Particle Accelerators, 1995, Vol. 50, pp. 177–187
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ON LANDAU DAMPING OF DIPOLE MODES BY NON-LINEAR SPACE CHARGE AND OCTUPOLES

D. MÖHL

CERN-PS, CH-1211 Geneva 23, Switzerland

(Received 3 March 1995; in final form 3 March 1995)

The joint effect of space-charge non-linearities and octupole lenses is important for Landau damping of coherent instabilities. The octupole strength required for stabilisation can depend strongly on the sign of the excitation current of the lenses. This note tries to extend results, previously obtained for coasting beams and rigid bunches, to more general head–tail modes.

KEY WORDS: Space-charge, Landau damping, coherent instabilities, head-tail

1 INTRODUCTION

The present note deals with the influence of octupole lenses and space-charge non-linearities on the collective motion. We find that the spread of the incoherent tune shift alone has no stabilizing effect on the dipolar type of instabilities. If, however, external non-linearities, e.g. octupole lenses, are included, then the incoherent tune spread can also become effective for Landau damping. Qualitatively this can be understood on the basis of the following arguments: The incoherent force acting on a particle is given by its distance from the *beam centre*. If the particles, and hence the beam centre, undergo a coherent oscillation, the incoherent force has no effect, as long as all particles have the same coherent eigenfrequency. Particles just do not change their distance from the beam centre during their common motion. However, in the presence of a tune spread introduced by forces that are determined by the distance from the *centre of the chamber*, the beam no longer responds fully coherently and then the incoherent tune spread is also "mixed in".

It is then important to choose the "right" sign of excitation of the octupole lenses so that they add to the internal tune spread. Usually space-charge forces lead to a tune depression that is largest in the beam centre. The "right" octupoles then introduce a tune increase with amplitude, preferably in both transverse planes, which requires two families of lenses (at large β_x and β_y, respectively). In fact, for beams where direct space charge dominates the wake fields, the external spread required for stability is equal to about 3/4 of the incoherent tune shift $[(3/4)\Delta Q_{ic}]$ for the good sign, but $(5/4)\Delta Q_{ic}$ for the bad sign of the octupoles.

The problem has been addressed repeatedly in the literature. Recently Blaskievicz and Weng[1] have treated the influence of space charge on bunched-beam stability, but their model does not include the combined effect of tune spread introduced by octupoles and by space charge. For a coasting beam and for rigid bunch modes, the combined effect was discussed a long time ago.[2] This note extends the results[2] (in an approximate manner) to head–tail modes. Based on these considerations, we recommended the use of a moderate number of octupoles in the LHC.

2 EQUATIONS OF MOTION

In writing the equation of motion of a test particle (i) we take three types of forces into account:

(1) the "external" focusing forces (quadrupoles and higher order multipole lenses) that depend on the deviation x_i of the particle from a fixed reference (the centre of the chamber, say);

(2) the "coherent" space-charge forces that depend on the deviation of the beam centre $\bar{x}$ from the centre of the chamber at the azimuth of the test particle. The corresponding tune shift is written as $-\Delta Q_c$;

(3) the "incoherent" space-charge forces that depend on the the deviation $(x_i - \bar{x})$ of the particle from the beam centre. The corresponding tune shift is $-\Delta Q_{ic}$.

Then

$$\ddot{x}_i + \Omega_i^2(Q_i^2 - 2Q_i \Delta Q_{ic})x_i = 2\Omega_i^2 Q_i(\Delta Q_c - \Delta Q_{ic})\bar{x} \ . \tag{1}$$

Here we have combined all terms given by the motion of the beam centre $\bar{x}$ on the r.h.s. of the equation, treating them as "driving terms" in the equation of the test particle. In a linear approximation, ΔQ_c and ΔQ_{ic} are the coherent and incoherent Laslett tune shifts. We generalized them here to include wake fields (due to resistivity, cross-sectional variation etc.) for the oscillation mode under consideration; Ω_i is the angular revolution frequency of the particle. We note in passing that the transverse coupling impedance $Z_t(\omega)$ and the (generalized) tune shifts are related by:

$$Z_t(\omega) = -i\, Z_0 \frac{2\pi Q \gamma}{N r_p} (\Delta Q_c - \Delta Q_{ic})_{(\omega)} \ , \tag{2}$$

where $Z_0 = 377\ \Omega,\ r_p = 1.54 \times 10^{-18}$ m for protons and N is the number of particles in the coasting beam.

