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ON SOME GEOMETRICAL PROPERTIES OF GAUGE THEORIES*

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Gauge theories have become the universal language of fundamental interactions. To this discovery, Martinus J.G. Veltman played a major role. In this short note, dedicated to his memory, we try to understand some of their geometrical properties. We show that a d-dimensional SU(N) Yang–Mills theory can be formulated on a $(d+2)$-dimensional space, with the extra two dimensions forming a surface with non-commutative geometry. The non-commutativity parameter is proportional to $1/N$ and the equivalence is valid to any order in $1/N$. We study explicitly the case of the sphere and the torus.

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1. Introduction

The most important discovery in high-energy physics of the last decades was undoubtedly the realisation that all interactions among elementary particles are described by gauge-invariant theories. The concept of gauge invariance has a very long history, it spans almost two centuries, and cannot be attributed to a single man. In the next section, we will sketch a very brief history of the evolution of these ideas by indicating its main milestones. We will see that a very important one is marked by Martinus Justinus Gode-

* A tribute to Martinus J.G. Veltman.

friedus Veltman (1931–2021)[1] who, in the late 1960s, decided to launch a frontal attack on the renormalisation problem of a quantum field theory involving charged, massive, vector bosons. It is the success of this enterprise that made possible the building of what became known as *The Standard Model* of particle physics.

Today gauge theories are part of our scientific culture and are taught in basic university courses. They brought to physics novel ideas which include concepts and techniques from differential geometry. We are not yet at the end of this adventure and we continuously discover new paths leading to unexplored territories. In this note, we want to present some speculations, although we do not yet know whether they correspond to paths still to be explored or to mere dead ends. We thought it was appropriate to devote this essay to Veltman's memory because we try to follow his example. As we will see in the next section, Veltman attacked the problem of gauge field theories by direct computations [2], hoping that whichever problems are there they will eventually show up.

2. Brief historical notes

The concept of gauge invariance can be traced to classical electromagnetic theory [3]. The vector potential was introduced during the first half of the nineteenth century, either implicitly or explicitly, by several authors independently. It appears in some manuscript notes by C.F. Gauss as early as 1835, and it was fully written by G. Kirchoff in 1857, following some earlier work by F. Neumann. It was soon noticed that it carried redundant variables and several "gauge conditions" were used[2]. However, for internal symmetries, the concept of gauge invariance, as we know it today, belongs to the quantum theory [4]. It is the phase of the wave function, or that of the quantum fields, which is not an observable quantity and produces the internal symmetry transformations. The local version of these symmetries are the gauge theories of the Standard Model. The first person who realised that the invariance under local transformations of the phase of the wave function in the Schrödinger theory implies the introduction of an electromagnetic field was Fock in 1926 [5], just after Schrödinger wrote his equation. Naturally, one would expect non-Abelian gauge theories to be constructed following

[1] Martinus ("Tini") Veltman started his scientific career relatively late: he obtained his Ph.D. from the University of Utrecht in 1963 [1] under the direction of Leon Van Hove, but he had moved to CERN already in 1961 following his advisor who was named leader of the CERN Theory Division. At CERN, van Hove studied mainly hadronic physics but Veltman became interested in weak interactions (he even joined Bernardini's neutrino experiment for a while) and current algebras. It is in these fields that he made his most important and lasting contributions.

[2] The condition, which in modern notation is written as $\partial_\mu A^\mu = 0$, was proposed by the Dane L.V. Lorenz in 1867.

the same principle immediately after Heisenberg introduced the concept of isospin in 1932. However, in this case, history took a totally unexpected route.

The development of the General Theory of Relativity offered a new paradigm for a gauge theory. It exerted such a fascination on an entire generation of physicists that for many years, local transformations could not be conceived independently of general coordinate transformations. Two attempts were made to construct an SU(2) gauge theory: the first by Klein in 1938 [6] and the second by Pauli in 1953 [7]. The amazing fact is that both, fifteen years apart one from the other, decided to construct the SU(2) gauge theory for strong interactions and both choose to follow a totally counter-intuitive method. They started from a theory of general relativity formulated in a six-dimensional space-time[3], they compactified the extra dimensions *à la* Kaluza–Klein, and took the flat space limit. Yang and Mills [8] were the first to understand that the gauge theory of an internal symmetry takes place in a fixed background space which can be chosen to be flat, in which case general relativity plays no role.

