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Updating QCD_2

E. Abdalla* and M.C.B. Abdalla†

CERN-TH
CH-1211 Geneva 23
Switzerland

Abstract

We review two-dimensional QCD. We start with the field theory aspects since 't Hooft's $1/N$ expansion, arriving at the non-Abelian bosonization formula, coset construction and gauge-fixing procedure. Then we consider the string interpretation, phase structure and the collective coordinate approach. Adjoint matter is coupled to the theory, and the Landau–Ginzburg generalization is analysed. We end with considerations concerning higher algebras, integrability, constraint structure, and the relation of high-energy scattering of hadrons with two-dimensional (integrable) field theories.

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* Permanent address: Instituto de Física - USP, C.P. 20516, S. Paulo, Brazil.
† Permanent address: Instituto de Física Teórica - UNESP, R. Pamplona 145, 01405-000, S. Paulo, Brazil.

CONTENTS

1. Introduction

Over several years, gauge theories have proved their efficacy to describe phenomena in the high energy domain and QCD is a important part of such a theoretical framework. The high energy behaviour of strong interactions, as analyzed by means of renormalization group (RG) and Callan–Symanzik (CS) equations can be investigated by a perturbative expansion, permitting to confront this facet of the theory with experiment, with excellent results.[1] Indeed, the momentum-dependent running coupling constant characterizes the strength of the interaction, in such a way that because of the negative β-function of QCD, perturbation theory is legitimate.

But this is only part of the development of gauge theories, after the long-standing success of quantum electrodynamics.[2] Indeed, classical solutions (monopoles or instantons) have been obtained, showing the complexity of the theory and the importance of topology. In this way, more abstract branches of mathematics came to play an important role in the unravelling of the structural properties of gauge theories.[3]

For the time being, the use of more sophisticated gauge theories, namely the use of larger gauge groups, combined with the idea of symmetry breaking and the Higgs mechanisms, has led to the development of unified theories to describe the high energy interactions, albeit not including gravity at a first step. Thus the electroweak $SU(2) \otimes U(1)$ symmetric gauge interaction, and the strong interaction described by the $SU(3)$ colour gauge group, could in principle be unified into the framework of higher gauge groups, such as $SU(5)$, $O(10)$ or even more sophisticated ones.[4]

The inclusion of gravity in such a scheme is provided by the introduction of supersymmetry, which is, moreover, potentially sufficient to solve the hierarchy problem.[5] However, further troubles concerning the introduction of gravity in such a unifying scheme is only solved in the framework of string theory.[6,7]

On the other hand, string theory was developed as a model for strong interactions, as a consequence of ideas related to dual models. Strong interactions as described by quantum chromodynamics must, in a sense, exist in two phases, an infrared phase, with confined quarks bound into mesons, evolving according to a (non-critical) string theory, and a high energy phase, described by the perturbative expansion as previously mentioned.

In fact, we have to point out that two kinds of string theories exist, associated to different types of phenomena. Critical strings are appropriate to implement the unification of all gauge theories, but are not the subject of concern here. Non-critical strings, on the other hand, should provide the framework necessary for the description of strong interactions, being presumably related to gauge theories. These ideas, although appealing, and in conformity with the general intuition about high energy interactions have not yet been fully accomplished, in a realistic gauge theory.

In this scenario two-dimensional gauge theories are configurated as a laboratory[8] where ideas may be tested, and the relation between string concepts and field theory may be analyzed in detail. Moreover, results such as the spectrum, less orthodox, perturbative approaches as the one based on the large-N limit, the algebraic structure of the theory, and the recently stressed duality properties may be analyzed in detail, and some of them exactly.

The prototype of two-dimensional gauge theories is QED_2, or the Schwinger model.[9] In this model the gauge field acquires a mass via a dynamical Higgs mechanism induced by the fermions. There is a spontaneous breakdown of the chiral $U(1)\times\widetilde{U}(1)$ symmetry, where the first factor refers to charge and the second to chirality of the fermion. The ground state exhibits a double infinite degeneracy labelled by fermion number and chirality, analogous to four-dimensional quantum chromodynamics.[10] The so-called "infrared slavery" of QCD_4 is naturally described in QED_2 as a consequence of the linear rise of the Coulomb potential, characteristic of one space dimension.

The question of confinement is however not settled in this way, since asymptotic states corresponding to screened quarks might exist as well. By all means the Fock space is that of a free pseudoscalar bosonic field $\Sigma(x)$ of mass $e/\sqrt{\pi}$, where e is the electric charge. This is a manifestation of confinement in the Schwinger model.[10] All states in the physical Hilbert space $\mathcal{H}_{phys}$ can be constructed by applying functionals of the bosonic field $\Sigma(x)$ on the irreducible vacuum state. However, confinement cannot be fully understood before a flavour quantum number is assigned to the fermion, since it is otherwise not possible to distinguish between neutral bosonic states and screened fermionic states.[11]

Bosonization of the Schwinger model[12] provides further insight; the mass content of the theory can be easily read from the diagonalization of the quadratic Lagrangian, once we substitute the gauge field for $A_\mu = -\frac{\sqrt{\pi}}{e}\widetilde{\partial}_\mu(\Sigma+\eta)$. In such a case one can also verify that the massive Schwinger model, which is not soluble, can be written in terms of a modified sine-Gordon equation, in which the periodic symmetry is broken by the electromagnetic interaction. The massive theory is far more complicated than the massless case. The confinement issue can be understood semi-classically; as it turns out, the screening effects are more violent in this case, since such computation leads us to a potential whose linear rise is sacrificed for the advantage of the screening picture. For a long discussion see chapter 10 of ref. [8]. Moreover one finds that the field η obeys a massive equation. A similar outcome will be true in the non-Abelian case.

Several properties of the Schwinger model provide a realization of analogous features expected to characterize four-dimensional quantum chromodynamics (QCD_4). Besides the spontaneous breakdown of the chiral symmetry without Goldstone bosons, infinite degeneracy of the vacuum and confinement, an enormous amount of physical implications followed. However, the model is certainly still too simple, and the first missing issue is the generalization of the mesonic bound state, which in QCD is believed to present itself according to the Regge behaviour, while in QED_2, owing to the simplicity of the $U(1)$ symmetry, only functionals of the bosonic field $\Sigma(x)$ appear in the physical spectrum.

Therefore, it is natural to include colour as the next step towards more realistic models, implementing the non-Abelian character of the fundamental fields, and consider quantum chromodynamics in two dimensions (QCD_2). In this case, bosonization leads to a simpler theory, but the spectrum is still rather complex, even if it can be worked out in the large-N limit. The Hilbert space contains a larger class of bound states, connected with the Regge behaviour of the mass spectrum, thus leading to a large number of states. The case of massless fermions is no longer soluble in terms of free fields. Even pure gauge QCD_2 is non-trivial, and cannot be completely solved in terms of simple fields, see section 3. The situation is even more difficult in the case of massive fermions, since the exact fermion determinant, necessary to obtain the bosonized action, is not available, and the argument leading to bosonization must be based on the principle of form invariance. However, it is from the latter model that we expect a more realistic description of the string behaviour, thought to be the most important character of four-dimensional non-Abelian gauge theories in their description of the strong interaction. Still the question of confinement must be analyzed, especially if one considers that it is possible to construct operators such as the gauge-invariant part of the bound state of two fermions, by means of the inclusion of strings of the type $\exp\left[ie\int_x^y \mathrm{d}z^\mu\, A_\mu\right]$.

Several authors made efforts in the direction of solving such a difficult model,[13−27] some of them giving useful results, but an exact solution is still missing. We mention the $1/N$ expansion introduced by 'tHooft,[13,14] from which one obtains some information about the spectrum of the theory, and the computation of the exact fermion determinant[15] in terms of a Wess–Zumino–Witten (WZW) model,[16,17] by which one arrives at an equivalent bosonic action.[17−24]

Quantum chromodynamics in two dimensions is a super-renormalizable field theory with finite field and coupling constant renormalization. As already mentioned it was first studied by 'tHooft[14] who, working in the light-cone gauge, ($A_- = 0$), and formulating the problem in terms of light-cone variables, obtained a non-linear equation for the fermion self-energy, from which he obtained the approximate spectrum of the theory. This procedure is however ambiguous, as pointed out by Wu,[25] and implies a tachyon for small bare fermion masses (therefore also in the massless fermions case), see also ref. [26]. This situation clearly requires that a non-perturbative and explicitly gauge invariant approach should be used to obtain information based on firm grounds. We will also see that Wu's and 'tHooft's results, for the fermion self-energy, are different and lead to different physical pictures: while 'tHooft's two-point fermion function corresponds to a simple free Fermi behaviour[1)], Wu's result presents an anomalous branch cut reflecting the fact that all planar rainbow graphs contribute, in his scheme, to the self-energy. Differently from the Abelian case, fermion loops are suppressed in the large-N limit, in a way that at lowest order in $1/N$ the gluon field remains massless, leading to speculations that QCD_2 might exist in two phases, associated to the weak or strong coupling regime, where the weak coupling phase – or 'tHooft's phase – would be associated with massless gluons, with a Regge trajectory for the mesons, while in the strong coupling regime – or Higgs phase – the gluons would be massive and the $SU(N)$ symmetry would be broken to its maximal Abelian subgroup.

The nature of such strong or weak limits is very delicate. In fact, the theory is asymptotically free, as it should since it is super-renormalizable. In the strong coupling, it should be a confining theory. The introduction of an explicit infrared cut-off in fact selects the confining properties. In that case, quarks disappear from the spectrum, which consists of mesons with a Regge behaviour. As pointed out by Callan, Coote and Gross,[27] for gauge-invariant quantities one can interpret all integrals as principal values, and we are led to the solution $\Sigma_{SE}(p) = \frac{e^2 N}{\pi}\frac{1}{p_-}$ for the self-energy (SE), with the fermion two-

[1)] It is nevertheless infrared-cut-off-dependent, due to quark confinement. The confining properties are made manifest since quark poles are pushed to infinity as the cut-off disappears. For gauge-independent quantities, such a procedure is equivalent to interpreting all integrals as principal values.

point function $S_F(p) = \frac{\not p + m + \frac{e^2 N}{2\pi}\frac{\gamma_-}{p_-}}{p^2 - m^2 + \frac{e^2 N}{\pi}}$. Such a procedure is useful to analyze properties connected to the high-energy scattering amplitudes, displaying properties connected to what is known as parton-like properties; that is, in the high-energy limit, the quarks behave like free particles.

This does not mean, however, that confinement does not take place. To see the confinement mechanism, one has to examine the current two-point function, and as a result, quark continuum states do not appear, a fact confirming confinement (for a more precise discussion see refs. [8] and [11]).

Such a procedure has some advantages, namely one can study the high energy behaviour of the theory, which, because of asymptotic freedom, must exhibit a free field structure, which is not the case with an infrared cut-off, since the high-energy limit and the zero infrared cut-off limit do not commute.

In the massless case, there are several non-perturbative results available. In particular, the external field problem for the effective action has been analyzed, the computation of gauge current and fermionic Green's functions can be reduced to the calculation of tree diagrams. There are also features not covered by 'tHooft's method. As an example, if we take the pseudo-divergence of the Maxwell equation we arrive at

$$\left(\nabla^2 + \frac{e^2}{\pi}\right) F_{01} = 0 \quad ,$$

where F is the gauge field strength; this equation generalizes the analogous result obtained for the Schwinger model to the non-Abelian case. This suggests that an intrinsic Higgs mechanism, analogous to the one well-known in QED_2, can also characterize the non-Abelian theory. This is, nevertheless, not contained in 'tHooft's approach, since the mass arises from a fermion loop, the same which contributes to the axial anomaly, and it is suppressed in the $1/N$ expansion.

In spite of difficulties, QCD_2 served as a laboratory for gaining insight into various phenomenological aspects of four-dimensional strong interactions, such as the Brodsky–Farrar scaling law[28] for hadronic form factors, the Drell–Yan–West relation or the Bloom–Gilman duality[29] for deep inelastic lepton scattering.

The next important step towards understanding this theory is its relation to string theory, or SCD_2. It concerns one of the most important applications of the theory of non-critical strings.[30]

The general problem of strong interactions did not progress substantialy until recently as far as it concerns low-energy phenomena. Such a problem should be addressed using non-perturbative methods, since perturbation theory of strong interactions is only appropriate for the high-energy domain, missing confinement, bound-state structure and related phenomena. In fact, several properties concerning hadrons are understandable by means of the concept of string-like flux tubes, which are consistent with linear confinement and Regge trajectories, as well as the approximate duality of hadronic scattering amplitudes, which are the usual concepts of the string idea. In fact, a similar idea is already present in the construction of the dipole of the Schwinger model, in which case it is, however, far too simple to be realistic.

The large-N limit of QCD_2 is smooth and provides a picture of the string in the Feynman diagram space. The large-N limit is expected to provide most of the qualitative pictures of the low-energy limit of the theory. In certain low-dimensional systems, the $1/N$ expansion turns out to be *the* correct expansion, for models with problematic infrared behaviour, such as $\mathbb{C}P^{N-1}$ and Gross–Neveu models,[31,32] where properties such as confinement and spontaneous mass generation are straightforwardly derived in the large-N approximation, and the S-matrices can be explicitly checked.[33,34]

In short, these ideas support the suggestion that the understanding of the theory of strong interactions requires the study of the large-N limit of QCD. Although several models mimic such a theory in two dimensions, concerning the confinement aspects, a more thorough comprehension by a simplified two-dimensional model cannot be complete without the inclusion of QCD_2.

For pure QCD_2 the $1/N$ expansion of the partition function can be obtained to arbitrary order, and may be interpreted as a sum over surfaces, thus describing a string theory. The string action is not exactly known, but it is described, in the zero area limit, by a topological field theory. Area corrections are given by the Nambu action and possibly terms in the extrinsic geometry, forbidding folds. It is clear that the theory without matter is yet too simple. Even the introduction of fundamental fermions is not sufficient to describe certain realistic aspects of the higher dimensional theory.

Matter in the adjoint representation of the gauge group provides fields which mimic the transverse degrees of freedom characteristic of gauge theories in higher dimensions, and may show more realistic aspects of strong interactions. The main new point consists in the presence of a phase transition indicating a deconfining temperature.

There are algebraic structures in QCD_2, indicated by canonical methods, and such algebras point to the integrability of the model. In particular, there are spectrum-generating algebras of the same type as that appearing in the quantum Hall effect.[35] Moreover, the relation to Calogero systems and the $c = 1$ matrix model confirms such integrability properties, which can finally be proved by the construction of a Lax pair. This opens the possibility of a closed solution, at least at the S-matrix level (on-shell physics[8]).

Finally, the high energy description of four-dimensional QCD is also described by an integrable two-dimensional model, opening a possibility of more realistic results from such a study. Indeed, at high energies Feynman diagrams simplify and become effectively two dimensional. The theory may be described in the impact parameter space, and in the case of QCD_4, the Reggeized particles scatter according to an integrable Hamiltonian.

2. QCD_2 as a field theory

We start with the definition of the theory, from very early developments, and concentrate on non-perturbative results. Since the theory confines the fermionic degrees of freedom, it is usefull to integrate over the fermions. This amounts to the computation of the fermionic determinant. In terms of the original gauge fields it is given by an infinite series, as given by eq. (2.15). Later the fermionic determinant will be computed in terms of the potentials used to define the gauge fields, leading to the Wess–Zumino–Witten action.

Before entering in the full bosonization of the theory (via the WZW action) it is possible to investigate the mesonic spectrum through the $1/N$ expansion of the theory, where the gauge group is $SU(N)$, or $U(N)$. In such a case there is a simplification in the light-cone gauge, where the gauge fields do not have self interactions, and the ghosts decouple. The self-energy of the quarks can be exactly computed. A word of cautious concerning this method has to be said concerning the infrared cut-off (see section 2.2).

After computing the fermionic determinant and writing the theory in an appropriate form to be discussed in section 5 in the framework of integrable models, we discuss the strong coupling limit, where the $1/N$ expansion possibly breaks down, and later the gauge (first class) constraints imposed to the theory in the WZW functional language. The case of chiral interactions is briefly discussed in the last subsection.

The theory is defined by the Lagrange density

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_i(i\,\partial\!\!\!/ + e\,A\!\!\!/)\psi_i \quad , \tag{2.1}$$

with the notation defined in Appendix A. The fermions ψ_i are in the fundamental representation of the gauge group. The field equations derived from this Lagrangian are

$$\nabla_\mu^{ab}F_b^{\mu\nu} + e\overline{\psi}\gamma^\nu\tau^a\psi = 0 \quad , \tag{2.2a}$$

$$i\gamma^\mu\partial_\mu\psi + e\gamma^\mu\tau^a A_\mu^a\psi = 0 \quad . \tag{2.2b}$$

The current $J_\mu^a = \overline{\psi}\gamma_\mu\tau^a\psi$ is covariantly conserved as a consequence of (2.2b), namely

$$\left(\partial_\mu\delta^{ab} + ef^{acb}A_\mu^c\right)\left(\overline{\psi}\gamma^\mu\tau^b\psi\right) = 0 \quad . \tag{2.3}$$

For a gauge-invariant regularization this equation holds true in the quantum theory. We consider in general an external field current

$$J_\mu^a(x|A) = \langle\overline{\psi}(x)\gamma_\mu\tau^a\psi(x)\rangle_A \quad , \tag{2.4}$$

which depends on the external gauge field A_μ. It is obtained by differentiating the functional

$$W[A] = -i\ln\frac{\det i\,D\!\!\!\!/[A]}{\det i\,\partial\!\!\!/} \quad , \tag{2.5}$$

with respect to A_a^μ, i.e. the current (2.4) is given in terms of (2.5) by the expression

$$eJ_\mu^a(x|A) = \frac{\delta W}{\delta A_a^\mu(x)} \quad . \tag{2.6}$$

The functional $W[A]$ represents an effective action for A_μ.

By the Fujikawa method,[37] making small transformations in $i\,D\!\!\!\!/$, corresponding to classical symmetry transformations, we can analyze the change in the integration variable (since the action is supposed to be invariant under symmetry transformation) and search for anomalies. The measure $\mathcal{D}\overline{\psi}\mathcal{D}\psi$ is $U(1)$-invariant, in such a way that the current J_μ^a is covariantly conserved. However, it is not invariant under a chiral transformation. Let us consider the pseudo current

$$J^b_{5\mu}(x|A) = \langle\overline{\psi}(x)\gamma_\mu\gamma_5\tau^b\psi(x)\rangle_A = \epsilon_{\mu\nu}J^{\nu b}(x|A) \quad . \tag{2.7}$$

In such a case, use of the Fujikawa method with a gauge-invariant regulator leads to the anomaly equation

$$\nabla^{ab}_\mu J^{5\mu b} = -\widetilde{\nabla}^{ab}_\mu J^{\mu b} = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu a} \quad . \tag{2.8}$$

The first consequence of the above anomaly equation is obtained from the pseudo-divergence of the first Maxwell equation (2.2a), which yields (see notation in Appendix A)

$$\epsilon_{\nu\rho}\nabla^\rho\nabla_\mu F^{\mu\nu} + e\widetilde{\nabla}_\mu J^\mu = -\frac{1}{2}\left(\nabla^2 + \frac{e^2}{\pi}\right)\epsilon_{\mu\nu}F^{\mu\nu} = 0 \quad , \tag{2.9}$$

showing that one expects, as foreseen in the introduction, a mass generation for the gauge field, analogous to the Schwinger model.

Furthermore, it is possible to compute the external field current $J^a_\mu(x|A)$ integrating (2.8). Indeed, introducing the kernel $K^{ab}_\mu(x,y|A)$ by the equations

$$\begin{array}{c} \nabla^{ab}_\mu K^{\mu bc}(x,y|A) = 0 \\ \widetilde{\nabla}^{ab}_\mu K^{\mu bc}(x,y|A) = -\delta^{ac}\delta(x-y) \end{array} \quad \Longrightarrow \quad \begin{array}{c} \nabla^{ab}_- K^{bc}_+ = -\delta^{ac}\delta(x-y) \quad , \\ \nabla^{ab}_+ K^{bc}_- = \delta^{ac}\delta(x-y) \quad , \end{array} \tag{2.10}$$

we have

$$J^a_\mu = \frac{e}{2\pi}\int \mathrm{d}^2y\, K^{ab}_\mu(x,y|A)\epsilon_{\rho\sigma}F^{\rho\sigma b}(y) \quad , \quad \text{or} \tag{2.11a}$$

$$\begin{aligned} J^a_\pm &= \frac{e}{2\pi}\int \mathrm{d}^2y\, K^{ab}_\pm(x,y|A)\epsilon_{\rho\sigma}F^{\rho\sigma b}(y) \\ &= \mp\frac{e}{2\pi}\int \mathrm{d}^2y\, \widetilde{K}^{ab}_\pm(x,y|A)\epsilon_{\rho\sigma}F^{\rho\sigma b}(y), \end{aligned} \tag{2.11b}$$

where the kernel K_μ depends on the external gauge field A_μ. The kernel K_μ can be obtained as an expansion in the Lie algebra valued fields $\mathcal{A}^{ab}_\mu = f^{acb}A^c_\mu$ with the use of the function $D_\pm(x-y)$:

$$D_\pm = \partial_\pm D(x) \quad , \quad D(x) = -\frac{i}{4\pi}\ln\left(-x^2 + i\epsilon\right) \quad , \tag{2.12}$$

as[8]

$$\begin{aligned} \mp K^{ab}_\pm(x,y|A) &= \widetilde{K}^{ab}_\pm(x,y|A) = \delta^{ab}D_\pm(x-y) \\ &- i\sum_{n=1}^{\infty}(-e)^n\int \mathrm{d}^2x_1\cdots\mathrm{d}^2x_n\, D_\pm(x-x_1)\cdots D_\pm(x-x_n)\left[\mathcal{A}_\mp(x_1)\cdots\mathcal{A}_\mp(x_n)\right]^{ab} \end{aligned} \tag{2.13a}$$

It is sometimes useful to rewrite this expression in the fundamental representation, where $A_\mu = \sum_c A^c_\mu \tau^c$, obtaining

$$
\begin{aligned}
K^{ab}_\pm(x,y|A) = \delta^{ab} D_\pm(x-y)-& \\
&- \sum_{n=1}^{\infty} (-e)^n \int \mathrm{d}^2 x_1 \cdots \mathrm{d}^2 x_n \, D_\pm(x-x_1)\cdots D_\pm(x-x_n) \\
&\times \mathrm{tr}\, \left\{ \tau^a \left[A_\mp(x_1), [A_\mp(x_2), \cdots [A_\mp(x_n), \tau^b]] \cdots \right] \right\} \quad . \qquad (2.13b)
\end{aligned}
$$

Equation (2.11) can be linearized by means of (2.10), since the field strength F_{+-} may be alternatively written as $F_{+-} = -\partial_- A_+ + \nabla_+ A_-$ or $F_{+-} = -\nabla_- A_+ + \partial_+ A_-$, in such a way that after a partial integration (and use of (2.10)) one has

$$
\begin{aligned}
J^a_\pm(x|A) &= \frac{e}{2\pi} A_\pm(x) - \frac{e}{2\pi} \int \mathrm{d}^2 y \, K^{ab}_\pm(x,y|A) \partial_\pm A^b_\mp \\
&= \frac{e}{2\pi} \left[A_\pm - \int \mathrm{d}^2 y \, \partial_\pm D_\pm(x-y) A^a_\mp(y) \right] \\
&\quad + \frac{i}{2\pi} \sum_{n=2}^{\infty} (-ie)^n \int \mathrm{d}^2 x_1 \cdots \mathrm{d}^2 x_n \, D_\pm(x-x_1) \cdots D_\pm(x-x_n) \\
&\quad \times \mathrm{tr}\, \{ \tau^a \left[A_\mp(x_1), [\cdots [A_\mp(x_{n-1}), \partial_\pm A_\mp(x_n)]] \cdots \right] \} \quad . \qquad (2.14)
\end{aligned}
$$

Notice that we now have a sum of tree diagrams, although this is a one-loop result. This is a consequence of two-dimensional space-time integration, where one-loop diagrams can be computed in terms of tree diagrams (see ref. [8]). In any case, care must be taken about divergences, in order not to lose anomalous contributions.

Using such expressions, one can compute once more the fermionic determinant as an expansion in terms of the gauge fields, or following refs. [38, 39] one finds the effective action functional

$$
\begin{aligned}
W[A] = W[0] &- \frac{ie^2}{2\pi} \int \mathrm{d}^2 x \, \delta^{ab} A^a_\mu \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \right) A^b_\nu(x) \\
&+ \frac{i}{2} \sum_{n=2}^{\infty} \frac{(ie)^{n+1}}{n+1} \int \mathrm{d}^2 x \, \mathrm{tr} \, \left[A_-(x) T^{(n)}_+(x|A) + A_+(x) T^{(n)}_-(x|A) \right] \quad , \qquad (2.15)
\end{aligned}
$$

where

$$
\begin{aligned}
T^{(n)}_\pm(x|A) = &- \frac{1}{2\pi} (-1)^n \int \mathrm{d}^2 x_1 \cdots \mathrm{d}^2 x_n \, D_\pm(x-x_1) \cdots D_\pm(x-x_n) \\
&\times [A_\mp(x_1), [\cdots [A_\mp(x_{n-1}), \partial_\pm A_\mp(x_n)]] \cdots] \quad . \qquad (2.16)
\end{aligned}
$$

An alternative to such an expression for the effective action is the Polyakov–Wiegmann method, where a closed expression can be obtained. On the other hand it depends on potentials used to define the gauge field A_μ, and even in such a case the result is non-local as a function of those potentials. Nevertheless the Polyakov–Wiegmann result will prove to be useful in a wider sense than the above result.

2.1 The $1/N$ expansion and spectrum

The first successful attempt to obtain an insight into the dynamical structure of QCD_2 was undertaken by 'tHooft, who considered the limit where the number of colours N is large, i.e. the $1/N$ expansion. In such a case one considers quarks interacting via an $U(N)$ colour gauge group[2)]. In the large-N limit, one considers the contributions of graphs with the same topology. Therefore, the diagrams with the same number and type of external lines are classified according to its (non-)planarity, or the number of handles and holes – the Euler characteristic. Moreover, in two dimensions it is useful to work in the light-cone gauge, since the gauge field strength has only one component; in that gauge the field strength is linear in the fields, since

$$\epsilon^{\mu\nu}F_{\mu\nu} = F_{+-} = \partial_+ A_- - \partial_- A_+ - ie[A_+, A_-] = -\partial_- A_+ \quad , \tag{2.17}$$

for the gauge $A_- = 0$, and all self interactions of the gauge fields disappear. In such a case the Lagrangian boils down to

$$\mathcal{L} = \frac{1}{8}\mathrm{tr}\,(\partial_- A_+)^2 + \overline{\psi}\left(i\gamma^\mu\partial_\mu + \frac{e}{2}\gamma_- A_+ - m\right)\psi \quad , \tag{2.18}$$

where a mass is allowed for the quarks.

The ghosts decouple in such a gauge. Notice also that using light-cone quantization one readily sees that the momentum canonically associated to A_- is zero, and that it is not a dynamical field. The Feynman rules are very simple. We have for the gauge field, and fermion propagator, respectively

$$\langle A_+(k)A_+(-k)\rangle = \frac{4i}{k_-^2} \quad , \tag{2.19a}$$

$$\left\langle \psi(k)\overline{\psi}(-k)\right\rangle = \frac{i}{\not k - m} = i\frac{\not k + m}{k^2 - m^2} \quad , \tag{2.19b}$$

and the only vertex is

$$\left\langle \psi\overline{\psi}A_+\right\rangle = \frac{ie}{2}\gamma_- \quad . \tag{2.19c}$$

Since $\gamma_-^2 = 0$, and $\gamma_+\gamma_-\gamma_+ = 4\gamma_+$, the γ-algebra is extremely simple. The fermion propagator simplifies to $\frac{ik_-}{k^2-m^2}$, and the vertex to ie.

The limit $N \to \infty$ has to be taken for $\alpha = e^2 N$ fixed. In such a case one generates, as usual, the planar diagrams with no fermion loops in lowest order in $1/N$, further corrections being classified by the Euler characteristic of the diagram in Feynman rules space. Therefore we are left with ladder diagrams, with self-energy insertions for the

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[2)] Some authors have considered the case of the $SU(N)$ gauge group. In the large-N limit, for the lowest order, the difference is irrelevant.

fermion lines. It is thus possible to write the full fermion propagator in terms of the unknown function $\Sigma_{SE}(k)$, the self-energy, to be computed later. Adding contributions as $\frac{ik_-}{k^2-m^2+i\epsilon}\sum_{n=0}^{\infty}\left[\frac{k_-}{k^2-m^2+i\epsilon}\Sigma_{SE}(k)\right]^n$, we find[3)]

$$\left\langle\psi(k)\overline{\psi}(-k)\right\rangle_{\rm full} \equiv S_F(k) = \frac{ik_-}{k^2-m^2-k_-\Sigma_{SE}(k)} \quad . \tag{2.20}$$

For the planar approximation it is possible to derive a simple bootstrap equation for $\Sigma_{SE}(k)$, since following the external line one finds a vertex, which is connected to the second outgoing fermion line by the gauge-field propagator (2.19a), exact in the large-N limit, and by the full fermion propagator (2.20); the equation obtained is (see Fig. 1)

$$-i\Sigma_{SE}(k) = -2e^2N\int\frac{{\rm d}p_+\,{\rm d}p_-}{(2\pi)^2}\,\frac{i}{p_-^2}\,\frac{i(k_-+p_-)}{(k_++p_+)(k_-+p_-)-\Sigma_{SE}(k+p)(k_-+p_-)-m^2+i\epsilon} \quad . \tag{2.21}$$

[3)] Notice the absence of the factors 1/2 from the γ matrices. There is a cancellation of the factor 4 due to $\gamma_+\gamma_-\gamma_+ = 4\gamma_+$ and the factors 1/2 coming from the fermion propagator and the vertex. A final factor 1/2 for each outgoing fermion will be implicitly taken into account.

Fig. 1: Fermion self-energy equation.