At this stage we can already address the question, under which condition is the 'driven motion' of all particles the same, i.e. $x_i = \bar{x}$? From Eq. (1) this is a consistent solution when the zero-intensity tunes $\Omega_i Q_i$ and the *coherent* tune shifts ΔQ_c are the same for all particles. Then the incoherent force $\Delta Q_{ic}(x_i - \bar{x})$ drops out, even if ΔQ_{ic} is different for different particles. It is readily verified from Eq. (1) that the common motion then has the frequency $\Omega(Q - \Delta Q_c)$ (small $\Delta Q_c \ll Q$ assumed). Superimposed on this driven motion is the homogeneous solution of Eq. (1) which has the frequency $\Omega(Q - \Delta Q_{ic})$. It does not contribute to the centre of gravity $\bar{x}$ to the extent that the phases (due to the initial conditions) are random. We then identify the "driven" as the "coherent" and the "homogeneous solution" as the "incoherent" motion.

3 COASTING-BEAM DIPOLE MODES

Assume that the beam centre oscillates harmonically in time and space with

$$\bar{x} = \bar{A}\ \exp\left[i(-\omega t + n\theta)\right],$$

where θ is the azimuthal position around the ring and $n = 0,\ \pm1,\ \pm2$ is the mode number. The influence of the wake fields can be calculated either in the laboratory or in a moving-frame that goes around with the particle. In the present section we take the second (hydrodynamic) view. Then the derivative in Eq. (1) has to be taken along the orbit of the particles, i.e. $d/dt = \partial/\partial t + \dot{\theta}_i \partial/\partial\theta$, where for the coasting beam $\theta = \Omega_i t$, i.e. $\dot{\theta}_i = \Omega_i$.

For an approximate solution we convert the non-linear x and y dependence of the force into an amplitude dependence of the particle's tune using the method of the harmonic balance (see Appendices A and B). In essence this amounts to averaging the force over the incoherent motion. Then the steady-state solution of the test particle has the same t and θ dependence as the "driving term":

$$x_i = A_i\ \exp\left[i(-\omega t + n\theta)\right].$$

Superimposed on this is the incoherent betatron motion (solution of the homogeneous equation) $a_i \sin(Q_i \Omega_i t + \phi_i)$. Looking at the instability threshold we can assume that the coherent amplitudes are vanishingly small so that $a_i \gg A_i$.

If the tune dependence is due to a non-linearity in the other plane (e.g. $Q_x = Q_x(b_i)$ where $b_i \sin(Q_y \Omega_i t)$ is the incoherent motion in the y-plane), then the response of the particle to a driving force $F\ \exp\{i(-\omega t + n\theta)\}$ is simply

$$A_i = \frac{F}{\omega_{\beta i}^2 - (\omega - n\Omega_i)^2} \approx \frac{F}{2\omega_{\beta 0}[\omega_{\beta i} - (\omega - n\Omega)]}. \tag{3}$$

Here $\omega_{\beta i} = \omega_\beta(b) = \Omega_i[Q(b_i) - \Delta Q_{ic}(b_i)]$. In the present case [see Eq. (1)] F is given by

$$F = 2\Omega_i^2(\Delta Q_c - \Delta Q_{ic}) \cdot \bar{A}.$$

Using the fact that $\bar{A} = \langle A_i \rangle$ we obtain the dispersion relation for the mode frequency ω as

$$1 = \left\langle \frac{2\Omega_i^2 Q_i\,(\Delta Q_c - \Delta Q_{ic})}{\omega_{\beta i}^2 - (\omega - n\Omega_i)^2} \right\rangle \approx \left\langle \frac{\Omega_i(\Delta Q_c - \Delta Q_{ic})}{\omega_{\beta i} - (\omega - n\Omega_i)} \right\rangle. \tag{4}$$

If the non-linearity is in the plane of the coherent motion then — as shown by Hereward[3] — the steady state is more involved than the simple-minded response given by Eq. (3). To first order in the amplitude dependence $Q(a)$, namely for $K = (dQ/da)/(Q/a) \ll 1$, one has:

$$A_i = \frac{F}{\omega_{\beta 0}} \frac{d}{da^2} \left(\frac{\omega_\beta(a) \cdot a^2}{\omega_\beta^2(a) - (\omega - n\Omega)^2} \right) . \tag{5}$$

Here $\omega_\beta(a) = \Omega_i[Q(a_i) - \Delta Q_{ic}(a_i)]$ gives the dependence of the tune on the amplitude (a_i) of the incoherent oscillation.

