ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB231



# COOLING OF AN ANNULAR BEAM BY USING NONLINEAR EFFECTS

F. Capoani*[1], A. Bazzani, Bologna University, Bologna, Italy
M. Giovannozzi, R. Tomás, CERN Beams Department, Geneva, Switzerland
[1]also at CERN Beams Department, Geneva, Switzerland

*Abstract*

In recent years, nonlinear effects have been used to modify the transverse beam distribution by adiabatic trapping into nonlinear resonances. This allows generating transversally split beams, in which the initial single Gaussian is divided into several ones depending on the order and stability type of the crossed resonance. An adiabatic modulation in presence of nonlinear effects could be used to reduce the beam emittance by acting on its transverse beam distribution. In this paper, we present and discuss the special case of a beam with an annular distribution, showing how the resulting emittance could be reduced by means of nonlinear effects.

## INTRODUCTION

Nonlinear effects introduce new phenomena in beam physics. In recent years, they have been used extensively to design novel beam manipulations in which the transverse beam distribution is modified in a controlled way for different purposes. This is the case for the beam splitting that is at the heart of the CERN Multiturn Extraction (MTE) [1–4]. It is also possible to redistribute the invariants between the two transverse degrees of freedom [5]. It is therefore natural to study also whether nonlinear effects can be used to reduce the linear invariants for a beam distribution, hence cooling the beam emittance.

Nonlinear effects do not preserve the linear invariant, i.e. the linear action or the so-called Courant-Snyder invariant. In this sense, they can be used to provide a reduction of the linear invariant without violating the Liouville character of the Hamiltonian dynamics.

In this paper we discuss an initial step towards the possibility of using nonlinear effects to cool a particle distribution. Here we develop a framework to cool an annular beam distribution, i.e. a distribution with non-zero density in an interval of radii $r_1 < r < r_2$, $r_1 > 0$. The annular beam distributions appear as a result of applying a single transverse kick to a centred beam in presence of decoherence. Hence, a potential application of cooling annular beams could be to restore the initial centred distribution after a kick.

## THE HAMILTONIAN MODEL

Transverse motion in the presence of an AC dipole is described by a Hamiltonian of a generic oscillator with a nonlinearity and a dipolar time-dependent exciter [6]

$$H(x, p_x, t) = \omega_0 \frac{x^2 + p_x^2}{2} + \frac{k_3}{3} x^3 + \varepsilon x \cos \omega t . \qquad (1)$$

* federico.capoani@cern.ch

In the unperturbed ($\varepsilon = 0$) action-angle coordinates $(\phi, J)$ the Hamiltonian reads

$$H(\phi, J) = \omega_0 J + \frac{\Omega_2}{2} J^2 + \varepsilon \sqrt{2J} \cos \phi \cos \omega t ,$$

where we introduce the detuning term $\Omega_2 = O(k_3^2)$.

In the rotating-frame reference, using the angle $\gamma = \phi - \omega t$, one obtains

$$H(\gamma, J) = (\omega_0 - \omega) J + \frac{\Omega_2}{2} J^2 + \varepsilon \sqrt{2J} \cos(\gamma + \omega t) \cos \omega t ,$$

taking into account the time derivative of the generating function $\partial F / \partial t = -\omega J$.

We average the perturbation term over the fast variable $\omega t$, to obtain the slow-dynamics Hamiltonian

$$H(\gamma, J) = (\omega_0 - \omega) J + \frac{\Omega_2}{2} J^2 + \frac{\varepsilon}{2} \sqrt{2J} \cos \gamma ,$$

and after re-scaling of the action, one obtains

$$H(\gamma, J) = 4J^2 - 2 \lambda J + \mu \sqrt{2J} \cos \gamma , \qquad (2)$$

where the parameters are defined as

$$\lambda = \frac{4}{\Omega_2} (\omega - \omega_0), \qquad \mu = \frac{4\varepsilon}{\Omega_2} ,$$

which can be changed upon acting on $\varepsilon$ and $\omega$.

