Universality in Grey-Body Radiance: Extending Kirchhoff's Law to the Statistics of Quanta

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An ideal grey body is a macroscopic object with definite temperature which absorbs only a fraction of the radiation incident on it. Assuming that a grey body always emits in a mixed state, and that the radiation density matrix factors into matrices for the various frequency modes, we employ general arguments to derive the complete statistics of grey-body radiance elicited by incident radiation. These depend only on the temperature and the absorptivities for the various frequencies (hence are of universal form), and coincide with the statistics of black-hole radiance first derived a decade and a half ago.

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The origin of quantum mechanics, one of the two pillars of contemporary theoretical physics, is deeply rooted in the universality of the form of blackbody radiation. However, blackbodies are but idealizations. What we mostly observe in nature are thermal grey bodies—hot systems that reflect part of the incident radiation. The issue of grey bodies has usually been regarded as a marginal question in theoretical physics, perhaps because of the widespread feeling that they are not subject to universal rules. In this Letter we show that in a very real sense, grey-body radiation is universal. By this we mean that the statistics of grey-body radiance depend only on the grey body's temperature and on the absorptivities for the various modes, but on no other properties.

As is well known, a blackbody at temperature T emits in each radiation mode of frequency ω a mean energy (we use units such that Boltzmann's $k = 1$)

$$\langle E\rangle = \frac{\hbar\omega}{e^{\hbar\omega/T} - 1}. \tag{1}$$

This can be viewed as a consequence of the canonical probability distribution for the number of quanta in the mode in question

$$p(m) = (1 - e^{-\hbar\omega/T})e^{-m\hbar\omega/T}. \tag{2}$$

Together with the stipulation that the different frequency modes are statistically independent, and that the density matrix for each is diagonal in the occupation number representation, this last formula gives the complete statistical description of blackbody radiance, including the fluctuations therein.

Macroscopic objects in real life are often in well defined thermal states, and absorb only part of the radiation that falls on them; i.e., the absorptivities a for various frequency modes may fall below unity. Because of its thermal state, such an object emits radiation in a mixed state, whatever the state of the radiation incident on it (except in the case that the body is absolutely cold). Further, one expects the corresponding density matrix to factor, to a good approximation, into matrices for the separate frequency modes. Indeed, an exactly factorizable mixed-state density matrix is a hallmark of blackbody radiance. This does not apply to radiation from individual atoms or molecules which can be in a pure state. Further, in the latter case intermode correlations are rampant. Since ordinary macroscopic objects are complex systems in which the repeated interactions of the radiation with the many constituent atoms are expected to erase phase relations and couplings between frequencies, these objects should resemble more the blackbody case than that of individual atoms in regards to the nature of the density matrix. We may thus define an ideal grey body as one possesing the following properties.

(a) An ideal grey body absorbs in the mean a definite fraction a of the radiation incident on it in each frequency mode.

(b) It has a well defined temperature T.

(c) Unless $T = 0$, the radiation emerging from it is always described by a mixed-state density matrix.

(d) This density matrix factors into matrices for each frequency mode.

Evidently grey-body radiance must be more complicated than blackbody radiance because the a's are not all equal to unity. Thus the formula analogous to formula (2) must refer to the conditional probability $p_a(m|n)$ for the emission, in a mode with absorptivity a, of m quanta, given that n are incident in the same mode.

When the existence of spontaneous black-hole radiance was theoretically ascertained by Hawking [1], it was immediately clear that a black hole radiates as a grey body (as we have defined it): it reflects part of the radiation falling on it, part of the emergent intrinsically thermal radiation is returned into the black hole by the curvature

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barrier that surrounds it, and only part of it gets out [1], and the spontaneous radiance density matrix factors into matrices for the separate frequency modes [2]. Taking note of the fact that the blackbody radiance described by formula (2) is the maximal entropy radiance, given the mean emitted energy per mode, Eq. (1), Bekenstein and Meisels [3] derived the $p_a(m|n)$ for a Kerr-Newmann black hole by *assuming* that, when exposed to incident thermal radiation, it will return outward maximal entropy radiance, mode for mode, given the emitted and reflected mean energies. For a Schwarzschild (nonrotating and uncharged) black hole, their result reduces to [3,4]

$$p_a(m|n) = \frac{(e^x-1)e^{xn}a^{m+n}}{(e^x-1+a)^{m+n+1}} \sum_{k=0}^{\min(m,n)} \frac{(-1)^k(m+n-k)!}{k!(n-k)!(m-k)!}\left[1-4\frac{1-a}{a^2}\sinh^2(x/2)\right]^k, \tag{3}$$