To work out the averages, [Eq. (4) for the response (3) and a similar relation for the response (5)], we now introduce functions $h_1(a)$, $h_2(b)$ and $h_3(p)$ to describe the distribution of the (incoherent!) amplitudes and the momentum distribution of the particles. The latter enters since both Ω_i and Q_i are momentum-dependent. We take the three distribution functions as uncorrelated and choose the normalization

$$\int_0^\infty h_1(a) a da = 1; \quad \int_0^\infty h_2(b) b db = 1 \quad \text{and} \quad \int_0^\infty h_3(p) dp = 1 .$$

Noting that $\bar{A} = \int_0^\infty A_i h_1(a) h_2(b) h_3(p) a da \, b db \, dp$ and inserting the responses (3) and (5) we obtain the following dispersion relation for the mode frequency ω.

$$1 = \int \frac{2Q_0\Omega_{(p)}^2(\Delta Q_c - \Delta Q_{ic})}{\omega_\beta^2(a,b,p) - (\omega - n\Omega_{(p)})^2} \left(\frac{-h_1'(a)}{2} a^2 \right) \cdot h_2(b) b \cdot h_3(p) \cdot da \, db \, dp . \tag{6}$$

Here we have used integration by parts of the response (5) assuming that $a^2 h(a)$ is zero at the limits $a = 0$ and $a = \infty$. We have also approximated the numerator by its average value. This approximation becomes questionable when the external Q spreads are much weaker than the space-charge non-linearity.

Equation (6) is the dispersion relation already derived by Schönauer and Möhl.[2] The spread due to space-charge non-linearity and octupoles is included in $\omega_\beta(a, b, p)$ in the denominator.

We do not attempt to solve Eq. (6) here but content ourselves with the result that, as a rule of thumb, a spread in the frequency ω_β (traditionally this spread is denoted ΔS) larger than the frequency shift $|\Omega(\Delta Q_c - \Delta Q_{ic})|$ insures stability. Here the spread is given by

$$\Delta S = \left[\left(\Omega a_{\mathrm{rms}}^2 \frac{\partial (Q - \Delta Q_{ic})}{\partial a^2} \right)^2 + \left(\Omega b_{\mathrm{rms}}^2 \frac{\partial (Q - \Delta Q_{ic})}{\partial b^2} \right)^2 + \left(\Delta p_{\mathrm{rms}} \frac{\partial (\Omega Q)}{\partial p} \right)^2 \right]^{1/2} .$$

We see that the tune spreads due to octupoles $[a_{\mathrm{rms}}^2 (\partial Q / \partial a^2)$ etc.] and due to space-charge non-linearity add with the proper sign. Further details are given in Appendices A and B.

4 BUNCHED-BEAM DIPOLAR MODES

There are two main differences from the coasting beam case:

(i) both the arrival time and the betatron frequency of a particle are modulated by its synchrotron motion and

(ii) for a short bunch many betatron bands with frequencies near $(n \pm Q)\Omega$ contribute to driving a mode.

Let the synchrotron oscillation be described by the "time-of-arrival difference"

$$\tau = \hat{\tau}\cos(\omega_s t + \psi) \ . \tag{7}$$

For simplicity we take a "hollow-bunch model"[4,5] for the longitudinal density, i.e. we assume that all particles have the same synchrotron amplitude $\hat{\tau}$ (but different phases ψ!). To first order the betatron frequency is.[4,5]

$$\omega_\beta = Q_0\Omega_0[1 - \dot{\tau}(1 - \xi/\eta)] \ , \tag{8}$$

where

$$\xi = \frac{dQ/Q}{dp/p}$$

is the chromaticity and

$$\eta = -\frac{d\Omega/\Omega}{dp/p} = \frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}$$

is the off-momentum function of the ring. Then under a number of simplifying assumptions the equation of motion of a test particle is

$$\ddot{x} + \omega_\beta^2(t)x = WT_0\tilde{\delta}(t - \tau - kT_0) = W\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-in\Omega(t-\tau)} \ . \tag{9}$$

Here W is the "wake field" acting on the particle. It depends on the position of the particle in the bunch. We assume that the wake is induced on a short structure. It is experienced by a particle once per revolution period T_0 as expressed by the periodic δ-function $\tilde{\delta}(t - \tau - kT_0) = \sum \delta(t - \tau - kT_0)$, i.e. by a sum of ordinary Dirac functions spaced by T_0. In the second step the Fourier expansion of this function is used in Eq. (9).