Equation (2) is a well-known Hamiltonian [7, 8], and can be cast in the following form:

$$H(X, Y) = (X^2 + Y^2)^2 - \lambda (X^2 + Y^2) + \mu X , \qquad (3)$$

using the Cartesian co-ordinates $X = \sqrt{2J} \cos \gamma$, $Y = \sqrt{2J} \sin \gamma$.

When $\lambda > (3/2) \mu^{2/3}$, a hyperbolic fixed point exists at $Y = 0$ and

$$X = x_c = \frac{\sqrt{6\lambda}}{3} \cos \left[ \frac{\pi}{6} + \frac{1}{3} \operatorname{asin} \left( \frac{3\sqrt{6}}{4} \frac{\mu}{\lambda^{3/2}} \right) \right] .$$

The separatrix divides the phase space in three regions, as shown in Fig. 1, whose areas $A_i$ can be computed analytically. Let $H_c = H(x_c, 0)$, then the separatrix curve $H(\gamma, J) = H_c$ can be explicitly computed as

$$J(\gamma) = \frac{\lambda - 2x_c^2}{2} - 2x_c \sqrt{\lambda - 2x_c^2} \sin \gamma + 2x_c^2 \sin^2 \gamma ,$$

and $J(\gamma) = 0$ for $\gamma = \gamma_0 = \operatorname{asin}\left( \sqrt{\lambda - 2x_c^2} / 2x_c \right)$. The areas enclosed are given by the following integrals:

$$A_1(\lambda, \mu) = \int_{-\gamma_0}^{\pi - \gamma_0} \mathrm{d}\gamma \, J(\gamma) = \frac{\pi \lambda}{2} - K_1 - K_2,$$

$$A_3(\lambda, \mu) = \int_{-\pi - \gamma_0}^{\gamma_0} \mathrm{d}\gamma \, J(\gamma) = \frac{\pi \lambda}{2} + K_1 + K_2,$$

$$A_2(\lambda, \mu) = A_3(\lambda, \mu) - A_1(\lambda, \mu) = 2(K_1 + K_2),$$

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB231

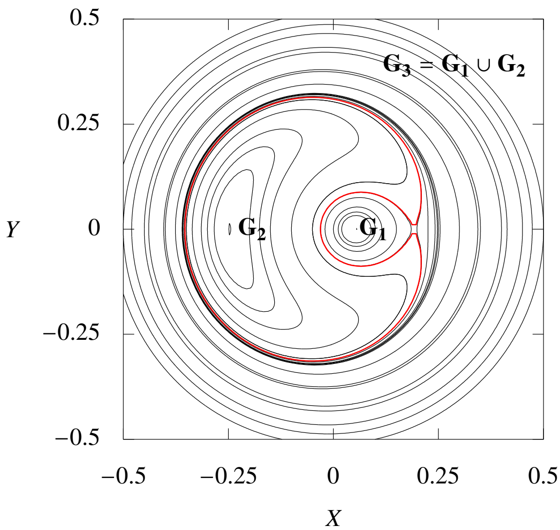


Figure 1: Phase-space portrait of the Hamiltonian (3) with parameters $\lambda = 0.1$, $\mu = 0.01$. The red line is the separatrix.

where

$$K_1 = \lambda \operatorname{asin} \frac{\sqrt{\lambda - 2x_c^2}}{2x_c}, \qquad K_2 = \frac{3}{2}\sqrt{(\lambda - 2x_c^2)(6x_c^2 - \lambda)}\,.$$