where T_{bh} is the black-hole temperature, and a and $x = \hbar\omega/T_{\mathrm{bh}}$ are the black-hole absorptivity and dimensionless frequency for the mode in question. Panangaden and Wald [5] checked this distribution by extending Hawking's quantum field theoretic calculation of mean occupation numbers to the statistics. Earlier Page [6] had concluded that a large collection of oscillators all at one temperature reacts to incident radiation in accordance with this same distribution. The purpose of this Letter is to show that the distribution (3) is not confined to black holes and particular models of matter, but applies to *any* ideal grey body, as here defined. In other words, in the occupation number representation, the diagonal part of the conditional grey-body radiance density matrix is always given by distribution (3).

The distribution (3) yields expected results in some easy limits. Thus when $a \to 0$ (perfect reflection) the dominant term in the sum makes a contribution to $p_a(m|n)$ proportional to $\lim_{a\to 0} a^{m+n-2\min(m,n)} \propto \delta_{m,n}$. Working out the prefactors we find $p_0(m|n) = \delta_{m,n}$ as expected. In the limit $a \to 1$ (perfect absorption) the square bracket in distribution (3) is unity. Although we have not succeeded in evaluating it analytically, the sum over the combinatoric factors is shown to be unity in all cases by numerical evaluation. Hence, $p_1(m|n) = (1-e^{-x})e^{-mx}$ for all n, as expected for blackbody radiance. For intermediate a in the $T \to 0$ ($x \to \infty$) limit, the dominant term in the sum makes a contribution to $p_a(m|n)$ proportional to $\lim_{x\to\infty} e^{\min(m,n)x-mx}$. For $m > n$ this vanishes so that $p_a(m|n) = 0$: as expected, for $T = 0$, no more quanta go out than went in. For $m \le n$ the contribution is evidently nonzero. Working out the combinatoric factors for $k = m$ shows that $p(m|n) = \binom{n}{m}(1-a)^m a^{n-m}$ which is the expected distribution of reflected indistinguishable quanta for albedo $1-a$.

Since the derivation of formula (3) in Ref. [3] makes almost no use of specific black-hole properties, it could be construed as establishing formula (3) for grey bodies. However, the assumption is made in Ref. [3] that incident thermal radiation at temperature Θ in a given mode elicits the perfectly thermal (maximum entropy) emission in that mode for the available energy. Reasonable though it seems, it actually amounts to assuming that the radiative properties of each grey-body mode are determined by a single parameter. For the $p_a(m|n)$ of Ref. [3] are those that maximize the mode entropy subject to the constraint

$$\sum_{m,n} \hbar\omega p_a(m|n)(1-e^{-\hbar\omega/\Theta})e^{-n\hbar\omega/\Theta} = \frac{\hbar\omega a}{e^x-1} + \frac{\hbar\omega(1-a)}{e^{\hbar\omega/\Theta}-1}. \tag{4}$$

In the language of information theory, the probabilities $p_a(m|n)$ are the least biased given the one piece of information represented by a. If two numbers rather than just a were necessary to describe the radiative properties of each mode, two constraints would be needed, e.g., mean energy and its variance, and the $p_a(m|n)$ obtained would be different. Thus to show by an *alternative* procedure that formula (3) is correct is tantamount to proving that all statistical properties of the radiation of an ideal grey body are completely determined by its absorptivities a in all modes. This we do now.

In view of the factorization of the density matrix of grey-body radiance with respect to the frequency modes, we concentrate on a single mode, and on its conditional probability distribution $p_a(m|n)$, where we display the dependence on absorptivity explicitly. Implicit is a dependence on the grey-body temperature T, and perhaps on other parameters describing the mode. We shall assume the trivial conditions of positivity of the $p_a(m|n)$ and normalization in the sense $\sum_m p_a(m|n) = 1$. We further note that for a pair of grey bodies of temperature T possessing absorptivities a and a' for the mode in question, the conditional probabilities will automatically satisfy three additional conditions, namely (henceforth $x = \hbar\omega/T$), (1) $\sum_m m p_a(m|n) = b + (1-a)n$ and similarly with $a \to a'$; (2) $p_a(m|n)e^{-xn} = p_a(n|m)e^{-xm}$ and similarly with $a \to a'$; and (3) $\sum_k p_a(m|k)p_{a'}(k|n) = p_{a''}(m|n)$.