We assume for the moment that W has only a single frequency component $W_0 \exp\{i(-\omega_d t + \varphi_d)\}$ with

$$\omega_d \approx (n - Q_0)\Omega - m\omega_s \tag{10}$$

near a "synchrotron sideband" m of a betatron harmonic. If W is small so that the frequency shift is small compared with ω_s, the coherent response of (9) is (Appendix C):

$$x = \frac{J_m\left((\omega_\beta - n\Omega - \omega_\xi)\hat{\tau}\right)}{2\omega_\beta(\omega_\beta - n\Omega - m\omega_s - \omega_d)}\frac{W_0}{2\pi}\exp\left[-i\left((\omega_d + m\omega_s)t + m\psi + \varphi_d\right)\right] \ . \tag{11}$$

Here the J_m are Bessel functions of order m, the index $m = 0,\ \pm1,\ \pm2$ ... distinguishes different head–tail modes, and $\omega_\xi = (\xi/\eta)\omega_\beta$ is the "chromatic frequency". If the driving force has several harmonics, then the sum over the corresponding n-values has to be taken, where negative and/or positive n are possible. To drive a given head–tail mode m, the excitation must be synchronized to the bunch such that

$$m\psi + \varphi_d = 0\,. \tag{12}$$

In this case, the response (11) is independent of the synchrotron phase ψ of the particles and thus coherent. For external excitation (beam-transfer function measurement) this synchronization condition must be fulfilled together with the frequency condition (10), to drive a pure head–tail mode. If the bunch interacts with itself via the wake fields, this phase condition is automatically fulfilled for mode m.

As in the coasting beam case we now relate the "driving force" to the mode pattern of the beam and the coupling impedance. We can construct the mode by summing over the particles. The sum is easy if we take the "hollow-bunch model"[4,5] already used above. In the weak wake-field limit the oscillation of a particle seen at a fixed azimuth is:

$$\begin{aligned} x &= A\exp\left(-i\int \omega\, dt\right)\tilde{\delta}(t-\tau-kT_0) \\ &= \sum_{n,m=-\infty}^{\infty} J_m((\omega_\beta - n\Omega - \omega_\xi)\hat{\tau})A\exp[-i((\omega_{\beta c} + m\omega_{sc})t)]\,. \end{aligned}$$

Then the transverse dipole moment (bunch current $\times$ displacement) for mode m is approximated by:

$$d = \frac{N_b e\Omega}{2\pi} J_m((\omega_\beta - n\Omega - \omega_\xi)\hat{\tau})\bar{A}\exp[-i((\omega_{\beta c} + m\omega_{s0})t)]\,. \tag{13}$$

The transverse acceleration is $-iZ_t(\omega)d/m\gamma$. Inserting this on the r.h.s. of Eqs. (9)–(11) we finally obtain the dispersion relation:

$$1 = \left\langle \sum_n \frac{-iN_b r_p\Omega^2}{\pi\gamma Z_0}\frac{J_m^2((\omega_\beta - n\Omega - \omega_\xi)\hat{\tau})}{2\omega_\beta(\omega_\beta + m\omega_s - \omega_{\beta c} - m\omega_{sc})} Z_t(\omega_\beta + m\omega_s - n\Omega)\right\rangle. \tag{14}$$

Here we assume that the coupling impedance covers several harmonics n with $\omega_n = \omega_\beta + m\omega_s - n\Omega$ and take the sum of the response (11).

We now have to remember that in the case of non-linearities in the plane of the motion, the response is somewhat more involved than Eq. (11) [see Eq. (5) for the coasting-beam case]. This, however, does not alter our conclusions which are meant to be qualitative only.

