(r^2+b^2)dr}{r^2\Delta}, \tag{A1}$$

we may introduce retarded coordinates (u, ϕ_-, ψ_-), where

$$du = dt - dr_*, \qquad d\phi_- = d\phi - \frac{a(r^2+b^2)dr}{r^2\Delta}, \qquad d\psi_- = d\psi - \frac{b(r^2+a^2)dr}{r^2\Delta}. \tag{A2}$$

In terms of these, the metric (2.1) then becomes

$$\begin{aligned} ds^2 = &-du^2 - 2dr(du - a\sin^2\theta d\phi_- - b\cos^2\theta d\psi_-) + \rho^2 d\theta^2 \\ &+ \frac{2M}{\rho^2}(du - a\sin^2\theta d\phi_- - b\cos^2\theta d\psi_-)^2 + (r^2+a^2)\sin^2\theta d\phi_-^2 + (r^2+b^2)\cos^2\theta d\psi_-^2. \end{aligned} \tag{A3}$$

This form of the metric is regular in the neighborhood of future null infinity.

We may also introduce advanced coordinates (v, ϕ_+, ψ_+) by

$$dv = dt + dr_*, \qquad d\phi_+ = d\phi + \frac{a(r^2+b^2)dr}{r^2\Delta}, \qquad d\psi_+ = d\psi + \frac{b(r^2+a^2)dr}{r^2\Delta}, \tag{A4}$$

with respect to which the metric (2.1) becomes

$$ds^2 = -dv^2 + 2dr(dv - a\sin^2\theta d\phi_+ - b\cos^2\theta d\psi_+) + \rho^2 d\theta^2 + \frac{2M}{\rho^2}(dv - a\sin^2\theta d\phi_+ - b\cos^2\theta d\psi_+)^2 + (r^2+a^2)\sin^2\theta d\phi_+^2 + (r^2+b^2)\cos^2\theta d\psi_+^2. \quad (A5)$$

It can be seen from this form of the metric that it is regular as one crosses the future horizon.

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