The first condition expresses the expectation that the grey body returns outward, in the mean, a fraction $(1-a)$ of the number n of incident quanta, plus those it spontaneously emits, here quantified by the mean emission coefficient b, which may in principle be a function of T, a, and some other parameters implicit in the $p_a(m|n)$'s. The first condition is evidently necessary for a grey body to attain equilibrium with thermal radiation at the same temperature. However, it is not sufficient; from Einstein's work [7] we know that detailed balance must hold in addition. This is exactly what our second condition stands for: the probability for the absorption of n quanta by a grey body in a blackbody environment with consequent emission of m quanta [see Eq. (2)] equals the probability for the absorption of m quanta with emission of n quanta.

The third condition claims that the effect of two ideal grey bodies at the same temperature T, but having different a's, which process incoming radiation in sequence, is equivalent to the effect of a third ideal grey body at temperature T, but having some other absorptivity. This is a reasonable expectation because, as we now show, the combined system satisfies all the defining conditions for an ideal grey body at the common temperature T. First, since the radiation incident on the second grey body from the first is partially reflected back, the combined system must have absorptivity below unity. Second, since each grey body emits in a mixed quantum state, so should the combined system. And since the radiation density matrix emerging from each of the two bodies factors, the matrix describing their joint action must also factor. Finally consider the probability distribution $q_{aa'}(m|n) \equiv \sum_k p_a(m|k)p_{a'}(k|n)$ generated by the two grey bodies acting in sequence. Obviously $\sum_m q_{aa'}(m|n) = 1$ and $q_{aa'}(m|n) \geq 0$. By using the detailed balance (second condition) for both p_a and $p_{a'}$, and exchanging n and m, one easily sees that $q_{aa'}(m|n)e^{-xn} = q_{a'a}(n|m)e^{-xm}$. Thus the combined system can be put in detailed balance with blackbody radiation at temperature T (the exchange of a and a' is necessary because inverse processes involve the grey bodies in opposite orders). Thus the combined system behaves exactly as expected of a system at temperature T. In summary, a system of two ideal grey bodies at like temperature in tandem posesses all the defining properties of an ideal grey body at temperature T, and the probability distribution $q_{aa'}(m|n)$ describing the statistics of its radiation should thus be identical to the grey-body conditional probability distribution $p_{a''}(m|n)$ for temperature T, for some effective absorptivity a'', and for some values of the extra parameters implicit in the individual $p_a(m|n)$'s. This validates the assumed condition (3).

Before demonstrating the universality of the grey-body radiation, let us extract some preliminary results. The first one is that the mean spontaneous emission by an ideal grey body is uniquely determined by its absorptivity and temperature. Indeed, summing the second condition over n and recalling the normalization of probability, we obtain

$$e^{-xm} = \sum_n p_a(m|n)e^{-xn}. \tag{5}$$

Now, if we multiply both sides by m, sum over m, and use the first condition, we get

$$\sum_m me^{-xm} = \sum_n \left[b + (1-a)n\right] e^{-xn}. \tag{6}$$

After a trivial algebra, if follows that

$$b = \frac{a}{e^x - 1}. \tag{7}$$

This result for the mean number of spontaneously emitted quanta reproduces Kirchhoff's law that the emissivity of a hot body equals its absorptivity a: the mean number of quanta spontaneously emitted, b, depends only on T and a, and is a fraction a of the Planckian value.

To deduce the probability distribution for the number of spontaneously emitted quanta, we appeal to the second condition. For $n = 0$

$$p_a(m|0) = p_a(0|m)e^{-xm}. \tag{8}$$

Now the probability $p_a(0|1)$ that a quantum is incident on the grey body and none reemerges should be decomposable as $p_a(0|0)e^{-\mu}$, where $e^{-\mu}$ is the actual probability of absorption which has to be multiplied by the probability that no quantum gets emitted in the mode, $p_a(0|0)$, to get $p_a(0|1)$. By analogy since the absorption of various quanta are uncorrelated events, $p_a(0|m) = p_a(0|0)e^{-m\mu}$. Substituting this in Eq. (8) and determining $p_a(0|0)$ by the requirement of normalization we obtain

$$p_a(m|0) = (1 - e^{-\beta})e^{-\beta m}, \tag{9}$$

where $\beta \equiv \mu + x$. Now comparing the mean number of quanta spontaneously emitted implied by Eq. (9) with our previous result, Eq. (7), we find that β is determined by

$$\frac{1}{e^\beta - 1} = \frac{a}{e^x - 1}. \tag{10}$$

It is easy to check that the spontaneous emission distribution $p_a(m|0)$ is that which maximizes the entropy emitted given the mean energy $b(x, a)$. It is precisely by using the maximum entropy principle that the results (9) and (10) were earlier derived in the context of black holes [8]. Our alternative derivation here certifies that the spontaneous emission of an ideal grey body is fully determined by the absorptivities (and temperature) alone, and that there are no second parameters.