With the work of Yang and Mills, gauge theories entered particle physics[4]. Although the initial motivation was a theory of the strong interactions, the first semi-realistic models aimed at describing the weak and electromagnetic interactions and used the group SU(2) $\times$ U(1) [10]. The only one among these early attempts which has survived the test of time is due to Glashow [11] who understood that a rich algebraic structure is obtained if you allow for a mixing between the two neutral generators, that of U(1) and the neutral component of SU(2). The resulting mixing angle is known as θ_W. In the same year, Gell-Mann and Glashow published a paper [12] in which they extend the Yang–Mills construction to direct products of Lie algebras. The well-known result of associating a coupling constant to every simple factor in the algebra is obtained there. In addition, they correctly identify various phenomenological problems, such as the absence of strangeness violating neutral currents.

In all these early works, two questions were left aside: *(i)* the mass of the vector bosons was put in by hand thus breaking gauge invariance and, *(ii)* there was no attempt to study these theories as quantum field

[3] Klein starts from a 5-d theory but his fifth dimension has, in fact, two components.

[4] The term *gauge* is due to Weyl [9]. He is more known for his 1918 unsuccessful attempt to enlarge diffeomorphisms to local scale transformations but, in fact, a byproduct of this work was a different form of unification between electromagnetism and gravitation. In his 1929 paper, which contains the gauge theory for the Dirac electron, he introduced many concepts which have become classic, such as the Weyl two-component spinors and the vierbein and spin-connection formalism. Although the theory is no more scale invariant, he still used the term *gauge invariance*, a term which has survived ever since.

theories by computing higher orders in the perturbation expansion. The first was answered in 1964 [13] with the concept of spontaneous symmetry breaking. It was incorporated into the $\mathrm{SU}(2) \times \mathrm{U}(1)$ model in 1967 [14] and, supplemented with the GIM mechanism [15], gave rise to the electroweak sector of the Standard Model [16]. The second question is the one first addressed by Veltman.

Around 1966, Veltman was trying to understand the deeper origin of the conservation, or near conservation, of the weak currents. In particular, he tried to throw some light on the general confusion which prevailed at that time concerning the so-called "Schwinger terms" in the commutators of two current components. While he was on a visit from CERN to Brookhaven, he wrote a paper in which he suggested a set of divergence equations which generalised the notion of the covariant derivative of quantum electrodynamics. This fundamental idea was taken up next year and developed further by Bell [17]. At that time, people had postulated the existence of a pair of charged, massive vector bosons $W^{\pm}$ as intermediaries of the weak interactions, so motivated by these divergence equations, Veltman decided to study their field theory properties. This study turned out to be very complicated, both conceptually, because the correct Feynman rules were not known, as well as practically because the number of terms grew very fast. Veltman had to develop a computer program to handle them. He called it SCHOONSCHIP[5] and it was the first program of symbolic manipulations applied to theoretical high-energy physics.

We find here Veltman's method of direct attack. No matter which form the final theory will turn out to have, it will certainly contain the photon and the charged Ws. Thus, he started from the electrodynamics of charged vector bosons. It was known [18] that electromagnetic gauge invariance allows to express the vector boson's charge e, magnetic moment μ, and quadrupole moment Q in terms of only two parameters e and κ, as $\mu = e(1+\kappa)/2m_W$ and $Q = -e\kappa/m_W^2$. The resulting theory is highly divergent but Veltman noticed that many divergences cancel for the particular value $\kappa = 1$. It is the value predicted by a theory in which $W^{\pm}$ and the photon form a Yang–Mills triplet. For Veltman, this was a clear signal that the theory of weak and electromagnetic interactions must obey a Yang–Mills gauge invariance.

In order to go further, he needed to have the correct Feynman rules. We know today that they are not the ones you would guess naïvely by generalising QED. A few years earlier, Feynman had made the right guess [19]. It is worth noticing that Feynman did not use the path integral method to derive the rules. This was done later by Faddeev and Popov. However,

[5] "Clean ship" in Dutch.

Feynman knew better than anybody else that in perturbation theory the Feynman rules are not God-given by some Lagrangian; the rules *define* the theory. So, he introduced the ghost fields for internal consistency at the one-loop level. Veltman had to learn all this by himself, by trial and error. The experience he had acquired in his thesis working with diagrams in which the particles in the intermediate lines were on their mass shells, the so-called "cutting rules", was precious. In 1969, he was joined in Utrecht by Gerard 't Hooft, a graduate student with whom he shared the 1999 Nobel Prize. Their work was a real "tour de force" [20]. The citation of the Nobel Prize reads "...for elucidating the quantum structure of electroweak interactions in physics." The importance of this work cannot be overestimated. Although the citation refers to the electroweak interactions, their result made possible the subsequent discovery of QCD. Since that time, gauge theories became the universal language of fundamental physics.