Equation (14) is then the dispersion relation for the coherent frequency $\omega_{\beta c} - m\omega_{sc}$ of head–tail mode m. One notes the equivalence with Eq. (4) for the coasting beam. The incoherent space-charge and the octupolar spreads are to be included in ω_β. If the space charge is independent of the longitudinal position in the bunch, then no spread further than the transverse amplitude dependence enters. In the more realistic case of a longitudinal variation of the density, the tune shift depends also on the synchrotron amplitude. This has the tendency to increase the spread and at the same time enhances the mode pattern [and hence, e.g., the "weighting functions" J_m in (14)]. A self-consistent treatment would have to determine the modes and the tune shifts from a given distribution. We content ourselves with the observation that (for bunches long compared to the transverse aperture of the vacuum chamber) the tune spread is typically 1/4 of the peak space-charge tune shift in the bunch centre. This, together with the spread due to transverse non-linearity of space charge, becomes effective for Landau damping if octupoles inducing (at least) a similar tune spread are used. The octupoles can then be used to study stability thresholds and to cure a large class of potential beam instabilities.

5 CONCLUSION

Octupoles capable of providing a tune spread at injection equal to about the Laslett tune shift are very valuable for diagnostics and as a general-purpose tool to stabilize coherent instabilities.

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APPENDIX A: OCTUPOLE LENSES

Magnetic vector potential $\{0,\ 0,\ A_s\}$ of an octupole lens (see texts, e.g. E. Wilson, CERN Accelerator school, CERN report 85–19, p. 99):

$$A_s = -A(x^4 - 6x^2y^2 + y^4)\ .$$

Magnetic field $B_y = -\partial A_s/\partial x$:

$$B_y = 4A(x^3 - 3xy^2)\ .$$

Betatron equation including this field:

$$x''(s) + Kx + \frac{B_y}{B_0\rho_0} = 0\ . \tag{A.1}$$

Here $K(s)$ represents the linear focusing. The quantity $B_y/B_0\rho_0 = K_{xx}(x^3 - 3xy^2)$ with

$$K_{xx} = K_{xx}(s) = \frac{4A}{B_0\rho_0} = \frac{(1/6)(\partial^3 B_y/\partial x^3)}{B_0\rho_0} \tag{A.2}$$

represents the octupoles (where $B_0\rho_0$ [tesla m] = 3.33... p [GeV/c] is the magnetic rigidity). Frequently the octupole strength K_{xx} is also denoted by K'', which should not be confused with an azimuthal derivative.

We insert for x and y the free betatron oscillation $x = a_i \sin(Q_{ix}\theta + \psi_{ix})$, $y = b_i \sin(Q_{iy}\theta + \Phi_{iy})$. We use $\sin^3(\varphi) = 3/4\sin(\varphi) - 1/4\sin(3\varphi)$ and $\sin^2(\varphi) = 1/2 - 1/2\cos(2\varphi)$. In accordance with the method of the harmonic balance we neglect the influence of the higher harmonics with frequencies $3Q$ and $2Q$ on the basic harmonic Q. We obtain:

$$\frac{B_y}{B_0\rho_0} \approx K_{xx}\left(\frac{3a_i^2}{4} - \frac{3b_i^2}{2}\right) a_i \sin(Q_{ix}\theta + \psi_{ix})\ .$$

This term is equivalent to a linear focusing term (quadrupole) $\Delta K\, x_i$ in (A.1) with

$$\Delta K = K_{xx}\left(\frac{3a_i^2}{4} - \frac{3b_i^2}{2}\right)\ .$$

The corresponding Q change is (see e.g. Courant and Snyder, Ann. Phys. 3 (1958) pp. 1–48, Eq. (4.31) or textbooks on synchrotron theory):

$$\Delta Q_x = \frac{1}{4\pi}\int_0^{2\pi R} \Delta K\, \beta_x\, ds = \frac{1}{4\pi}\int_0^{2\pi R} K_{xx}\left(\frac{3a_i^2}{4} - \frac{3b_i^2}{2}\right)\beta_x\, ds\ .$$

In a similar way we obtain the vertical tune shift as

$$\Delta Q_y = -\frac{1}{4\pi}\int_0^{2\pi R} K_{xx}\left(\frac{3b_i^2}{4} - \frac{3a_i^2}{2}\right)\beta_y\, ds\ .$$