Moving on we note that there is a useful relation between the absorptivities in a mode of two ideal grey bodies at the same temperature and that of the grey body obtained by combining them. Multiplying the third condition by m and summing over m gives

$$\sum_m m \sum_k p_a(m|k)p_{a'}(k|n) = \sum_m mp_{a''}(m|n). \tag{11}$$

Interchanging the order of the summations, making repeated use of the first condition in the left hand side of this equation, and recalling Eq. (7) gives

$$\sum_m mp_{a''}(m|n) = \frac{a' + (1-a')a}{e^x - 1} + (1-a)(1-a')n. \tag{12}$$

A comparison of this result with the first condition and Eq. (7) reveals that the effective absorptivity of the composite grey body is

$$a'' = a + a' - aa'. \tag{13}$$

Note that if both a and a' lie in the range (0,1), so does a''.

We now prove that formula (3) is the universal $p_a(m|n)$ given x and a. In the third condition, let us take $n = 0$ thus obtaining

$$p_{a''}(m|0) = \sum_n p_a(m|n) p_{a'}(n|0), \tag{14}$$

which with the aid of Eqs. (9) and (10) reads

$$(1 - e^{-\gamma})e^{-\gamma m} = \sum_n p_a(m|n)(1 - e^{-\alpha})e^{-\alpha n} \tag{15}$$

where α and γ are determined by

$$\frac{1}{e^{\alpha} - 1} = \frac{a'}{e^{x} - 1} \tag{16}$$

and

$$\frac{1}{e^{\gamma} - 1} = \frac{a''}{e^{x} - 1}. \tag{17}$$

Recalling the law of composition of absorptivities, Eq. (13), we can alternatively express γ as a function of α:

$$\frac{1}{e^{\gamma} - 1} = \frac{a}{e^{x} - 1} + \frac{1 - a}{e^{\alpha} - 1}. \tag{18}$$

Equations (15) and (18) say that each mode of an ideal grey body responds to thermally distributed incident radiation by emitting thermally distributed radiation (at a different effective temperature). Let us regard x and a in the last five equations as fixed, while α, or alternatively $z \equiv e^{-\alpha}$, is variable. Then Eq. (15) is equivalent to the identity of two functions of z:

$$\sum_n p_a(m|n) z^n = \frac{1 - e^{-\gamma(z)}}{1 - z} e^{-\gamma(z) m} \equiv f(z). \tag{19}$$

The conditional probabilities now follow from repeated differentiations of this expression at the origin:

$$p_a(m|n) = \frac{1}{n!} \frac{d^n f(z)}{dz^n}\bigg|_{z=0}. \tag{20}$$

The calculation was done in [3] and led uniquely to Eq. (3) with $x \equiv \hbar\omega/T$.

The above shows that the statistical properties of greybody radiance depend only on T and the absorptivities, and are in this sense universal. This extends Kirchhoff's law—the absorptivities completely determine the *mean* emission—to the statistics of quanta.

As an application, suppose a spherical grey body is enclosed inside an also spherical perfectly reflecting cavity perforated with a tiny hole through which radiation of arbitrary character is allowed to enter. This radiation will be absorbed and reflected by the grey body which will also emit. The combined radiation will then be reflected back by the cavity towards the grey body thus iterating the process. In the limit the distribution of the radiation is that obtained by processing the initial radiation through an infinite convolution of identical $p_a(m|n)$'s. Now it is easy to see from Eq. (13) that after N reflections, the effective absorptivity is $a_N = 1 - (1 - a)^N$. In the limit $N \to \infty, a_N \to 1$. But, as mentioned earlier, for $a = 1$ the $p_a(m|n)$ in Eq. (3) reduces, for any n, to the Planckian distribution, Eq. (2). Thus over a sufficiently long time, any radiation gets converted by interaction with a grey body into blackbody radiation.

All the above results are for boson radiation. Fermions, if emitted by the grey body, will also be described by a conditional probability distribution $p(m|n)$, with $m, n = 0, 1$ because of the Pauli principle. However, as shown in Ref. [3], the Fermi-Dirac analog of Eq. (1) forces the result

$$p(1|n) = \frac{a}{e^{x} + 1} + (1 - a)\delta_{1n}, \qquad p(0|n) = 1 - p(1|n), \tag{21}$$

without any special assumptions. For fermions we thus get universality in a trivial way.

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