The right-hand side does not depend on k_+, as one readily sees by changing variables as $p'_+ = k_+ + p_+$. Therefore $\Sigma_{SE}(k)$ depends only on k_-, and as a consequence the "+" integral may be performed independently of the function $\Sigma_{SE}(k)$ itself, and the equation simplifies to

$$\Sigma_{SE}(k_-) = \frac{ie^2 N}{2\pi^2} \int \frac{\mathrm{d}p_- \,(k_- + p_-)}{p_-^2} \int \frac{\mathrm{d}p_+}{p_+(k_- + p_-) - (k_- + p_-)\Sigma_{SE}(k_- + p_-) - m^2 + i\epsilon} \quad . \tag{2.22}$$

There are two kinds of divergences in such an integral. Ultra-violet divergences are soft, since the theory is super-renormalizable, and in the present integral it is only logarithmic and disappears using symmetric integration; for the p_+ integration we find, by simple contour integration, the result $\frac{-\pi i}{|k_- + p_-|}$, independent of $\Sigma_{SE}(p)$. Therefore the solution follows straightforwardly after substituting the result of the p_+ integration, that is

$$\Sigma_{SE}(k_-) = \frac{e^2 N}{2\pi} \int \frac{\mathrm{d}p_-}{p_-^2} \varepsilon(k_- + p_-) \quad . \tag{2.23}$$

The onus of such a solution is that it is infrared-divergent, as a consequence of the choice of gauge and of super-renormalizability. There are procedures to regularize the infrared divergence in $\Sigma_{SE}(k_-)$ above, and obtain sensible results. The original strategy followed by 'tHooft was to cut-off a slice in momentum space around $k_- = 0$, with width λ, and take $\lambda \to 0$ when computing physical (gauge-invariant) quantities. A second strategy followed by the authors of refs. [26-28] is to define the light-cone gauge propagator by means of a principal-value prescription as

$$P\,\frac{1}{q_-^2} = \frac{1}{2}\left[\frac{1}{(q_- - i\epsilon)^2} + \frac{1}{(q_- + i\epsilon)^2}\right] \quad . \tag{2.24}$$

In the first case, the self-energy is cut-off-dependent, with the result

$$\Sigma_{SE}(k) = \frac{e^2 N}{\pi}\left[\frac{\varepsilon(k_-)}{\lambda} - \frac{1}{k_-}\right] \quad , \tag{2.25}$$

while in the latter case one obtains the finite value

$$\Sigma_{SE}(k) = -\frac{e^2 N}{\pi}\frac{1}{k_-} \quad . \tag{2.26}$$

In ref. [40] λ was interpreted, in the limit $\lambda \to 0$, as a gauge parameter. In 'tHooft's procedure, confinement was interpreted in terms of λ, since quark propagator's poles are removed to infinity, while in the regular cut-off prescription the fermion propagator is

$$S_F(k) \sim \frac{ik_-}{k^2 - m^2 + \frac{e^2 N}{\pi} + i\epsilon} \quad . \tag{2.27}$$

In this latter case, confinement is obtained from the fact that quark continuum states do not appear. Coloured operators vanish with the use of the cut-off procedure in the limit $\lambda \to 0$, since there presumably exist no finite-energy coloured states. However, as discussed in [27] it is sometimes useful to consider coloured states in order to understand the interplay between confinement and the high-energy limit, since the zero cut-off and high-energy limits do not commute.

Using the singular cut-off we arrive at the dressed propagator

$$S_F^\lambda(k) = \frac{ik_-}{k^2 - m^2 + \frac{e^2 N}{\pi} - \frac{e^2 N}{\pi\lambda}|k_-| + i\epsilon} \quad , \tag{2.28}$$

which displays the above-mentioned fact that the pole is shifted towards $k_+ \to \infty$ excluding the physical single-quark state. Independently of keeping or not the λ-dependence, we can proceed and set a Bethe–Salpeter equation in order to find information about the bound states. We consider a blob representing a two quark bound state. In the planar limit, such a wave functional obeys the equation pictorially depicted in Fig. 2, where a homogeneous term has been abandoned. The structure of this equation is extremely simplified in the light-cone gauge, where as mentioned, the gauge field has no self interaction, and due to planarity only ladder-type diagrams as in the right-hand side of Fig. 2 survive.

Fig. 2: Bethe–Salpeter equation.

This leads to the integral equation for the blob $\varphi(p;q)$

$$\varphi(p;q) = \frac{e^2N}{(2\pi)^2} \frac{i(p_- - q_-)p_-}{\left[(p-q)^2 - M^2 - \frac{e^2N}{\pi\lambda}|p_- - q_-| + i\epsilon\right]} \times \frac{1}{\left[p^2 - M^2 - \frac{e^2N}{\pi\lambda}|p_-| + i\epsilon\right]} \int \mathrm{d}k_+ \, \mathrm{d}k_- \, \frac{\varphi(p+k;q)}{k_-^2} \quad , \tag{2.29}$$

where $M^2 = m^2 - e^2N/\pi$. Notice also that $\varphi(p;q)$ is a function of the momenta, that is $\varphi(p_+, p_-; q) \equiv \varphi(p_+, p_-; q_+, q_-)$. We do not need the full solution in order to obtain the spectrum; we consider a simplified equation obeyed by

$$\varphi(p_-;q) \equiv \varphi(p_-;q_+,q_-) = \int \mathrm{d}p_+ \, \varphi(p_+, p_-; q) \quad . \tag{2.30}$$

The k_+ integral in eq. (2.29) corresponds to the definition (2.30), and we can furthermore integrate over p_+, obtaining

$$\varphi(p_-;q) = i\frac{e^2N}{(2\pi)^2} \int \mathrm{d}p_+ \left[p_+ - q_+ - \frac{M^2}{p_- - q_-} + \left(-\frac{e^2N}{\pi\lambda} + i\epsilon\right) \varepsilon\,(p_- - q_-)\right]^{-1} \times \left[p_+ - \frac{M^2}{p_-} - \left(\frac{e^2N}{\pi\lambda} - i\epsilon\right) \varepsilon\,(p_-)\right]^{-1} \int \mathrm{d}k_- \, \frac{\varphi(p_- + k_-;q)}{k_-^2} \quad . \tag{2.31}$$

The p_+ integral is zero if $\varepsilon\,(p_- - q_-)$ and $\varepsilon\,(p_-)$ are equal, since we have to integrate between the poles, to get a non-zero result. For $q_- > 0$, we can satisfy this condition only for $0 < p_- < q_-$, in which case the integral picks up the contribution of one pole, with the result

$$\varphi(p_-;q) = \frac{e^2N}{2\pi}\theta(p_-)\theta(q_- - p_-) \left[\frac{M^2}{p_-} + \frac{M^2}{q_- - p_-} + \frac{2e^2N}{\pi\lambda} + q_+\right]^{-1} \int \mathrm{d}k_- \, \frac{\varphi(p_- + k_-;q)}{k_-^2} \quad . \tag{2.32}$$

Had we used a regular cut-off, such an integral equation would be finite, owing to the absence of the term $\frac{e^2}{\pi\lambda}$, and to the principal-value prescription for the distribution $1/k_-^2$. Using the singular cut-off, we have to separate the divergent piece

$$\int \mathrm{d}k_- \, \frac{\varphi(p_- + k_-; q)}{k_-^2} = \frac{2}{\lambda}\varphi(p_-; q) + \int \mathrm{d}k_- \, \varphi(p_- + k_-; q) \frac{P}{k_-^2} \quad , \tag{2.33}$$

where P is the principal-value prescription for the quadratic singularity near the origin, equation (2.24) (the first term on the right-hand side arises from $\varphi(p_-; q) \int_{-\lambda}^{\lambda} \mathrm{d}k \frac{P}{k^2}$). One finds, for $\lambda \to 0$, that the cut-off disappears after inserting (2.33) back into (2.32). We arrive at the integral equation

$$-q_+ \varphi(p_-; q) = M^2 \left(\frac{1}{p_-} + \frac{1}{q_- - p_-} \right) \varphi(p_-; q) - \frac{e^2 N}{2\pi} P \int_{p_-}^{q_- - p_-} \mathrm{d}k_- \, \frac{\varphi(p_- + k_-; q)}{k_-^2} \quad , \tag{2.34a}$$

which in its turn, upon use of

$$\tau = \frac{\pi M^2}{e^2 N} = \frac{\pi m^2}{e^2 N} - 1 \quad , \quad q^2 = e^2 N \mu^2 \quad , \quad \text{and} \quad p_-/q_- = x \quad , \tag{2.34b}$$

where μ is the mass of the two-particle state in units of $e/\sqrt{\pi}$, can be rearranged into

$$\mu^2 \varphi(x) = \tau \left(\frac{1}{x} + \frac{1}{1-x} \right) \varphi(x) + P \int_0^1 \mathrm{d}y \, \frac{\varphi(y)}{(x-y)^2} \quad . \tag{2.35}$$

Although it is not possible to solve such an equation, the approximate spectrum can be obtained. The right-hand side of (2.35) is interpreted as a Hamiltonian action on the "wave function" $\varphi(x)$. We suppose that the eigenstates behave as x^β for $x \approx 0$. Using

$$\int_0^\infty \mathrm{d}x \, \frac{x^\beta}{(x-1)^2} = \beta\pi \operatorname{cotg} \beta\pi \quad , \tag{2.36}$$

we verify that such a solution of eq. (2.35) can be found if

$$\pi\beta \operatorname{cotg} \pi\beta + \tau = 0 \quad , \tag{2.37}$$

for functions that vanish on the boundary, $\varphi(0) = \varphi(1) = 0$, the "Hamiltonian" is Hermitian. However there are still problems when we require eigentates to be mutually orthogonal. Nevertheless the boundary condition respects completeness. It is thus natural to consider periodicity. For a periodic function, the second term in the Hamiltonian is approximated by

$$\int_0^1 \mathrm{d}y \, \frac{\mathrm{e}^{iwy}}{(y-x)^2} \approx \int_{-\infty}^{\infty} \mathrm{d}y \, \frac{\mathrm{e}^{iwy}}{(y-x)^2} = -\pi |w| \mathrm{e}^{iwx} \quad , \tag{2.38}$$

and for the above-discussed boundary conditions the eigenfunctions are $\varphi_k = \sin k\pi x$ for $\tau \approx 0$, with eigenvalues $\mu_k^2 = \pi^2 k$, leading to a Regge trajectory, without continuum part in the spectrum. This is a good approximation for large values of k.

It is important to know whether the $1/N$ expansion gives trustworthy results. As remarked in ref. [27], it does. The next-to-leading corrections are simplified by the fact that quarks are confined, and we need to take the $\lambda \to 0$ limit. The gauge-field propagator does not get corrections in such a case. The quark–antiquark scattering amplitude was also studied in detail, and computed in terms of the eigenfunctions $\varphi_k(x)$.

The consideration of the equation obeyed by the quark–antiquark scattering amplitude is a straightforward generalization of the previous results. Consider the quark–antiquark scattering amplitude, which we denote by $T_{\alpha\beta,\gamma\delta}(p,p';r)$, as given in the left-hand side of Fig. 3. The quark lines with momenta p and p' are connected by products of γ_- and the quark propagator. Since $\gamma_-^2 = 0$, only a simple γ-type factor survives, and we have

$$T_{\alpha\beta,\gamma\delta}(p,p';q) = \gamma_{-\alpha\gamma}\gamma_{-\beta\delta}\, T(p,p';q) \quad . \tag{2.39a}$$

In the large-N limit it obeys the equation graphically displayed in Fig. 3, which translates into

$$\begin{aligned} T_{\alpha\beta,\gamma\delta}(p,p';q) &= \frac{ie^2}{(p_- - p'_-)^2}(\gamma_-)_{\alpha\gamma}(\gamma_-)_{\beta\delta} \\ &+ ie^2 N \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{(\gamma_-)_{\alpha\epsilon}(\gamma_-)_{\beta\lambda}}{(k_- - p_-)^2} S(k)_{\epsilon\mu} S(k-q)_{\lambda\nu} T_{\mu\nu,\gamma\delta}(k,p';q) \quad . \end{aligned} \tag{2.39b}$$

Fig. 3: Bethe–Salpeter equation for the quark–antiquark scattering.

Using eq. (2.39a), and substituting (2.28) in (2.39b), after the γ-matrix algebra, one gets rid of all γ_- factors. We now use the kind of trick introduced in (2.30), since the (+) variables are essentially spectators, a fact derived from the instantaneous interaction in one light-cone variable. We define the functional

$$\varphi(p_-,p'_-;q) = \int \mathrm{d}p_+\, S_F(p) S_F(p-q) T(p,p';q) \quad , \tag{2.40}$$

from which eq. (2.39b) is solved for $T(p,p';q)$ in terms of $\varphi(k_-,p';q)$ as

$$T(p,p';q) = \frac{2ie^2}{(p_- - p'_-)^2} + \frac{2ie^2 N}{\pi^2}\int \mathrm{d}k_-\, \frac{\varphi(k_-,p';q)}{(k_- - p_-)^2} \quad . \tag{2.41}$$

We can suppose that the masses of quarks and antiquarks in the two propagators in (2.40) are different, but we take a simplified point of view where they are the same. The integral

$$\begin{aligned}
&\int \mathrm{d}p_+ \, S_F(p) S_F(p-q) \\
&= \int \mathrm{d}p_+ \, \frac{i}{p_+ - A(p_-)} \frac{i}{p_+ - q_+ - A(p_- - q_-)} \\
&= \int \mathrm{d}p_+ \, \frac{1}{A(p_-) - A(p_- - q_-) - q_+} \left[\frac{1}{p_+ - A(p_-)} - \frac{1}{p_+ - q_+ - A(p_- - q_-)} \right] \quad (2.42a)
\end{aligned}$$

is non-zero if it is performed between the poles. With $A(p_-)$ given by

$$A(p_-) = \frac{e^2 N}{\pi} \left[\frac{1}{p_-} - \frac{\varepsilon(p_-)}{\lambda} \right] + \frac{m^2 - i\epsilon}{p_-} \quad ,$$

we find the result

$$\int \mathrm{d}p_+ \, S_F(p) S_F(p-q) = \frac{1}{\frac{M^2}{p_-} + \frac{M^2}{q_- - p_-} + \left(\frac{e^2 N}{\pi \lambda} - i\epsilon \right) \varepsilon(p_-)} \theta(p_-) \theta(q_- - p_-) \quad . \quad (2.42b)$$

We use (2.39a) in (2.39b), multiply it by $S_F(p)S_F(p-q)$, integrate over p_+, and substitute definitions (2.34b), in the final equation, arriving at

$$\mu^2 \varphi(x, x'; q) = \frac{\pi^2}{N q_- (x - x')^2} + \tau \left(\frac{1}{x} - \frac{1}{1-x} \right) \varphi(x, x'; q) + \int_0^1 \mathrm{d}y \, \frac{\varphi(x, x'; q) - \varphi(y, x'; q)}{(x-y)^2} \, . \quad (2.43a)$$

The homogeneous equation is of the Schrödinger type, that is

$$H \varphi_k = \mu_k^2 \varphi_k = \tau \left(\frac{1}{x} + \frac{1}{1-x} \right) \varphi_k(x) + \int_0^1 \mathrm{d}y \, \frac{\varphi_k(x) - \varphi_k(y)}{(x-y)^2} \quad , \quad (2.43b)$$

and the eigenfunctions $\varphi_k(x)$ can be found as before. In terms of $\varphi_k(x)$, the authors of ref. [27] constructed

$$\varphi(x, x'; q) = - \sum \frac{\pi e^2}{q^2 - q_k^2} \frac{1}{q_-} \int_0^1 \mathrm{d}y \, \frac{\varphi_k(x) \varphi_k^*(y)}{(x' - y)^2} \quad , \quad (2.44)$$

and subsequently the quark–antiquark scattering amplitude

$$\begin{aligned}
T(x', x; q) &= \frac{i e^2}{q_-^2 (x' - x)^2} - \frac{i e^2 (e^2 N)}{\pi q_-^2} \sum_k \frac{1}{(q^2 - q_k^2)} \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}y' \, \frac{\varphi_k^*(y') \varphi_k(y)}{(y' - x')^2 (y - x)^2} \\
&= \frac{2 i e^2}{q_-^2 (x' - x)^2} \\
&\quad - \sum_k \frac{2i}{(q^2 - q_k^2)} \left\{ \varphi_k^*(x') \frac{2e}{\lambda} \left(\frac{e^2 N}{\pi} \right)^{\frac{1}{2}} \left[\theta(x'(1-x')) + \frac{\lambda}{2|q_-|} \left(\frac{\tau}{x'} + \frac{\tau}{1-x'} - \mu_k^2 \right) \right] \right\} \\
&\quad \times \left\{ \varphi_k(x) \frac{2e}{\lambda} \left(\frac{e^2 N}{\pi} \right)^{\frac{1}{2}} \left[\theta(x(1-x)) + \frac{\lambda}{2|q_-|} \left(\frac{\tau}{x} + \frac{\tau}{1-x} - \mu_k^2 \right) \right] \right\} \, , \quad (2.45)
\end{aligned}$$

which shows no continuum states, but again only bound-state poles at $q^2 = q_k^2 = \pi k e^2 N$, making once more the confinement features explicit. The normalized bound state has also been computed, and it can be shown to be of order $1/\lambda$ as $\lambda \to 0$, compensating the fact that the quark propagator vanishes in the same limit, which finally leads to finite-bound-state amplitudes.

The results of 'tHooft represented a profound breakthrough, since they were precursors of more recent attempts to write down differential equations to be obeyed by bound states or collective excitations. The results about the Regge behaviour of the physical spectrum lead to a strong link to string theory developments, which had to wait almost two decades to be followed further.

2.2 Ambiguity in the self-energy of the quark

The basis of 'tHooft's result about the $1/N$ expansion is an infrared cut-off procedure, which consists in drilling a hole in momentum space around the infrared region ($k \sim 0$), the size of the hole (λ) being the cut-off. For $\lambda \to 0$, when calculating some gauge-invariant objects, one observes that λ-dependent constants cancel. Such a procedure is equivalent to defining the light-cone propagator with a principal-value prescription, that is, following ref. [40]

$$\partial_-^{-1} = \frac{1}{4}\int \mathrm{d}^2 y\, \varepsilon(x_+ - y_+)\delta(x_- - y_-) \quad , \tag{2.46a}$$

$$\partial_-^{-2} = \frac{1}{8}\int \mathrm{d}^2 y\, |x_+ - y_+|\delta(x_- - y_-) \quad , \tag{2.46b}$$

or, in momentum space

$$P\,\frac{1}{k_-} = \frac{1}{2}\left(\frac{1}{k_- + i\epsilon} + \frac{1}{k_- - i\epsilon}\right) \quad , \tag{2.47}$$

$$P\,\frac{1}{k_-^2} = \frac{1}{2}\left(\frac{1}{(k_- + i\epsilon)^2} + \frac{1}{(k_- - i\epsilon)^2}\right) \quad . \tag{2.48}$$

The difference between 'tHooft's procedure, and the principal-value prescription for $\lambda \to 0$, is not dificult to obtain. Using the cut–off procedure and Fourier transforming we obtain the expression

$$\partial_-^{-2} = \frac{1}{8}\left(|x_+| - \frac{2}{\pi\lambda}\right)\delta(x_-) \quad , \tag{2.49}$$

which differs from eq. (2.46b) by the extra term $-\frac{1}{4\pi\lambda}\delta(x) \equiv -\frac{1}{4\pi\lambda}\delta(x^+)$ corresponding to a gauge ambiguity in the Coulomb equation

$$\partial_-^2 A_+ = -J_- \quad . \tag{2.50}$$

The cut–off dependence may be gauged away. The procedure is by all means ambiguous. Indeed, using the principal-value prescription, the momentum integrals in a Feynman diagram do not commute, but rather obey the Poincaré–Bertrand formula[25]

$$\int \frac{\mathrm{d}k'}{k-k'} \int \frac{\mathrm{d}k''}{k'-k''} f(k',k'') - \int \mathrm{d}k'' \int \mathrm{d}k' \frac{1}{(k-k')(k'-k'')} f(k',k'') = -\pi^2 f(k,k) \quad . \tag{2.51}$$

The choice made by Wu[25] was to Wick-rotate the Feynman integral, going to Euclidian space, in order to compute the fermion self-energy. In such a case, the fermion propagator is

$$S_F(k) = \frac{i(k_1 - ik_2)}{k^2 + m^2 - (k_1 - ik_2)\Sigma_{SE}(k)} \quad , \tag{2.52}$$

where the self-energy $\Sigma_{SE}(k)$ satisfies the integral equation

$$\Sigma_{SE}(p) = -\frac{e^2 N}{\pi^2} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_1 \mathrm{d}k_2}{(k_1 - ik_2)^2} \frac{k_1 + p_1 - i(k_2 + p_2)}{(k+p)^2 + m^2 - [k_1 + p_1 - i(k_2 + p_2)]\Sigma_{SE}(k+p)} \quad . \tag{2.53}$$

With the ansatz

$$\Sigma_{SE}(k) = (k_1 + ik_2) f(k^2) \quad , \tag{2.54}$$

one can integrate over the angular variables and obtain the algebraic equation

$$f(k^2) = \frac{e^2 N}{\pi^2} \frac{1}{k^2[1 - f(k^2)] + m^2} \quad , \tag{2.55}$$

with the solution

$$f(k^2) = \frac{1}{2k^2} \left\{ k^2 + m^2 - \left[(k^2 + m^2)^2 - \frac{4}{\pi} e^2 N k^2 \right]^{1/2} \right\} \quad . \tag{2.56}$$

Therefore, $\Sigma_{SE}(p)$ (and hence the fermion propagator) has a square-root-type singularity. The propagator pole has not appeared. The principal-value result is obtained in the large momentum limit.

This result is significantly different from the one obtained by 'tHooft. In fact, there are further inconsistencies, as pointed out in ref. [26], in a detailed analysis of the infrared cut-off procedures defined by either the cut-off λ around the origin in momentum space, or by the principal-value prescription. Those authors analyzed general gauges of the type $n^\mu A_\mu = 0$, where n^μ is a fixed vector. In the case $n^2 = -1$, the most singular terms in the gauge-field propagator are of the form $1/(n^\mu A_\mu)^2$, with integrals of the form

$$\int \frac{\mathrm{d}(n \cdot k)}{(n \cdot k)^2} \, f\left(n \cdot (k-p)\right) \quad . \tag{2.57}$$

Notice that 'tHooft's analyses were done in the light-cone gauge, where $n^2 = 0$.

Although, as discussed in ref. [27], the use of either 'tHooft's regularization or the principal-value prescription leads to the same bound spectrum, the interpretation in terms of fermion propagators is yet unclear. The authors have proved that there is no solution for the self-energy equation if the bare mass of the quark is small and the principal-value prescription is used. On the other hand, the Euclidian symmetric integration was applied to the Schwinger model, and the results obtained agree with the exact solution of the theory.[42]

Since the quark is after all not an observable, the question seems to be one of interpretation of the results. In ref. [43] it is argued that the tachyonic character of the $1/N$ correction computed in the Minkowski light-cone gauge can be seen as a reinterpretation of Nambu's pion in terms of spontaneously broken chiral symmetry. Wu's solution, in the zero quark-mass limit, transforms into 'tHooft's result in the infinite momentum frame. By all means, the theory as presented in Minkowskian version has an appealing interpretation when one analyzes the confined quarks as compared to the high-energy behaviour, providing a model of hadrons with the expected properties of confinement as displayed by the spectrum, and description by a parton model in the high-energy limit.

A QCD_2 physical interpretation of the results obtained from the large-N limit, using also the experience with the Schwinger model, can already be drawn. Such a question was studied in detail in [27], where the authors presented QCD_2 as a good model for confined quarks in spite of the huge simplification due to the reduced number of dimensions.

The fundamental property of quarks, which evaded solution in terms of a realistic theory such as four-dimensional QCD, is that they are confined, in the sense that only colour singlets appear in the Hilbert space; nevertheless, high-energy scattering is described in terms of the parton model, where quarks are essentially free – although forming a bound state. Therefore one needs apparently contradictory ideas, the infrared slavery leading to confinement, and asymptotic freedom describing the parton model. The string idea seems correct to describe the mesonic spectrum or to explain the Regge behaviour; but the ultraviolet scattering of strings is too soft, since amplitudes fall off too quickly with energy.[7] On the other hand, Yang–Mills theory has been proved to be asymptotically free, in excellent agreement with experiment at high energy, but its description of bound states is far from being realized.

It is very exciting that some of these gaps have been covered over the years in two-dimensional QCD. In fact, in two dimensions, the theory is "infrared enslaving" due to the properties of the Coulomb law. If the experience with the (soluble) Schwinger model can be taken into account, several phenomenologically appealing properties of long-distance physics of hadrons may be described. Moreover we will see that the best description of QCD_2 with fermions is achieved by the non-Abelian bosonization, where the bosons are bound states of the fermions, providing a natural description of the mesons. Besides that, two-dimensional Yang–Mills theory is far better than the Schwinger model, since it is highly non-trivial, and can provide a more realistic model of interacting particles, while the "meson" of the Schwinger model is free. Moreover, although in two dimensions it is not possible to analyze important questions such as large-angle scattering, hadronic scattering amplitudes and further short-distance properties of high-energy, scattering of bound states are satisfactorily described by QCD_2, since the coupling constant has positive mass dimension, leading to obvious asymptotic freedom.

In ref. [27] it is argued that the confinement properties found using the infrared cut-off in the $1/N$ expansion, by means of which the quark pole is shifted towards infinity in the vanishing cut-off limit, hold true in the principal-value prescription. Therefore, in spite of the possible criticisms presented in the previous section, the interpretation of the theory in terms of confined quarks is solid, since every approach leads to a rather well defined bound-state structure, although the quark propagator itself is an ill-defined quantity, depending on the cut-off procedure. The second question is whether such properties hold true at higher orders in $1/N$, and the answer is positive.

The large-N limit of QCD is suitable to describe the dual resonance model, as is also clear from the topological structure seen in the Feynman diagrams in such a limit. This issue will be studied in detail in section 3 for the pure gauge theory. The effectiveness of such an expansion depends on the relative size of the higher corrections. The correction to the gluon propagator for large N is given by the fermion loop. If the full fermion propagator is used with the cut-off procedure, such a correction is seen to vanish, since the quark propagator itself vanishes in the limit $\lambda \to 0$. A more careful analysis shows that one has to take into account terms of order λ in the gluon propagator, but one can prove that they do only change the infrared-divergent part of the propagator. Likewise, an analysis of the quark–antiquark gluon vertex as well as of the quark self-energy shows, by arguments based on rescaling of the momentum variables, that the relevant properties obtained at lowest order in $1/N$ remain unchanged as $\lambda \to 0$.

If it is obvious that quarks are confined in the cut-off procedure due to the shift of the quark propagator pole towards infinity in the $\lambda \to 0$ limit, this is less transparent in the principal-value prescription. On the other hand, the free-field structure at high energies turns out to be clearer in the principal-value prescription. We shall now see how this works.

Using the regular cut-off (principal-value prescription) the final Bethe–Salpeter equation (2.43a) remains unchanged since the λ-terms disappear in favour of the principal-value terms. Quark continuum states do not appear in the solution, signalling confinement. On the other hand, one can consider also coloured operators and the corresponding expectation values, such as the two-point expectation $\langle 0|T\overline{\psi}_i\psi_j(x)\overline{\psi}_j\psi_i(y)|0\rangle$, which vanishes using the singular cut-off procedure; for the regular cut-off case it gives a finite result, compatible with free-field perturbation theory. The results are correct, and are not contradictory, since the high-energy and zero cut-off limits do not commute, as exemplified for the explicit case of the integral

$$q^2 \int \frac{\mathrm{d}x}{q^2 x(1-x) - e^2 N/\pi\lambda} \quad , \tag{2.58}$$

which appears in the coloured two-point function. The integral (2.58) approaches a constant for large q^2, but vanishes if the small cut-off limit is taken first.

Therefore, the principal-value procedure (regular cut-off) is appropriate to describe the high-energy behaviour of the theory, i.e. reproduces parton model results. Green functions containing currents and computed using such a procedure show a short-distance behaviour compatible with the free-quark model, showing the interplay between asymptotic freedom and confinement.

Form factors have been discussed by Einhorn.[40] A "parton model" has been constructed there, and hadronic form factors have been shown to be power-behaved, with a power determined by the coupling constant. For meson scattering amplitudes see ref. [41].

2.3 Polyakov–Wiegmann formula and gauge interactions

In the case of $U(1)$, namely the Schwinger model, only the first term in eq. (2.13) survives, and one just obtains the gauge-field mass generation, well known in the model. In the general non-Abelian case, the computation of the fermion determinant is accomplished (see eqs. (2.5–16)), but is given in terms of an infinite series. A detailed account of the method is described in chapter 11 of ref. [8].

By all means, the most interesting and clear way of computing the fermion determinant is the Polyakov–Wiegmann method, starting from the equations obeyed by the current, and ending up with the WZW functional, which represents a summation of the series. As a bonus, we obtain the bosonized version of the fermionic action. The implementation of the bosonization techniques of a non-Abelian symmetry is well known[15−19] and we shall review the Polyakov–Wiegmann identities and the consequent form of the action using functional methods.

In two dimensions we can locally[4)] write the gauge field in terms of two matrix-valued fields U and V as

$$A_+ = \frac{i}{e} U^{-1}\partial_+ U \quad , \quad A_- = \frac{i}{e} V\partial_- V^{-1} \quad , \tag{2.59a}$$

or

$$eA_\mu = \frac{i}{2}(g_{\mu\nu} + \epsilon_{\mu\nu}) V\partial^\nu V^{-1} + \frac{i}{2}(g_{\mu\nu} - \epsilon_{\mu\nu}) U^{-1}\partial^\nu U \quad . \tag{2.59b}$$

Since the fermionic determinant must be gauge-invariant, there is no loss in choosing $V = 1\,(A_- = 0)$. We start out of the current conservation and anomaly equation

$$\partial_\mu J^\mu - ie[A_\mu, J^\mu] = 0 \quad , \tag{2.60a}$$

$$\widetilde{\partial}_\mu J^\mu - ie[\widetilde{A}_\mu, J^\mu] = -\frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \quad , \tag{2.60b}$$

for which, using the gauge potentials (2.59), we find the solution

$$J_\pm = \pm\frac{i}{2\pi} U^{-1}\partial_\pm U \quad . \tag{2.61}$$

Notice that eq. (2.60a) was given by (2.3), while (2.60b), resp. (2.8), is a one-loop effect, but maintained to all orders.