## THEORY OF ADIABATIC SEPARATRIX CROSSING

At first, an initial condition from an annular-shape distribution evolves in the outer region with an initial action $J_0$, and $\lambda$ and $\mu$ are slowly varied. At time $t^*$, $\lambda = \lambda^*$, $\mu = \mu^*$, and $A_3 = 2\pi J_0$, and according to adiabatic separatrix-crossing theory [7, 9], having defined $\xi = \dfrac{\mathrm{d}A_i/\mathrm{d}t}{\mathrm{d}A_3/\mathrm{d}t}$, the orbit is trapped in region $G_i$ ($i = 1, 2$) with a probability $P_i$

$$P_i = \xi \;\text{ if }\; \xi \in ]0,1[, \quad P_i = 0 \;\text{ if }\; \xi < 0, \quad P_i = 1 \;\text{ if }\; \xi > 1\,,$$

and with an action value after trapping given by $A_i/2\pi$. Given a distribution of initial conditions with action $J \in [J_0 - \Delta, J_0 + \Delta]$, the expected value of their final action after trapping, if $\Delta$ is sufficiently small, is $\langle J \rangle_{\mathrm{f}} = (A_1 P_1 + A_2 P_2)/2\pi \le J_0$, which means that the separatrix-crossing process reduces the emittance of the annular distribution.

To optimise the cooling process, two protocols have been considered: one consists in trapping all particles in $G_1$, the other in trapping all particles in $G_2$, and then, for both processes, adiabatically moving the resonance island to the origin of phase space (videos illustrating these two protocols can be found at [10, 11]).

### *Trapping in $G_1$*

To achieve complete trapping in $G_1$, one needs, at the separatrix-crossing point $t = t^*$, to fulfil the condition $P_2 = 0$, i.e.

$$\frac{\mathrm{d}A_2}{\mathrm{d}t} = \frac{\partial A_2}{\partial \lambda} + \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}\frac{\partial A_2}{\partial \mu}\Big|_{\lambda^*,\mu^*} = 0\,,$$

and a solution $\mu' = \mathrm{d}\lambda/\mathrm{d}\mu\,\big|_{\lambda^*,\mu^*}$ is obtained. At the separatrix-crossing point, it determines the coupled variation of $\lambda$ and $\mu$ needed to achieve complete trapping in $G_1$, and it can be shown that $\partial A_2/\partial\lambda > 0$, $\partial A_2/\partial\mu > 0$, and $\mu' < 0$.

The trapping protocol is made of two stages: in the first one, $\lambda$ is kept at zero and $\mu$ is slowly increased up to $\mu_{\max} = \mu^* - \lambda^*\mu'$, to match the initial distribution to the phase-space topology with the AC-dipole. Then, $\lambda$ is increased linearly and $\mu$ decreased, according to the law $\mu(t) = \mu_{\max} + \mu'\,\lambda(t)$, which ensures that when $\lambda = \lambda^*$, $\mu = \mu^*$ then $\mathrm{d}\lambda/\mathrm{d}\mu = \mu'$. The process ends when $\mu = 0$, since in this state $G_2$ disappears, as the perturbation provided by the AC-dipole has been switched off, and the particles trapped in $A_1$ have been transported to the centre of phase space.

Given an initial annular beam distribution at $J = J_0$, the final expected action is $J_{\mathrm{f}} = A_1/2\pi = \lambda^* - 2J_0$, where $\lambda^*(\mu^*)$ solves the implicit equation $A_3(\lambda,\mu) = 2\pi J_0$. This solution exists for $2 \le \lambda^*/J_0 \le 4$ and $0 \le \mu^*/J_0^{3/2} \le 8/\sqrt{27}$. For $\lambda^* = 2J_0$, the final expected action reaches zero, but then both $|\mu'|$ and $\mu_{\max}$ diverge, so that an infinite perturbation strength is needed for a 100% cooling efficiency [12].

In Fig. 2 (left) the expected cooling ratio $\langle J \rangle_{\mathrm{f}}/J_0$ for an annular distribution at $J_0 = 0.05$ is shown as a function of $\mu^*$, while in Fig. 2 (right) the $J_0$ dependence of the cooling ratio is shown for different values of $\mu^*$, to identify the range of actions for which it is possible to cool the beam. The simulated data are obtained by means of a symplectic integration of Eq. (1), with $\omega_0 = 0.414 \times 2\pi$, $k_3 = 1$, and give a detuning coefficient $\Omega_2 = -0.3196$, using a FFT evaluation [13].