In 1973, 't Hooft and Veltman published a review article as a CERN yellow report [21]. In the introduction, we read: "... (Feynman) diagrams form the basis from which everything must be derived. They define the operational rules, and tell us when to worry about Schwinger terms, subtractions, and whatever other mythological objects need to be introduced." This is part of Veltman's legacy: when you are short of intuition, sit down and compute. If you are smart, ideas may be revealed from the computations.

3. The large-N limit of SU(N) gauge theories

The limit when the number of fields becomes very large often reveals interesting properties of quantum field theories. They include the O(N) vector model of a $\lambda(\phi^i\phi^i)^2$ $i = 1,\ldots,N$ theory which becomes soluble in the large-N limit, and the SU(N) gauge theory which reduces to the sum over planar diagrams. In both cases, we must rescale the coupling constant in order to take into account the fact that, in this limit, we sum over an infinite number of graphs. The bare coupling constant goes to zero and we keep fixed $\tilde{\lambda} = \lambda N$ and $\tilde{g}^2 = g^2 N$, respectively.

Here, we shall consider the SU(N) gauge theory but in a different limit because we are interested in some geometrical properties. The main remark which underlines our approach is the following: Let $\phi^i(x)$ $i = 1,\ldots,N$ $N \to \infty$ be an N-component field defined in a d-dimensional space. At large N, we can write $\phi^i(x) \to \phi(\sigma, x)$ $0 \leq \sigma \leq 2\pi$, *i.e.* we can consider $\phi^i(x)$ as the Fourier components of a field in $(d+1)$ dimensions with the extra dimension being a circle. In this case, we have

$$\sum_{i=1}^{\infty} \phi^i(x)\phi^i(x) \to \int_0^{2\pi} \mathrm{d}\sigma (\phi(\sigma, x))^2 \,. \tag{1}$$

However, the interaction term will be non-local in σ:

$$\phi^4 \to \left(\int_0^{2\pi} \mathrm{d}\sigma (\phi(\sigma, x))^2 \right)^2 . \tag{2}$$

The crucial remark is that, for a Yang–Mills field $A_\mu(x)$ belonging to the adjoint representation of an SU(N) group, the resulting expression at large N is local [22].

Written explicitly we have: Given an SU(N) Yang–Mills theory in a d-dimensional space $A_\mu(x) = A^a_\mu(x) t_a$, there exists a reformulation in $(d+2)$ dimensions $A_\mu(x) \to \mathcal{A}_\mu(x, \sigma_1, \sigma_2)$ and $F_{\mu\nu}(x) \to \mathcal{F}_{\mu\nu}(x, \sigma_1, \sigma_2)$ with σ_1 and σ_2 appropriately chosen coordinates on a compact 2-dimensional surface, such that, at $N \to \infty$, the matrix commutators become the usual Poisson brackets with respect to σ_1 and σ_2

$$\begin{aligned} [A_\mu(x), A_\nu(x)] &\to \{\mathcal{A}_\mu(x, \sigma_1, \sigma_2), \mathcal{A}_\nu(x, \sigma_1, \sigma_2)\} , \\ [A_\mu(x), \mathit{\Omega}(x)] &\to \{\mathcal{A}_\mu(x, \sigma_1, \sigma_2), \mathit{\Omega}(x, \sigma_1, \sigma_2)\} , \end{aligned} \tag{3}$$

where $\mathit{\Omega}$ is an element of the gauge group. We can show that the Yang–Mills action becomes

$$\int \mathrm{d}^4x \mathrm{Tr} \left(F_{\mu\nu}(x) F^{\mu\nu}(x) \right) \to \int \mathrm{d}^4x \, \mathrm{d}\sigma_1 \, \mathrm{d}\sigma_2 \, \mathcal{F}_{\mu\nu}(x, \sigma_1, \sigma_2) \mathcal{F}^{\mu\nu}(x, \sigma_1, \sigma_2) . \tag{4}$$

The SU(N) gauge invariance has become invariance under area-preserving diffeomorphisms of the 2-dimensional surface spanned by σ_1 and σ_2. The proof of this statement is essentially algebraic. A direct way [23] is to prove that at the limit $N \to \infty$, the SU(N) structure constants, appropriately rescaled, go to those of [SDiff$_2$]. We can also show it explicitly for the sphere [22] and the torus [22, 24]. We shall present briefly these derivations in the following section.