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4) In general we are considering the fields as maps from Minkowski space to the $SU(N)$ algebra and such a decomposition holds globally. In section 3, where we find a more general situation, one has to be cautions about this choice.

The effective action $W[A]$ is obtained by noticing that it is exactly its variation with respect to the gauge field that leads to the current, that is

$$J_- = \frac{2}{e}\frac{\delta W}{\delta A_+} \quad , \tag{2.62}$$

where the factor 2 comes from the definitions in Appendix A. Therefore

$$\delta W = \frac{e}{2}\int \mathrm{d}^2x\, J_-\delta A_+ \quad . \tag{2.63}$$

In terms of the U-variation, we have

$$e\delta A_+ = iU^{-1}\partial_+\delta U - iU^{-1}\delta U U^{-1}\partial_+ U = i\nabla_+(U^{-1}\delta U) \quad , \tag{2.64}$$

where the operator ∇_+ acts as

$$\nabla_+ f = \partial_+ f + [U^{-1}\partial_+ U, f] \quad . \tag{2.65}$$

Therefore we find for the variation of the effective action, after integrating ∇_+ by parts:

$$\delta W = -\frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\, U^{-1}\delta U \nabla_+(U^{-1}\partial_- U) \quad . \tag{2.66}$$

We can use the simple identity $\nabla^\mu(U^{-1}\partial_\mu U) = \partial^\mu(U^{-1}\partial_\mu U)$, as well as the relation $\partial_+(U^{-1}\partial_- U) = \nabla_-(U^{-1}\partial_+ U)$, to rewrite δW as

$$\delta W = -\frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\, U^{-1}\delta U \partial_-(U^{-1}\partial_+ U) \quad . \tag{2.67}$$

Such an equation may be integrated in terms of the WZW action. Consider the action of the principal σ-model $S_{P\sigma M}$:

$$S_{P\sigma M} = \frac{1}{8\pi}\int \mathrm{d}^2x\, \mathrm{tr}\ \partial^\mu U^{-1}\partial_\mu U \quad , \tag{2.68}$$

and the Wess–Zumino term S_{WZ}

$$S_{WZ} = \frac{1}{4\pi}\mathrm{tr}\int_0^1 \mathrm{d}r \int \mathrm{d}^2x\, \epsilon^{\mu\nu}\widetilde{U}^{-1}\dot{\widetilde{U}}\widetilde{U}^{-1}\partial_\mu\widetilde{U}\widetilde{U}^{-1}\partial_\nu\widetilde{U} \quad , \tag{2.69}$$

where $\widetilde{U}(r,x)$ is the extension of $U(x)$ to a space having the Euclidian two-dimensional space as a boundary, such that $\widetilde{U}(0,x) = 1$ and $\widetilde{U}(1,x) = U(x)$. The variation of $S_{P\sigma M}$ is trivially performed

$$\delta S_{P\sigma M} = -\frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\, U^{-1}\delta U \partial_\mu(U^{-1}\partial^\mu U) \quad , \tag{2.70}$$

while the variation of S_{WZ} is a total derivative in r, such that we can integrate over that auxiliary parameter, which leads to

$$\delta S_{WZ} = -\frac{1}{4\pi}\int_0^1 \mathrm{d}r \int \mathrm{d}^2x\, \frac{\partial}{\partial r}\epsilon^{\mu\nu}\,\mathrm{tr}\,\widetilde{U}^{-1}\delta\widetilde{U}\partial_\mu(\widetilde{U}^{-1}\partial_\nu\widetilde{U}) \quad ,$$
$$= -\frac{1}{4\pi}\epsilon^{\mu\nu}\int \mathrm{d}^2x\,\mathrm{tr}\,U^{-1}\delta U\partial_\mu(U^{-1}\partial_\nu U) \quad . \tag{2.71}$$

Adding $S_{P\sigma M}$ and S_{WZ} we obtain the corresponding variation which matches (2.67); therefore we find the solution to the effective action

$$W[A] = -\frac{1}{8\pi}\int \mathrm{d}^2x\,\partial^\mu U^{-1}\partial_\mu U - \frac{1}{4\pi}\int_0^1 \mathrm{d}r\int \mathrm{d}^2x\,\epsilon^{\mu\nu}\widetilde{U}^{-1}\dot{\widetilde{U}}\widetilde{U}^{-1}\partial_\mu\widetilde{U}\widetilde{U}^{-1}\partial_\nu\widetilde{U} \quad ,$$
$$= -\Gamma[U] \quad , \tag{2.72}$$

defining the Wess–Zumino–Witten (WZW) functional.[16,17,44] The WZW theory has a right- and a left-moving component of Noether current, which generate an affine Lie algebra. Witten proved that the minimal theory (i.e. with unit coefficient) is equivalent to free fermions, and the above-mentioned currents turn into the fermionic currents. The elementary fields of the theory build a representation of the affine algebra. The equation of motion can be interpreted, in the quantum theory, as equations defining the realization of the current algebra – or the so-called null states of conformal field theory, for whose detailed definition we refer to the original literature.[45] In the case of quantum chromodynamics, the gauge-field self interaction will correspond to an off-critical perturbation of such a model.

From the gauge invariance, we recover the full U, V dependence, replacing U by the gauge-invariant $\Sigma = UV$. Notice, at this point, that the determinant does not factorize in terms of chiral and anti-chiral components[5)].

Such a problem motivates us to consider the relation between the WZW functional computed as a function of the product $\Sigma = UV$, namely $\Gamma[UV]$, and $\Gamma[U]$, $\Gamma[V]$. Each part of the WZW functional can be considered as follows. For the principal sigma model piece, we have, using the cyclicity of the trace:

$$S_{P\sigma M}[UV] = \frac{\mathrm{tr}}{8\pi}\int \mathrm{d}^2x\,\partial^\mu(V^{-1}U^{-1})\partial_\mu(UV)$$
$$= \frac{\mathrm{tr}}{8\pi}\int \mathrm{d}^2x\,\partial^\mu U^{-1}\partial_\mu U + \frac{\mathrm{tr}}{8\pi}\int \mathrm{d}^2x\,\partial^\mu V^{-1}\partial_\mu V + \frac{\mathrm{tr}}{4\pi}\int \mathrm{d}^2x\,U^{-1}\partial_\mu UV\partial^\mu V^{-1}$$
$$= S_{P\sigma M}[U] + S_{P\sigma M}[V] + \frac{\mathrm{tr}}{4\pi}\int \mathrm{d}^2x\,U^{-1}\partial_\mu UV\partial^\mu V^{-1} \quad . \tag{2.73}$$

For the WZ term (2.69) we use

$$(UV)^{-1}\partial_\mu(UV) = V^{-1}\left[U^{-1}\partial_\mu U + \partial_\mu VV^{-1}\right]V \quad , \tag{2.74}$$

5) This is expected on grounds of vector gauge invariance, since it forces us into a definite type of regularization. If one defines the determinant as factorizing into definite chiral components, vector gauge invariance is not respected. The difference is a contact term, as a result of eq. (2.78). See refs. [18,19] for a detailed discussion.

and after some calculation we find

$$S_{WZ}[UV] = S_{WZ}[U] + S_{WZ}[V] + \frac{1}{4\pi}\,\mathrm{tr}\int_0^1 \mathrm{d}r \int \mathrm{d}^2x\, \epsilon_{\mu\nu} W^{\mu\nu} \quad , \tag{2.75}$$

where

$$W_{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}r}\,\widetilde{U}^{-1}\partial_\mu \widetilde{U}\widetilde{V}\partial_\nu \widetilde{V}^{-1} - \partial_\mu\left[\widetilde{V}\partial_\nu\widetilde{V}^{-1}\widetilde{U}^{-1}\dot{\widetilde{U}}\right] - \partial_\nu\left[\widetilde{U}^{-1}\partial_\mu\widetilde{U}\widetilde{V}\dot{\widetilde{U}}^{-1}\right] \quad . \tag{2.76}$$

Since the last two terms are total derivatives, they drop out, while the first one turns out to be local, not depending on the extensions $\widetilde{U}$, $\widetilde{V}$. Therefore

$$S_{WZ}[UV] = S_{WZ}[U] + S_{WZ}[V] + \frac{1}{4\pi}\mathrm{tr}\int_0^1 \mathrm{d}r \int \mathrm{d}^2x\, \epsilon^{\mu\nu} U^{-1}\partial_\mu U V \partial_\nu V^{-1} \quad . \tag{2.77}$$

Adding eqs. (2.73) and (2.77) we find

$$\Gamma[UV] = \Gamma[U] + \Gamma[V] + \frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\,(g^{\mu\nu} + \epsilon^{\mu\nu})U^{-1}\partial_\mu U V \partial_\nu V^{-1} \quad , \tag{2.78a}$$

$$= \Gamma[U] + \Gamma[V] + \frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\, U^{-1}\partial_+ U V \partial_- V^{-1} \quad , \tag{2.78b}$$

which we call the Polyakov–Wiegmann formula from now on. The last term is an obstacle for the factorizability of chiral left and right parts of the fermionic determinant.

In order to implement the change of variables (2.59), in the quantum theory, we still have to compute its Jacobian, that is

$$\mathcal{D}A_+\mathcal{D}A_- = J\,\mathcal{D}U\mathcal{D}V \quad , \tag{2.79}$$

where

$$J = \det\frac{\delta A_+}{\delta U}\frac{\delta A_-}{\delta V} = \det\nabla \quad . \tag{2.80}$$

Notice that ∇ is the covariant derivative in the adjoint representation. Therefore it corresponds to the previously computed determinant in fundamental representation, raised to the power given by the quadratic Casimir (c_V), leading to the result

$$J = \mathrm{e}^{-ic_V\Gamma[UV]} \quad . \tag{2.81}$$

It is well known that the invariances of the fermionic part of the Lagrangian (2.1) are local gauge transformations $SU(N)$, as well as $SU(N)_L \times SU(N)_R$, for both right (R) and left (L) components, namely

$$\psi_{\substack{R\\L}} \to w_{\substack{R\\L}}\psi_{\substack{R\\L}} \quad , \tag{2.82}$$

$$A_\pm \to w_{\substack{R\\L}}\left(A_\pm + \frac{i}{e}\partial_\pm\right)w^{-1}_{\substack{R\\L}} \quad , \tag{2.83}$$

corresponding to pure vector gauge transformation when $w_R = w_L = w$, while for $w_R = w_L^{-1} = w$ it corresponds to a pure axial vector transformation. If we use the change of variables (L), the transformations

$$\psi \to \mathrm{e}^{i\gamma_5\theta}\psi \quad , \tag{2.84}$$

$$A_+ \to wA_+w^{-1} + \frac{i}{e}w(\partial_+ w^{-1}) \quad , \tag{2.85}$$

$$A_- \to w^{-1}A_-w + \frac{i}{e}w^{-1}(\partial_- w) \quad , \tag{2.86}$$

reduce to $U \to Uw^{-1}$ and $V \to w^{-1}V$.

The above transformations are not symmetries of the effective action $W[A]$ due to the axial anomaly. This non-invariance may be understood in terms of a new bosonic action $S_F[A, w]$ for the fermions in a background field A_μ. Indeed

$$S_F[A, g] \equiv \Gamma[UgV^{-1}] - \Gamma[UV] \quad , \tag{2.87}$$

in such a way that using the invariance of the Haar measure, we find

$$\det i\not{D} \equiv \mathrm{e}^{iW[A]} = \int \mathcal{D}g\, \mathrm{e}^{iS_F[A,g]} \quad . \tag{2.88}$$

In fact $S_F(A, g)$ plays the role of an equivalent bosonic action and its explicit form may be obtained by repeated use of the Polyakov–Wiegmann formula (2.78)

$$S_F[A,g] = \Gamma[g] + \frac{1}{4\pi}\int \mathrm{d}^2x\, \mathrm{tr}\, \left[e^2A^\mu A_\mu - e^2A_+gA_-g^{-1} - eiA_+g\partial_-g^{-1} - eiA_-g^{-1}\partial_+g\right] \quad , \tag{2.89}$$

representing $\det i\not{D}$ in terms of bosonic degrees of freedom. Therefore we recover a formulation in terms of A_μ and of the independent field g.

The advantage of such a result over (2.16) is plural, and largely compensates the problems posed by the fact that a local formulation does not exist for the WZW fields g. As we have seen, (2.89) represents a bosonized version of QCD_2, and the inclusion of gauge interactions was natural. Moreover, algebraic properties of the Polyakov–Wiegmann functional will later permit to obtain drastic simplifications of the theory, with a thorough separation of some gauge fields, whose appearance will be effective only by means of the BRST constraints. Therefore such results point to an extraordinary parallel to the line of development of the Schwinger model, as presented for instance in [8].

Observe that we may reobtain Witten's non-Abelian bosonization formulae

$$j_+ = -\frac{i}{2\pi}g^{-1}\partial_+g \quad , \tag{2.90a}$$

$$j_- = -\frac{i}{2\pi}g\partial_-g^{-1} \quad , \tag{2.90b}$$

from eq. (2.88), by first writing down the vacuum expectation value of products of the currents j^μ of the free fermion theory and then functionally differentiating it with respect to A_μ and setting $A_\mu = 0$. For the correlators of $j_\pm$ we have

$$\begin{aligned}\langle j_+^{i_1 j_1}(x_1)\cdots j_+^{i_n j_n}(x_n)\rangle_F& \\ = \left(\frac{-1}{2\pi}\right)^n &\left\langle [g^{-1}(x_1) i\partial_+ g(x_1)]^{i_1 j_1}\cdots [g^{-1}(x_n) i\partial_+ g(x_n)]^{i_n j_n}\right\rangle_B \quad , \qquad (2.91a)\\ \langle j_-^{i_1 j_1}(x_1)\cdots j_-^{i_n j_n}(x_n)\rangle_F& \\ = \left(\frac{-1}{2\pi}\right)^n &\left\langle [g(x_1) i\partial_- g^{-1}(x_1)]^{i_1 j_1}\cdots [g(x_n) i\partial_- g^{-1}(x_n)]^{i_n j_n}\right\rangle_B \quad . \qquad (2.91b)\end{aligned}$$

In the $U(1)$ case, where the Wess–Zumino term in (2.72) vanishes, only the principal σ-model is left, namely

$$W[A] = -\frac{1}{8\pi}\int \mathrm{d}^2x\,(\partial_\mu \Sigma^{-1})(\partial_\mu \Sigma) \quad . \qquad (2.92)$$

Recall that $\Sigma \equiv UV$. Now if we substitute $U = \mathrm{e}^{i(\varphi-\phi)}$ and $V = \mathrm{e}^{-i(\varphi+\phi)}$, in eqs. (2.59) one obtains for A_μ and $F_{\mu\nu}$ respectively

$$eA_\mu = (\partial_\mu \varphi + \widetilde{\partial}_\mu \phi) \quad , \qquad (2.93)$$
$$eF_{\mu\nu} = (\partial_\mu \widetilde{\partial}_\nu - \partial_\nu \widetilde{\partial}_\mu)\phi = -\epsilon_{\mu\nu}\Box\phi \quad . \qquad (2.94)$$

In the above set of equations, (2.94) can be solved for ϕ in terms of $F_{\mu\nu}$, and we get

$$\phi = \frac{e}{2}\int \mathrm{d}^2y\, D(x-y)\epsilon^{\mu\nu}F_{\mu\nu}(y) \quad , \qquad (2.95)$$

where $D(x-y)$ is the massless propagator. If we make the identification $\Sigma = \mathrm{e}^{2\phi}$, and use that $\Box D(x-y) = \delta^2(x-y)$ we obtain

$$W[A] = \frac{1}{2\pi}\int \mathrm{d}^2x\,(\partial_\mu\phi)^2 = \frac{1}{8\pi}\int \mathrm{d}^2x\,\epsilon^{\mu\nu}F_{\mu\nu}(x)D(x-y)\epsilon^{\lambda\rho}F_{\lambda\rho}(y) \quad . \qquad (2.96)$$

If now we return to the equivalent bosonic action $S_F[A,g]$, eq. (2.89), in the general non-Abelian case, we rewrite the integrand of the second term as

$$e^2A^\mu A_\mu - e^2A^\mu g\,(g_{\mu\nu}+\epsilon_{\mu\nu})\,A^\nu g^{^{-1}} - ieA^\mu\,(g_{\mu\nu}+\epsilon_{\mu\nu})\,g\partial^\nu g^{^{-1}} - ieA^\mu\,(g_{\mu\nu}-\epsilon_{\mu\nu})\,g^{^{-1}}\partial^\nu g\,,$$

take the variational derivative with respect to A_μ, and we obtain the current

$$J_\mu = \frac{e}{2\pi}A_\mu - \frac{ei}{4\pi}\left\{(g_{\mu\nu}+\epsilon_{\mu\nu})\,gD^\nu g^{-1} + (g_{\mu\nu}-\epsilon_{\mu\nu})\,g^{-1}D^\nu g\right\} \quad . \qquad (2.97)$$

Using eq. (2.59), this current may also be written in the form

$$J_+ = \frac{i}{2\pi}\left\{U^{-1}\partial_+ U - (Ug)^{-1}\,\partial_+ (Ug)\right\} \quad , \tag{2.98}$$

$$J_- = \frac{i}{2\pi}\left\{V^{-1}\partial_- V - \left(Vg^{-1}\right)^{-1}\partial_-\left(Vg^{-1}\right)\right\} \quad . \tag{2.99}$$

Under local gauge transformations in the extended bosonic space, $U \to U\omega$, $V \to V\omega$ and $g \to \omega^{-1}g\omega$, the above currents (2.98) and (2.99) transform covariantly: $J_\pm \to \omega^{-1}J_\pm\omega$, and the effective action realizes the local symmetry, $S_F[A, \omega^{-1}g\omega] = S_F[A, g]$.

Only in the Abelian case, the components (2.98) and (2.99) reproduce the form of the Witten current (2.90). In the non-Abelian case, however, it involves the gauge field itself. This leads to an effective action $S_F[A, g]$, which in terms of the current (2.97) reads

$$S_F[A, g] = \Gamma[g] + \int \mathrm{d}^2x\,\mathrm{tr}\left\{eJ_\mu A_\mu - \frac{1}{4\pi}\left(A_+A_- - g^{-1}A_+gA_-\right)\right\} \quad . \tag{2.100}$$

The second term in the integrand cancels only in the Abelian case, where $S_F[A, \omega]$ reduces to the conventional form.

We now turn to the partition function related to the QCD_2 Lagrangian, given by eq. (2.1). It reads

$$\mathcal{Z}\left[\overline{\eta}, \eta, i_\mu\right] = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}A_\mu\, \mathrm{e}^{i\int \mathrm{d}^2z\,\mathcal{L} + i\int \mathrm{d}^2z\,(\overline{\eta}\psi + \overline{\psi}\eta + i_\mu A_\mu)} \quad , \tag{2.101}$$

where $\eta, \overline{\eta}$ are the external sources for the fermions $\overline{\psi}, \psi$, and i_μ is the external source for the gauge field A_μ.

The bosonized version of the theory was obtained by rewriting the fermionic determinant $\det i\,\not{D}$ as a bosonic functional integral as eq. (2.88), where now we identify $W[A] \equiv -\Gamma[UV]$. The external sources have been used to redefine the fermionic field as

$$\overline{\psi} i\,\not{D}\psi + \overline{\eta}\psi + \overline{\psi}\eta = \left[\overline{\psi} + \overline{\eta}(i\,\not{D})^{-1}\right] i\,\not{D}\left[\psi + (i\,\not{D})^{-1}\eta\right] - \overline{\eta}(i\,\not{D})^{-1}\eta \quad . \tag{2.102}$$

The non-linearity in the gauge-field interaction can also be disentangled by means of the identity

$$\mathrm{e}^{-\frac{i}{4}\int \mathrm{d}^2z\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}} = \int \mathcal{D}E\,\mathrm{e}^{-i\int \mathrm{d}^2z\left[\frac{1}{2}\mathrm{tr}\,E^2 + \frac{1}{2}\mathrm{tr}\,EF_{+-}\right]} \quad , \tag{2.103}$$

where E is a matrix-valued field. Taking into account the previous set of information we arrive at

$$\mathcal{Z}\left[\overline{\eta}, \eta, i_\mu\right] = \int \mathcal{D}E\mathcal{D}U\mathcal{D}V\mathcal{D}g$$
$$\times\, \mathrm{e}^{i\Gamma[UgV] - i(c_V+1)\Gamma[UV] - i\int \mathrm{d}^2z\,\mathrm{tr}\left[\frac{1}{2}E^2 + \frac{1}{2}EF_{+-}\right] + i\int \mathrm{d}^2z\,i_\mu A_\mu - i\int \mathrm{d}^2z\mathrm{d}^2w\,\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)} \quad . \tag{2.104a}$$

We should also include a term $m\,\mathrm{tr}\,(g+g^{-1})$ in the effective action,[23] if we were considering massive fermions, but we shall avoid such a complication and consider only the massless case. We should mention, repeating the introduction, that already in the case of the Schwinger model the inclusion of mass for the fermion couples the previously free and massless "η" excitations to the theory, rendering results technically more complicated to be obtained.

Gauge fixing is another ingredient and, in fact, the process of introducing ghosts here is standard. We perform the procedure implicitly, until it is necessary to explicitly take into account the ghost degrees of freedom. Up to considerations concerning the spectrum, our manipulations do not explicitly depend on the gauge fixing/ghost system, and we keep it at the back of our minds and formulae. We return to this problem in section 2.6.

It is not dificult to see that the field $\widetilde{g}$ decouples (up to the BRST condition) after defining a new gauge-invariant field $\widetilde{g}=UgV$. Using the invariance of the Haar measure, $\mathcal{D}g=\mathcal{D}\widetilde{g}$, the partition function turns into

$$\mathcal{Z}\left[\overline{\eta},\eta,i_\mu\right]=\int\mathcal{D}\widetilde{g}\,\mathrm{e}^{i\Gamma[\tilde{g}]}\int\mathcal{D}E\mathcal{D}U\mathcal{D}V\mathcal{D}(\mathrm{ghosts})$$
$$\times\,\mathrm{e}^{-i(c_V+1)\Gamma[UV]-i\int\mathrm{d}^2z\,\mathrm{tr}\,[\frac{1}{2}E^2+\frac{1}{2}EF_{+-}]+iS_{\mathrm{ghosts}}+i\int\mathrm{d}^2z\,i_\mu A_\mu-i\int\mathrm{d}^2z\mathrm{d}^2w\,\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)}\,, \qquad (2.104b)$$

where the A_μ variables are given in terms of the two-matrix-valued fields U and V, as in eq. (2.59).

In the way the gauge-field strength F_{+-} is presented, it hinders further developments; however if we write it in terms of the U and V potentials, we arrive at the identity aqui

$$\mathrm{tr}\,EF_{+-}=\frac{i}{e}\mathrm{tr}\,UEU^{-1}\partial_+(\Sigma\partial_-\Sigma^{-1}) \quad . \qquad (2.105)$$

We have used the variable $\Sigma=UV$ and this will imply a further factorization of the partition function. In fact, Σ is a more natural candidate for representing the physical degrees of freedom, since U and V are not separately gauge-invariant. We redefine E, taking once more advantage of the invariance of the Haar measure, in such a way that the effective action depends only on the combination Σ. The variables U and V will then appear separately only in the source terms, which are gauge-dependent, as they should; there the gauge fields may be described as

$$A_+=\frac{i}{e}U^{-1}\partial_+U \quad , \quad A_-=\frac{i}{e}(U^{-1}\Sigma)\partial_-(\Sigma^{-1}U) \quad . \qquad (2.106)$$

If we eventually choose the light-cone gauge, we will have e.g. $U=1$, $A_-=\frac{i}{e}\Sigma\partial_-\Sigma^{-1}$ and $A_+=0$. From the structure of (2.105), it is natural to redefine variables as $\widetilde{E}'=UEU^{-1},\mathcal{D}E=\mathcal{D}\widetilde{E}'$. Notice that already at this point the E redefinition implies, in terms of the gauge potential, an infinite gauge tail, which captures the possible gauge transformations. It is also convenient to make the rescaling $\widetilde{E}'=2e(c_V+1)\widetilde{E}$, with a constant Jacobian. In terms of the field $\widetilde{E}$, consider the change of variables

$$\partial_+\widetilde{E}=\frac{i}{4\pi}\beta^{-1}\partial_+\beta \quad , \quad \mathcal{D}\widetilde{E}=\mathrm{e}^{-ic_V\Gamma[\beta]}\mathcal{D}\beta \quad , \qquad (2.107)$$

introducing β, the analogous of a Wilson-loop variable. Now we use the identity (2.78) to transform the $\beta\Sigma$ interaction into terms that can be handled in a more appropriate fashion. Writing both steps separately, we have

$$\mathcal{Z}\,[\overline{\eta},\eta,i_\mu] = \int \mathcal{D}\widetilde{g}\,\mathrm{e}^{i\Gamma[\tilde{g}]}\mathcal{D}U\mathcal{D}(\text{ghosts})\,\mathrm{e}^{iS_{\text{ghosts}}}\int \mathcal{D}\Sigma\mathcal{D}\widetilde{E}\mathrm{e}^{-i(c_V+1)\Gamma[\Sigma]}$$
$$\times\mathrm{e}^{-(c_V+1)\mathrm{tr}\int \mathrm{d}^2z\partial_+\widetilde{E}\Sigma\partial_-\Sigma^{-1}-2ie^2(c_V+1)^2\int \mathrm{d}^2z\mathrm{tr}\,\widetilde{E}^2+i\int \mathrm{d}^2z\, i_\mu A_\mu - i\int \mathrm{d}^2z\mathrm{d}^2w\,\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)} \tag{2.108a}$$

in such a way that after replacement of (2.107) in (2.108a) and using (2.78) for $\Gamma[\beta\Sigma]$, we arrive at

$$\mathcal{Z}\,[\overline{\eta},\eta,i_\mu] = \int \mathcal{D}\widetilde{g}\,\mathrm{e}^{i\Gamma[\tilde{g}]}\mathcal{D}U\mathcal{D}(\text{ghosts})\,\mathrm{e}^{iS_{\text{ghosts}}}\int \mathcal{D}\Sigma\mathcal{D}\beta$$
$$\times\mathrm{e}^{-i(c_V+1)\Gamma[\beta\Sigma]+i\Gamma[\beta]+\frac{2ie^2(c_V+1)^2}{(4\pi)^2}\mathrm{tr}\int \mathrm{d}^2z[\partial_+^{-1}(\beta^{-1}\partial_+\beta)]^2+i\int \mathrm{d}^2z i_\mu A_\mu - i\int \mathrm{d}^2z\mathrm{d}^2w\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)} \tag{2.108b}$$

Defining the massive parameter $\mu = (c_V+1)e/2\pi$ and the field $\widetilde{\Sigma} = \beta\Sigma$, the partition function reads

$$\mathcal{Z}\,[\overline{\eta},\eta,i_\mu] = \int \mathcal{D}\widetilde{g}\,\mathrm{e}^{i\Gamma[\tilde{g}]}\mathcal{D}U\mathcal{D}(\text{ghosts})\,\mathrm{e}^{iS_{\text{ghosts}}}\int \mathcal{D}\widetilde{\Sigma}\,\mathrm{e}^{-i(c_V+1)\Gamma[\widetilde{\Sigma}]}$$
$$\times\int \mathcal{D}\beta\,\mathrm{e}^{i\Gamma[\beta]+\frac{\mu^2 i}{2}\mathrm{tr}\int \mathrm{d}^2z\,[\partial_+^{-1}(\beta^{-1}\partial_+\beta)]^2}\,\mathrm{e}^{i\int \mathrm{d}^2z\, i_\mu A_\mu - i\int \mathrm{d}^2z\,\mathrm{d}^2w\overline{\eta}(z)(i\not{D})^{-1}(z,w)\eta(w)}\quad, \tag{2.108c}$$

where now $A_+ = \frac{i}{e}U^{-1}\partial_+U$, $A_- = \frac{i}{e}(U^{-1}\beta^{-1}\widetilde{\Sigma})\partial_-(\widetilde{\Sigma}^{-1}\beta U)$, and we used the Haar invariance of the Σ measure.

Up to BRST constraints and source terms, the above generating functional factorizes in terms of a conformal theory for $\widetilde{g}$, which represents a gauge-invariant bound state of the fermions, of a second conformal field theory for $\widetilde{\Sigma}$, which represents some gauge condensate, and of an off-critically perturbed conformal field theory for the β field, which also describes a gauge-field condensate, which we interpret as an analogue of the Wilson-loop variable in view of the change of variables (2.107). The conformal field theory representing $\widetilde{\Sigma}$ has an action with a negative sign (see (2.108c)). Therefore we have to carefully take into account the BRST constraints in order to arrive at a positive metric Hilbert space. This is reminiscent of the commonly encountered negative metric states of gauge theories, and appeared already in the Schwinger model.[8] In that case the requirement that the longitudinal current containing the negative metric field vanishes, implies the decoupling of the unwanted fields from the physical spectrum. The only trace of such massless fields is the degeneracy of the vacuum. In that case, the chiral densities commute with the longitudinal part of the current, and it is possible to build operators carrying non-vanishing fermion number and chirality. They are, however, constant operators commuting with the Hamiltonian, and the ground state turns out to be infinitely degenerate. There are definite vacua superpositions where the above states are just phases – the so-called θ-vacua.

Notice that in eq. (2.108a) the only place where the charge shows itself is in the $\widetilde{E}^2$ term. The limit $e \to 0$ corresponds to a topological theory, since $\widetilde{E}$ integration would restrain the $F_{\mu\nu}$ field to be a pure gauge. This will be used later in connection with the string formulation. Moreover, we might also generalize such a term to $f(\widetilde{E})$, for an arbitrary function f not quadratic in $\widetilde{E}$. In such a case, possible in two dimensions, we arrive at a Landau–Ginzburg generalization (see section 4.4).