### *Trapping in $G_2$*

All particles could be trapped in $G_2$, varying $\mu$ while keeping $\lambda$ constant, as $\mathrm{d}A_2/\mathrm{d}\mu = 2\,\mathrm{d}A_3/\mathrm{d}\mu$ so $P_2 = 1$.

This ensures that we have $J_{\mathrm{f}} = A_2/2\pi = 2J_0 - \lambda^*/2$ as final action, and the cooling is possible in the range $2 \le \lambda^*/J_0 \le 4$. Also in this case, depending on $\lambda^*$, the desired cooling target is a free parameter.

The first phase of our protocol consists, for time $t \in [0, t_1]$, in keeping $\lambda(t) = \lambda^*$, while ramping $\mu(t) = \mu_1 \epsilon t$, $\epsilon t_1 = 1$, the only condition being $\mu_1 > \mu^*$, which solves the equation $A_3(\lambda^*, \mu^*) = 2\pi J_0$.

After trapping in $G_2$, the particle has a smaller action, but the definition of the adiabatic invariant is not $J = (x^2 + p_x^2)/2$ as $\mu \ne 0$. Thus, an adiabatic transport process to reduce $\mu$ to 0, keeping fixed the area $A_2$, needs to be envisaged to avoid particle detrapping, i.e.

$$\frac{\mathrm{d}A_2}{\mathrm{d}t} = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\left(\frac{\partial A_2}{\partial \lambda} + \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\frac{\partial A_2}{\partial \mu}\right) = 0\,. \qquad (4)$$

12th Int. Particle Acc. Conf. IPAC2021, Campinas, SP, Brazil JACoW Publishing
ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB231



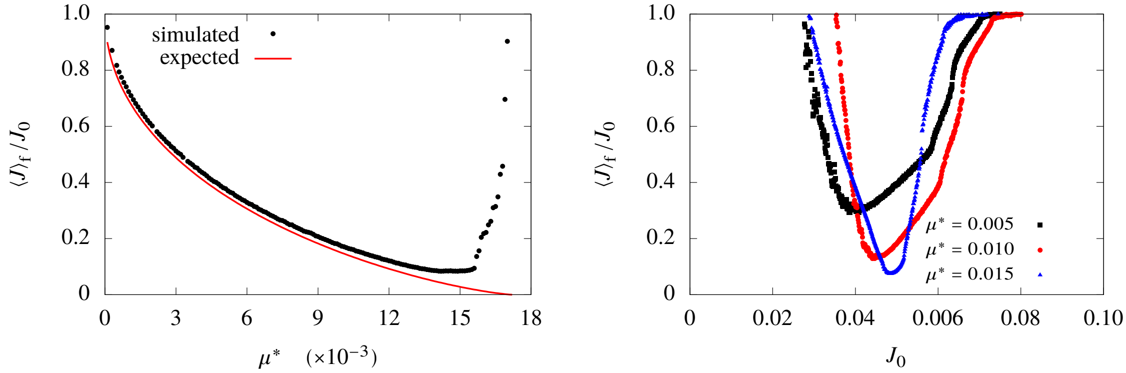


Figure 2: Left: expected and simulated cooling ratio for trapping in $G_1$ as a function of $\mu^*$. Initial distribution is an annulus at $J_0 = 0.05$. Right: cooling ratio, for different values of $\mu^*$, as a function of the initial annular distribution $J_0$. The Hamiltonian of Eq. (1) has been used, with $k_3 = 1$, $\omega_0 = 0.414 \times 2\pi$, $\Omega_2 = -0.3196$ for both plots.