A final remark: Equivalence (4) is established for the classical theories. To go to the quantum theory, we should first find a suitable gauge and this can be done. However, then we are facing a second problem: The quadratic part of the new 6-dimensional action has no derivatives with respect to the variables σ_1 and σ_2. As a result, the perturbation expansion cannot be defined. This is not surprising. In proving (4), we have not imposed 't Hooft's rescaling condition in which g^2N is kept fixed and we recover, already at lowest order, the infinite number of graphs. It is possible, although we have no explicit proof, that we can absorb these divergences in a clever renormalisation scheme, but it is not clear whether any new insight can be obtained this way. The 4-dimensional theory we started from is renormalisable for any finite N. A different approach would be to expand around a non-trivial solution which, hopefully, captures part of the non-perturbative dynamics of the Yang–Mills theory. Such a "master field" has not yet been found.

4. Gauge theories and non-commutative geometry

There may be several motivations to study a quantum theory in a space with non-commutative geometry and we will list some of them here.

— Not surprisingly, the first proposal goes back to Heisenberg who, in a letter to Peierls in 1930 [25], suggested that non-commutativity among space coordinates could eliminate the short-distance singularities. He tried to convince Peierls and Pauli to work on this problem, but, apparently, Pauli did not think much of the idea[6]. He talked instead to Oppenheimer [27] who, apparently, gave it as a problem to his former Ph.D. student Snyder, known from their common paper on gravitational collapse [28]. Snyder published a paper in 1947 [29] in which he tries to enlarge the quantum mechanical commutation relations to a system involving the commutators among space components. His main concern seems to have been compatibility with Poincaré invariance. He wrote a system of rather obscure commutation relations of the form of

$$\begin{aligned} [x,y] &= \left(ia^2/\hbar\right) L_z\,, & [t,x] &= \left(ia^2/\hbar c\right) M_x\,, \\ [x,p_x] &= i\hbar\left[1+(a/\hbar)^2 p_x^2\right]\,, & [x,p_y] &= i\hbar(a/\hbar)^2 p_x p_y\,, \end{aligned} \tag{5}$$

where a is the parameter defining space non-commutativity and L and M are the generators of Lorentz transformations. As far as we know, neither Snyder nor anybody else attempted to write a field theory based on these relations and there was no follow-up for many years[7]. Snyder himself left non-commutative geometry and had a successful career as an accelerator engineer [30]. In fact, as history evolved, Pauli was probably right. The motivation based on short-distance singularities did not prove fruitful for elementary particle physics. With the development of the renormalisation program in the framework of quantum field theories, the problem of ultraviolet divergences took a completely different turn. While a space cut-off makes all theories finite, the renormalisation program applies to very few and very specific field theories. It is a most remarkable fact that they are precisely the ones chosen by Nature. It is not finiteness but rather lack of sensitivity to unknown physics at very short distances that turned out to be the important criterion. The geometry of physical space may still produce an ultraviolet cut-off, but its presence does not seem to be relevant for the calculation of physical processes among elementary particles.

[6] In a letter to Bohr he commented: "... it seems to be a failure for reasons of physics." [26].

[7] It is also strange that Oppenheimer's name is never mentioned in the paper.

— However, almost at the same time, a new motivation for studying theories in a non-commutative space appeared, although only recently, it was fully appreciated. In 1930, Landau [31] solved the problem of the motion of an electron in an external constant magnetic field and, besides computing the energy levels, the so-called "Landau levels", he showed that the components of the velocity operator of the electron do not commute. A simple way to visualise this result is to think of the classical case where the electron follows a spiral trajectory whose projection on a plane perpendicular to the field is a circle. In Landau's quantum mechanical solution, the centre's coordinates are

$$x_c = \frac{cp_y}{eH} + x \,; \qquad y_c = -\frac{cp_x}{eH} \tag{6}$$

which shows that the two coordinates do not commute. The magnetic field has induced a non-commutative structure on space itself. Following Heisenberg's suggestion, Peierls [32] showed that, at least the lowest Landau level, can be obtained by using this space non-commutativity. Since the presence of non-vanishing magnetic-type external fields is a common feature in many modern supergravity and string models, the study of field theories formulated on spaces with non-commutative geometry has become quite fashionable [33].

— A new element was added a few years ago with the work of Seiberg and Witten [34] who showed the existence of a map between gauge theories formulated in spaces with commuting and non-commuting coordinates.