Reobtaining the $U(1)$ case

If we write the β field as an exponential, $\beta = \mathrm{e}^{2i\sqrt{\pi}\vartheta}$, we find for the β Lagrangian

$$L = \frac{1}{2}\left(\partial_\mu \vartheta\right)^2 - \frac{1}{2}\frac{e^2}{\pi}\vartheta^2 \quad , \tag{2.109}$$

which describes a bosonic excitation of mass

$$m_\vartheta = \frac{e}{\sqrt{\pi}} \quad , \tag{2.110}$$

which is well known from the Schwinger-model analysis. The remaining fields are massless excitations, and the full Lagrangian reads

$$L = \frac{1}{2}\left(\partial_\mu \vartheta\right)^2 - \frac{1}{2}\frac{e^2}{\pi}\vartheta^2 - \frac{1}{2}\left(\partial_\mu \eta\right)^2 + \frac{1}{2}\left(\partial_\mu \varphi\right)^2 \quad , \tag{2.111}$$

which describes the Schwinger model.

2.4 Strong coupling analysis

The 'tHooft analysis is well suited for the weak coupling limit of the theory, namely for very heavy quarks. For strong coupling there is a problem signalled by the presence of the tachyonic pole in the quark propagator for light quarks. Since the quarks should not appear asymptotically, the issue of the strong coupling remains. Indeed, the question of whether we have the screening/confinement picture characteristic of the Schwinger model is not yet clear.

Some authors tried to give an answer to such a question.[20,22] The only available method to study strongly coupled fields is bosonization. Bosonization of fermions in a representation of non-Abelian symmetry groups is trivial and leads to complicated σ-model interactions. Borrowing methods used in the Abelian case, one is led to non-local terms, and the symmetry is not preserved. However, some calculations may still be performed, and it was used in this case to arrive at a generalized sine-Gordon interaction rendering some non-trivial results. Abelian bosonization methods have been used for the first time in non-Abelian gauge theories in ref. [48].

The analysis of gauge theories is rather complex by itself. First there is the question of gauge fixing. In fact there are procedures which simplify our work enormously. In ref. [49] it has been proved that it is possible to choose a gauge where the electric field is diagonal and traceless, and where the diagonal gauge fields are in the Coulomb gauge. In such a case the ghosts decouple.[49] We shall use such a procedure.

The Lagrangian

$$\mathcal{L} = -\frac{1}{2}F^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\,\partial\!\!\!/ + e\,A\!\!\!/)\psi - m\overline{\psi}\psi \quad (2.112)$$

leads to canonical momenta given by the expressions

$$\Pi_0 = 0 \quad , \quad \Pi_1 = F^{01} \quad , \quad \text{and} \quad \Pi_\psi = -i\psi^+ \quad . \quad (2.113)$$

The Hamiltonian is easily computable, and one finds

$$H = -\frac{1}{2}\Pi_1^2 + i\overline{\psi}\gamma_1\partial_1\psi + m\overline{\psi}\psi - eA^\mu\overline{\psi}\gamma_\mu\psi - \Pi_1(\partial_1 A_0 + ie[A_0, A_1]) \quad . \quad (2.114)$$

The Gauss law, which is a consequence of the constraint ($\Pi_0 = 0$), is written

$$\partial_1\Pi_1 - ie[A_1, \Pi_1] - e\overline{\psi}\gamma_0\psi = 0 \quad . \quad (2.115)$$

We define new fields e_i in terms of the canonically conjugated momenta, which are diagonal in the group index

$$\sqrt{\pi}\Pi_1^{ii} = e_i - \frac{1}{N}\sum_{i=1}^{N} e_i \quad , \quad (2.116a)$$

from which it follows that

$$\sum_i \left(\Pi_1^{ii}\right)^2 = \frac{N}{\pi}\sum_{i,j}(e_i - e_j)^2 \quad . \quad (2.116b)$$

This is very useful due to the gauge fixing discussed before eq. (2.112): non-diagonal momenta Π_1^{ij}, for $i \neq j$, can be chosen as zero. With such a choice, the Gauss law (2.115) is very simple for off-diagonal terms, and determines A_1. One finds

$$-\frac{ie}{\sqrt{\pi}}A_1^{ik}(e_k - e_i) - e\overline{\psi}_i\gamma_0\psi_k = 0 \quad , \quad i \neq k \quad . \quad (2.117)$$

Therefore $A_1^{ik} = \frac{\sqrt{\pi}i}{e_k - e_i}\overline{\psi}_i\gamma_0\psi_k$; upon its insertion into the Hamiltonian (2.114), using (2.116b) and the Gauss law explicitly for the last term, we obtain

$$H = -\frac{N}{2\pi}\sum_{i,j}(e_i - e_j)^2 + i\overline{\psi}\gamma_1\partial_1\psi + m\overline{\psi}\psi - ie\sqrt{\pi}\,\overline{\psi}_i\gamma_0\psi_k\frac{1}{e_k - e_i}\overline{\psi}_k\gamma_1\psi_i \quad . \quad (2.118)$$

The last term may be Fierz-transformed in order to obtain only the diagonal terms of the current, by means of the formula

$$\overline{\psi}_i\gamma_0\psi_j\overline{\psi}_j\gamma_1\psi_i = -\frac{1}{2}\left[\overline{\psi}_i\gamma_0\gamma^\mu\gamma_1\psi_i\overline{\psi}_j\gamma_\mu\psi_j + \overline{\psi}_i\gamma_0\gamma_5\gamma_1\psi_i\overline{\psi}_j\gamma_5\psi_j + \overline{\psi}_i\gamma_0\gamma_1\psi_i\overline{\psi}_j\psi_j\right]$$
$$= -\frac{1}{2}\left[\overline{\psi}_i\gamma_1\psi_i\overline{\psi}_j\gamma_0\psi_j + \overline{\psi}_i\gamma_0\psi_i\overline{\psi}_j\gamma_1\psi_j + \overline{\psi}_i\psi_i\overline{\psi}_j\gamma_5\psi_j - \overline{\psi}_i\gamma_5\psi_i\overline{\psi}_j\psi_j\right] \quad . \tag{2.119}$$

The first two terms of the above equation vanish, since they are symmetric for $i \leftrightarrow j$, while we sum over i,j, multiplying by $\frac{1}{e_i - e_j}$, which is antisymmetric; using further such symmetry properties we obtain

$$ie\sqrt{\pi}\sum_{i,j} M_+^i M_-^j \frac{1}{e_i - e_j} \quad , \tag{2.120}$$

where $M_\pm^i = \overline{\psi}_i(1 \pm \gamma_5)\psi_i$.

At this point one uses the Abelian bosonization procedure, by means of which the diagonal part of the current is written as

$$j_\mu^i = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\phi^i \quad . \tag{2.121}$$

Using the fact that Π_1 is diagonal, in the diagonal part of the Gauss law, and substituting (2.120) therein for the current, one identifies ϕ^i and e_i, that is $e_i = \phi^i$.

The last term in eq. (2.118) depends on the regularization procedure employed to define the severe divergences appearing in the product of fields as given, but it is nevertheless clear that such a term depends on the difference $\phi_i - \phi_j$. Baluni[20] was able to obtain an integral representation for such a term, but we just write the bosonic version of the Hamiltonian as

$$H = \sum_i\left[\frac{1}{2}\Pi_{\phi_i}^2 + \frac{1}{2}(\phi_i')^2 + m'\cos\phi_i\right] - \frac{N}{2\pi}\sum_{i,j}(\phi_i - \phi_j)^2 + \sum_{i,j} f(\phi_i - \phi_j) \quad , \tag{2.122}$$

where Π_{ϕ_i} is the momentum canonically conjugated to the bosonized field ϕ_i, $f(\phi)$ is a function of the differences and contains regularization-dependent constants. Notice that the terms $\Pi_{\phi_i}^2$ and $\phi_i'^2$ correspond, in the Hamiltonian procedure, to the bosonized version of the kinetic term. Recently,[142] it has been argued that the bosonized field ϕ_i fulfills an equation of motion, in the presence of (background) gauge fields, which corresponds to the Bethe–Salpeter equation (2.35). In such a case, the integral term arises from the Vandermonde determinant. We wish here to obtain also information about the bosonic spectrum in the strong coupling limit, using the Hamiltonian (2.122). It is useful to define a new basis of fields as

$$\varphi = \frac{1}{\sqrt{N}}\sum \phi_i \text{ and } \quad \chi_l = \sum_{i=1}^N t_{ii}^l\phi_i \quad , \quad \text{with inverse} \quad \phi_{N-i} = \frac{\varphi}{\sqrt{N}} + \sum M_{ij}\chi_{N-j} \quad , \tag{2.123}$$

where $l = 1, \cdots, N$ and the above constants are

$$t^l_{ii} = \begin{cases} 0 & i > l+1 \quad , \\ -\sqrt{l(l-1)} & i = l \quad , \\ 1/\sqrt{l(l-1)} & i < l \quad , \end{cases} \tag{2.124a}$$

and

$$M_{ij} = \begin{cases} 0 & i < j-1 \quad , \\ -\sqrt{\frac{N-j}{N-j+1}} & i = j-1 \quad , \\ \frac{1}{\sqrt{(N-j)(N-j+1)}} & i > j-1 \quad . \end{cases} \tag{2.124b}$$

In terms of the above fields one has

$$\sum_{i,j} (\phi_i - \phi_j)^2 = N \sum \chi_l^2 \quad , \tag{2.125}$$

and the last term in the Hamiltonian does not depend on the "centre-of-mass" coordinate φ. In the strong coupling limit φ has a very small mass compared to the $(N-1)$-plet described by the field χ_l. For the cosine term one has

$$\cos\left[2\sqrt{\pi}\left(\frac{\varphi}{N} + \sum M_{ij}\chi_j\right)\right] = \cos\left[2\sqrt{\frac{\pi}{N}}\varphi\right] \prod_j \cos\left(2\sqrt{\pi} M_{ij}\chi_j\right) + \sin \text{terms} \quad . \tag{2.126}$$

For very massive χ fields (strong coupling limit) we shall put $\chi \sim 0$, and the terms in sine disappear. According to the usual renormalization procedure for the cosine terms, it must be renormalized according to the mass of the fields (see refs. [8] and [12]), which leads to

$$m\Lambda \left(\frac{e\sqrt{\pi}}{\Lambda}\right)^{\frac{N-1}{N}} \cos\left[2\sqrt{\frac{\pi}{N}}\varphi\right] \quad , \tag{2.127}$$

where Λ is an arbitrary renormalization parameter, providing φ with an interaction of the form defined by the Hamiltonian

$$H = N_{m'}\left[\frac{1}{2}\Pi^2_\varphi + (\partial_1\varphi)^2 - {m'}^2 \cos\left(2\sqrt{\frac{\pi}{N}}\varphi\right)\right] \quad , \tag{2.128}$$

where

$${m'}^2 = \left\{ Nm \left(\frac{e}{\sqrt{\pi}}\right)^{\frac{N-1}{N}}\right\}^{\frac{2N}{2N-1}} \quad , \tag{2.129}$$

and $N_{m'}$ is a normal product with respect to the renormalized mass m'.

The baryon, interpreted as a soliton with quark number N, thus has a mass

$$M_B = \frac{2}{\pi} 2\sqrt{\frac{\pi}{N}} m'(2N-1) \underset{N\to\infty}{\sim} \frac{8}{\sqrt{\pi}}\sqrt{\frac{me}{\sqrt{\pi}}} N \quad . \tag{2.130}$$

Therefore it vanishes for small quark mass, in accordance with the idea that its mass approaches zero in the chiral limit, and such baryons are reminiscent of Goldstone states. Baryonic masses may also be interpreted as soliton–antisoliton bound states of the sine-Gordon field. Further properties of the strong coupling limit can be found in ref. [50].

2.5 A Lagrangian realization of the coset construction

As we have seen, fermionic gauge theories are naturally written in terms of gauged WZW theories. These in turn provide a Lagrangian realization of the coset construction.[51,52]

For a moment we delete the "mass term" μ in (2.108c). We are left with a gauged WZW theory as explained in section 2.3. The pure WZW functional is invariant under a $G \times G$ symmetry transformation given by

$$g(x^+,x^-) \to G(x^-)g(x^+,x^-)G_+(x^+) \quad . \tag{2.131}$$

In general, the anomaly-free vector subgroup $H \subset G\times G$ (in the QCD_2 case, H corresponds to G) can be gauged by adding the term

$$\frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x \left[e^2A_+A_- - e^2A_+gA_-g^{-1} + ieA_-g^{-1}\partial_+g + ieA_+g\partial_-g^{-1}\right] \quad . \tag{2.132}$$

Such a gauging procedure introduces constraints in the theory.[53] In order to understand this point in more detail, we have to consider the effect of the ghost sector. In general, ghosts are implemented by considering a gauge-fixing function $\mathcal{F}(A)$ and introducing a factor

$$\det\left(\frac{\partial\mathcal{F}}{\partial A_\mu}\frac{\partial A_\mu}{\partial\epsilon}\right)\delta\left(\mathcal{F}(A)\right) \tag{2.133}$$

in the partition function, where ϵ is the gauge parameter. However, if we are to render explicit the conformal content of the theory, it is more useful here to represent all possible chiral determinants in terms of ghost integrals. The reparametrization invariance is thus explicit and one can verify that the gauge-fixing procedure, as outlined above, and which is more frequently used in the gauge-field literature, is trivial in the sense that one is led to a unit Faddeev–Popov determinant.

Therefore we assume that ghosts are introduced by writing determinants in terms of ghost systems decoupled from the gauge fields by a chiral rotation, a procedure which is possible in two-dimensional space-time. This is equivalent to writing all determinants as

$$\det\nabla_+ = \mathrm{e}^{-ic_V\Gamma[U]}(\det\partial_+)^{c_V} \quad , \quad \det\nabla_- = \mathrm{e}^{-ic_V\Gamma[V]}(\det\partial_-)^{c_V} \quad , \tag{2.134}$$

and substituting the free Dirac determinant in terms of ghosts as

$$(\det\partial_+)^{c_V} = \int \mathcal{D}b_{--}\mathcal{D}c_+\,\mathrm{e}^{i\mathrm{tr}\int\mathrm{d}^2x\, b_{--}\partial_+c_+} \quad , \tag{2.135}$$

$$(\det\partial_-)^{c_V} = \int \mathcal{D}b_{++}\mathcal{D}c_-\,\mathrm{e}^{i\mathrm{tr}\int\mathrm{d}^2x\, b_{++}\partial_-c_-} \quad . \tag{2.136}$$

In fact the determinant of the Dirac operator does not factorize as in eq. (2.134) because of the regularization ambiguity. At every step, one has to ensure vector current conservation. Such determinants cancel out by changing some of the variables (as in eq. (2.107)) but do not cancel in (2.108c), from which we are led to the contribution

$$\int \mathcal{D}b_{--}\mathcal{D}b_{++}\mathcal{D}c_+\mathcal{D}c_-\, \mathrm{e}^{i\mathrm{tr}\int \mathrm{d}^2x\,(b_{++}\partial_- c_- + b_{--}\partial_+ c_+)} \quad . \tag{2.137}$$

Although decoupled at the Lagrangian level, such terms are essential due to constraints arising in the zero total conformal charge sector, and lead to BRST constraints on physical states. The constraints are obtained in a system of interacting conformally-invariant sectors $(g, \Sigma, b_{++}, b_{--}, c_+, c_-)$ described by the partition function

$$\mathcal{Z} = \int \mathcal{D}g\mathcal{D}\Sigma\mathcal{D}b_{++}\mathcal{D}b_{--}\mathcal{D}c_-\mathcal{D}c_+\, \mathrm{e}^{ik\Gamma[g]-i(c_V+k)\Gamma[\Sigma]+i\mathrm{tr}\int \mathrm{d}^2x\,(b_{++}\partial_- c_- + b_{--}\partial_+ c_+)} \quad . \tag{2.138}$$

Such a construction is a particular one out of a general equivalence[51,53] between the algebraic construction of G/H coset models, and an H-gauged WZW theory on a group manifold G.

Indeed, starting out of the WZW functional $\Gamma[G]$ one can gauge the anomaly-free vector subgroup H by means of the gauged WZW functional

$$\Gamma[g, A] = \Gamma[g] + \frac{1}{4\pi}\mathrm{tr}\int \mathrm{d}^2x\, \left\{-iA_+\partial_- g g^{-1} + iA_- g^{-1}\partial_+ g - A_+ g A_- g^{-1} + A_+A_-\right\} \quad , \tag{2.139}$$

where A_μ belongs to the adjoint representation of H; we rewrite the gauge field in terms of the potentials as in (2.59) and use the Polyakov–Wiegmann identity (2.78), regainning (2.87). Taking into account the ghost system and the Jacobian, and moreover using the gauge $V = 1$, we arrive at

$$Z = \int \mathcal{D}g\mathcal{D}h\mathcal{D}b_{++}\mathcal{D}b_{--}\mathcal{D}c_-\mathcal{D}c_+\, \mathrm{e}^{ik\Gamma[g]-i(k+c_H)\Gamma[U]+i\mathrm{tr}\int \mathrm{d}^2x(b_{++}\partial_- c_- + b_{--}\partial_+ c_+)} \quad , \tag{2.140}$$

where c_H is the quadratic Casimir for the subgroup H and arises from the Jacobian induced by (2.59). As before, the partition function factorizes into non-interacting sectors at the Lagrangian level. However, the BRST condition couples them. The reason is the existence of constraints in the theory. They can be derived by coupling all the Lagrangian fields to an external gauge field by means of the minimal coupling such as exemplified by (2.139), with $A_+^{\mathrm{ext}} = \partial_+ U^{\mathrm{ext}} U^{\mathrm{ext}\,-1}$, $A_-^{\mathrm{ext}} = \partial_- V^{\mathrm{ext}} V^{\mathrm{ext}\,-1}$, which again, by means of eq. (2.78), leads to a partition function independent of U^{ext} and V^{ext} due to the vanishing of the total central charge. By computing the derivative of the partition function with respect to A_+ and A_- we obtain currents which must vanish for consistency.

Let us prove this assertion. The interaction of the fields from the WZW theory with such external gauge fields is equivalently obtained from (2.87), that is

$$ik\Gamma[g, A] = ik\Gamma[U_{\mathrm{ext}} g V_{\mathrm{ext}}] - ik\Gamma[U_{\mathrm{ext}} V_{\mathrm{ext}}] \quad , \tag{2.141}$$

$$-i(c_H + k)\Gamma[\Sigma, A] = -i(c_H + k)\Gamma[U_{\mathrm{ext}}\Sigma V_{\mathrm{ext}}] + i(c_V + k)\Gamma[U_{\mathrm{ext}} V_{\mathrm{ext}}] \, , \tag{2.142}$$

and

$$i\mathrm{tr}\int \mathrm{d}^2x\,[b_{++}\overset{\mathrm{ext}}{D_-}c_- + b_{--}\overset{\mathrm{ext}}{D_+}c_+] = i\mathrm{tr}\int \mathrm{d}^2x\,[b_{++}V_{\mathrm{ext}}\partial_-(V_{\mathrm{ext}}^{-1}c_-) + b_{--}U_{\mathrm{ext}}^{-1}\partial_+(U_{\mathrm{ext}}c_+)], \tag{2.143}$$

where k is the central charge. We recall that in section 2.3 we had $k = 1$. In the first two cases, namely eqs. (2.141) and (2.142), the invariance of the Haar measure permits a change of variables as $\widetilde{g} = U_{\mathrm{ext}}gV_{\mathrm{ext}}$, $(\mathcal{D}\widetilde{g} = \mathcal{D}g)$ and $\widetilde{\Sigma} = U_{\mathrm{ext}}\Sigma V_{\mathrm{ext}}$ $\left(\mathcal{D}\Sigma = \mathcal{D}\widetilde{\Sigma}\right)$, while in the latter case (2.143) a chiral rotation can be done, leaving back the free ghost system and a WZW term, $c_H\Gamma[U_{\mathrm{ext}}V_{\mathrm{ext}}]$. Therefore, the $\Gamma[U_{\mathrm{ext}}V_{\mathrm{ext}}]$ term cancels, owing to the balance of central charges, and the partition function does not depend on the external gauge fields. This implies, in particular, that the functional derivative of the partition function with respect to the external gauge fields vanishes, and therefore

$$\left.\frac{\delta\mathcal{Z}(\overset{\mathrm{ext}}{A_+},\overset{\mathrm{ext}}{A_-})}{\delta\overset{\mathrm{ext}}{A_+}}\right|_{\overset{\mathrm{ext}}{A_+},\overset{\mathrm{ext}}{A_-}=0} = 0 = \left.\frac{\delta\mathcal{Z}(\overset{\mathrm{ext}}{A_+},\overset{\mathrm{ext}}{A_-})}{\delta\overset{\mathrm{ext}}{A_-}}\right|_{\overset{\mathrm{ext}}{A_+},\overset{\mathrm{ext}}{A_-}=0} \quad , \tag{2.144}$$

which are equivalent, because of the minimal coupling (see eq. (2.89)), to the set of constraints

$$\left\langle kg^{-1}\partial_+g - (c_H+k)\Sigma^{-1}\partial_+\Sigma - 4\pi[b_{++},c_-]\right\rangle = 0 = \left\langle J_{+g} + J_{+\Sigma} + J_{+\mathrm{ghost}}\right\rangle \tag{2.145}$$

and

$$\left\langle k\partial_-gg^{-1} - (c_H+k)\partial_-\Sigma\Sigma^{-1} - 4\pi[b_{--},c_+]\right\rangle = 0 = \left\langle J_{-g} + J_{-\Sigma} + J_{-\mathrm{ghost}}\right\rangle \quad . \tag{2.146}$$

Each of the above currents satisfies a current algebra with a central charge. One can build up a BRST charge Q as

$$Q^{(+)} = \sum :c^i_{-\,-n}\left(J^i_{+g\,n} + J^i_{+\Sigma\,n}\right): -\frac{i}{2}f^{ijk}\sum :c^i_{-\,-n}b^j_{++\,-m}c^k_{-\,n+m}: \quad , \tag{2.147}$$

where the indices i, j, k refer to the adjoint representation of the symmetry group, f^{ijk} are the structure constants, and the mode expansion of the fields reads

$$c^i_- = \sum c^i_{-\,n}x^{+\,-n} \quad , \tag{2.148a}$$

$$b^i_{++} = \sum b^i_{++\,n}x^{+\,-n-1} \quad , \tag{2.148b}$$

$$J^i_{+g,\Sigma} = \sum \left(J^i_{+g,\Sigma}\right)_n x^{+\,-n-1} \quad . \tag{2.148c}$$

As it should, the charge (2.147) is nilpotent: ${Q^{(+)}}^2 = \frac{1}{2}\{Q^{(+)},Q^{(+)}\} = 0$. This implies that the above system is a set of first-class constraints (a similar set of constraints $Q^{(-)}$ is obtained for J_-, b_{--}, and c_+).

The stress tensor can be computed in terms of such currents, and we have three contributions, namely $T^{\rm tot}(z) = T_g(z) + T_\Sigma(z) + T^{\rm ghost}(z)$, that is

$$T^{\rm tot}(x^+) = \frac{1}{k + c_G} : \left(J_+^g\right)^2 : - \frac{1}{k + c_H} : \left(J_+^\Sigma\right)^2 : - : b_{++}^i \partial_+ c_-^i = T^G + T^H + T^{\rm gh} \quad . \quad (2.149)$$

The central charges corresponding to the right-hand side of eq. (2.149) can be computed, being respectively $c(G,k)$, $c(H, -k - c_H)$ and $c_{\rm gh} = -2d_H$; adding them we obtain the total central charge which is

$$\begin{aligned} c^{\rm tot} &= \frac{2kd_G}{2k + c_G} + \frac{2(-k - c_H)d_H}{2(-k - c_H) + c_H} - 2d_H \\ &= \frac{2kd_G}{2k + c_G} - \frac{2kd_H}{2k + c_H} \quad . \end{aligned} \quad (2.150)$$

Therefore the total central charge coincides with the one found using the coset construction. The energy momentum tensor, when written in the form

$$T^{\rm tot} = T^G - T^H + T' = T^{\rm coset} + T' \quad , \quad (2.151)$$

is such that one can prove that the central charge c' corresponding to T' vanishes, while T' itself commutes with $T^{\rm coset}$. The unitary representations of T' are thus trivial,[54] and there is a strong equivalence with the coset construction, once the unitarity of the physical spectrum is established. This has been done in ref. [53].

The gauged WZW model is equivalent to the coset construction of G/H conformal field theories. The physical subspace is generated by a product of matter and ghost sectors, obeying the equation $Q|{\rm phys}\rangle = 0$. This also solves the problem of the sector with negative central charge, which should not be considered separately, being coupled through the BRST condition. Had we no such condition we would expect problems concerning negative metric states. Therefore we cannot consider each sector separately.

In the case of the inclusion of QCD_2 in such a scheme, we shall see that there are further constraints. Although the new constraints seem to be of the first-class type when considered alone, there is a combination that is second-class due to the cancellation of the ghost contribution. Therefore, in the case of QCD_2 we have to deal with a Dirac quantization procedure of second-class constraints![55]

However, we shall see (in section 5) that several interesting properties, characteristic of the model, as well as part of the conformal structural relations, still hold, and the QCD_2 problem can be understood as an integrable perturbation of a coset construction of conformal field theory.

2.6 Chiral interactions

Fermionic gauge theories with chiral coupling of the fermions to the gauge field exhibit an anomaly in the covariant divergence of the external field gauge current, $J^a_{\mu,ch}(x|A)$, which is referred to as the non-Abelian anomaly.[56] Such an anomaly implies, at first sight, an inconsistency with the gauge field equations of motion, and a breakdown of gauge invariance. The requirement that the physical particles belong to safe representations, where the anomaly has a vanishing group theoretic factor leads to the prediction of a certain number of quarks (balancing the leptons). Although such predictions seem to be successful, the study of anomalous theories reveals a consistent field theoretic structure.[8]

The issue can be better understood from the fact that because of the non-invariance of the fermionic measure under chiral transformations, all dynamical variables are observable, and gauge fixing is neither required nor allowed. However, one can follow the line drawn by refs. [57, 58], introducing the unity in the "Faddeev–Popov form" (see chapter 13 of ref. [8] for details), namely

$$\Delta_{\mathcal{F}}[A]\int \mathcal{D}g\,\delta[\mathcal{F}(A^g)] = 1 \quad , \tag{2.152}$$

where $\mathcal{F}$ is an arbitrary gauge fixing function. We are lead to a gauge-invariant formulation in terms of a larger set of fields, where the partition function is

$$Z[J,\eta,\overline{\eta}] = \int \mathcal{D}g\mathrm{d}\mu[A]\mathcal{D}\psi\mathcal{D}\overline{\psi}\,\mathrm{e}^{iI[A,g,\psi,\overline{\psi}]+i\int[J\,{}^gA+\overline{\eta}^g\psi+{}^g\overline{\psi}\eta]} \quad , \tag{2.153}$$

and the gauge measure is now given by the usual Faddeev–Popov procedure,

$$\mathrm{d}\mu[A] = \mathcal{D}A_\mu\Delta_{\mathcal{F}}[A]\delta(\mathcal{F}(A)) \quad , \tag{2.154a}$$

and the notation is defined as

$${}^gA_\mu \equiv A_\mu^{g^{-1}} = g\left(A_\mu + \frac{i}{e}\partial_\mu\right)g^{-1} \quad , \tag{2.154b}$$

$$I[A,g,\psi,\overline{\psi}] = S[A,\psi,\overline{\psi}] + \alpha_1(A,g^{-1}) \quad ; \tag{2.154c}$$

$\alpha_1(A,g^{-1})$ is the 1-cocycle defined by the Wess–Zumino consistency condition. The usual "unitary gauge" $g=1$, leads to the "ordinary" (in the sense of naïve) discussion of the theory. Such a gauge defines the so-called gauge-non-invariant formulation.[57--59] The "classical" equation of motion acquires a modification due to the 1-cocycle above and reads

$$\nabla_\mu F^{\mu\nu} + J^\nu + \frac{\delta\alpha_1}{\delta A_\nu} = 0 \quad , \tag{2.155}$$

displaying no inconsistency. Now the theory displays a quantum extended gauge symmetry. Both representations, namely the full gauge theory, and the theory at unitary gauge, are equivalent only after integration over the gauge field A_μ. In such a case, we can see that the previously mentioned inconsistency of the equations of motion disappears.

The integration of the fermionic sector may be performed, leading to a bosonized action, obtained integrating the fermions. The effective action reads

$$L_{\rm eff} = -\frac{1}{4}{\rm tr}\, F_{\mu\nu}F^{\mu\nu} + \Gamma^{(R)}[A] \quad , \tag{2.156}$$

where

$$i\Gamma^{(R)}[A] = \ln\ \det \left(i\, \partial\!\!\!/ + e\, A\!\!\!/ \frac{1-\gamma_5}{2}\right) \quad , \tag{2.157}$$

which is equal to the usual WZW action in the gauge $A_+ = 0$. However, we cannot apply the vector gauge symmetry as before to select a particular combination of both potentials, and we must allow for a regularization arbitrariness which, as shown by Jackiw and Rajaraman,[60] has the form of the square of the gauge field with an arbitrary coefficient (a). One therefore obtains a partition function given by

$$Z = \int \mathcal{D}A_\mu \mathcal{D}g\, {\rm e}^{iS_{\rm eff}[A,g]} \quad , \tag{2.158}$$

with

$$S_{\rm eff}[A,g] = \int {\rm d}^2x \left[-\frac{1}{4}{\rm tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{ae^2}{8\pi}A^\mu A_\mu \right] - \Gamma[g] - \frac{ie}{4\pi}\int {\rm d}^2x\, {\rm tr}\ \left[g^{-1}\partial_+ g A_- \right] \quad . \tag{2.159}$$

The Euler–Lagrange equations derived from above read

$$(g^{\mu\nu} - \epsilon^{\mu\nu})\,\partial_\mu(g^{-1}\partial_\nu g) + ie\,(g^{\mu\nu}+\epsilon^{\mu\nu})\,\nabla_\mu A_\nu = 0 \quad , \tag{2.160a}$$

$$\nabla_\mu F^{\mu\nu} + \frac{ae^2}{4\pi}A^\nu - \frac{ie}{4\pi}\,(g^{\nu\mu} - \epsilon^{\nu\mu})\,{\rm tr}\,(g^{-1}\partial_\mu g\, t) = 0 \quad , \tag{2.160b}$$

where $\nabla_\mu \chi = \partial_\mu \chi + [g^{-1}\partial_\mu g, \chi]$ is the adjoint covariant derivative. The canonical quantization may be performed using the Dirac method. In the Abelian case the theory simplifies. A full account of such developments is beyond the scope of the present review; it is presented in chapters 13 and 14 of ref. [8], to which we refer, as well as in the references presented therein.