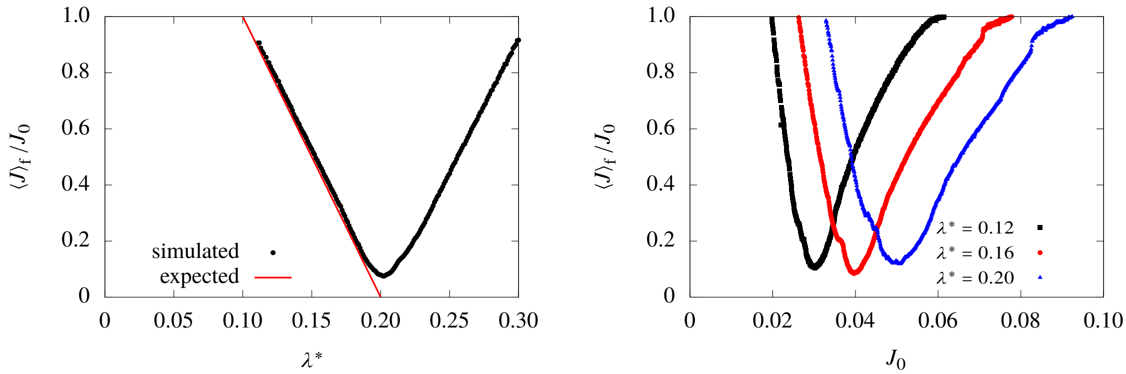


Figure 3: Left: expected and simulated cooling ratio for trapping in $G_2$ as a function of $\lambda^*$. Initial distribution is an annulus at $J_0 = 0.05$. Right: cooling ratio, for different values of $\lambda^*$, as a function of the initial annular distribution $J_0$. The Hamiltonian of Eq. (1) has been used, with $k_3 = 1$, $\omega_0 = 0.414 \times 2\pi$, $\Omega_2 = -0.3196$, $\mu_1 = 0.02$ for both plots.

From Eq. (4) we obtain a differential equation for $\mu(\lambda)$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = -2x_\mathrm{c}\sqrt{\frac{\lambda - 2x_\mathrm{c}^2}{6x_\mathrm{c}^2 - \lambda}}\,\mathrm{asin}\,\frac{\sqrt{\lambda - 2x_\mathrm{c}^2}}{2x_\mathrm{c}}\,. \tag{5}$$

For $t \in [t_1, t_2]$, we set $\lambda(t) = \lambda^* - \epsilon(t - t_1)$, and as $\mathrm{d}\lambda/\mathrm{d}t = -\epsilon$, from Eq. (5) we have a Cauchy problem for $\mu(t)$ that reads:

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = -\epsilon\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}, \qquad \mu(t_1) = \mu_1, \qquad t \in [t_1, t_2]\,,$$

and can be solved numerically. This ensures that while $\lambda$ is reduced, $\mu$ is increased, maintaining $A_2$ constant and reducing $A_1$ to zero. The final time $t_2$ is determined from the equation $\mu(t_2) = (2\lambda(t_2)/3)^{3/2}$, which is the existence condition of the saddle points and thus of the islands. For $t \in [t_2, t_2 + t_1]$ the perturbation can be switched off while ramping down $\lambda(t) = \lambda(t_2)(1 - \epsilon(t - t_2))$ and keeping $\mu(t) = (2\lambda(t)/3)^{3/2}$, so that no resonance island is present.

In Fig. 3 (left) the expected cooling ratio $\langle J\rangle_\mathrm{f}/J_0$ is shown for an annular distribution at $J_0 = 0.05$ as a function of $\lambda^*$, whereas in Fig. 3 (right) the $J_0$ dependence of the cooling ratio, for different values of $\lambda^*$ , is shown, which identifies the range of actions in which it is possible to cool the beam.

## CONCLUSIONS AND OUTLOOK

The two protocols proposed for cooling annular beam distributions feature an upper bound on the performance, and the simulated data show clearly that this is around 90% of cooling. A key result of the analyses carried out is that both protocols have a significant cooling range, which means that it is possible to cool a finite-thickness distribution.