This permits us to calculate the matrix giving $(\Delta Q_x,\ \Delta Q_y)$ as a function of the amplitudes $a_i,\ b_i$, using the K_{xx} values and the beta functions at the position of the octupoles. If the octupoles are located at large β_x (F-) and large β_y (D-octupoles), respectively, then for most particles (except for the few with zero amplitude)

$$\Delta Q_x \approx \frac{1}{4\pi}\int_0^{2\pi R} K_{xx,\mathrm{F}}\left(\frac{3a_i^2}{4}\right)\beta_x\, ds, \qquad \Delta Q_y \approx 0$$

$$\Delta Q_y \approx \frac{1}{4\pi}\int_0^{2\pi R} -K_{xx,\mathrm{D}}\left(\frac{3b_i^2}{4}\right)\beta_y\, ds, \qquad \Delta Q_x \approx 0$$

for the F- and D-octupoles, respectively.

As the amplitudes (a, b) change with the beta functions, we use the "single-particle emittances" $a_i^2/\beta_x = \varepsilon_{xi},\ b_i^2/\beta_y = \varepsilon_{yi}$ which are constant around the ring. The r.m.s. Q-spreads are then obtained in terms of the beam emittances $\varepsilon_{x,\mathrm{rms}} = a_{\mathrm{rms}}^2/\beta_x,\ \varepsilon_{y,\mathrm{rms}} = b_{\mathrm{rms}}^2/\beta_y$, as

$$\Delta Q_{x,\mathrm{rms}} \approx \frac{\varepsilon_{x,\mathrm{rms}}}{4\pi}\int_0^{2\pi R} K_{xx,\mathrm{F}}(3/4)\beta_x^2\, ds$$

$$\Delta Q_{y,\mathrm{rms}} \approx \frac{\varepsilon_{y,\mathrm{rms}}}{4\pi}\int_0^{2\pi R} -K_{xx,\mathrm{D}}(3/4)\beta_y^2\, ds\ .$$

Remember that for both the F- and D-octupoles the strength K_{xx} is defined by (A.2).

APPENDIX B: SPACE CHARGE

The tune shift for a beam of elliptical cross-section and parabolic density

$$\rho(x, y) = \frac{\lambda}{\bar{a}\bar{b}}\left(1 - \frac{x^2}{2\bar{a}^2} - \frac{y^2}{2\bar{b}^2}\right)$$

was calculated by Schönauer and Möhl.[2] Neglecting image forces the result is:

$$\Delta Q_{ic}(a, b) = -\frac{N}{B_{\mathrm{F}}}\frac{r_p R}{Q_0\pi\beta^2\gamma^3}\frac{1}{\bar{a}(\bar{a}+\bar{b})}\left(1 - \frac{2\bar{a}+\bar{b}}{8(\bar{a}+\bar{b})}\frac{a^2}{\bar{a}^2} - \frac{\bar{b}}{4(\bar{a}+\bar{b})}\frac{b^2}{\bar{b}^2}\right)\ . \tag{B.1}$$

This expresses the tune depression as a function of the (incoherent) betatron amplitudes (a, b) of a particle, $\bar{a}\sqrt{2}$ and $\bar{b}\sqrt{2}$ are the maximum amplitudes, $N/B_{\mathrm{F}}(\phi)$ is the particle number divided by the ratio of average current to local current at azimuthal position ϕ: $B_{\mathrm{F}}(\phi) = I_{\mathrm{av}}/I(\phi)$. Note that the shift for small amplitudes, $\Delta Q_{ic}(0, 0)$, Eq. (B.1), is the usual Laslett tune shift.

To make a simple estimate for the space-charge Q-spread, we take the difference in Q-shift between zero-amplitude particles and particles with amplitude $\bar{a}$ and $\bar{b}$, i.e. $1/\sqrt{2}$ times the beam radius. We obtain from (B.1):

$$\delta Q_a = \frac{\partial Q}{\partial a^2}\bar{a}^2 = |\Delta Q_{ic}(0, 0)| \left(\frac{2\bar{a} + \bar{b}}{8(\bar{a} + \bar{b})}\right)$$

$$\delta Q_b = \frac{\partial Q}{\partial b^2}\bar{b}^2 = |\Delta Q_{ic}(0, 0)| \left(\frac{\bar{b}}{4(\bar{a} + \bar{b})}\right) .$$

For the special case of a round beam this yields:

$$\delta Q_a = |\Delta Q_{ic}(0, 0)| \frac{3}{16}$$

$$\delta Q_b = |\Delta Q_{ic}(0, 0)| \frac{1}{8} .$$

These spreads have to be added to the octupolar spread.