3. Pure QCD$_2$ and string theory

3.1 Introduction

Quantum electrodynamics in four dimensions hit enourmous successes after the establishment of renormalized perturbation theory. Due to the smallness of the fine structure constant, perturbation theory led to results that could be tested experimentally, and the relative errors were ten orders of magnitude smaller than unity. The renormalization prescription, although very awkward, was later precisely defined in the mathematical sense, in such a way that all predictions were reliable. Important lessons were drawn for general quantum field theories off the perturbative scheme, such as the LSZ formalism[61] or the axiomatic approach.[62] However, dynamical calculations were restricted to perturbation theory, and results concerning the strong interactions remained unreliable. In particular, information about the spectrum of the theory was only accessible via approximative, often non-unitary schemes, as the Bethe–Salpeter equation in the ladder approximation. Weak interactions, although well described by perturbation theory, was known to be, in the case of the Fermi theory, ill-behaved in the high-energy domain. Therefore, quantum field theory fell into stagnation due to the difficulty in going beyond QED.

This motivated the works on the S-matrix theory, which subsequently played a dominant role.[63] It was thought that the bootstrap idea might substitute the dynamical principles and provide a more fundamental formulation, implying a very radical position towards conventional developments. This led to the concept of duality.[64] The explicit realization of such ideas was implemented by the Veneziano formula,[65] leading to full development of dual models. However, its predictive power was very low, due to the lack of an underlying dynamical principle, since the idea of having a Lagrangian was abandoned, or at least avoided. A number of incorrect results concerning the description of strong interactions led physicists to discard the dual model formulation. In particular, the high-energy behaviour of strong interactions is extremely well described by perturbative QCD if use is made of the RG and CS equations to impose perturbation theory.[1] On the other hand, string theory was reinterpreted as a theory of unified interactions.

Nonetheless the problem of strong interactions could still not advance for the understanding of low-energy phenomena, which should only be addressed using a non-perturbative method. In fact, several properties concerning hadrons are understandable by means of the concept of string-like flux tubes, consistent with linear confinement and linear Regge trajectories, properties derived also for the large-N limit of QCD_2 with fermions. As we have already observed, the $1/N$ expansion classifies the Feynman diagrams according to their topology. For a fixed topology, the sum of diagrams is like a sum of triangulated surfaces, with a structure very similar to string theory. But it is only recently that more concrete results were obtained with a direct relation between the large-N expansion of two-dimensional QCD without fermions and the string expansion. It should be clear that the string dynamics is not that of critical strings, or even Liouville strings, and that terms depending on the extrinsic geometry must be present.

3.2 Wilson loop average and large-N limit

The biggest difficulty in the analysis of strong interactions is the question of the large-distance behaviour, which cannot be understood by perturbation theory. In this way, important phenomena in the description of strong interactions, such as confinement, θ-vacua structure, as well as the bound-state spectrum, are poorly understood. On the other hand one knows that the high-energy theory is well described by perturbation theory, which is especially revigorated by the use of asymptotic freedom in connection with the Callan–Symanzik and regularization-group equations. In such a description quarks are called partons, and are essentially free, contrasting with the confinement picture. Such different behaviours point to different pictures.

There are some attempts to deal with the strong limit by means of the discretization of space-time, where one can obtain strong coupling expansion, treating the system by methods borrowed from statistical mechanics.

The fact that phases are described, in general, in terms of such local-order parameters facilitates the understanding of statistical systems displaying a complex phase behaviour. However, Elitzur[66] proved that every non-gauge-invariant local quantity has a vanishing expectation value at all temperatures. But a phase transition should be described by a parameter that would indicate spontaneous symmetry breaking and, due to the above results, such a local quantity does not exist, and local observables cannot be used to indicate the possibility of different phases.

There are however, in gauge theories, observables that are not local. Indeed, the Aharanov–Bohm effect shows that the exponential of the path-ordered integration of the gauge field is a meaningful physical quantity, and contains non-trivial information. Actually Feynman had already used phases as meaningful objects in quantum mechanics, in order to describe amplitudes. Even in the absence of the ("physical") electric and magnetic field, the effect of the phase $\mathrm{e}^{ie\int \mathrm{d}x^\mu A_\mu}$ can be measured in an electron wave function, which gives such a phase a physical meaning by itself.

The Wilson loop[67]

$$W[C] = \mathrm{tr}\; P\,\mathrm{e}^{ie\oint_C \mathrm{d}x^\mu A_\mu} \tag{3.1}$$

may be defined as a function of the loop C for any gauge theory, where P means that we have to order the group indices according to the loop location. It is colourless and will have a definite rôle in the description of confinement, as we shall see. The Wilson line attached to a fermion $\mathrm{e}^{ie\int_y^x A_\mu dx^\mu}\psi(y)$, has an amplitude interpretation; such a phase may describe the fermionic interaction.

It is not a very easy task to obtain information concerning Wilson-loop expectation values in four-dimensional gauge theories. A strong coupling expansion[68] is available in lattice gauge theories, but general results are very hard to obtain. The situation in two-dimensional Yang–Mills theories in the absence of dynamical fermions is drastically simplified, and by a clever choice of gauge the Wilson-loop expectation value can be exactly computed in terms of the gauge group parameters. Let us consider the Wilson-loop expectation value for the pure gauge theory

$$W[C] = \mathcal{N}^{-1}\mathrm{tr}\; P\int \mathcal{D}A_\mu\, \mathrm{e}^{ie\oint_C \mathrm{d}x^\mu A_\mu}\,\mathrm{e}^{-\frac{i}{4}\mathrm{tr}\int \mathrm{d}^2x\, F_{\mu\nu}F^{\mu\nu}} \quad . \tag{3.2}$$

If we consider the Coulomb gauge $A_0 = 0$, there is no ghost contribution; it partially cancels with the multiplicative normalization of the functional integral $\mathcal{N}^{-1}$, and we are left with

$$W[C] = \mathcal{N}'^{-1}\mathrm{tr}\; P\int \mathcal{D}A_1 \mathrm{e}^{ie\oint_C \mathrm{d}x^1\, \tau^a_{x_1} A^a_1}\mathrm{e}^{\frac{i}{2}\mathrm{tr}\int \mathrm{d}^2x\,(\partial_0 A_1)^2} \quad . \tag{3.3}$$

Here we have written explicitly the generators of the group τ^a. Introducing now the Green function $G(x,y)$ satisfying

$$\partial_{x_0}{}^2 G(x,y) = \delta^{(2)}(x-y) \quad , \tag{3.4}$$

with solution

$$G(x,y)=\frac{1}{2}|x^0-y^0|\,\delta(x^1-y^1)\quad ; \tag{3.5}$$

completing the square in the functional integral we find[69]

$$W[C]=\mathrm{tr}\ P\,\mathrm{e}^{\frac{i}{2}e^2\oint\oint \mathrm{d}x^1\mathrm{d}y^1\,\tau^a_{x_1}G(x,y)\,\tau^a_{y_1}}\quad . \tag{3.6}$$

Since the Green function is local in (x^1-y^1), the indices carried by $\tau^a_{x_1},\tau^a_{y_1}$ are unimportant, and we obtain their square, which is the quadratic Casimir, namely $C_2(R)=\tau^a\tau^a$. Therefore the P symbol can be immediately deleted, and the trace operation leads to the dimension of the representation of the gauge group under consideration. In formulas we write

$$W_R[C]=(\dim R)\ \mathrm{e}^{\frac{i}{2}e^2C_2(R)A(C)}\quad , \tag{3.7}$$

where R is the representation, and $A(C)$ the area enclosed by the loop C, obtained in

$$\oint \mathrm{d}x_1\oint \mathrm{d}y_1\,G(x,y)=2\times\frac{1}{2}|x_0-y_0|\oint \mathrm{d}x_1=|x_0-y_0||x_{\mathrm{init}}-x_{\mathrm{fin}}|=A\quad . \tag{3.8}$$

The fact that the result is given by the area is equivalent to saying that, while computing the Wilson loop, the potential at a point of the loop, which is due to the quark at other points, is proportional to the distance, signalling a confining situation. Were the Wilson-loop exponent is proportional to the perimeter, as is the case for a typical three-dimensional massless propagator instead of (3.5), we would have no confinement, as in weakly coupled QED in four dimensions. Such is the Wilson criterion for confinement,[67] and such is also the use of the Wilson loop as an order parameter of the theory.

The language used here, as well as the introduction of a lattice, makes it more natural to use Euclidean rather than Minkowski space. Thus until (and including) section 3.3 we work on Euclidean space.

As discussed in great detail in ref. [68] there is a natural interpretation of the theory on the lattice in terms of a string theory, which describes the flux tubes, and the string tension is defined by

$$k=\lim_{C\to\infty}-\frac{1}{A(C)}\ln W[C]\quad . \tag{3.9}$$

The non-confining case thus corresponds to $k=0$, which is the case if the Wilson loop displays the perimeter behaviour. It follows that the Wilson loop is a good parameter to verify whether we have a confinement phase or not.

This may be considered as the starting point of extremely important results relating pure QCD_2 and string theory. The large-N limit of two-dimensional pure Yang–Mills theory can be obtained as a consequence of some exact results concerning Wilson loops. First we have the fact that the expectation value of the Wilson loop in two dimensions was exactly computed in terms of the quadratic Casimir, eq. (3.7). We refer the reader to Appendix C as well as refs. [70-77] for more details concerning group theory. Here we shall repeat only some of the main results, recalling that given the representation, a $1/N$ expansion of such a result for the expectation of a Wilson loop when using a gauge group $G = SU(N)$ or $G = U(N)$ can be obtained from group theoretical values of the dimension and the Casimir of the representation, which is computed as a function of N and the lengths $n_1 > n_2 \cdots > n_r$ of the horizontal lines of the Young tableau $Y(R)$, defining the representation R. Given a representation defined by a Young tableau with rows $(n_1, \cdots, n_r)$, $\sum n_i = n$, the result[75] for $G = U(N)$ is:

$$C_2^{U(N)}(R) = Nn + \sum n_i(n_i + 1 - 2i) = Nn + \widetilde{C}_2^{U(N)}(R) \quad , \tag{3.10}$$

where $\sum\limits_{i=1}^{r} n_i = n$. For $SU(N)$ one substitutes n_i by $n_i - \frac{n}{N}$, and drop the first term (arising from $N \sum n_i$), obtaining

$$C_2^{SU(N)}(R) = Nn + \sum n_i(n_i + 1 - 2i) - \frac{n^2}{N} = Nn - \widetilde{C}_2^{SU(N)}(R) \quad . \tag{3.11}$$

For the sake of completeness we also recall that the dimension of the representation is

$$\dim R = \frac{\prod\limits_{i \le j \le N} (n_i - i - n_j + j)}{\prod\limits_{i \le j \le N} (i - j)} \quad . \tag{3.12}$$

Notice that when we use (3.12) and (3.11) in the expression for the Wilson loop (3.7), we end up, after a suitable redefinition of the charge ($e^2 = \alpha/N$), with a result that can be analyzed for large values of N. Such an analysis is still premature, and we should first rewrite the theory in a convenient way for string interpretation.

That step is realized from results arising from the lattice formulation.[68] The introduction of gauge fields on a lattice may be done analysing a general pure matter action of the type $\mathcal{L} \sim \varphi_i^\dagger \varphi_j$, where (i,j) are two sites on the lattice. In a case where the interaction is invariant under a symmetry group acting linearly on φ, such a symmetry can be raised to a local symmetry by introducing a gauge field taking values on the link (i,j) and by writing the interaction as $\mathcal{L} \sim \varphi_i^\dagger U_{ij} \varphi_j$, so that U transforms as $U \to gUg^{-1}$ under an element g of the symmetry group G, while $\varphi \to g\varphi$, and $\varphi^\dagger \to \varphi^\dagger g^{-1}$. A gauge field self-interaction can be taken as a trace of U on an elementary closed loop: a plaquette, namely $\mathcal{L}_U \sim \mathrm{Re}\,(\mathrm{tr}\; U_{ij} U_{jk} U_{kl} U_{li})$.

Thus, in general, one works with group-valued objects defined on an elementary link U, after dividing the space in elementary plaquettes, with such links as edges. The field U will represent the gauge degrees of freedom, and in the continuum limit, where the size of the link goes to zero, we have $U = 1 + ieAa$.

The so-called Wilson action reproduces correctly the Yang–Mills theory in the continuum limit; it is advantageous for certain lattice computations, and can be described very easily in terms of group-valued elements. Indeed, if one considers a lattice, which we suppose here for simplicity to be a regular square lattice, each scalar field is defined on a site, while the natural definition of a gauge field is on a link, due to the vectorial character.

One thus takes U as above to describe the gauge field, and the action as given by the cyclic product of gauge fields as above, that is[68]

$$S_W = \sum_p \frac{N}{2\alpha a^2} \mathrm{tr}\ \left(U_p + U_p^\dagger\right) \quad . \tag{3.13}$$

The fact that the Wilson-loop average displays a string-like behaviour in the large-N limit, has been known for some time, due to the possibility of relating it to a sum over surfaces with minimal area. Thus representing Yang–Mills theory in terms of Wilson-loop averages is the corner-stone of its relation to string theory. But there is a further technical point, which in practice is a crucial device for performing some exact computations, which is the introduction of the heat-kernel action. We start discussing this latter aspect of the problem.

Using the lattice formulation, the Wilson loop is defined as the trace of the product of group-valued operators over the edges of a plaquette. Consider the (Euclidian) Wilson action, as given before. Loop averages, such as

$$W[U] = \int \mathrm{d}U\ \mathrm{e}^{-\frac{N}{2\alpha a^2}\mathrm{tr}\ (U+U^+)} \mathrm{tr}\ U \quad , \tag{3.14}$$

have been considered in the literature, see refs. [78–80]. In two dimensions Migdal[69] found a way to systematically integrate the above quantity over a given edge variable. In other words, in a partition function representation in terms of such quantities, one integrates out a link that is common to two plaquettes, in such a way that the action does not acquire modifications from such a procedure. This means that one has to modify the lattice action, keeping the continuum limit, arriving at a renormalization-group-invariant action. As a result of such requirements one arrives at the heat kernel action, to be properly defined below. Such an improved action is thus exact in the sense that it describes either small or large lattices, implying the important property of being almost independent of the type of triangulation of the two-dimensional world-sheet. One considers the Boltzman factor $Z[U, e^2, a^2]$ and first develops it in an expansion on characters of the group $\chi_R[U]$, which for a single plaquette of area a^2 reads

$$Z[U, e^2, a^2] = \sum_R f_a(R)\,(\dim R)\,\chi_R[U] \quad , \tag{3.15}$$

where the sum is taken over all representations R and $f_a(R)$ are the coefficients of the expansion. Such a series is the Fourier representation in terms of the group characters, being therefore very general, and is valid for any arbitrary area A. Therefore we are allowed to write

$$Z[U, e^2, A] = \sum_R f(R)\,(\dim R)\,\chi_R[U] \quad . \tag{3.16}$$

The heat kernel formulation is established once one knows the coefficients $f(R)$. We compute them by imposing that the product $Z[UL]\,Z[L^\dagger V]$ [6)], after integration over the

[6)] Here we shortened the notation, but the Boltzman factor $Z[U, e^2, a^2]$, denoted shortly by Z[U], is still a function of e^2 and a^2, consequently $Z[UV]$ is a function of the charge e^2 and the sum of both areas.

common link, namely after integrating out the variable L, is identical to the Boltzman factor $Z[UV]$. Therefore we have

$$Z[UV] = \int \mathcal{D}L \sum_{R,S} f(R)\, f(S)\, (\dim \mathrm{R})\, (\dim \mathrm{S})\, \chi_R[UL]\, \chi_R[L^\dagger V] \quad . \tag{3.17}$$

Integration over characters is a common procedure in group theory. The rules are summarized in Appendix C. Using eq. (C.12) we have

$$Z[UV] = \sum_R (f(R))^2\, (\mathrm{dimR})\, \chi[UV] \quad . \tag{3.18}$$

Such a procedure can be repeated indefinitely, and one can start out of a single plaquette of area a^2, ending up with a macroscopic area A. One thus obtains as a result

$$\mathrm{e}^{-S} = Z[U] = \sum_R (f(R))^{A/a^2}\, (\dim \mathrm{R})\, \chi[U] \quad . \tag{3.19}$$

The form of $f(R)$ must be such that it goes to unity as $a^2 \to 0$, that is

$$f(R) \sim 1 - a^2 \epsilon_R \quad ; \tag{3.20}$$

therefore, for a finite area $f(R) = \mathrm{e}^{-A\epsilon_R}$, and the Boltzman factor reads

$$Z[U] = \sum_R \mathrm{e}^{-A\epsilon_R}\, (\dim \mathrm{R})\, \chi[U] \quad . \tag{3.21}$$

The form of ϵ_R is obtained upon computation of the expectation value of the Wilson loop $W[C] = \mathrm{tr}_{\,C}\, U$, using $Z[U]$ as a Boltzman factor. The (Euclidian counterpart of the) result (3.7) should be reproduced. We use the fact that the Wilson loop, being the trace over the group-valued field, projects out a character from the expansion (3.19) and one obtains

$$\langle W[C] \rangle = \mathrm{e}^{-A\epsilon_R}\, (\dim \mathrm{R}) = (\dim \mathrm{R})\, \mathrm{e}^{-\frac{\alpha^2}{2N} C_2(R) A} \quad , \tag{3.22}$$

thus fixing $Z[U]$ as given by

$$Z[U] = \sum_R \mathrm{e}^{-\frac{\alpha^2}{2N} C_2(R) A} (\dim \mathrm{R})\, \chi_R[U] \quad . \tag{3.23}$$

The heat-kernel action is the corner-stone of all subsequent developments. It is not difficult to see that the heat-kernel action diagonalizes the Hamiltonian operator in a given representation. The Hamiltonian for the pure QCD_2 model corresponds to the square of the momentum operator, that is, in the temporal gauge

$$H = \frac{e^2}{2} \int \mathrm{d}x\, \frac{\delta^2}{\delta A_1^{a\,2}} \quad . \tag{3.24}$$

It acts on functionals of the loop operator on a compact space ($0 < t < L$), $U = P\,\mathrm{e}^{ie\int_0^L \mathrm{d}x\,A_1}$, as

$$H = \frac{e^2 L}{2}\mathrm{tr}\,\left(U\frac{\partial}{\partial U}\right)^2 \quad , \tag{3.25}$$

where we have considered a space of total length L. Now $\left(U\frac{\partial}{\partial U}\right)^a \chi_R(U) = \chi_R(T^a U)$, and the diagonalized Hamiltonian is just

$$H = \frac{1}{2}e^2 L C_2(R) \quad . \tag{3.26}$$

Before proceeding, notice that these issues are not inherent in the lattice formulation, but rather in the fact that gauge theory can be expressed in terms of loops averages, upon defining the matrix U_{xy} along an arbitrary contour C_{xy} connecting the points x and y as

$$U_{xy} = \mathrm{tr}\;P\,\mathrm{e}^{ie\int_{C_{xy}}\mathrm{d}x_\mu\,A^\mu(x)} \quad , \tag{3.27}$$

and for a closed contour C, the expectation value of a Wilson loop in the continuum is

$$\langle W[C]\rangle = Z^{-1}\int \mathcal{D}A_\mu\,\mathrm{e}^{iS}\mathrm{tr}\;P\,\mathrm{e}^{ie\oint \mathrm{d}x^\mu\,A_\mu(x)} \quad . \tag{3.28}$$

In particular, making small variations of the loop, with respect to its area, it is possible to obtain a differential equation obeyed by the Wilson loop described above, the Makeenko–Migdal equation.[81]

Such results may be generalized for the product of two neighbouring Wilson loops. We consider two neighbouring loops with a common link l, to which we associate the group element L. Suppose we have a loop (lp_1), and another $(p_2 l^\dagger)$, where $l^\dagger$ runs in the opposite direction to l. Integration over the link is

$$\begin{aligned} W[p_1\,p_2] &= \int \mathrm{d}L\;W[l\,p_1]\,W[p_2\,l^\dagger] = \\ &= \sum_{R_1}\sum_{R_2}(\dim R_1)(\dim R_2) f(R_1) f(R_2)\int \mathrm{d}L\,\chi_{R_1}[U_{p_1}L]\,\chi_{R_2}[L^\dagger U_{p_2}]\,. \end{aligned} \tag{3.29}$$

In view of the orthogonality relation of the characters (C.12) we obtain

$$W[p_1\,p_2] = \sum_R (\dim R)\,f(R)^2\,\chi_R[U_1 U_2] \quad . \tag{3.30}$$

The form of $f(R)$ was obtained before, that is $f(R) = \mathrm{e}^{-\frac{\alpha^2}{2N}C_2(R)A}$. Furthermore, it is not difficult to obtain, using such heat-kernel action, the partition function in the case of a genus g surface. We consider the particular case of a sphere with n holes, and cut it into parts without holes. Subsequently we integrate over the link variables used in the cutting procedure, as in Fig. 4.

Fig. 4: Cutting procedure.

We find

$$W_{A_1 A_2} = \sum_{R_1, R_2} \dim R_1 \dim R_2 \, \mathrm{e}^{-A_1 C_2(R_1) - A_2 C_2(R_2)} \int \mathrm{d}L_1 \mathrm{d}L_2 \chi_{R_1}[L_1 U_1 L_2 W_1] \chi_{R_2}[U_2 L_2^\dagger W_2 L_1^\dagger] \,. \tag{3.31}$$

The integration is now performed using again (C.12), and we find

$$W_{A_1 A_2} = \sum_R \mathrm{e}^{-\frac{\alpha^2}{2N}(A_1{+}_2) C_2(R)} \chi_R[U_1 U_2] \, \chi_R[W_1 W_2] \quad . \tag{3.32}$$

Notice the absence of one factor of $\dim R$ in such a integral. We can now proceed with the gluing procedure as before, attaching with further elements to glue over the link variables $U_1 U_2$. We have to take into account that the presence of a hole lowers by a unit the power of $\dim R$. Moreover we are left with a factor $\chi[W]$, where W represents the "Wilson loop" of the hole. We thus obtain, for genus g

$$W_A^{(g)} = \sum_R (\dim R)^{2-2g} \, \mathrm{e}^{-\frac{\alpha^2}{2N} C_2(R) A} \prod_1^h \chi_R[W_i] \quad , \tag{3.33}$$

where the handles can be obtained gluing holes and repeating the argument.

3.3 String interpretation

The partition function thus obtained can be expanded in powers of $1/N$, as is clear from its form. The idea developed by Gross,[70] to be reviewed in what follows, is that such an expansion is equivalent to a string theory expansion, where the string coupling is identified with $1/N$, and the string tension is related to the coupling constant ($\alpha = e^2 N$).

The general intuition comes from the simplicity of the geometrical properties of the pure Yang–Mills action. The Euclidian Yang–Mills partition function over a manifold $\mathcal{M}$ is

$$Z_{\mathcal{M}} = \int \mathcal{D} A^\mu \, \mathrm{e}^{-\frac{1}{4} \mathrm{tr} \int \mathrm{d}^2 x \sqrt{g} F_{\mu\nu} F_{\mu\nu}} \quad . \tag{3.34}$$

Here $\hat{g}$ is the determinant of the induced metric, i.e. $\hat{g} = \det g^{\alpha\beta} = \det \left(\frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} G_{\mu\nu} \right)$.

The dynamics is very simple in the absence of quarks, since gauge-vector fields in two dimensions have zero degrees of freedom, as a simple counting reveals; nevertheless the theory is non-trivial if $\mathcal{M}$ contains non-contractible loops. Indeed, if C is such a loop, the Wilson loop $\mathrm{tr}\, P\mathrm{e}^{ie \oint_C \mathrm{d}x^\mu A_\mu}$ cannot be gauged to unity. Gross[70] argued that this is due to the fact that in two dimensions the field strength can be defined in terms of a scalar field F by

$$F_{\mu\nu} = \epsilon_{\mu\nu} F \quad , \tag{3.35}$$

in terms of which the action is geometrically very simple:

$$S = \frac{1}{2} \mathrm{tr} \int \mathrm{d}^2 x\, F^2 \sqrt{\hat{g}} \quad , \tag{3.36}$$

being independent of the metric, except for the volume form. The model is invariant under area-preserving diffeomorphisms (W_∞). Therefore the resulting theory can only depend on the topology of the manifold, its area, and the parameters N and e, that is

$$Z_{\mathcal{M}} = Z_{\mathcal{M}}[G, N, e^2 A] \quad , \tag{3.37}$$

where G is the genus of the target manifold.

The above partition function could be mapped to the partition function of a string theory with target space $\mathcal{M}$. Such a conjecture reads

$$\begin{aligned} \ln Z_{\mathcal{M}}[G, N, e^2 A] &= Z_{\mathcal{M}}^{\mathrm{string}}[g_{st} = 1/N, \alpha = e^2 N] \\ &= \sum_{\mathrm{genus} \equiv g} (g_{st})^{2g-2} \int \mathcal{D}x^\mu \mathrm{e}^{-\int \mathrm{d}^2 \xi \sqrt{\hat{g}} + \cdots} \quad , \end{aligned} \tag{3.38}$$

where the Nambu action is used to define the string theory. In such a relation we have to define the theory on the right-hand side and relate the genera g and G, in such a way as to obtain equality. In fact, as we will see later, the Nambu action turns out to be only part of the full story. Indeed, for vanishing area the right-hand side is a topological theory, and the Nambu action will describe the effect of the area.[73] We do not consider the topological theory; in any case, the action appearing in the right-hand side will not be considered dynamically. Thus the dots in the right-hand side of eq. (3.38) are momentarily not relevant.

The Nambu–Goto action is certainly invariant under area-preserving diffeomorphism, but it is very difficult to be quantized. The Polyakov action on the other hand also displays a W_∞ symmetry, but this is realized non-linearly. Moreover, besides the difficulty of quantizing the Polyakov action, there are further facts that make the direct use of the Nambu–Goto theory more appealing. Indeed, the question of the singularity of the maps defining the embedding of the (two-dimensional) world-sheet into a two-dimensional target space is clearly treated in the case of the Nambu–Goto theory, while such singularities are unseen in the conformal gauge. In fact, the area of the surface described by the string is only non-trivial due to folds. The Nambu–Goto action for non-singular maps (non-vanishing Jacobians $|\partial x^\mu/\partial\xi^\alpha|$) is a topological number, measuring how many times one covers the target space. This raises doubts about the equivalence with the usual Liouville description of non-critical strings for two-dimensional target spaces if one does not take folds into account, since in that case one uses a conformal gauge. The singularities would presumably describe the sources of the theory. Moreover, string theory usually contains also graviton and dilaton fields, a nuisance for the string theory interpretation of QCD_2, since the latter does not contain those fields. The absence of folds may be a cure of such a problem, since we have to include terms in the extrinsic geometry forbidding them; this would presumably also prevent gravitons and dilatons, which is necessary, since this is a theory of strong interactions without gravity. This mechanism is also a way to prevent the tachyon, which is a centre-of-mass degree of freedom; but since there are no propagating particles due to the absence of maps with zero winding number, tachyons disappear as well.

Now, given the partition function of the pure Yang–Mills theory, performing the $1/N$ expansion is just a matter of computation of group theoretical factors expanded in a Taylor series in $1/N$, which is technically feasible. One thus has to interpret the terms by relating them to a sum over geometric objects.

$1/N$ expansion of the Yang–Mills partition function

In order to keep track of several terms and be able to interpret them, it is useful to rewrite the quadratic Casimir eigenvalues as[75]

$$\frac{1}{N}C_2^{U(N)}(R) = \sum_i n_i + \frac{1}{N}\widetilde{C}(R) \quad , \tag{3.39a}$$

$$\frac{1}{N}C_2^{SU(N)}(R) = \frac{1}{N}C_2^{U(N)}(R) - \frac{1}{N^2}\left(\sum_i n_i\right)^2 \quad , \tag{3.39b}$$

$$\widetilde{C}(R) = -\,\widetilde{C}(\overline{R}) \quad , \tag{3.39c}$$

$$\left(\dim_{-N}\overline{R}\right)^2 = \left(\dim_N R\right)^2 \quad , \tag{3.39d}$$

where $\overline{R}$ is the representation conjugate to R and we made the N dependence of the dimension of the representation explicit.

The exponent can be expanded as a series in $1/N$, with the result

$$\mathrm{e}^{-\frac{\alpha A}{N}C_2(R)} = \sum_{i,j}\mathrm{e}^{-\alpha A n}\frac{1}{i!}\left(-\frac{\alpha A}{N}\widetilde{C}(R)\right)^i\frac{1}{j!}\left(-\frac{n^2}{N^2}\right)^j \quad , \tag{3.40}$$

where the last term drops out in the case where the group is $U(N)$, instead of $SU(N)$. Moreover, summing over the representation and its conjugate one finds a factor

$$\sum\frac{1}{i!}\left(-\frac{\alpha A}{N}\widetilde{C}(R)\right)^i\left[\left(\dim_N R\right)^{2-2G} + (-1)^i\left(\dim_{-N}\overline{R}\right)^{2-2G}\right] \quad . \tag{3.41}$$

Only even powers of $1/N$ survive. This is a correct result for a theory of closed orientable strings.

