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RADIATION DAMPING AND LAGRANGE INVARIANTS

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A B S T R A C T

A general formula is presented for the damping of small oscillations, about closed orbits in classical mechanics, by dissipative perturbations. It is based on the variation of Lagrange invariants. It is applied to rederive the standard results for the effects of classical radiation damping on storage ring orbits.

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1. - INTRODUCTION

In the literature [1-3] the discussion of classical radiation damping of storage ring orbits has been along ad hoc - but perfectly sound ! - lines. We will show here how the results can be obtained by straightforward calculation from a rather general formula.

The starting point is the constancy, for any Hamiltonian system, of the expression (related to 'Lagrange brackets' [4,5])

(1)

where q_n and p_n are canonical co-ordinates and momenta, the summation is over degrees of freedom, and δ_1 and δ_2 denote variations from a given solution of the equations of motion to nearby solutions. In what follows the given solution will always be a closed orbit. It will be assumed that any dependence of the Hamiltonian on the independent variable - distance s measured along the closed orbit - has the periodicity of that orbit.

For example, from a given variation $(\delta_1 q, \delta_1 p)$, a second can be obtained simply by looking one (or more than one) revolution further on :

(2)

Here $T(s)$ is the matrix which propagates small displacements one revolution around the machine, from the point s to $s+s_o$, where s_o is the length of the closed orbit. Substitution of (2) into (1) gives a constant of the motion involving only the single displacement δ_1. This constant of motion has been much used in accelerator theory [6-10], under various names or none. It has been applied for example to the question of adiabatic variation [6], and to the problem of 'twist' instabilities [7,11].

The constancy of (1) along the orbit is readily verified by differentiation with respect to the independent variable s (supposed here to be left unchanged by the variations δ_1 and δ_2) and invocation of the Hamilton equations where K is the Hamiltonian for independent variable s [4,7,12]. Suppose now that these equations are perturbed to

(3)

The $\mathcal{F}_n$ represent additional forces that we cannot, or do not wish to, incorporate into the Hamiltonian K. Then one readily finds

(4)

The validity of (4) depends only on the linearized equations of motion for small $(\delta q, \delta p)$. These remain valid when real solutions are combined to form complex solutions, as is often convenient. Let $(\delta_1 q, \delta_1 p)$ be identified with some complex solution $(\delta q, \delta p)$, and let $(\delta_2 q, \delta_2 p)$ be identified with the complex conjugate solution $(\delta q^*, \delta p^*)$. Then (4) becomes

(5)

Integrating round the ring we have the change in one turn

(6)

Consider in particular one of the characteristic solutions which change only by an over-all factor in one revolution

(7)

where μ and d are real. Using this in (6) and with the approximation

we have our main result

(8)

The quantities on the right are evaluated along the unperturbed orbit ; no particular argument s need then be specified for D, for it is independent of s.

2. - THE RING

The notation will be essentially that of Sands [2]. The device is supposed to have a plane of symmetry, called 'horizontal' in what follows, and a closed orbit lying in that plane. As independent variable we take the distance s measured along the closed orbit. As dependent variables we take vertical displacement z, horizontal displacement x outward and perpendicular to the closed orbit, and time delay τ (opposite sign to Sands), with respect to the particle on the closed orbit at the given s. The corresponding canonical momenta are (Appendix A4)

(9)

where m_o is particle rest mass ; t is time ;

where v is particle velocity and c the velocity of light ; A and ϕ are electromagnetic potentials. A prime (') denotes differentiation with respect to s.

In what follows we consider for simplicity only the extreme relativistic approximation

(10)

where ρ is the radius of curvature of the closed orbit. Moreover we will work only to the first order in small deviations from the closed orbit. Then

(11)

where E_o is the energy of the closed orbit, and ... indicates potential terms which do not contribute to the Lagrange invariant (Appendix A), i.e., to D in (8).

3. - RADIATION REACTION

The classical radiation reaction on a particle of charge e and velocity $\vec{v}$ in a magnetic field B is [13]

(12)

in the extreme relativistic case

with (in mks units)

(13)

where ε_o is permittivity of free space.

In what follows we need $\vec{F}$ only to first order in small deviations from the closed orbit, where v.B is zero. So for our purposes

(14)

Note that the energy loss per turn on the closed orbit is

(15)

To first order in small quantities the curvilinear components of F are (Appendix B16)

(16)

In (8) then

(17)

where

(18)

Note that (12) allows only for the magnetic guide field and not for the rf accelerating fields. The latter are regarded as of the same order as the radiation reaction itself, for which they have to compensate, and the corresponding terms in (12) would be a perturbation of higher order.

4. - VERTICAL OSCILLATIONS

For the vertical betatron oscillations

The summations in (8) then reduce to single terms. From (11) and (17)

(19)

(20)

Remembering that D is a constant of the motion, independent of s, the quotient is

(21)

5. - SYNCHROTRON OSCILLATIONS

The rf forces being supposed weak, a complete synchrotron oscillation requires many revolutions

(22)

and over any single revolution the energy varies little :

(23)

The energy shift δE induces a change in radial position

(24)

where $\eta(s)$ is the 'off-energy function' [2]. At increased radius the particle takes longer to traverse a given ds, and falls behind in time

(25)

The left-hand side is also, for the characteristic solution,

(26)

So for small μ_s, and δE constant over one revolution,

(27)

Because of the large factor $(1/\mu_s)$ here, the δx terms can be neglected in comparison with the $\delta\tau$ terms in N and D of (8). [But in $\delta\mathcal{F}_\tau$, (17), one must not neglect the δx term in comparison with the δE term.]