— An independent line of approach has been initiated by Connes [35] and co-workers, and aims at constructing a gauge theory with spontaneously broken symmetry using the techniques of non-commutative geometry. The result which relates the symmetry breaking parameter to the distance between different branes has been first obtained in this approach.

— A different but related motivation comes from SU(N) gauge theories at large N and matrix models, to which we turn next.

4.1. A fuzzy sphere

For the classical sphere, a convenient choice of coordinates is given by the usual angles θ and ϕ. We can write $x_1 = \cos\phi\ \sin\theta$, $x_2 = \sin\phi\ \sin\theta$ and $x_3 = \cos\theta$. The corresponding spherical harmonics are given by

$$Y_{l,m}(\theta,\phi) = \sum_{\substack{i_k=1,2,3 \\ k=1,\ldots,l}} \alpha^{(m)}_{i_1\ldots i_l}\ x_{i_1}\ldots x_{i_l}\,, \tag{7}$$

where $\alpha^{(m)}_{i_1 \dots i_l}$ is a symmetric and traceless tensor. For fixed l, there are $2l+1$ linearly-independent tensors $\alpha^{(m)}_{i_1 \dots i_l}$, $m = -l, \dots, l$.

We choose, inside SU(N), an SU(2) subgroup[8] whose generators we call S_i. They satisfy the commutation relations: $[S_i, S_j] = i\epsilon_{ijk} S_k$. We can use them as a basis to build the N^2-1 generators of SU(N) in the fundamental representation

$$S^{(N)}_{l,m} = \sum_{\substack{i_k=1,2,3 \\ k=1,\dots,l}} \alpha^{(m)}_{i_1 \dots i_l}\ S_{i_1} \dots S_{i_l} \Rightarrow \left[S^{(N)}_{l,m},\ S^{(N)}_{l',m'}\right] = i f^{(N)l'',m''}_{l,m;\ l',m'}\ S^{(N)}_{l'',m''}\,, \quad (8)$$

where the constants $f^{(N)}$ are the SU(N) structure constants in a somehow unusual notation. It is now clear that the three SU(2) generators S_i, rescaled by a factor proportional to $1/N$, will have well-defined limits as N goes to infinity

$$S_i \to T_i = \frac{2}{N} S_i \quad \text{implies} \quad [T_i, T_j] = \frac{2i}{N} \epsilon_{ijk} T_k$$

and

$$T^2 = T_1^2 + T_2^2 + T_3^2 = 1 - \frac{1}{N^2}\,.$$

In other words: under the norm $\|x\|^2 = \operatorname{Tr} x^2$, the limits as N goes to infinity of the generators T_i are three objects x_i, which commute and are constrained by $x_1^2 + x_2^2 + x_3^2 = 1$. This, in turn, shows that the classical Yang–Mills theory becomes the theory invariant under area-preserving diffeomorphisms of equation (4) with the closed surface being a sphere S^2.

So much for the large-N limit. For any value of N, we can parametrise the three operators T_i in terms of two operators, z_1 and z_2 as follows[9]:

$$\begin{aligned} T_+ &= T_1 + iT_2 = \mathrm{e}^{\frac{iz_1}{2}}\ \left(1 - z_2^2\right)^{\frac{1}{2}}\ \mathrm{e}^{\frac{iz_1}{2}}\,, \\ T_- &= T_1 - iT_2 = \mathrm{e}^{-\frac{iz_1}{2}}\ \left(1 - z_2^2\right)^{\frac{1}{2}}\ \mathrm{e}^{-\frac{iz_1}{2}}\,, \\ T_3 &= z_2\,. \end{aligned} \quad (9)$$

Then it is straightforward algebra [22] to prove the following algebraic statement:

$$[z_1, z_2] = \frac{2i}{N} \Leftrightarrow [T_i, T_j] = \frac{2i}{N} \epsilon_{ijk} T_k\,, \quad (10)$$

in other words, if z_1 and z_2 satisfy the Heisenberg algebra, the operators T_i satisfy the SU(2) algebra and the opposite is also true, the SU(2) algebra for the operators T_i imply the Heisenberg algebra among z_1 and z_2.

[8] For a more precise definition of this choice, see reference [36].

[9] A similar parametrisation has been used by Holstein and Primakoff in terms of creation and annihilation operators [37].