Gross uses some usefull relations. Consider the dimension of the representation ($\dim R$), given by eq. (3.12), and the dimension (d_R) of the representation of the symmetric group of $n = \sum n_i$ objects,

$$d_{[n_1 \cdots n_r]} = n! \frac{\prod_{i \leq j \leq r} (h_i - h_j)}{\prod_{i \leq j \leq r} (i - j)} \quad , \quad h_i = n_i - i + N \quad . \tag{3.42}$$

One finds the relation

$$\dim R = \frac{d_R}{n!} \prod_1^r \frac{(N + n_i - i)!}{(N - i)!} \quad , \tag{3.43}$$

which is useful to find a $1/N$ expansion since

$$\frac{(N + n_i - i)!}{(N - i)!} = N^{n_i} \prod_{k=1}^{n_i} \left(1 + \frac{k - i}{N}\right) \quad , \tag{3.44}$$

where $k(i)$ runs over the columns (rows) of the Young tableau. The dimension is thus given by

$$\dim R = d_R \frac{N^n}{n!} \prod_v \left(1 + \frac{\Delta_v}{N}\right) \quad , \tag{3.45}$$

where Δ_v is, for each cell of the Young tableau, the column index minus the row index. Finally, as a function of N, we have the relation

$$|\dim \overline{R}| = d_R \frac{N^n}{n!} \prod_v \left(1 - \frac{\Delta_v}{N}\right) = |\dim R(-N)| \quad . \tag{3.46}$$

The large-N analysis of the pure Yang–Mills results has to be done in three qualitatively different cases, depending on the genus, due to the term $(\dim R)^{2-2G} \sim N^{-2n(G-1)}$. Indeed, for $G > 1$, there is a simplification, and the leading term in the $1/N$ expansion of the Yang–Mills partition function is

$$Z_G = \sum_{n=0}^{\infty} N^{-2n(G-1)} \mathrm{e}^{-n\alpha A} \sum_{\mathrm{rep}\, S_n} \left(\frac{n!}{f_r}\right)^{2(G-1)} \quad , \tag{3.47}$$

where $\mathrm{rep}\, S_n$ are the representations of the symmetric group S_n. There are corrections from the dimension of the group as well as from the Casimir eigenvalue. For genus $G = 1$, the torus, the result can be computed in closed form in the large-N limit. One has

$$\begin{aligned} Z_{G=1} &= \sum_R \mathrm{e}^{-\frac{\alpha A}{N} C_2(R)} = \sum_{l_1 \geq l_2 \geq \cdots \geq l_n \geq 0} \mathrm{e}^{-\alpha A \sum l_i} \\ &= \sum_{k_i = l_{i+1} - l_i \geq 0} \mathrm{e}^{-\alpha A \sum n k_n} = \prod_{l=1}^{N} \frac{1}{1 - \mathrm{e}^{-n\alpha A}} = \eta\left(\mathrm{e}^{-\alpha A}\right) \quad , \end{aligned} \tag{3.48}$$

that is, one obtains the Dedekind function.

If we consider the logarithm of the above result, which should, as conjectured, be interpreted as the string partition function, we find

$$Z_{str} = \ln \eta \left(e^{-\alpha A}\right) = \sum_n e^{-n\alpha A} \sum_{ab=n} (a+b) \quad . \tag{3.49}$$

The coefficient of $e^{-n\alpha A}$ counts the number of different maps of a torus onto a torus n times, where the two above cycles are winding a and b times around the two cycles of the target space torus.

The specific case of the sphere ($G = 0$) is more delicate due to the positive power of the group dimension, and we refer for the specific treatment of this case to the original publication, since it involves the technique of discrete orthogonal polynomials.

The case of physical interest is the torus, where we have a flat target manifold. There, as shown above, and according to the string interpretation to be given below, every term can be simply understood. For higher genus there are corrections due to the dimension of the representation of higher order in $1/N$, which are not given a natural interpretation, and one needs corrections to the Nambu–Goto action.

Before delving further into these points, it is natural to consider, with some further detail, the $1/N$ expansion of pure Yang–Mills theory. We have, with the considerations already stated about the dimension of the representation, the expansion

$$\begin{aligned} Z(G,\alpha A,N) &= \sum_{n=0}^{\infty} \sum_{R\in Y_n} (\dim R)^{2-2G} e^{-\frac{\alpha A}{2N} C_2(R)} \\ &= \sum_n \sum_{R\in Y_n} \left(\frac{n!}{d_R}\right)^{2G-2} e^{-\frac{n\alpha A}{2}} \sum_{i=0}^{\infty} \Big[\frac{1}{2^i i!} \left(-\alpha A \widetilde{C}(R)\right)^i N^{n(2-2G)-i} + \mathcal{O}\left(N^{n(2-2G)-i-1}\right) \Big] \ . \end{aligned} \tag{3.50}$$

Such an expansion corresponds to considering the partition function of the genus G surface with no holes

$$Z(G,\alpha A,N) = \sum_R (\dim R)^{2-2G} e^{-\frac{\alpha A}{2} n} e^{-\frac{\alpha A}{N} \widetilde{C}(R)} e^{\frac{\alpha A}{2N^2} n^2} \quad , \tag{3.51}$$

and expanding the second exponent in powers of $1/N$ (the third in a feature of $SU(N)$ theory). Notice that the first exponential corresponds to the exponential of the area of the string, which winds n times around the target space area A. We will interpret it together with the next exponential as the branched covering. The last term will be understood in terms of tubes and collapsed handles.

However, one has to be cautious about counting the large-N contributions at this point. Expression (3.51) is correct. When one expands, and takes the sum over Young tableaux with n boxes, one loses, however, several important representations - in fact, it has been argued that this only contains half of the theory, the so-called chiral perturbation. The order in $1/N$ depends on the factor $(\dim R)^{-2(G-1)}$, which is of order $N^{-2n(G-1)}$, for Young tableaux with n boxes.

In general, representations are obtained from symmetrizing and antisymmetrizing tensor products of the fundamental representation. For large N, we have to take the leading-order contributions for quadratic Casimir and dimension, as we shall do. However, there are also representations obtained from products of a smaller representation R and the conjugated of another representation, $\overline{S}$. Such "composite" representations are defined by the Young tableau as in Fig. 5,

Fig. 5: A composite representation.

where S, R are two given representations, $\overline{S}$ is the adjoint of S. As it turns out, the quadratic Casimir (almost) factorizes; consider the product representation $T = \overline{S}\, R$, for which the quadratic Casimir is

$$C_2(T) = C_2(R) + C_2(S) + \frac{2 n_R n_S}{N} \quad . \tag{3.52}$$

For the dimension (at large N), we have

$$\dim T = (\dim R)(\dim S)(1 + \mathcal{O}(N^{-1})) \quad . \tag{3.53}$$

Therefore, at large N, the total partition function factorizes into two chiral contributions, as defined by (3.50), with a coupling term $\mathrm{e}^{-\frac{\alpha A}{N^2} n_R n_S}$. In fact, the problem of taking into account all representations is very delicate. (For the sphere, Douglas and Kazakov[82] showed that for $\alpha A < \pi^2$ there must be further contributions due to the phase transition at $\alpha A = \pi^2$.) From such counting of "composite" representations, we obtain

$$Z(G, \alpha A, N) = \sum_{n_R} \sum_{n_S} \sum_{R \in Y_{n_R}} \sum_{S \in Y_{n_S}} (\dim \overline{S} R)^{-2(G-1)} \mathrm{e}^{-\frac{\alpha A}{N}\left[C_2(R) + C_2(S) + 2\frac{n_R n_S}{N}\right]} \quad , \tag{3.54}$$

allowing the interpretation of the above in terms of two coupled chiral sectors, one being orientation-preserving and the other orientation-reversing. We will interpret first the simplest case of a single chiral sector, and then couple the two sectors. We write the chiral sector as

$$Z(G, \alpha A, N) = \sum_{g=-\infty}^{\infty} \sum_{n} \sum_{i} \zeta_{g,G}^{n,i}\, \mathrm{e}^{-\frac{n \alpha A}{2}} (\alpha A)^i N^{-2(g-1)} \quad , \tag{3.55}$$

where

$$2(g-1) = 2n(G-1) + i \tag{3.56}$$

is the Kneser formula (see Appendix B), and

$$\zeta_{g,G}^{n,i} = \sum_{R} \left(\frac{n!}{d_R}\right)^{-2(G-1)} \frac{1}{i!} \left(\frac{\widetilde{C}(R)}{2}\right)^{i} \quad . \tag{3.57}$$

Still we are looking at the $U(N)$ theory, since the $SU(N)$ term contains the contribution $\mathrm{e}^{\frac{\alpha A}{2}\frac{n^2}{N}}$. Notice that the sum over the base space genus g goes from $-\infty$ to ∞, whose meaning is the inclusion of disconnected diagrams. The relation (3.56) is crucial and we present a pedestrian proof in Appendix B.

The interpretation of $\zeta_{g,G}^{n,i}$ in terms of maps becomes clear when considering maps with a given winding number n, singular at a finite set of points. These are the branch points, such as those appearing in the maps $w = z^n$. Actually, this is the most general case, as one finds from the following result.[83] Consider a non-constant holomorphic mapping between Riemann surfaces, $f : \mathcal{M} \to \mathcal{N}$. Let $P \in \mathcal{M}$, and choose local coordinates $\tilde{z} \in \mathcal{M}$ vanishing at P and $w \in \mathcal{N}$ vanishing at $f(P)$. Thus we have $w = f(\tilde{z}) = \sum_{k \geq n} a_k \tilde{z}^k$, $n > 0$, and $w = [\tilde{z}h(\tilde{z})]^n$, where $h(\tilde{z})$ is holomorphic and $h(0) \neq 0$, which is equivalent to $w = z^n$. The value of n is the ramification number, or equivalently $(n-1)$ is the branch number of f at P. The theorem of Riemann–Hurwitz states that[83]

$$2(g-1) = 2n(G-1) + B \quad , \tag{3.58}$$

where $B = \sum_i (n_i - 1)$ is the total branch number. Gross and Taylor proved that for $2(g-1) = 2n(G-1) + i$, the coefficient $(\zeta_{g,G}^{n,i})$ is given in terms of the set $\sum(G, n, i)$ of n-fold covers of $\mathcal{M}_G$ with i branch points, and $\nu \in \sum(G, n, i)$, $\nu : \mathcal{M}_g \to \mathcal{M}_G$. To every cover there is a symmetry factor S_ν, which is the number of distinct homomorphisms from $\mathcal{M}_g$ to itself leaving ν invariant. The above-mentioned relation is

$$i! \zeta_{g,G}^{n,i} = \sum_{\nu \in \sum(G,n,i)} \frac{1}{|S_\nu|} \quad . \tag{3.59}$$

It is important to note that there is a very natural interpretation of the numerical coefficient in (3.55) in terms of maps. In the case of the torus this is the number of partitions of (n), and we are led to the result

$$Z = \sum_n p(n) \mathrm{e}^{-\frac{n\alpha A}{2}} = \eta \left(\mathrm{e}^{-\frac{\alpha A}{2}} \right) \quad . \tag{3.60}$$

In that case the geometrical interpretation is quite direct.

The last piece deserving interpretation concerns the remaining part of the Casimir, and the coupling of the two chiral sectors. It is given by (the exponential of)

$$\frac{\alpha A}{2N^2} n^2 = \frac{\alpha A}{2N^2} n + \frac{\alpha A}{2N^2} n(n-1) \quad . \tag{3.61}$$

Such a partition is useful, because each term has a definite interpretation; the first one in terms of handles in $\mathcal{M}_g$ mapped entirely onto a single point in target space, and which is in one of the n sheets of the cover, whose position has to be integrated, giving a factor of the area. Moreover each handle in string perturbation theory comes with a factor $1/N^2$ since the genus is increased by 1. Since the handles are infinitesimal their positions are the only moduli. Finally, the factor 1/2 accounts for the indistinguishability of the ends. Shrinking the length of the handle, the points coalesce; nevertheless the factor 1/2 remains. This contribution to the free energy accounts for the first factor above, $\mathrm{e}^{\frac{\alpha A}{2N^2}n}$. Moreover, if we have n_h handles, there is a symmetry factor $\frac{1}{n_h!}$ leading to exponentiation of such a term. For the second term in eq. (3.61), we have the interpretation of pinched tubes. Since the tubes now connect two different sheets, we have a factor $\frac{1}{2}n(n-1)$ arising from counting the number of them.

The whole previous interpretation in terms of maps preserves orientation and is consistent with the chiral component partition. There remains to understand the coupling between the two chiral components, namely $\mathrm{e}^{-\frac{\alpha A}{N^2}n_R n_S}$. A coupling between the chiral components of the form $\left(\frac{\alpha A}{N^2}n\widetilde{n}\right)$ can be interpreted as a gluing of the two surfaces by removing two disks and connecting them by an orientation-preserving cylinder. An example of such a cylinder is[71] $C = S^1 \times [0,1] = \{(z,x),\, |z|=1,\, a < x \le 1\}$ such that the map

$$\nu(z,x) = \begin{cases} z(1-2x) & \text{for } x \le 1/2 \\ \overline{z}(2x-1) & \text{for } x \ge 1/2 \end{cases} \tag{3.62}$$

is orientation-preserving, while

$$\nu(z,x) = z(1-2x) \tag{3.63}$$

is orientation-reversing.

The factor of the area accounts for the arbitrariness in the location of the tube. As before, the genus increases by a unit, and there is a factor -1 for each. The symmetry factors lead to exponentiation.

One may now ask which string theory is being described by such maps. Although it is not possible to get a full account of the result, some conclusion may be drawn. As anticipated, it must contain a W_∞ symmetry. The free energy is given as an expansion in $\mathrm{e}^{-\frac{\alpha}{2}A}$, or more precisely by terms $\mathrm{e}^{-\frac{n\alpha}{2}A}$; we can thus interpret this as an expansion in terms of the exponential of an action proportional to the area. It is not quite the Nambu action. Indeed, the result nA in the exponent signals maps wrapping n times over the target space, but unlike the Nambu action there are no folds: the area of the maps $\xi \to x$ is $\int \mathrm{d}^2x \,\det \frac{\partial x^\mu}{\partial \xi^\alpha}$, while the Nambu action is $\int \mathrm{d}^2x \left|\det \frac{\partial x^\mu}{\partial \xi^\alpha}\right|$.

Therefore, if we write an ansatz beginning with the Nambu action, we are obliged to have terms suppressing folds. This forbids, in particular, maps with zero winding number. This is a fact in accordance with pure QCD_2, which contains no particle, while terms with zero winding would describe particles. However, at the moment there is not much more to be said about such a formulation.

There are ways of rewriting the expansion of the partition function in terms of known group theoretic factors, which will be convenient to arrive at a further interpretation and at the possible Lagrangian formulation. First, the Frobenius formula relates the observable

$$\Upsilon_\sigma[U] = \prod_{j=1}^{s} \operatorname{tr} U^{n_j} \quad , \tag{3.64}$$

where σ is an element of the permutation group with cycles $n_1 \cdots n_s$, to the characters, since the above functions also build a complete set. We have

$$\chi_R[U] = \sum_{\sigma \in S_n} \frac{\chi_R[\sigma]}{n!} \Upsilon_\sigma[U] \quad , \tag{3.65a}$$

$$\Upsilon_\sigma[U] = \sum_{R \in Y_n} \chi_R[\sigma] \chi_R[U] \quad , \tag{3.65b}$$

where S_n is the permutation group of n elements and Y_n is the Young tableau of dimension n. In particular, for $U = 1$

$$\dim R = \sum_{\sigma \in S_n} N^s \chi_R[\sigma] \quad , \tag{3.66}$$

holds true, where s is the number of cycles.

The partition function of the chiral contribution may be expanded as

$$Z[N, \alpha A, G] = \sum_{n=0}^{\infty} \sum_{R \in Y_n} \sum_{i,t,h} \mathrm{e}^{-n\alpha A} \frac{(\alpha A)^{i+t+h} (-)^i n^h [n(n-1)]^{t+i}}{i! t! h! (n!)^{2-2G} 2^{i+t+h} d_R^i}$$
$$\times \{\chi_R[T_2]\}^i N^{n(2-2G)-i-2(t+h)} \left(\sum_{\sigma \in S_n} \frac{\chi_R(\sigma)}{N^{n-s}} \right)^{2-2G} \quad , \tag{3.67}$$

where i is the number of branch points, t is the number of orientation-preserving tubes, and h is the number of handles mapped to points. Moreover the Casimir has been computed in terms of the character $\chi_R[T_2]$, of the element T_2 containing a single cycle of length 2 and $(n-2)$ cycles of length 1 as

$$\widetilde{C}[R] = \frac{n(n-1)\chi_R[T_2]}{d_R} \quad . \tag{3.68}$$

The single cycle of length 2 is a building block and will be used in the collective field interpretation of the theory. Some rearrangement is still required to achieve the final appropriate formulation. Frobenius relation for unit matrix gives the expression for the dimension, leading to

$$\dim R = \frac{1}{n!} \sum_{\sigma \in S_n} N^{S_\sigma} \chi_R[\sigma] = \frac{N^n}{n!} \chi_R \left(\sum_{\sigma \in S_n} N^{S_\sigma - n} \sigma \right) \quad , \tag{3.69}$$

where S_σ is the number of cycles in the permutation σ. The leading term for large N is simply $\frac{N^n}{n!} d_R$, where d_R is the dimension of the representation of the permutation group. We define the group element

$$\Omega_n = \sum_{\sigma \in S_n} N^{S_\sigma - n} \sigma \quad , \tag{3.70}$$

which will correspond to extra twists on the covering space. From the combination formula for characters, eq. (C.13), we get

$$(\chi_R[\Omega_n])^{-1} = d_R^{-2}\chi_R[\omega_n] \quad , \tag{3.71}$$

from which, for any l (positive or negative) one finds

$$(\dim R)^l = \left(\frac{N^n d_R}{n!}\right)^l \frac{\chi_R[\Omega_n^l]}{d_R} \quad . \tag{3.72}$$

Using again eqs. (C.13) and (C.12), we derive

$$\sum_{\sigma,\varphi\in S_n} d_R^{-1}\chi(\sigma\varphi\sigma^{-1}\varphi^{-1}) = \frac{n!}{d_R^2}\sum_{\varphi}\chi(\varphi)\chi(\varphi^{-1}) = \left(\frac{n!}{d_R}\right)^2 \quad . \tag{3.73}$$

Therefore, the dimension may be written in terms of factors of N and d_R times a character associated with the special operator Ω_n,

$$(\dim R)^{2-2G} = \left(\frac{N^n d_R}{n!}\right)^{2-2G} \chi_R(\Omega_n^{2-2G}) \quad ; \tag{3.74}$$

moreover, the counting factor associated with the given Young tableau is

$$\left(\frac{n!}{d_R}\right)^{2G} = \sum \frac{1}{d_R}\prod_1^G \chi_R(\sigma_j\varphi_j\sigma_j^{-1}\varphi_j^{-1}) \quad . \tag{3.75}$$

All characters in the chiral partition function can be combined using eq. (C.13) $(G+i+1)$ times, and one arrives at

$$\begin{aligned} Z^+(G,\alpha A,N) = &\sum_{n,i,t,h} \mathrm{e}^{-n\alpha A/2}\frac{(\alpha A)^{i+t+h}}{i!t!h!}N^{n(2-2G)-i-2(t+h)}\frac{(-1)^i n^h(n^2-n)^t}{2^{t+h}} \\ &\times \sum_{p_1,\cdots,p_i\in T_2}\sum_{s_1,t_1,\cdots,s_G,t_G\in S_n}\left[\frac{1}{n!}\delta(p_1\cdots p_i\Omega_n^{2-2G}\prod_{j=1}^G s_jt_js_j^{-1}t_j^{-1})\right], \end{aligned} \tag{3.76}$$

where $\delta(\sigma) = \frac{1}{n!}\sum_R d_R\chi_R(\sigma)$.

The terms Ω_n^{2-2G} in the delta function show extra twists (permutations of sheets) in the covering at $(2-2G)$ points. For $2-2G>0$, this is easier to understand.

Such a result can be generalized along the same lines of reasoning to the full theory, that is for the coupled chiral and antichiral composite partition function. Details concerning such a derivation are too long to be reported here, and we refer to [72] and [73]. The final result reads

$$\begin{aligned}
&Z(G,\alpha A,N)\\
&\sim \sum_{n^\pm,i^\pm=0}^{\infty} \sum_{p_1^\pm,\cdots,p_{i^\pm}^\pm\in T_2\subset S_{n^\pm}} \sum_{s_1^\pm,t_1^\pm,\cdots,s_G^\pm,t_G^\pm\in S_{n^\pm}} \left(\frac{1}{N}\right)^{(n^++n^-)(2G-2)+(i^++i^-)}\\
&\times \frac{(-)^{(i^++i^-)}}{i^+!i^-!n^+!n^-!}(\alpha A)^{(i^++i^-)}\mathrm{e}^{-\frac{1}{2}((n^+)^2+(n^-)^2-2n^+n^-)\alpha A/N^2}\\
&\times \delta_{S_{n^+}\times S_{n^-}}\left(p_1^+\cdots p_{i^+}^+ p_1^-\cdots p_{i^-}^-\,\Omega_{n^+,n^-}^{2-2G}\prod_{j=1}^{G}[s_j^+,t_j^+]\prod_{k=1}^{G}[s_k^-,t_k^-]\right) \quad , \qquad (3.77)
\end{aligned}$$

where $[s,t]=sts^{-1}t^{-1}$. Here δ is the delta function on the group algebra of the product of symmetric groups $S_{n^+}\times S_{n^-}$, T_2 is the class of elements of $S_{n^\pm}$ consisting of transpositions, and Ω_{n^+,n^-}^{-1} are certain elements of the group algebra of the symmetric group $S_{n^+}\times S_{n^-}$ with coefficients in $\mathbb{R}(1/N)$. Let us remain with the chiral component, eq. (3.76). The theory defined thereafter is related to a topological field theory, as shown in ref. [73]. In order to see this, one first considers the limit of the partition function (3.76) for vanishing area, which is still non-trivial. One considers the homotopy group of a punctured surface (L punctures), and the homomorphism of such a group into S_n. One also defines the Hurwitz space of branched coverings: consider $H(n,B,G,L)$, the set of equivalence classes of the manifold Σ_t[7)] with degree n, branching number B; S is a set of points on Σ_t; $H(n,B,G,L)$ is the equivalence class of branched coverings of Σ_t.

The chiral amplitude at zero area

$$Z(G,0,N)=\sum_{n=0}^{\infty}N^{n(2-2G)}\sum_{s_1\varphi_1\cdots s_G\varphi_G\in S_n}\frac{1}{n!}\delta\left(\Omega_n^{2-2G}\prod_1^G s_j\varphi_j s_j^{-1}\varphi_j^{-1}\right) \qquad (3.78)$$

has been shown[73] to be given by the simpler expression

$$Z(G,0,N)=\sum_{n=0}^{\infty}\sum_{B=0}^{\infty}\sum_{L=0}^{B}\left(\frac{1}{N}\right)^{2G-2} d(2-2G,L)\sum_{f\in H(n,B,G,s)}\frac{1}{|\mathrm{Aut}\, f|} \quad , \qquad (3.79)$$

where $|\mathrm{Aut}\, f|$ is the order of the automorphism group of the branched covering map f, and

$$d(2-2G=\chi_G,L)=\frac{\chi_G!}{(\chi_G-L)!L!} \qquad (3.80)$$

is an Euler character (see ref. [73] for details); moreover one finds for the last factor in (3.79) the result

$$d(2-2G,L)\sum\frac{1}{|\mathrm{Aut}\, f|}=\chi(H(n,B,G,L)) \quad . \qquad (3.81)$$

7) Recall that one considers here the maps from world sheet to the target space: $\Sigma_{ws}\to\sigma_t$.

This leads to a topological theory described by (3.76). To fully demonstrate such results a rather heavy mathematical instrumentation is needed, which goes far beyond the scope of the present review. It is useful to quickly state some results. The topological theory describing the above partition function is the topological gravity, with a topological sigma model, and a further topological term, the so-called co-σ sector. The area is restored by perturbing the topological action with the Nambu action, which does reappear in this context as a perturbation of the topological theory. The full theory can also be discussed along these lines.

The area in the above formulation is always multiplied by the charge, for dimensional reasons. The limit $e \to 0$ corresponds to a topological theory (see discussion after eq. (2.108c)).

3.4 Collective coordinates approach

Further information and insight in two-dimensional string theory can be obtained by the method of collective coordinates, which are useful to characterize the properties of string theory as a whole. Indeed, non-critical strings have very rich and detailed descriptions by means of either the two-dimensional Liouville theory or matrix models (see ref. [30] and references therein). In the Liouville approach, the d-dimensional string corresponds effectively to a $(d+1)$-dimensional theory, since the Liouville field itself plays the role of the extra coordinate. Therefore a two-dimensional string is described by a $c=1$ model. In the matrix-model approach, this is equivalent to considering the dynamics of a Hermitian matrix $M(t)$, depending on a single coordinate, with Lagrange density

$$\mathcal{L} = \mathrm{tr}\, \frac{1}{2}\dot{M}^2 - \mathrm{tr}\; V(M) \quad , \tag{3.82}$$

where the kinetic term is actually a simplification of the exponential propagator,[84] and the potential $V(M)$ is an arbitrary function of the matrix $M(t)$. The current

$$J = i[M, \dot{M}] \tag{3.83}$$

is conserved. Diagonalizing the matrix $M = \mathrm{diag}\, \lambda(t)$, the eigenvalues describe a system of free fermions. The analogue of the Wilson-loop average, the trace of the exponential of M, is computable as a simple sum:

$$\widetilde{\phi}_k(t) = \mathrm{tr}\; \mathrm{e}^{ikM} = \sum_{j=1}^{N} \mathrm{e}^{ik\lambda_j(t)} \quad , \tag{3.84}$$

whose Fourier transform $\phi(x,t)$ can be interpreted as a density of fermions. We can study the theory in terms of the two-dimensional scalar field $\phi(x,t)$. Therefore, the model defined by eq. (3.82) will be replaced by an effective two-dimensional field theory. Such a description has several advantages; in particular it provides a global description of the string.

As it stands, the problem of computing wave functionals is very complicated. Jevicki and Sakita[85] handled a similar problem by introducing the change of variables (3.84), in such a way that a Schrödinger wave equation in terms of the $\phi(x,t)$ variables is simpler. After the transformation (3.84) one finds that the Laplacian operator appearing in the Hamiltonian corresponding to eq. (3.82) is given by

$$\frac{\partial^2}{\partial M^2} = \frac{\partial^2 \phi}{\partial M^2}\frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial M}\right)^2 \frac{\partial^2}{\partial \phi^2} = -k^2 \phi_k \frac{\partial}{\partial \phi_k} - k k' \phi_{k+k'} \frac{\partial}{\partial \phi_k}\frac{\partial}{\partial \phi_{k'}} \, . \tag{3.85}$$

Therefore, it is possible to find a Hamiltonian in terms of the one-component field $\phi(x,t)$ and its conjugate $\Pi(x,t) \sim -i\frac{\partial}{\partial \phi(x,t)}$; it reads

$$H = \int \mathrm{d}x \left\{ \frac{1}{2}\partial_x \Pi \phi \partial_x \Pi + \frac{1}{6}\Pi^2 \phi^3 + V(x)\phi \right\} \quad . \tag{3.86}$$

The Hamiltonian shows a cubic interaction and a tadpole term, indicating a string-type interaction and annihilation into the vacuum. Notice the fact that ϕ is a two-dimensional field; therefore one is describing strings in a two-dimensional target space, as already observed in the description of the relation between Liouville and matrix models. One can also introduce chiral components

$$\alpha_\pm(x,t) = \partial_x \Pi \pm \pi \phi(x,t) \quad , \tag{3.87}$$

with Poisson brackets

$$\{\alpha_\pm(x), \alpha_\pm(y)\} = \pm 2\pi \delta'(x-y) \quad , \tag{3.88}$$

in terms of which the Hamiltonian (3.86) turns into

$$H_{coll} = \int \frac{\mathrm{d}k}{2\pi} \left\{ \frac{1}{6}(\alpha_+^3 - \alpha_-^3) + \left(V(x) + \frac{1}{2}\mu\right)(\alpha_+ - \alpha_-) \right\} \quad , \tag{3.89}$$

where μ is a constant indicating the energy level. The simplifying case of a harmonic oscilator potential $V(x) = -\frac{1}{2}x^2$ has been largely studied.[84] An infinite number of conservation laws is found for such a model.

It is not difficult to derive a Das–Jevicki-type[86] Hamiltonian describing the string interaction directly from the $SU(N)$ representation in terms of a sum of representations in the case of QCD_2. Consider the observables defined by

$$\Upsilon_\sigma(U) = \prod_{j=1}^{s} \mathrm{tr}\, U^{n_j} \quad , \tag{3.90}$$

in terms of the group variable U, where we considered as usual the partition $n = \sum_{i=1}^{s} n_i$.

Such objects satisfy the basic assumption that as one goes around a Wilson loop one induces a permutation of the sheets covering the loop. Therefore, to any Wilson loop can be associated an element σ of the permutation group S_n, where n is the number of sheets of the world sheet covering the Wilson loop. Consider a manifold with special points, and loops that possibly go around such points; as one encircles any (homotopically non-trivial) loop, the n sheets on the basis space loop interchange, defining an element in the permutation group.