APPENDIX C: DRIVING HEAD–TAIL MODES

It is useful to review the solution of the following equation

$$\ddot{x} + \omega^2(t)x = F(t) \tag{C.1}$$

for the case where the change in ω in one oscillation period is small compared to ω, i.e.

$$\frac{\dot{\omega}}{\omega^2} \ll 1 \qquad \text{and} \qquad \frac{\ddot{\omega}}{\omega^3} \ll 1 . \tag{C.2}$$

For the homogeneous equation (C.1) i.e. for $F(t) = 0$, there are two independent solutions which to first order in $\dot{\omega}/\omega^2$ can be written as

$$x_{h1,2}(t) = \sqrt{\frac{1}{\omega(t)}} \exp\left(\pm i \int_0^t \omega(t')dt'\right) . \tag{C.3}$$

The Wronskian $W(x_{h1},\ x_{h2},\ t)$ is given by

$$W(x_{h1},\ x_{h2},\ t) = \begin{vmatrix} x_{h1} & x_{h2} \\ \dot{x}_{h1} & \dot{x}_{h2} \end{vmatrix} = -2i . \tag{C.4}$$

The complete solution to Eq. (C.1) is given by

$$x = x_c + c_1 x_{h1} + c_2 x_{h2} \tag{C.5}$$

with c_1 and c_2 constants that are chosen to fit the initial conditions, and the particular solution $x_c(t)$ is obtained by performing the following integration

$$x_c(t) = \int_0^t \frac{x_{h1}(t')x_{h2}(t) - x_{h1}(t)x_{h2}(t')}{W(x_{h1},\ x_{h2},\ t')} F(t')dt' \ . \tag{C.6}$$

For the special case where $\omega(t) = \omega_0 + \Delta\omega\cos(\omega_s t + \psi)$, the equation for x_c reduces to

$$\begin{aligned} x_c(t) \ = \ & \frac{i}{2\omega^{1/2}(t)} e^{-i\int_0^t \omega(t')dt'} \sum_{m=-\infty}^{\infty} e^{-im\psi} J_m\left(\frac{\Delta\omega}{\omega_s}\right) \int_0^t dt' \frac{F(t')e^{i(\omega_0 - m\omega_s)t'}}{\omega^{1/2}(t')} \\ & - \frac{i}{2\omega^{1/2}(t)} e^{i\int_0^t \omega(t')dt'} \sum_{m=-\infty}^{\infty} e^{im\psi} J_m\left(\frac{\Delta\omega}{\omega_s}\right) \int_0^t dt' \frac{F(t')e^{-i(\omega_0 - m\omega_s)t'}}{\omega^{1/2}(t')} \end{aligned} \tag{C.7}$$

where we have used the fact that

$$e^{i(\Delta\omega/\omega_s)\sin(\omega_s t - \psi)} = \sum_{m=-\infty}^{\infty} J_m\left(\frac{\Delta\omega}{\omega_s}\right) e^{+im(\omega_s t - \psi)} \ . \tag{C.8}$$

In order to perform the integration over t', we restrict the driving term $F(t)$ to be given by

$$F(t) = F\cos(\omega_d t + \varphi_d) = \frac{F}{2}\left(e^{i(\omega_d t + \varphi_d)} + e^{-(i\omega_d t + \varphi_d)}\right) \ . \tag{C.9}$$

For the case when $\omega_d \approx (\omega_0 - m\omega_s)$, we can neglect the rapidly oscillating terms in performing the integration and obtain under the additional assumption $\Delta\omega \ll \omega_s$ the following expression for x_c:

$$x_c(m) = \frac{F J_m(\Delta\omega/\omega_s)}{2\omega_0(\omega_0 - m\omega_s - \omega_d)} \cos(\omega_d t + \varphi_d - m(\omega_s t - \psi)) \ . \tag{C.10}$$