From that point, we can go on and show a formal equivalence: Given an SU(N) Yang–Mills theory in a d-dimensional space, there exists a reformulation in $(d+2)$ dimensions in which

$$A_\mu(x) \to \mathcal{A}_\mu\left(x, z_1, z_2\right) , \qquad F_{\mu\nu}(x) \to \mathcal{F}_{\mu\nu}\left(x, z_1, z_2\right) \quad \text{and} \quad [z_1, z_2] = \frac{2i}{N}$$

such that

$$[A_\mu(x), A_\nu(x)] \ \to \ \{\mathcal{A}_\mu\left(x, z_1, z_2\right), \mathcal{A}_\nu\left(x, z_1, z_2\right)\}_{\text{Moyal}} , \tag{11}$$

$$[A_\mu(x), \mathit{\Omega}(x)] \ \to \ \{\mathcal{A}_\mu\left(x, z_1, z_2\right), \mathit{\Omega}\left(x, z_1, z_2\right)\}_{\text{Moyal}} , \tag{12}$$

where the brackets are the symmetrised Moyal brackets [38] with respect to the operators z_1 and z_2, and the action becomes

$$\int \mathrm{d}^4x \, \mathrm{Tr}\left(F_{\mu\nu}(x) F^{\mu\nu}(x)\right) \to \int \mathrm{d}^4x \, \mathrm{d}z_1 \, \mathrm{d}z_2 \ \mathcal{F}_{\mu\nu}\left(x, z_1, z_2\right) * \mathcal{F}^{\mu\nu}\left(x, z_1, z_2\right) \tag{13}$$

with a *-product which should be appropriately defined for the sphere. Note that this is not the one induced from the variables z_1 and z_2 which, as we shall see in the following section, is the *-product appropriate to the torus. We do not have a closed expression for the sphere *-product expressed in terms of the canonically conjugate variables $\cos\theta$ and ϕ, but it is instructive to understand better the form of the $N \times N$ matrices which reproduce the sphere at the large-N limit.

We start with the SU(2) generators $S_\pm$ and S_3 in an $(N = 2s+1)$-dimensional, unitary, irreducible representation. We choose S_3 to be diagonal

$$(S_3)_{kj} = \delta_{kj}(s - k + 1) \quad \text{or} \quad S_3 = \mathrm{Diag}(s, s-1, \ldots, -s) . \tag{14}$$

S_- has $N-1$ non-zero elements, those of the first lower diagonal

$$(S_-)_{kj} = \delta_{k-j,1} \ a_j^{(1)} , \qquad a_j^{(1)} = \sqrt{(2s - j + 1)\, j} , \tag{15}$$

where $k = 2, \ldots, N$ is the row index and $j = 1, 2, \ldots, N-1 = 2s$ the column index.

Using the three S operators, we construct $N^2 - 1$ operators $\hat{Y}_{lm}$. They are traceless $N \times N$ matrices built as linearly-independent polynomials of S_3, S_+ and S_-. They are SU(2) tensor operators and they can be viewed as matrix spherical harmonics defining a *fuzzy sphere* with the non-commutativity parameter $\hbar \sim 1/N$. The range of the indices l and m is: $l = 1, \dots, N-1$ and $m = -l, \dots, l$. For fixed N, the resulting $N^2 - 1$ matrices $\hat{Y}_{lm}$ represent the generators of SU(N).

Since $\hat{Y}_{lm}$ are SU(2) tensor operators, their commutation relations with the generators S_3 and $S_\pm$ are

$$\left[S_3, \hat{Y}_{lm}\right] = m\hat{Y}_{lm}\,, \qquad \left[S_\pm, \hat{Y}_{lm}\right] = \sqrt{(l \mp m)(l \pm m + 1)}\;\hat{Y}_{l,m\pm1}\,. \quad (16)$$

Following Schwinger, we can construct explicitly the $\hat{Y}_{lm}$s as polynomials in the SU(2) generators $S_\pm$ and S_3: Let z_+ and z_- be two independent complex variables, then we have

$$\frac{\left(-z_+^2 S_+ + z_-^2 S_- + 2z_+z_-S_3\right)^l}{2^l\; l!} = \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} \frac{z_+^{l+m} z_-^{l-m}}{\sqrt{(l+m)!(l-m)!}} \hat{Y}_{lm}\,. \quad (17)$$

This relation is an identity in the complex numbers z_+ and z_-.