Then from (11) and (17)

(28)

(29)

So

(30)

where

(31)

6. - HORIZONTAL BETATRON OSCILLATION

From the way in which radiation reaction perturbs Liouville's theorem it is easily shown that [1]

(32)

It follows from (21) and (30) that

(33)

However, it is perhaps of some interest to see how brutal application of (8) gives this result. We again have $\delta z = 0$ and again coupled oscillation of δx and $\delta\tau$. This coupling has to be followed in more detail than before, because μ_x is not supposed small. Once again

(34)

From

follows

(35)

and then

(36)

(37)

where we have used

We introduce now the standard form [14] (with $a \equiv \varepsilon_x^{1/2}$)

(38)

where a and δ are s-independent amplitude and phase, the function $\beta_x(s)$ is characteristic of the focussing system, and

(39)

Then

where μ_x is the phase change per revolution, or

(40)

where

(41)

and $\lambda(s)$ is defined likewise with the cos under the integral sign replaced by sin. It is important to recognize [e.g., Eq. (3.6) in Ref. [8]], that this $\eta(s)$ is the same 'off-energy function' already used in (24).

Strictly speaking, the time variation $\delta\tau$, in conjuction with the rf field, implies an energy variation δE. But since we regard the rf field as of the same order of magnitude as the radiation reaction for the zero order trajectories we have

(42)

Then in (8)

(43)

[using (15)]

(44)

The denominator is, from (8) and (11),

(45)

Dividing (44) by (45) gives (33), as expected.

7. - CONCLUDING REMARKS

We think it is already of some interest to see how the familiar results fit into the more general framework related to the formula (8). If it were necessary to improve on the extreme relativistic approximation, or (more likely) on the weak rf approximation, the present method would probably be less painful than those in the literature.

In comparing the present treatment with others it might be thought strange that the discussion of vertical oscillation damping in §4 makes no explicit reference to the accelerating cavity. In other approaches [2] it is said sometimes that the damping occurs 'at the cavity' as a result of the acceleration. But this is dependent on the variable that is considered. The slope z' is not changed directly by the radiation reaction, which does not change directly the direction of the particle. This slope changes at the rf cavity when the total momentum E/c changes but the transverse momentum Ez'/c does not. On the other hand this transverse momentum is changed by the radiation reaction - which changes E. This last picture, considering transverse momentum rather than slope, is the more closely related to our considerations.

APPENDIX A - ROLE OF THE POTENTIALS

The canonical momenta are not just velocities, but contain also the vector and scalar potentials. However, we will see that the potentials do not contribute to the relevant Lagrange invariants - in the present problem. The Cartesian components of velocity are

(A1)

Then from (B1) and (B7)

(A2)

Write in this

(A3)

where $\tau = 0$ for the reference orbit. Then

(A4)

where

(A5)

(A6)

Thus (as is well known) p_z and p_x are the usual canonical momenta, and p_τ is the negative of the total energy.

Consider now how the potential parts of the p's contribute to the Lagrange invariant

(A7)

where the summation is over q_n (= z,x,τ).

The simplest case is that of purely vertical oscillation. The only potential contribution is

(A8)

Quite generally contributions from $(\partial A_x/\partial x)$ and $(\partial\phi/\partial\tau)$ cancel out in this same way. The remaining contributions involve the combinations

(A9)

[the last sign depending on whether the (x,s,z) system is right- or left-handed.]

In this paper we assume the reference orbit to be a plane of symmetry. Then E_z and B_s are zero on the reference orbit (where the derivatives in question have to be evaluated). There could be a horizontal electric field E_x (in the accelerating cavity) - but it will be assumed negligible (for the cavity is designed to accelerate rather than deflect the particle).

APPENDIX B - FORCES WITH s AS INDEPENDENT VARIABLE

With the usual Lagrangian

(B1)

(where ϕ and $\vec{A}$ are potentials for the applied fields, and $\vec{u} = \dot{\vec{r}}$ is velocity) the Cartesian co-ordinate equations of motion are

(B2)

where $\vec{F}$ is the radiation reaction force. This implies that for variations δ, away from a solution of the equations, restricted to a finite part of the orbit,

(B3)

The variations $\delta r|_t$ are at fixed time. The variation $\delta r|_s$ at fixed value of some other variable s is

(B4)

Then an equivalent variation principle is

(B5)

or

(B6)

with

(B7)

(B8)

where the q_i are a set of arbitrary curvilinear co-ordinates including time but excluding the new independent variable s. The new Lagrangian equations of motion are where a prime denotes differentiation with respect to s :

Or equivalently, in Hamiltonian form,

(B9)

where the new Hamiltonian is

(B10)

and

(B11)

We take for q_i the radial and vertical displacements x and z from the reference orbit, and time delay τ with respect to the reference particle, all at given s, where s is the distance measured along the reference orbit. Then from (B8)

(B12)

In the text we consider explicitly only the extreme relativistic limit. Then

(B13)

where ρ_o is the reference orbit radius of curvature. On its actual trajectory the velocity is c, but with s measured along the reference orbit we have the extra factor $(1 + x/\rho_o)$. Moreover, we work only to first order in x, z, τ. Then with

(B14)

where

(B15)

we have finally

(B16)

where E is particle energy, B is applied magnetic field, and E_o and B_o refer to the reference orbit.

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