Therefore it is natural to assign an element in S_n to each string state. Moreover, the states are orthogonal if the elements characterizing the two given states are not in the same conjugacy class. The normalization of the observables (3.90) is given in terms of the characters as

$$\begin{aligned}\int \mathrm{d}U\, \Upsilon_\sigma(U)\Upsilon_\tau(U^\dagger) &= \int \mathrm{d}U \sum_{R,R'} \chi_R(\sigma)\chi_{R'}(\tau)\chi_R(U)\chi_{R'}(U^\dagger) \\ &= \sum_R \chi_R(\sigma)\chi_R(\tau) = \delta_{T_\sigma,T_\tau}\frac{n!}{C_\sigma} \quad ,\end{aligned} \tag{3.91}$$

where C_σ is the number of elements in the conjugacy class T_σ; we can define the scalar product

$$\langle s'|s\rangle = \sum_{t\in S_n} \delta_{s',tst'} \;=\; \begin{cases} \frac{n!}{C_\sigma} & \text{if } s\sim s' \quad , \\ 0 & \text{otherwise} \quad ,\end{cases} \tag{3.92}$$

where $s \sim s'$ means that the elements are in the same conjugacy class, and the number of elements in such a conjugacy class is C_s. As argued in ref. [87] a state in S_n is typically

$$|S\rangle = \prod_{l=1}^{n} (a_l^\dagger)^{n_l}|0\rangle \quad , \tag{3.93}$$

where $|0\rangle$ is the vacuum, and $a_l^\dagger$ is the creation of a string with wind l, satisfying

$$[a_l, a_m^\dagger] = |l|\delta_{l,m} \quad , \tag{3.94}$$

which leads to eq. (3.92) for the scalar product of two elements of the type (3.93). Since the "energy" is $\frac{1}{2}e^2 Ln$, the free part of the Hamiltonian is

$$H_0 = \frac{1}{2}e^2 L \sum a_l^+ a_l \quad . \tag{3.95}$$

The first interaction describes the joining of two strings; it should be represented by a sum over conjugacy classes, weighted by the factor $\frac{e^2 L}{2N}$; it must be described by the expectation of an operator belonging to the conjugacy class of S_n with one two-cycle, namely T_2, and the rest are one-cycles, as

$$H_{int}^{(3)} = \frac{e^2 L}{2N} \sum_{p\in S_{\tilde n}} \langle s'|p|s\rangle = \frac{e^2 L}{2N} \sum_{\substack{t\in S_n \\ p\in S_{\tilde n}}} \delta_{s'p,tst^{-1}} \quad , \tag{3.96}$$

where $S_{\tilde n} \subset S_n$ is in the conjugacy class with one two-cycle, namely the elementary permutation; the remaining elementary strands are one-cycles. Therefore, s has cycles n_1 and

n_2, while s' contains a cycle n_1+n_2, and $s'p = tst^{-1}$. A simple counting reveals that conjugation leads to $n_1 n_2$ elements, while one has n_1+n_2 distinct conjugates, which is correctly described by the operator

$$H^{(3)} = \frac{e^2 L}{2N} \sum_{\substack{n,n'>0 \\ n,n'<0}} (a^\dagger_{n+n'} a_n a_{n'} + \text{c.c.}) \quad . \tag{3.97}$$

The remaining interaction (quartic) is such that either two strings of the same chirality disappear and are subsequently generated or the opposite chirality strings interact, but the sign is opposite with respect to the previous (equal chirality) possibility. This is described by the interacting Hamiltonian

$$H^{(4)} = \frac{e^2 L}{2N^2} \left[\sum_{n>0} (a^\dagger_n a_n - a^\dagger_{-n} a_{-n}) \right]^2 \quad , \tag{3.98}$$

completing the total Hamiltonian. One can also show that relating a_k with the field $\varphi(x)$ and its momentum canonically conjugated $\pi(x)$ by

$$a_k = \frac{1}{2} \int \mathrm{d}x\, \mathrm{e}^{-ikx} \left[\varphi(x) + \frac{1}{\pi} \epsilon(k) \partial \Pi(x) \right] \quad , \tag{3.99}$$

one finds a slightly modified Das–Jevicki[86] Hamiltonian

$$H = \frac{4}{e^2 LN} \int \mathrm{d}x \left\{ \frac{1}{2} \partial\Pi \varphi \partial \Pi + \frac{1}{6} \pi^2 \varphi^3 - \left(\frac{e^2 LN}{4} \right)^2 \varphi \right\} + \Delta H \quad , \tag{3.100}$$

with the constraint

$$\int \mathrm{d}x\, \varphi(x) = N \quad , \tag{3.101}$$

which takes the zero-mode problem into account, and ΔH is the quantum correction to the free energy. Since one arrives at a $c=1$ matrix model, it is natural to ask what is the relation with the fermion picture of the latter. As a matter of fact, one can show that there is a simple description of pure QCD_2 on a circle of radius L in terms of free fermions.

In the gauge $A_0 = 0$, the pure QCD_2 Hamiltonian is just the square of the electric field

$$H = \frac{1}{2} \int_0^L \mathrm{d}x\, \mathrm{tr}\, F_{01}^2 = \frac{1}{2} \int_0^L \mathrm{d}x\, \mathrm{tr}\, \dot{A}_1^2 \quad . \tag{3.102}$$

One can define the gauge-invariant quantity[88]

$$V(x) = W[0,x] \dot{A}_1(x) W[x,L] \quad , \tag{3.103}$$

with the Wilson line given by

$$W[a,b] = P \mathrm{e}^{ie \int_a^b \mathrm{d}x\, A_1} \quad , \tag{3.104}$$

in such a way that the Gauss law

$$\nabla_1 F_{10} = \partial_1 \dot{A}_1 + ie[A_1, \dot{A}_1] = 0 \tag{3.105}$$

boils down to

$$\partial_1 V = 0 \quad . \tag{3.106}$$

Therefore we find $V(0) = V(L)$, implying, (from $V(x) = W[0,x]\dot{A}_1 W[x,L]$) that $\dot{A}_1$ commutes with $W = W[0,L]$. The time derivative of W may also be computed by means of the well-known formula

$$\dot{W} = ie \int_0^L \mathrm{d}x\, W[0,x]\dot{A}_1(x) W[x,L] = ie \int \mathrm{d}x\, V(x) \quad , \tag{3.107}$$

from which we can prove that W^{-1} commutes with $\dot{W}$; from the constancy of $V(x)$, we can also compute $\dot{A}_1(x) = W[x,0]\dot{A}_1(0)W[0,x]$, finding $\dot{A}_1(x) = \frac{1}{ieL} W[0,x]\dot{W}W^{-1}W[x,0]$, which permits us to rewrite the Hamiltonian as

$$H = \frac{1}{2}\int_0^L \mathrm{d}x\, \mathrm{tr}\, \left(W[x,0]\dot{A}_1(0)W[0,x]\right)^2 = -\frac{1}{2e^2 L}\mathrm{tr}\, \left(W^{-1}\dot{W}\right)^2 \quad . \tag{3.108}$$

This Hamiltonian describes a one-dimensional unitary matrix model. Since W and $\dot{W}$ commute, the problem is reduced to the consideration of eigenvalues of W.

Therefore we arrive at a system of N fermions on a circle with Hamiltonian[88,89]

$$H = -\left(\frac{e^2 L}{2}\right) \sum \frac{\partial^2}{\partial \theta_i^2} \quad . \tag{3.109}$$

The reduction of QCD_2 with matter to a simple dynamical system will be considered later. Moreover the infinite symmetry of the pure QCD_2 theory mirrors itself here in the fact that the relativistic fermions display also a W_∞ symmetry, corresponding to area-preserving diffeomorphisms of the Fermi sea.[35,36]

3.5 Phase structure of QCD_2

We have seen that QCD_2 may accommodate different phases, although we have not seen such a structure yet. The phase structure of the model has not been studied in the case where matter fields are present. However, in the large-N limit it is possible to prove that depending on the value of the "fine structure" constant, $\alpha = e^2 N$, the theory shows a different behaviour.

The first observation came long ago, when Gross and Witten[78] obtained a possible third-order phase transition for the large-N limit of lattice QCD_2. The argument relies on the large-N limit as employed by Brezin, Itzykson, Parisi and Zuber.[90] The leading term can be computed. One starts with the lattice formulation, where the Wilson action reads

$$S[U]=\sum_P \frac{1}{2e^2a^2}\mathrm{tr}\left(\prod_P U+\mathrm{h.\,c.}\right) \quad , \tag{3.110}$$

and a single plaquette action is the product

$$\prod_P U = U_{\vec{n},\hat{x}_0}U_{\vec{n}+\hat{x}_0,\hat{x}_1}U_{\vec{n}+\hat{x}_0+\hat{x}_1,\hat{x}_0}U_{\vec{n}+\hat{x}_1,-\hat{x}_1} \quad , \tag{3.111}$$

that is one starts out of the point $\vec{n}$, in the direction $\hat{x}_0$, returning back at the end of each round. Gross and Witten considered such a problem, used the invariance of the Haar measure, and gauge invariance, changing $U_{\vec{n},\hat{x}} \to V_{\vec{n}}U_{\vec{n},\hat{x}}V^\dagger_{\vec{n}+\hat{x}}$, choosing the gauge $U_{\vec{n},\hat{x}_0}=1$ for all $\vec{n}$, corresponding to $A_0=0$, after which the action reads

$$S[U]=\frac{1}{2e^2a^2}\sum \mathrm{tr}\left(U_{\vec{n},\hat{x}_1}U^\dagger_{\vec{n}+\hat{x}_0,\hat{x}_1}+\mathrm{h.\,c.}\right) \quad . \tag{3.112}$$

After such a procedure, one changes the variables to $W_{\vec{n}}$, defined as

$$U_{\vec{n}+\hat{x}_0,\hat{x}_1}=W_{\vec{n}}U_{\vec{n},\hat{x}_1} \quad , \tag{3.113}$$

which leads to a partition function that is the product of partition functions for each site, i.e.

$$Z=\int\prod_n \mathrm{d}W_n\, \mathrm{e}^{\sum_n \frac{1}{2e^2a^2}\mathrm{tr}\,(W_{\vec{n}}+W^\dagger_{\vec{n}})}=z^{V/a^2} \quad , \tag{3.114}$$

where z is the one-site partition function and V the total volume. Each integral is computable using the results of the appendix of ref. [91] and one obtains

$$Z(e^2,N)=\det M \quad , \tag{3.115}$$

$$M_{i,j}=I_{i-j}(1/e^2a^2) \quad , \tag{3.116}$$

where $I_i(x)$ is the Bessel function of order i. In terms of the eigenvalues of W, one can also write $W=TDT^\dagger$, where the diagonal matrix $D_{ij}=\delta_{ij}\mathrm{e}^{i\theta_j}$; since the angular piece is directly integrated, one is left with

$$\mathrm{d}W\sim\prod_1^N \Delta^2(\theta_i)\,\mathrm{d}\theta_i \quad , \tag{3.117}$$

where the Jacobian is a Vandermonde determinant

$$\Delta^2(\theta_i)=\prod\sin^2\frac{\theta_i-\theta_j}{2}=4^{-N}|\det\Delta|^2 \quad , \tag{3.118}$$

where $\Delta_{j,k} = \mathrm{e}^{ij\theta_k}$. The (one-site) partition function is

$$Z(e^2, N) = \int_0^{2\pi} \prod \mathrm{d}\theta_i \Delta^2(\theta_i)\, \mathrm{e}^{\frac{1}{e^2 a^2}\sum_i \cos\theta_i} \quad , \tag{3.119}$$

and the energy $E = -\frac{1}{N^2}\ln Z(e^2, N)$ can be computed in the limit $N \to \infty\, (\alpha = e^2 N)$ by the steepest-descent method, where the eigenvalues are given by the stationary condition

$$\frac{2}{\alpha a^2}\sin\theta_i = \sum_{j\neq i} \cot \left|\frac{\theta_i - \theta_j}{2}\right| \quad ; \tag{3.120}$$

we define a function $\theta(x)$ such that $\theta_i = \theta\frac{i}{N}$ since, for large N, $x_i \sim i/N$ can be seen as a continuous number, and find:

$$\begin{aligned} E(g) &= -\lim_{N\to\infty}\left\{\frac{1}{N\alpha a^2}\sum_1^N \cos\theta_i + \frac{1}{N^2}\sum_{i\neq j}\ln\left|\sin\frac{\theta_i - \theta_j}{2}\right|\right\} \\ &= \frac{1}{\alpha a^2}\int_0^1 \mathrm{d}x \cos\theta(x) + P\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \ln\left|\sin\frac{\theta(x)-\theta(y)}{2}\right| + \text{constant}\,, \end{aligned} \tag{3.121}$$

while

$$\frac{1}{\alpha a^2}\sin\theta(x) = P\int_0^1 \mathrm{d}y \cot\frac{\theta(x)-\theta(y)}{2} \quad . \tag{3.122}$$

At this point we introduce the density of eigenvalues

$$\rho(\theta) = \frac{\mathrm{d}x}{\mathrm{d}\theta} \geq 0 \quad . \tag{3.123}$$

There are two regions, depending now on the value of αa^2. For strong coupling one expects an eigenvalue distribution over the whole circle $[-\pi, \pi]$, and eq. (3.122) is solved using

$$\begin{aligned} \int_0^1 \mathrm{d}y \cos\frac{\theta(x)-\theta(y)}{2} &= \int_{-\pi}^{\pi} \mathrm{d}\beta\, \rho(\beta) \cot\frac{\theta-\beta}{2} \\ &= 2\sum_1^\infty \int \mathrm{d}\beta\, \rho(\beta)\,(\sin n\theta \cos n\beta - \cos n\theta \sin n\beta) \quad , \end{aligned} \tag{3.124}$$

which, together with the normalization condition $\int_{-\pi}^{\pi} \mathrm{d}\theta\, \rho(\theta) = 1$, fixes the density of eigenvalues

$$\rho(\theta) = \frac{1}{2\pi}\left[1 + \frac{1}{\alpha a^2}\cos\theta\right] \quad , \tag{3.125}$$

which is positive for $\alpha a^2 \geq 2$. For $\alpha a^2 \leq 2$, the integration over the "angle" β must be performed in a region $[-\theta_c, \theta_c]$. In ref. [78] the function

$$F(\zeta) = \int_{-\theta_c}^{\theta_c} \mathrm{d}\beta\, \rho(\beta) \cot\frac{\zeta - \beta}{2} \quad , \tag{3.126}$$

has been computed, and the eigenvalues distribution is found to be

$$\rho(\theta) = \frac{1}{\pi\alpha a^2}\cos\frac{\theta}{2}\left(\frac{\alpha a^2}{2} - \sin^2\frac{\theta}{2}\right)^{1/2} \quad , \tag{3.127}$$

showing that there is a phase transition at $\alpha a^2 = 2$, a point where the expressions (3.125) and (3.127) coincide.

The origin of the phase transition is the fact that the functional integral, in the strong coupling, has most contributions from the Vandermonde determinant, which shows a non-relativistic fermion character, and the fields interact repulsively due to Pauli's principle. Therefore the density distribution is almost constant, $\rho(\theta) \sim \frac{1}{2\pi}$. In the weak coupling limit, on the other hand, the Wilson action becomes important, and the interaction is attractive, therefore $\rho(\theta)$ is given by a semi-circle law:

$$\rho(\theta) = \frac{1}{\pi}\sqrt{\frac{1}{\alpha a^2}}\left(1 - \frac{\theta^2}{4\alpha a^2}\right)^{1/2} \quad , \quad |\theta|^2 \leq (2g)^2 \quad . \tag{3.128}$$

4. Generalized QCD_2 and adjoint-matter coupling

4.1 Introduction and motivation

The study of matrix models relevant for four-dimensional QCD is spoiled by the existence of the "barrier" at the value of the conformal central charge $c = 1$, as is clear from the expression of the dressed conformal dimension (see ref. [30] and references therein). The case $c = 1$ describes effectively two-dimensional string theory, since the time dimension is described by the Liouville field. Thus $c > 1$ describes, in an analogous way, $D > 2$ string theory. Such theories are in fact much richer than the two-dimensional counterpart due to the role played by the transverse oscillators. As it turns out, the most interesting case is also the most difficult sometimes: nature seems to hide itself in folds unreachable to perscrutation by the available means.

A matrix model in two dimensions ($c = 2$) can be defined by means of the Lagrangian

$$L = \mathrm{tr}\left[\frac{1}{2}\partial^\mu M\partial_\mu M + \frac{1}{2}\mu M^2 - \frac{\lambda}{3!\sqrt{N}}M^3\right] \quad , \tag{4.1}$$

where M is an $N \times N$ matrix field, and λ, μ arbitrary parameters. One should look for the singular limit in terms of the parameter λ/μ. Such a critical behaviour is difficult to obtain. As it stands, the model cannot correctly describe string effects. This assertion derives from the fact that the Fock space constructed out of the momentum space components of the M field, namely[92]

$$\mathcal{F} = \sum a^\dagger_{i_1 j_1}(k_1)\cdots a^\dagger_{i_n j_n}(k_n)|0\rangle \quad , \tag{4.2}$$

where

$$M_{ij}(x) = \frac{1}{\sqrt{2\pi}} \int \frac{\mathrm{d}k^+}{\sqrt{2k^+}} \left[a_{ij}(k) \mathrm{e}^{-ik^+x^-} + a^\dagger_{ji}(k) \mathrm{e}^{ik^+x^-} \right] \quad , \tag{4.3}$$

contains all multiplets, while closed string states should be singlets under global $SU(N)$ symmetry, whose action on M is defined as

$$M \to \Omega^\dagger M \Omega \quad ; \tag{4.4}$$

therefore the Hilbert space should be spanned by states of the form

$$\mathrm{tr}\, a^\dagger(k_1) \cdots a^\dagger(k_n) |0\rangle \quad . \tag{4.5}$$

Moreover, numerical results indicate also that one expects tachyons for purely bosonic models as above.

One possible cure of these problems would be a gauging of the symmetry, confining the non-singlet states, so that the singlets (4.5) span the physical Hilbert space of the theory. In fact, such a gauging procedure has further motivations, from the point of view of two-dimensional QCD, which is still very simple since the gauge field has no degree of freedom, by naïve counting. One might consider instead the more realistic case of three-dimensional QCD, and dimensionally reduce the $(2+1)$ dimensions to $(1+1)$, compactifying one of the spatial dimensions to a vanishingly small box. In such a case the third component of the gauge field becomes, in $(1+1)$ dimensions, bosonic matter in the adjoint representation.

4.2 Scalar and fermionic matter coupling; quantization

Light-cone quantization of $(1+1)$-dimensional QCD with adjoint matter fields in the light-cone gauge has been considered both for fermionic as well as bosonic matter.[92,93] We will restrain ourselves here to the fermionic case.

The procedure will be based on a choice of the light-cone gauge, in such a way that the action is quadratic in the remaining component of the gauge field, thus making its integration possible. Proceeding with the light-cone quantization, we see that one of the fermionic components obeys a constraint equation (no light-cone time derivative). Such a fact can be used in performing the light-cone quantization, choosing one light-cone variable as the "time" (or "light-cone time"). It is possible, in the Hamiltonian formalism, to fix the time, and the theory turns out to be described in terms of simple oscillators. The resulting Schrödinger equation leads to the bound-state structure. Thus we consider the action

$$S = \mathrm{tr} \int \mathrm{d}^2x \left[i\overline{\Psi}\, \not{D}\Psi + m\overline{\Psi}\Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] \quad , \tag{4.6}$$

and separate the chiral components of the Fermi field by means of the decomposition

$$\Psi = \begin{pmatrix} \psi \\ i\chi \end{pmatrix} \quad , \quad \chi^\dagger = \chi \quad , \quad \psi^\dagger = \psi \quad , \tag{4.7}$$

with the covariant derivative given as usual in the adjoint representation by the expression

$$D_\mu \psi = \partial_\mu \psi - ie[A_\mu, \psi] \quad . \tag{4.8}$$

One can also choose the light-cone gauge $A_- = 0$, in which the ghosts decouple. We find for the action the expression

$$S_f = \mathrm{tr} \int \mathrm{d}x^+ \, \mathrm{d}x^- \left[i\psi\partial_+\psi + i\chi\partial_-\chi - 2im\chi\psi + \frac{1}{8}(\partial_- A_+)^2 + J^+ A_+ \right] \quad , \tag{4.9}$$

where $J^+_{ij} = \mathrm{tr}\, \psi_{ik}\psi_{kj}$. If we choose x^+ as the time variable as described above, it is clear from (4.9) that χ does not obey any equation of motion in the Hamiltonian sense, but rather an equation of constraint, namely

$$\partial_- \chi - 2m\psi = 0 \quad , \tag{4.10}$$

since no light-cone time derivative is involved. The Gauss constraint, equivalent to the equation of motion of A_- is

$$\frac{1}{4}\partial_+\partial_- A_+ + J_+ = 0 \quad , \tag{4.11}$$

and is equivalent to the equation of motion for A_+, namely

$$\frac{1}{4}\partial_-^2 A_+ - J_- = 0 \quad , \tag{4.12}$$

once one uses current conservation. Moreover the gauge field A_+ is quadratic after gauge fixing, and one may perform the corresponding Gausssian integration, equivalent to the substitution of its equation of motion back into the action to obtain the light-cone components of the energy–momentum tensor

$$P^+ = \mathrm{tr} \int \mathrm{d}x^- \; i\psi\partial_-\psi \quad , \tag{4.13a}$$

$$P^- = \mathrm{tr} \int \mathrm{d}x^- \left[-4m^2 \psi \frac{1}{\partial_-} \psi + 2J^+ \frac{1}{\partial_-^2} J^+ \right] \quad . \tag{4.13b}$$

In the Hamiltonian formalism, working at fixed time, as we also mentioned, it is convenient to consider the mode expansion of the fermion field as given by

$$\psi_{ij}(x) = \int \frac{\mathrm{d}k^+}{2\sqrt{2\pi}} \, b_{ij} \mathrm{e}^{-ik^+ x^-} \quad . \tag{4.14}$$

The canonical procedure implies anticommutation relations for the fermions as given by

$$\{\psi_{ij}(x^-), \psi_{kl}(x'^-)\} = \frac{1}{4}\delta(x^- - x'^-)\delta_{il}\delta_{jk} \quad , \tag{4.15a}$$

$$\{b_{ij}(k^+), b_{kl}(k'^+)\} = \delta(k^+ + k'^+)\delta_{il}\delta_{jk} \quad . \tag{4.15b}$$

Therefore $b(k)$ are creation operators for $k \le 0$, while for $k \ge 0$ they are annihilation operators, and $b(k)_{k\ge 0}|0\rangle = 0$. Computation of the light-cone components of the energy–momentum tensor follows from (4.9) by the usual procedure, leading to (4.13). One substitutes (4.14), obtaining first the expressions for the current

$$\widetilde{J}_{ij}(k) = \int \mathrm{d}p\, b_{ik}(p) b_{kj}(k-p) \quad . \tag{4.16a}$$

We can compute now the energy momentum tensor components in terms of the creation and annihilation operators $b_{ij}(k)$ as

$$P^+ = \int_0^\infty \mathrm{d}k\, k b_{ij}(-k) b_{ij}(k) \quad , \tag{4.16b}$$

$$\begin{aligned} P^- &= m^2 \int_0^\infty \frac{\mathrm{d}k}{k} b_{ij}(-k) b_{ji}(k) - 2e^2 N \int_0^\infty \frac{\mathrm{d}k}{k} b_{ij}(-k) b_{ji}(k) \\ &+ \frac{e^2}{2\pi}\int_0^\infty \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3\, \mathrm{d}k_4 \Big\{ A\delta(k_1+k_2-k_3-k_4) b_{kj}(-k_3) b_{ji}(-k_4) b_{il}(k_1) b_{lk}(k_2) \\ &+ B\delta(k_1+k_2+k_3-k_4)[b_{kj}(-k_4)b_{ji}(k_3)b_{il}(k_2)b_{lk}(k_1) + b_{kj}(-k_1)b_{ji}(-k_2)b_{il}(-k_3)b_{kl}(k_4)]\Big\} \,, \end{aligned} \tag{4.16c}$$

where the coefficients are given, after eq. (4.13), by

$$A = \frac{-1}{(k_1-k_4)^2} + \frac{1}{(k_1+k_2)^2} \quad , \tag{4.17a}$$

$$B = \frac{-1}{(k_2+k_3)^2} + \frac{1}{(k_1+k_2)^2} \quad . \tag{4.17b}$$

A discretized version of the model was discussed by Dalley and Klebanov,[92] who found that for any value of e/m the spectrum is real, without any phase transition, contrary to a previous analysis of the pure-matrix-model case where they found numerical evidence of a tachyon. They also found that in the strong coupling limit ($e \to \infty$) the masses are pushed to infinity, and the theory becomes trivial. The conclusion is based upon the fact that for zero bare-fermion mass, there is still a non-vanishing mass gap in the spectrum. See also ref. [94] for numerical results.

It is quite remarkable that a wave functional obeying a Schrödinger equation with the Hamiltonian operator given by (4.16b) can give calculable eigenvalues for the high-energy part of the spectrum[8)]. Such an eigenvalue problem, namely

$$P^- \psi_B = \lambda \psi_B \quad , \tag{4.18}$$

is formidable, mainly due to fermion-number-changing terms in P^-; indeed, consider a general bosonic wave functional

$$\psi_B = \sum_{n=1}^\infty \psi_{2n} \quad , \tag{4.19}$$

8) In what follows one actually neglects pair creation and annihilation effects. The results can only be valid for the highly excited part of the spectrum.[93]

where

$$\psi_n = \int_0^1 \cdots \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \delta\left(\sum x_i - 1\right) \phi_n(x_1 \cdots x_n) \mathrm{tr}\, \psi(-x_1) \cdots \psi(-x_n)|0\rangle \quad , \quad (4.20)$$

and the total momentum P^+ has been chosen to be unity.

The fermion-changing part of P^- couples different wave functions ψ_n in (4.20), preventing a closed solution. In 'tHooft's solution one has also to make the symplifying assumption that one was computing the high-energy part of the spectrum, in which case the integral equation (2.35) simplifies, and in fact it has contribution for the left-hand side only from the singularity in (4.16c). The singular term is simpler, and one considers only such a contribution here, in which case the problem simplifies to the diagonalization of the operator:

$$P = -e^2 \int_0^\infty \mathrm{d}k_1 \cdots \mathrm{d}k_4 \, \frac{1}{(k_1 - k_4)^2} \delta(k_1 + k_2 - k_3 - k_4) \mathrm{tr}\, [b(-k_3)b(-k_4)b(k_1)b(k_2)] \quad , \quad (4.21)$$

which is effectively the result of the two-dimensional Coulomb force, and acts diagonally on the wave functionals ψ_n in (4.20). The eigenvalue equation $P\psi_B = \widetilde{\lambda}\psi_B$, can be written in terms of the wave functions ϕ_n, noticing that two annihilation operators in P produce anticommutator terms when acting on the wave functional, and the remaining terms are of the same form as the original function. It is enough then to change variable as $x_1 \to y_1$, $x_2 \to x_1 + x_2 - y_1$ as well as make cyclic permutations to arrive at the integral equation

$$\frac{\lambda}{e^2 N} \phi_n(x_1 \cdots x_n) = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{\mathrm{d}y_1}{(x_1 - y_1)^2} \phi_n(y_1, x_1 + x_2 - y_1, x_3, \cdots, x_n) \pm \text{cyclic permutations}, \quad (4.22)$$

where the sign on the right-hand side depends on how many times each fermion has jumped a fermion creation operator from the wave functional.

The general analysis of such an equation has been performed by Kutasov.[93] He found solutions for the bound states of two adjoint quarks of the type found already by 'tHooft, that is

$$\phi_2(x) = \sin \pi n_1 x \quad , \quad (4.23)$$

corresponding to the eigenvalue

$$\lambda_1 = 2e^2 N \pi^2 n \quad , \quad (4.24)$$

for n_1 even and large. But he found also higher bound states, whose wave function is a product of several symmetrized sine functions (see ref. [95] for further details), leading to the spectrum

$$M^2_{n_1 \cdots n_k} = 4e^2 N \pi^2 \sum_{i=1}^{k} n_i \quad , \quad (4.25)$$

for n_i even, and the sum large.

There is thus an exponentially growing density of states, and a consequent Hagedorn transition. However, we have to stress that not all states have been uncovered, as already mentioned.[93]

4.3 The Hagedorn transition; supersymmetry

The fact that the spectrum of the model points, once more, to the Regge behaviour, leads us to consider again the relation of QCD_2 to string theory. Moreover since we have the transverse degrees of freedom, whose role is played by the adjoint matter, we can consider a more realistic scenario. Effective interactions[96] foresee string type descriptions of the theory for the transverse oscillators. If such a string description is correct, one should expect a phase transition to occur, since the high-temperature theory is in a plasma phase, while the low-temperature physics is described by confinement. The subject is motivated by Polchinski's remark,[97] relating the statistical mechanics of string theory and large-N gauge theory, where in spherical topology the free energy is temperature independent, and one expects a transition at some critical temperature $1/\beta_c$, beyond which the leading order free energy is temperature-dependent.

The natural order parameters of the theory are Wilson loops, which at finite temperature $1/\beta$ wrap around the compactified "time" dimension used to describe temperature; therefore one considers the Wilson loop wrapped k times around the time, i.e.

$$W_k(x) = \frac{1}{N} \mathrm{tr}\, P\, \mathrm{e}^{ie\int_0^{k\beta} \mathrm{d}\tau\, A^2(\tau,x)} \quad . \tag{4.26}$$

One can consider the two-point function of the Wilson line. At low temperature it falls exponentially with distance, as we saw. Therefore we can write that

$$W_k(x) W_{-k}(0) \;\rightarrow\; \mathrm{e}^{-M_k(\beta)|x|} \tag{4.27}$$

as $|x| \rightarrow \infty$. The Wilson line for $k=1$ corresponds to an external quark, while for general k one has sources at higher representations. For low temperatures the theory confines, and one expects the area law to hold, and we have

$$M_k^2(\beta) \simeq (k\beta)^2 \quad , \tag{4.28}$$

as expected. If a phase transition to a plasma occurs, $M^2(\beta)$ will decrease, and at some critical temperature some winding modes will become tachyonic.[9)]

The study of gauge theory at finite temperature can be studied in the Coulomb gauge, which however cannot be reached, as usual ($A_0 = 0$), due to the boundary conditions;[10)] we are forced to generalize it to

$$A^0_{ab}(\tau, x) = \frac{1}{e\beta}\, \theta_a(x)\, \delta_{ab} \quad . \tag{4.29}$$

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9) In fact, a transition can occur even before the critical value is reached.

10) Since the time variable is compact, in the gauge $A_0 = 0$ one would have $\oint \mathrm{d}t\, A_0 = 0$, thus a trivial value for the Wilson loop.