Let us look at the r.h.s. of (17) for $m = -l$: $\sqrt{\frac{4\pi}{2l+1}} \frac{z_-^{2l}}{\sqrt{(2l)!}} \hat{Y}_{l,-l}$. Comparing with the l.h.s. we obtain

$$\hat{Y}_{l,-l} = \sqrt{\frac{2l+1}{4\pi}} \frac{\sqrt{(2l)!}}{2^l\; l!}\; S_-^l\,. \quad (18)$$

Therefore, we can construct all $\hat{Y}_{lm}$ applying on the matrix $\hat{Y}_{l,-l}$ relation (16)

$$\left[S_+, \hat{Y}_{lm}\right] = \sqrt{(l-m)(l+m+1)}\hat{Y}_{l,m+1}\,. \quad (19)$$

We define: $ad_+ \cdot \hat{Y}_{lm} \equiv [S_+, \hat{Y}_{lm}]$. It follows:

$$(ad_+)^{l+m} \cdot \hat{Y}_{l,-l} = \prod_{q=1}^{l+m} \sqrt{(2l-q+1)q}\; \hat{Y}_{l,m}\,. \quad (20)$$

We call the product: $C_{l,m} \equiv \prod_{q=1}^{l+m} \sqrt{(2l-q+1)q}$,

$$\hat{Y}_{l,m} = \frac{1}{C_{l,m}} (ad_+)^{l+m} \cdot \hat{Y}_{l,-l} = \frac{1}{C_{l,m}} \sqrt{\frac{2l+1}{4\pi}} \frac{\sqrt{(2l)!}}{2^l\; l!} (ad_+)^{l+m} \cdot S_-^l\,. \quad (21)$$

The spherical harmonics $Y_l^m(\theta, \phi)$ satisfy the complex conjugation relations $Y_l^{m*}(\theta, \phi) = (-)^m Y_l^{-m}(\theta, \phi)$. We can prove that the $\hat{Y}_{lm}$s satisfy the hermiticity relations

$$\hat{Y}_{lm}^\dagger = (-)^m\; \hat{Y}_{l,-m}\,. \quad (22)$$

In order to construct the matrices $\hat{Y}_{lm}$, we need the powers of S_-, see Eqs. (18) and (21).

$(S_-)^2$ has $N-2$ non-zero elements sitting at the second lower diagonal. We call them $a_j^{(2)}$ and they are given by $a_j^{(2)} = a_j^{(1)} a_{j+1}^{(1)}$, $j = 1, \ldots, N-2$.

$(S_-)^3$ has $N-3$ non-zero elements sitting at the third lower diagonal. $a_j^{(3)} = a_j^{(1)} a_{j+1}^{(1)} a_{j+2}^{(1)}$, $j = 1, \ldots, N-3$.

$(S_-)^{N-1}$ has 1 non-zero element, the element $(N, 1)$, given by $a^{(N-1)} = \prod_{j=1}^{N-1} a_j^{(1)}$.

According to (18), the various powers of S_- give the matrices $\hat{Y}_{l,-l}$. Commuting with S_+, we obtain all $\hat{Y}_{l,m}$. Commuting S_+ with a matrix having non-zero elements in the k^{th} lower (upper), diagonal gives a matrix having non-zero elements in the $(k-1)^{\text{th}}$ lower ($(k+1)^{\text{th}}$ upper) diagonal. Thus:

— $\hat{Y}_{l,m}$ *with negative* m *have non-vanishing elements in the lower diagonals.*

  We have: 1 matrix, $\hat{Y}_{N,-N}$, with non-vanishing elements in the $N-1$ lower diagonal; 2 matrices, $\hat{Y}_{N,-N+1}$ and $\hat{Y}_{N-1,-N+1}$, with non-vanishing elements in the $N-2$ lower diagonal; 3 matrices, $\hat{Y}_{N,-N+2}$, $\hat{Y}_{N-1,-N+2}$ and $\hat{Y}_{N-2,-N+2}$, with non-vanishing elements in the $N-3$ lower diagonal; $\ldots$; $N-1$ matrices with non-vanishing elements in the 1^{st} lower diagonal.

— $\hat{Y}_{l,m}$ *with positive* m *have non-vanishing elements in the upper diagonals.* The same numbers as above.

— $\hat{Y}_{l,0}$ *is diagonal.* There are $N-1$ such matrices, *i.e.* $\hat{Y}_{l,0}$ with $l = 1, \ldots, N-1$.

  This makes : $2 \times (1 + 2 + \cdots + N - 1) + N - 1 = N^2 - 1$ matrices.