The next step will focus on the study of a one-turn map model for Eq. (2) using a Normal Form interpolating Hamiltonian. The map is richer than that considered here and more realistic and it will provide a better insight on the actual performance of this cooling technique. A second step consists in developing cooling protocols based on stable islands that are created by magnetic nonlinear elements such as sextupoles and octupoles, without the use of an AC dipole. In this scenario, the cooling will be performed by acting on the linear tune and on the strength of nonlinear elements so to control position and surface of the stable resonance islands [12].

Note that the annular beam distribution considered in this studies can represent also the beam halo. Therefore, in future applications to halo manipulation will be considered, possibly including experimental tests at the LHC.

ISBN: 978-3-95450-214-1 ISSN: 2673-5490 doi:10.18429/JACoW-IPAC2021-TUPAB231

## REFERENCES

[1] R. Cappi and M. Giovannozzi, "Novel method for multiturn extraction: Trapping charged particles in islands of phase space", *Phys. Rev. Lett.*, vol. 88, p. 104 801, 2002. doi:10.1103/PhysRevLett.88.104801

[2] J. Borburgh *et al.*, "First implementation of transversely split proton beams in the CERN Proton Synchrotron for the fixed-target physics programme", *EPL*, vol. 113, no. 3, p. 34 001, 2016. doi:10.1209/0295-5075/113/34001

[3] S. Abernethy *et al.*, "Operational performance of the cern injector complex with transversely split beams", *Phys. Rev. Accel. Beams*, vol. 20, p. 014 001, 2017. doi:10.1103/PhysRevAccelBeams.20.014001

[4] A. Huschauer *et al.*, "Transverse beam splitting made operational: Key features of the multiturn extraction at the cern proton synchrotron", *Phys. Rev. Accel. Beams*, vol. 20, p. 061 001, 2017. doi:10.1103/PhysRevAccelBeams.20.061001

[5] A. Bazzani, F. Capoani, and M. Giovannozzi, "Manipulations of transverse invariants in 2D Hamiltonian systems", in preparation.

[6] R. Tomás, "Normal form of particle motion under the influence of an ac dipole", *Phys. Rev. ST Accel. Beams*, vol. 5, p. 054 001, 2002. doi:10.1103/PhysRevSTAB.5.054001

[7] A. I. Neishtadt, "Passage through a separatrix in a resonance problem with a slowly-varying parameter", *Journal of Applied Mathematics and Mechanics*, vol. 39, no. 4, pp. 594–605, 1975. doi:10.1016/0021-8928(75)90060-X

[8] A. I. Neishtadt, A. A. Vasilev, and A. V. Artemev, "Capture into resonance and escape from it in a forced nonlinear pendulum", *Regular and Chaotic Dynamics*, vol. 18, no. 6, pp. 686–696, 2013. doi:10.1134/S1560354713060087

[9] V. Arnol'd, V. Kozlov, and A. Neishtadt, *Mathematical aspects of classical and celestial mechanics. Dynamical systems III; 3rd rev. version*. Heidelberg, Germany: Springer, 2006. doi:10.1007/978-3-540-48926-9

[10] R. Tomás, "Cooling an annular beam with resonances", presented at the Non-Linear Beam Dynamics WG Meeting (Online), Zurich, Swtizerland, Nov. 2020, unpublished. https://indico.cern.ch/event/966240/.

[11] F. Capoani, "Latest results on cooling an annular beam distribution", presented at the Non-Linear Beam Dynamics WG Meeting (Online), Zurich, Switzerland, May. 2021, unpublished. https://indico.cern.ch/event/1014203/.

[12] A. Bazzani, F. Capoani, M. Giovannozzi, and R. Tomás, "Nonlinear cooling of an annular beam distribution", in preparation.

[13] R. Bartolini, M. Giovannozzi, W. Scandale, A. Bazzani, and E. Todesco, "Precise measurement of the betatron tune", *Part. Accel.*, vol. 55, pp. 1–10, 1996.