— *The first qualitative result is that the matrices* $\hat{Y}_{l,m}$*, which are the natural candidates to give a sphere at the large-*N *limit, fill upper and lower diagonals.*

We can go further and obtain more quantitative results because we know also the values of the matrix elements. At large N, the matrices $\hat{Y}_{l,m}$ become the spherical harmonics $Y_{l,m}(\cos\theta, \phi)$. For finite N, they should give an approximation of the surface for discrete values of the coordinates θ and ϕ. Therefore, we obtain a lattice which, as N grows, forms the surface of a sphere.

4.2. A fuzzy torus

The case of the fuzzy torus is even simpler. For the sphere, we had isolated inside SU(N) an SU(2) subgroup and express all the SU(N) generators in terms of the three generators of SU(2). For the torus, we isolate a quantum U(1) × U(1). Let us take first N odd (a similar construction applies to N even) and let ω be the N^{th} root of unity: $\omega = \mathrm{e}^{4\pi i/N}$. We define the two matrices

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix} ; \quad h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} . \tag{23}$$

They satisfy quantum group commutation relations

$$g^N = h^N = 1 ; \qquad hg = \omega gh . \tag{24}$$

We can use the integer mod$N$ powers of these matrices to express the SU(N) generators

$$S_{m_1,m_2} = \omega^{m_1 m_2/2} g^{m_1} h^{m_2} ; \qquad S^\dagger_{m_1,m_2} = S_{-m_1,-m_2} , \tag{25}$$

$$[S_{\boldsymbol{m}}, S_{\boldsymbol{n}}] = 2i \sin\left(\frac{2\pi}{N} \boldsymbol{m} \times \boldsymbol{n}\right) S_{\boldsymbol{m}+\boldsymbol{n}} \tag{26}$$

with $\boldsymbol{n} = (n_1, n_2)$ and $\boldsymbol{n} \times \boldsymbol{m} = n_1 m_2 - m_1 n_2$. We can show [39] that algebra (26) is indeed equivalent to that of SU(N) and at the limit $N \to \infty$ it becomes the algebra of the area-preserving diffeomorphisms of a 2-dimensional torus. This connection between SU(N) and [SDiff($T2$)] can be made explicit by choosing a pair of variables forming local symplectic coordinates on the torus, for example, the angles z_1 and z_2 of the two circles, and expanding all functions on the torus on the basis of the eigenfunctions of the Laplacian

$$h_{n_1,n_2} = \exp(i n_1 z_1 + 2\pi i n_2 z_2) , \qquad n_1, n_2 \in \mathbb{Z} . \tag{27}$$

Here, we are interested in the fuzzy torus, so we endow z_1 and z_2 with the commutation relations of the Heisenberg algebra (10). If we define the corresponding group elements h and g by

$$h = \mathrm{e}^{i z_1} , \qquad g = \mathrm{e}^{-2i\pi z_2} , \tag{28}$$

we can prove again the equivalence

$$[z_1, z_2] = \frac{2i}{N} \Leftrightarrow hg = \omega gh \tag{29}$$

for the set of group elements h^{n_1} and g^{n_2} with n_1 and n_2 integers mod$N$. Note that the latter imply the algebra of SU(N). The generators of the Heisenberg algebra z_i and the group elements h and g are infinite-dimensional operators, but we can represent the SU(N) algebra by the finite-dimensional ones (23) and (25). They form a discrete subgroup of the Heisenberg group and they have been used to construct quantum mechanics on a discrete phase space [40]. In this case, we can define two new operators $\hat{q}$ and $\hat{p}$, the first being the position operator on the discrete configuration space and the second its finite Fourier transform. They can be represented by $N \times N$ matrices, but, obviously, they do not satisfy anymore the Heisenberg algebra [41].

The Moyal bracket can be defined by symmetrising in z_1 and z_2, in which case only odd powers of $1/N$ appear. The *-product can be written as

$$f(z) * g(z) = \exp\left(i\xi\epsilon_{ij}\partial_z^i\partial_w^j\right) f(z)g(w)|_{w=z} \tag{30}$$

with $z = (z_1, z_2)$ and $\xi = 2/N$. The Yang–Mills action can be written again in the form of equations (11) and (13) and, as before for the sphere, this equivalence is exact at any order in the $1/N$ expansion.

Using matrices (25), we can construct again a lattice which, as N grows, forms the surface of a torus. For surfaces of higher genus, we could guess the form of the corresponding matrices. This way, we expect to gain some insight into the geometrical and topological properties of the surfaces which are associated with gauge theories.

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