As before, the other gauge-field component can be integrated out, and besides the fermionic self-interation, one is also left with a θ-field interaction. Such an integration leads to a quadratic term in the J_1 current as in the zero-temperature case, and we arrive at the effective Lagrangian

$$L_{\rm eff} = \frac{1}{e^2\beta^2}(\theta'_a)^2 + \overline{\psi}\gamma^\mu D^\theta_\mu\psi + m\overline{\psi}\psi - J_1(D^\theta_0)^{-2}J_1 \quad , \tag{4.30}$$

where

$$D^\theta_0 = \delta_{ab}\partial_0 - \frac{i}{\beta}(\theta_a - \theta_b) \quad , \quad D_1 = \partial_1 \quad . \tag{4.31}$$

It is not difficult to deal with such an effective Lagrangian in the high-temperature phase, where the charge and the mass are small, and one can sum the one-loop fermion diagrams. One has to sum over trajectories winding n times around the compact time, which leads to the expression[95]

$$S_{\rm eff} = -\frac{1}{2}\sum_{-\infty}^{\infty}{\rm tr}\int\frac{{\rm d}\tau}{\tau}\int_{{\rm periodic}\ x(t)}\mathcal{D}[x(t)]\,{\rm e}^{-\int_0^\tau {\rm d}\tau'\left[\frac{1}{4}\dot{x}^2+\frac{n^2\beta^2}{4\tau^2}-ieA_0\left(\dot{x}+\frac{n\beta}{\tau}\right)-m^2\tau\right]} \quad . \tag{4.32}$$

Since A_0 is τ-independent, the term containing it can be integrated, leading, for periodic $x(t)$, to the result tr $(-ien\beta A_0)$; taking the trace over the adjoint representation of $U(N)$, which is equivalent to the modulus squared of the trace in the fundamental representation, we obtain a factor

$$\sum_{a,b=1}^{N}{\rm e}^{in(\theta_a-\theta_b)} \simeq 2\sum_{a>b}\cos n\theta \quad . \tag{4.33}$$

Now the integration over x is a usual quantum mechanical procedure, and leads to the potential

$$V(\theta) = \frac{\beta L}{2\pi}\sum_{n=1}^{\infty}(-1)^n\int_0^\infty\frac{{\rm d}\tau}{\tau^2}\,{\rm e}^{-\frac{n^2\beta^2}{4\tau}-\tau m^2}\cos n\theta \quad , \tag{4.34}$$

with the effective Lagrangian

$$L_{\rm eff} = \frac{1}{e^2\beta}\theta'^2 + V(\theta) \quad . \tag{4.35}$$

We scale the integration variable $\tau \to \frac{1}{4}n^2\beta^2\tau$ in eq. (4.34), finding the relevant term in the high-temperature limit, contributing to the effective Lagrangian, which is given by

$$L_{\rm eff} = \frac{1}{e^2\beta}(\theta'_a)^2 + \frac{2}{\pi}\sum_{a,b=1}^{N}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\beta}\cos(n\theta_{ab}) \quad . \tag{4.36}$$

We could also follow the line of ref. [90] and introduce, at large N, the density of eigenvalues

$$\rho(\theta, x) = \frac{1}{N} \sum_a \delta(\theta - \theta_a(x)) \quad , \tag{4.37}$$

in terms of which the action becomes

$$S = \frac{N}{e^2 \beta} \int \mathrm{d}x \, \mathrm{d}\theta \, \rho^{-1} (\partial_\theta^{-1} \rho')^2 + \frac{N^2}{L} \int \mathrm{d}x \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \, \rho(\theta_1) \rho(\theta_2) V(\theta_{12}) \quad , \tag{4.38}$$

where one uses $\partial_\theta^{-1} \rho' = \frac{1}{N} \sum_a \frac{\partial \theta_a}{\partial x} \delta(\theta - \theta_a(x))$.

At large N both terms must be kept since $e^2 \sim 1/N$. The minimum of the potential implies that a possible classical configuration is where the eigenvalues are equal, that is $\rho \sim \delta(\theta - \theta_0)$, which breaks the $U(1)$ symmetry $\theta \to \theta + \epsilon$, while a symmetric phase would favour a constant distribution $\rho = 1/2\pi$. In order to study the stability of the latter, we expand ρ around it

$$\rho(\theta, x) = \frac{1}{2\pi} \left(1 + \sum_{n \neq 0} \rho_n(x) \, \mathrm{e}^{-in\theta} \right) \quad , \tag{4.39}$$

which leads to the action

$$S = N^2 \sum_{n \neq 0} \int \mathrm{d}x \, \left[\frac{1}{\beta e^2 n^2 N} (\rho_n')^2 + (-1)^n \frac{2}{\pi n^2 \beta} \rho_n^2 \right] \quad . \tag{4.40}$$

The mass of the winding states can be read from above, and we obtain

$$M_n^2(\beta \to 0) = \frac{2e^2 N}{\pi} (-1)^n \quad . \tag{4.41}$$

For n odd the winding states are tachyonic. In four-dimensional gauge theory this computation has been performed by Polchinski, who found for $V(\theta) \simeq \frac{1}{24\pi^2\beta^3} \sum \theta_{ab}^2 (2\pi - \theta_{ab})^2$, leading to a mass of the form

$$M_k^2(\beta \to 0) = - \frac{2e^2 N}{\pi^2 \beta^2 k^2} \quad . \tag{4.42}$$

These results imply deconfinement, since the area law is no longer attainable. A phase transition is related to the appearance of tachyons, since as argued in ref. [98] a divergence of the free energy in a given (Hagedorn) temperature takes place when a tachyon starts playing a role, as formed in a new mode, above that temperature.

The situation for the case of fermions in a fundamental representation seems to be different. Kutasov[93] computed the effective action for the distribution function ρ, and found no instability for any temperature, so that confinement seems to be settled for that case. For adjoint bosonic matter he found

$$M_k^2(\beta \to 0) = - \frac{22}{\pi} e^2 N \quad , \tag{4.43}$$

thus also exhibiting deconfinement.

Such a transition in adjoint matter indicates a rising density of single particle states at high energy, while for the fermion in the fundamental representation there is no such growth of the density of states with energy.

Supersymmetry

The QCD_2 action with fermions in the adjoint representation may be supersymmetric.[93] In fact, in the zero coupling limit, and with massless fermions, this can be immediately seen, the supersymmetry generator being given by

$$G = \frac{\mathrm{tr}}{3} \int \mathrm{d}x^- \, \psi\psi\psi \quad . \tag{4.44}$$

However, such a charge of fermionic character commutes with P^- as given by eq. (4.16*b*), when the theory is interactive as long as the fermions have a mass $m^2 = e^2 N$. Moreover

$$G^2 = N P^+ \quad . \tag{4.45}$$

In string theories with space-time fermions, infrared stability is achieved by a (rather fine) cancellation of bosonic and fermionic degrees of freedom, a fact requiring asymptotic supersymmetry (at high energy). Due to the enormous content of mesonic states in QCD_2 with adjoint matter the question is important as well.[93]

4.4 Landau–Ginzburg description; spectrum and string theory

One of the goals in the study of QCD_2 is to understand some technical details also existing in QCD_4, but which in two dimensions are rendered understandable or perhaps calculable, albeit not trivial. Several models serve as laboratories, as the two-dimensional non-linear σ-models, discussed in detail in [8], or QCD_2, where one finds a mass gap from dimensional transmutation, and confinement in some cases. However, as a two-dimensional counterpart of the theory of strong interactions, the pure Yang–Mills action $-\frac{1}{4}\mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}$ is not unique. It is possible to generalize such an interaction without losing several properties whose maintenance have been important up to now. Such a clever generalization, obtained in [99], maintains Migdal's heat-kernel formulation, where the partition function

$$Z[U] = \sum_R (\dim R)\, \mathrm{e}^{-\frac{ie^2}{N} a^2 C(R)} \chi_R[U] \tag{4.46}$$

has the self-reproducing property (3.18), thus being the best approximation of the continuum. Such a property is valid for an arbitrary function $C(R)$, and one can write a continuum action as

$$S = \mathrm{tr} \int \mathrm{d}^2 x \, [EF - f(E)] \quad , \tag{4.47}$$

generalizing the pure Yang–Mills action where

$$f_{YM}(E) = 2E^2 \quad . \tag{4.48}$$

According to the two-dimensional power counting, any arbitrary function $f(E)$ can be allowed instead of the quadratic term. Thus, in general we suppose that f is expandable as $f(E) = \sum_n f_n E^n$. In ref. [99], the theory of generalized QCD_2 with fermions in the fundamental representation was studied in the large-N limit. The pure gauge model was discussed in ref. [100], where they found a $1/N$ expansion similar to the one discussed in section 3, and a string interpretation.[101] Such a case is however still very simple, due to the absence of local degrees of freedom. Including matter fields, the theory is non-trivial, but still tractable since the coupling has dimension of mass, implying power-counting renormalizability.

In the $U(1)$ case one has a generalized Schwinger model, including a Landau–Ginzburg potential, and the Lagrangian density reads

$$L = E\epsilon_{\mu\nu}F^{\mu\nu} - f(E) + \overline{\psi} i \not{D}\psi - m\overline{\psi}\psi \quad . \tag{4.49}$$

Upon bosonization of the fermion and using the well known formulae[8]

$$\overline{\psi}\gamma^\mu\psi \simeq \frac{1}{\pi}\epsilon^{\mu\nu}\partial_\nu\phi \quad , \tag{4.50}$$

$$\overline{\psi} i \not{\partial}\psi \simeq \frac{1}{2\pi}\partial^\mu\phi\partial_\mu\phi \quad , \tag{4.51}$$

$$\overline{\psi}\psi = m\gamma \cos(2\phi) \quad , \tag{4.52}$$

one finds the equivalent bosonic action, that is

$$L = \frac{1}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu(E - e\phi) - f(E) + \frac{1}{2\pi}\partial^\mu\phi\partial_\mu\phi - m^2\gamma\cos(2\phi) \quad . \tag{4.53}$$

One can fix the light cone requiring that $A_1 = 0$. The A_0 equation of motion is a constraint equation (Gauss law), which demands

$$E = e(\phi + \theta/2) \quad , \tag{4.54}$$

where θ is a constant, interpreted as the label of the vacuum of the theory. One can substitute it back into the action, redefining $\phi \to \phi - \frac{\theta}{2}$ to obtain the Lagrangian

$$L = \frac{1}{2\pi}\partial^\mu\phi\partial_\mu\phi - f(e\phi) - m^2\gamma\cos(2\phi - \theta) \quad . \tag{4.55}$$

In the massless case we have a meson interacting via a Landau–Ginzburg potential $f(e\phi)$. Therefore, although there are still mesons (ϕ) that can be described as bound states of fermions, they now have complicated interactions dictated by the Landau–Ginzburg potential $f(e\phi)$.

The vacuum is described now by the potential

$$V(\phi) = f(e\phi) + m^2\gamma\cos(2\phi - \theta) \quad ; \tag{4.56}$$

notice the possibility of phase transitions at some critical points $e = e_c$.

Concerning the non-Abelian case, we have to consider the Lagrangian

$$L = \frac{N}{8\pi}\mathrm{tr}\, E\epsilon_{\mu\nu}F^{\mu\nu} - \frac{N}{4\pi}e^2\sum_{n=2}^{\infty} f_n \mathrm{tr}\left(\frac{E}{e}\right) + \overline{\psi}(i\, \not{D} - m)\psi \quad . \tag{4.57}$$

The light-cone gauge can be used as in section 2.1. The large-N limit is obtained as before once a generalized vertex E^n is included, the lines must inevitably finish in a quark line, where the f_2E^2 vertex is viewed as an interaction in the weak-coupling limit. The effect of a vertex f_nE^n as given in Fig. 6 is

$$\begin{aligned} I_n(p,p') = \int \frac{\mathrm{d}^2k_1}{(2\pi)^2}\cdots\frac{\mathrm{d}^2k_{n-1}}{(2\pi)^2}\frac{1}{p_- - k_{1\,-}}S(k_1)\frac{1}{k_{1\,-} - k_{1\,-}}S(k_2)\cdots \\ \cdots\frac{1}{k_{n-2\,-} - k_{n-1\,-}}S(k_{n-1})\frac{1}{k_{n-1\,-} - p'_-} \quad . \end{aligned} \tag{4.58}$$

Fig. 6: The generalized vertex.

Above $S(p)$ is given by (2.39c), and the self-energy generalizing (2.22) is

$$\Sigma_{SE}(p) = \sum(-i)^n n f_n I_n(p,p) \quad , \tag{4.59}$$

from which we find a two-particle irreducible kernel as given in Fig. 7,

$$K(q,q';p,p') = \sum_{n=2}^{\infty}(-i)^n f_n \sum_{l=1}^{n-1} I_l(q,q')I_{n-l}(p'-q';p-q) \quad . \tag{4.60}$$

Fig. 7: Two-particle scattering with a generalized vertex.

The important technical point used in the deduction of (2.25) from (2.22) was that it is possible to integrate over the (+) variables. This is also true above, and one arrives at

$$I_n(p,p') = \int_{\infty}^{\infty} \mathrm{d}k_1 \cdots \mathrm{d}k_{n-1} \frac{1}{p-k_1}\varepsilon(k_1)\frac{1}{k_1-k_2}\varepsilon(k_2)\cdots\varepsilon(k_{n-1})\frac{1}{k_{n-1}-p'} \quad . \tag{4.61}$$

At this point it is necessary to introduce the infrared regulator. For $n=2$ one finds

$$I_2(p,p') = \frac{2}{p-p'}\ln\left|\frac{p}{p'}\right| - \pi^2\varepsilon(p)\delta(p-p') \quad . \tag{4.62}$$

In ref. [99] the authors introduced the generating functional

$$u(p,p';z) = \sum_{n=0}^{\infty} z^{-n} I_n(p,p') \quad , \tag{4.63}$$

upon defining $I_0(p,p') = \varepsilon(p)\delta(p-p')$ and $I_1(p,p') = \frac{P}{p-p'}$. Such a function obeys a tractable integral equation. We multiply by z, and separate the first term, which is just the $I_0(p,p')$ contribution, as can be seen from eq. (4.61), that the remaining terms are of the form

$$I_{n+1}(p,p') = \int \mathrm{d}k \, \frac{P}{p-k}\varepsilon(k) I_n(k,p') \quad . \tag{4.64}$$

Therefore one arrives at an integral equation for the generating functional, as given by

$$z\, u(p,p';z) = z\,\varepsilon(p)\delta(p-p') + \int \mathrm{d}k \, \frac{P}{p-k}\varepsilon(k) u(k,p';z) \quad . \tag{4.65}$$

It is simple to find the result

$$\int_{-\infty}^{\infty} \mathrm{d}k \, \frac{P}{p-k}\varepsilon(k)|k|^{2\nu} = \pi\cot(\pi\nu)|p|^{2\nu} \quad . \tag{4.66}$$

We now use eqs. (4.61) and (4.66) to find, after consecutive integrations

$$\begin{aligned} z^{-n}\int I_n(p,k)\varepsilon(k)|k|^{2\nu}\mathrm{d}k &= z^{-n}\int \mathrm{d}k_1\cdots \mathrm{d}k_{n-1}\frac{1}{p-k_1}\cdots\frac{1}{k_{n-1}-k}\varepsilon(k)|k|^{2\nu}\mathrm{d}k \\ &= z^{-n}(\pi\cot\pi\nu)^n\varepsilon(p)|p|^{2\nu} \quad . \end{aligned} \tag{4.67}$$

Summing over z, we have

$$\varepsilon(p)\int \mathrm{d}k\, u(p,k;z)\varepsilon(k)|k|^{2\nu} = \frac{z}{z-\pi\cot\pi\nu}\varepsilon(p)|p|^{2\nu} \quad . \tag{4.68}$$

The solution to the integral equation (4.65) is unique, and can be found by inspection, with the help of eq. (4.66). One finds[99]

$$u(p,p';z) = \frac{z}{z^2+\pi^2}\left[P\frac{1}{p-p'}\left|\frac{p}{p'}\right|^{2\alpha(z)} + z\varepsilon(p)\delta(p-p')\right] \quad , \tag{4.69}$$

where $\alpha(z) = \frac{1}{\pi}\arctan\frac{\pi}{z}$. The functions $I_n(p,p')$ can now be found expanding (4.69) as a series in z^{-1}. We do not need a systematic computation for them. We consider the steps analogous to those used in (2.21) through (2.35). We can express the self-energy as

$$\Sigma_{SE}(p) = \sum_n (-1)^n n f_n I_n = \oint \frac{dz}{2\pi i} f'(z) u(p,p';iz) \quad , \tag{4.70}$$

since only the term $\sim z^{-1}$ survives in the right-hand side; therefore one finds the right-hand side of the integral equation

$$\left[q_+ - m^2\left(\frac{1}{p_-} + \frac{1}{q_- - p_-}\right)\right]\varphi(p_-;q)$$
$$= \left[\Sigma_{SE}(p_-) + \Sigma_{SE}(q_- - p_-)\right]\varphi(p_-;q) + \int_0^{q_-} dk_- K(p,k;q,q)\varphi(k_-,q) \quad ,$$

which is the generalized counterpart of (2.34a). Its right-hand side reads

$$\begin{aligned} 2e^2 \int \frac{dz}{2\pi i} f'(z) \Big\{ &- \left[u(q,q;iz) + u(p-q,p-q;iz)\right]\phi(q) \\ &+ 2\int_0^p dk\, u(q,k;iz) u(p-k,p-q;iz)\phi(k) \Big\} \\ = 2e^2 \int \frac{dz}{2\pi i} f'(z) \Big\{ &- iz \frac{z^2+\pi^2}{(z^2-\pi^2)^2} 2\alpha(iz) \left(\frac{1}{q} + \frac{1}{p-q}\right)\phi(q) \\ &- \frac{2z^2}{(z^2-\pi^2)^2} \int_0^p dk\, P\frac{1}{(q-k)^2}\left[\frac{q(p-k)}{(p-q)k}\right]^{2\alpha(iz)} \phi(k) \Big\} \quad , \end{aligned} \tag{4.71}$$

from which one finally obtains the bound-state equation in the form

$$\begin{aligned} &\left[\mu^2 - \tau\left(\frac{1}{x} + \frac{1}{1-x}\right)\right]\phi(x) \\ &= 2\pi \int \frac{dz}{2\pi i} f'(z) \Big\{ - iz \frac{z^2+\pi^2}{(z^2-\pi^2)^2} 2\alpha(iz) \left(\frac{1}{x} + \frac{1}{1-x}\right)\phi(x) \\ &- \frac{2z^2}{(z^2-\pi^2)^2} \int_0^1 dy\, P\frac{1}{(x-y)^2}\left[\frac{x(1-y)}{(1-x)y}\right]^{2\alpha(iz)} \phi(y) \Big\} . \end{aligned} \tag{4.72}$$

This result reveals that information can be obtained about the spectrum in generalized QCD_2. There are corrections to 'tHooft's equation. In ref. [99] such corrections have been exemplified for a quartic potential. In particular, the authors showed that for an arbitrary potential $f(z)$ there is always a massless eigenstate, arguing that it is a consequence of the chiral $U(1)$ symmetry in the large-N limit, where the $U(1)$ anomaly is suppressed in lowest order of $1/N$.

5. Algebraic aspects of QCD_2 and integrability

We saw that two-dimensional QCD, although not exactly soluble, in terms of free fields, is a theory from which some valuable results may be obtained. The $1/N$ expansion reveals a simple spectrum valid for weak coupling, while the strong coupling offers the possibility of understanding the baryon as a generalized sine-Gordon soliton. Moreover, the $1/N$ expansion of the pure-gauge case may be performed, and the partition function is equivalent to one of a string model described by a topological field theory, the Nambu–Goto string action, and presumably terms preventing folds.

All such results point to a relatively simple structure, which could be mirrored by an underlying symmetry algebra. In fact such algebraic structures do exist. In the above-mentioned case of the large-N expansion of pure QCD_2, one finds a W_∞-structure related to area-preserving diffeomorphisms of the Nambu–Goto action. A W_∞ structure for gauge-invariant bilinears in the Fermi fields is constructed[102] - (see section 5.1). Such is an algebra which appears also in fermionic systems, and in the description of the quantum Hall effect[35]. Moreover, as shown before, pure QCD_2 is equivalent to the $c = 1$ matrix model,[103] which has also a representation in terms of non-relativistic fermions,[104] and contains a W_∞ algebra[36,105,106] as well. The problem is also related to the Calogero–Sutherland models.[107] The mass eigenstates build a representation of the W_∞ algebra as found in [102].

After bosonizing the theory, further algebraic functions of the fields turn out to obey non-trivial conservation laws, as we will see. The theory can be related to a product of several conformally invariant WZW sectors, a perturbed WZW sector, all related by means of BRST constraints, which play a very important role in gauge theories, as described in section 2. A dual formulation exists and permits us to study the theory in two limits, both strong and weak couplings. Finally, once displayed, the relation to Calogero systems and further integrable models is also amenable to understanding in the previous framework.

5.1 W_∞ algebras for colourless bilinears

Let us consider the QCD_2 Lagrangian with massive fermions in the fundamental representation, as in section 2. Using light-cone coordinates, the Lagrangian is given by

$$\begin{aligned} L &= -\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\,\not{D} - m)\psi \\ &= \frac{1}{8}\mathrm{tr}\, F_{+-}^2 + \psi_-^\dagger(i\partial_+ + eA_+)\psi_- + \psi_+^\dagger(i\partial_- + eA_-)\psi_+ - m(\psi_+^\dagger\psi_- + \psi_-^\dagger\psi_+) \quad ,(5.1) \end{aligned}$$

where we use the notation $\psi = \begin{pmatrix}\psi_- \\ \psi_+\end{pmatrix}$.

As usual, we get a considerable simplification while working at the light-cone gauge, e.g. $A_+ = 0$, in which case ψ_- decouples, up to the mass term. Furthermore, we can also quantize the theory in the light cone, regarding x^+ as time and x^- as space, in which case the equation of motion of ψ_+ is actually a constraint equation, as we have done in section 4 for different reasons. The light-cone Hamiltonian, which corresponds to the + component of the energy–momentum tensor, is given by the expression

$$H \equiv P_+ = \int \mathrm{d}x^- \left[\frac{1}{2}\mathrm{tr}\, E^2 + m(\psi_+^\dagger \psi_- + \psi_-^\dagger \psi_+)\right] \quad , \tag{5.2}$$

where $E = \frac{1}{2}\partial_+ A_-$, the momentum canonically conjugated to A_-; therefore the canonical quantization is achieved by the commutation rule

$$\left[A_-(x^-, x^+), E(y^-, x^+)\right] = i\delta(x^- - y^-) \quad . \tag{5.3}$$

The equation of motion of A_+ is equivalent to the Gauss law

$$\partial_- E^{ab} - ie[A_-, E]^{ab} + 2e\left(\psi_-^{\dagger\, a}\psi_-^b - \frac{1}{N}\delta^{ab}\psi_-^{\dagger c}\psi_-^c\right) = 0 \quad , \tag{5.4}$$

where the last term drops out for the $U(N)$ case.

For the equation of motion of ψ_+, we obtain further constraints (no light-cone time derivative), that is

$$(i\partial_- + eA_-)\psi_+ - m\psi_- = 0 \quad , \tag{5.5}$$

and the corresponding complex conjugate equation. These constraints are important in the computation of correlators involving ψ_+, otherwise we do not need them. The Gauss law constraint can also be used to solve the electric field in terms of the Fermi fields, once the full gauge arbitrariness is fixed.

However we will follow another trend, defining the Wilson (open) operator by means of the bilinear[102]

$$M_{\alpha\beta,ij}(x^-, y^-; x^+) = \psi_{i\alpha}(x^-, x^+)\, \mathrm{e}^{ie\int_{x^-}^{y^-} A_-(z^-, x^+)\mathrm{d}z^-}\, \psi_{j\beta}^\dagger(y^-, x^+) \quad , \tag{5.6}$$

where $i, j = 1, \cdots, F$ are indices of flavour. In the Hamiltonian formalism one works at a given (light-cone) time x^+. Indeed, H is time-independent, and we can choose, for definiteness, a fixed time x_0^+. At that given fixed point, one can always choose $A_-(x^-, x_0^+) = 0$. The same can be done in the case of the algebra obeyed by the bilinear. We are going to compute the algebra at equal times, and in such a gauge one finds

$$M_{\alpha\beta,ij} = \psi_{i\alpha}(x^-, x^+)\psi_{j\beta}^\dagger(y^-, x^+) \quad , \tag{5.7}$$

for which it is simple to find the corresponding algebra, given the canonical equal time anticommutator of the fermions:

$$\{\psi_-(x^-, x^+), \psi_-^\dagger(y^-, x^+)\} = \delta(x^- - y^-) \quad , \tag{5.8a}$$

$$\begin{aligned}[M_{ij}(x^-, y^-; x^+), M_{kl}(x'^-, y'^-; x^+)] = {} & \delta_{jk}\delta(y^- - x^-)\, M_{il}(x^-, y'^-; x^+) - \\ & - \delta_{il}\delta(y'^- - x^-)\, M_{kj}(x'^-, y^-; x^+) \quad , \end{aligned} \tag{5.8b}$$

which is an infinite algebra of the type $W_\infty \otimes U(F)$. For the one-flavour case, it simplifies to the usual W_∞ algebra, of the type found in $c = 1$ matrix models. In fact, there are several types of W algebras obeyed by bilinears constructed out of representations of $U(F)$ groups. For off-critical perturbations of a free-fermion system one finds a rather elaborate $W_{1+\infty}$ algebra as obeyed by higher-spin currents.[108] Here it is very interesting to notice the rather simple structure obeyed by the above bilinear.

Still in the gauge $A_- = 0$, achieved at a definite time, the electric field is given by the expression

$$E^{ab}(x^-, x^+) = -e \int \mathrm{d}y^- \, \varepsilon\left(x^- - y^-\right) \, \psi_-^{\dagger\, b}(y^-, x^+)\psi_-^a(y^-, x^+) \quad , \tag{5.9}$$

as deduced from the Gauss law, which after substitution upon the Hamiltonian leads to the result

$$\begin{aligned} H = \frac{e^2}{4N} \int \mathrm{d}x^- \, \mathrm{d}y^- \, \Big[& \psi_{i-}^a(x^-)\psi_{j-}^{\dagger a}(x^-) \left|x^- - y^-\right| \psi_{j-}^b(y^-)\psi_{i-}^{\dagger b}(y^-) \\ & - \frac{1}{N}\psi_{i-}^a(x^-)\psi_{i-}^{\dagger a}(x^-) \left|x^- - y^-\right| \psi_{j-}^b(y^-)\psi_{j-}^{\dagger b}(y^-) \\ & - \frac{im^2}{4}\varepsilon\left(x^- - y^-\right) \psi_{i-}^a(x^-)\psi_{i-}^{\dagger a}(y^-) \Big] \quad . \end{aligned} \tag{5.10a}$$

It can be written solely in terms of the previously defined bilinears as

$$\begin{aligned} H = N \int \mathrm{d}x^- \, \mathrm{d}y^- \, \Big[& \frac{e^2}{4} M_{--,ij}(x^-, y^-) |x^- - y^-| M_{--,ij}(x^-, y^-) \\ & - \frac{e^2}{4N} M_{--,ii}(x^-, x^-) |x^- - y^-| M_{--,jj}(y^-, y^-) \\ & - \frac{im^2}{4}\varepsilon\left(x^- - y^-\right) M_{--,ii}(x^- - y^-) \Big] \quad . \end{aligned} \tag{5.10b}$$

Finally, one can use the fermionic constraints at the given time to compute M_{++} and M_{+-} in terms of M_{--}. The bilinears also obey quadratic constraints, and can be seen to imply the bound-state structure obtained from the large-N expansion as proposed by 'tHooft.

The realization of the W_∞ algebra in terms of the above fields $M(x, y)$ leads us to a close relation to string theory, and to 'tHooft's spectrum derived in section 2.

It is natural, in view of the results of section 2, which are derived for large N, to study the "string" field $M(x, y)$ in such a limit, where it becomes classical, being related to a (self-consistent Hartree-Fock) potential, where the fermions move. It is, in fact, not difficult to obtain the solution to the equation of motion obeyed by $M(x, y)$, fulfilling the constraint equation ($M^2(x, y) = M(x, y)$), obtained in ref. [102], to which we refer for details. This constraint is a consequence of colour invariance $E^{ab}(x^- = \infty) = E^{ab}(x^- = -\infty)$ (see eq. (5.9)). One can also define the baryon number $B = \mathrm{tr}\,(1 - M)$.

The classical solution of the constraint equations is given, in terms of the Fourier transform, for a single flavour as

$$M_0(k_-, k'_-; x^+) = \delta(k_- - k'_-)\theta(k_-) \quad , \tag{5.11}$$

where the Fermi level was choosen for $B = 0$. The fluctuations around (5.11) can be computed for large N, as

$$M = \mathrm{e}^{\frac{i}{\sqrt{N}} w}\, M_0\, \mathrm{e}^{-\frac{i}{\sqrt{N}} w} \quad , \tag{5.12}$$

where w represents a perturbation around the classical solution M_0. The main result found in ref. [102] is that the x^+-Fourier transform of $w^{-+}(k, k'; x^+) = w(k, -k'; x^+)$ for $k, k' > 0$ is exactly 'tHooft's wave function,

$$w^{-+}(k, k', x^+) = \int \frac{\mathrm{d}q^+}{2\pi} \varphi(q_- = k_- + k'_-, q_+; x^+) \mathrm{e}^{iq_+ x^+} \quad , \tag{5.13}$$

leading to (2.35) for $\varphi(q_-, q_+; x^+)$. This means that the mesons of two-dimensional QCD form a representation of the W algebra. As we mentioned, in the bosonized version it has been argued[142] that the mesons obey an equation of motion presumably corresponding to 'tHooft's equation, (2.35) which would imply a rather explicit realization of the given symmetry. For any number of flavours one has a representation of $W_\infty \otimes U(N)$.[11)]

5.2 Integrability and duality

We have seen in section 2 that after integrating out the fermions and performing a set of field transformations we arrive at a product of conformally invariant theories including a WZW theory with a non-local mass term. We suppose that the BRST constraints define the physical states, and at the Lagrangian level we consider the perturbed WZW action[46,47]

$$\begin{aligned} S &= \Gamma[\beta] + \frac{1}{2}\mu^2 \mathrm{tr} \int \mathrm{d}^2 z \, \left[\partial_+^{-1}(\beta^{-1}\partial_+\beta)\right]^2 \quad , \\ &= \Gamma[\beta] + \frac{1}{2}\mu^2 \Delta(\beta) \quad . \end{aligned} \tag{5.14}$$

We will look for the Euler–Lagrange equations for β. It is not difficult to find the variations:

$$\delta\Gamma[\beta] = \left[\frac{1}{4\pi}\partial_-(\beta^{-1}\partial_+\beta)\right]\beta^{-1}\delta\beta \quad , \tag{5.15a}$$

$$\delta\Delta(\beta) = 2\Big(\partial_+^{-1}(\beta^{-1}\partial_+\beta) - \left[\partial_+^{-2}(\beta^{-1}\partial_+\beta), (\beta^{-1}\partial_+\beta)\right]\Big)\beta^{-1}\delta\beta \quad . \tag{5.15b}$$

Collecting the terms, we find it useful to define the current components

$$\begin{aligned} J_+^\beta &= \beta^{-1}\partial_+\beta \quad , \\ J_-^\beta &= -4\pi\mu^2\partial_+^{-2} J_+^\beta = -4\pi\mu^2\partial_+^{-2}(\beta^{-1}\partial_+\beta) \quad , \end{aligned} \tag{5.16}$$

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11) Actually we have $W_{\infty+} \otimes W_{\infty-}$, see ref. [102] for details.

which summarize the β equation of motion as a zero-curvature condition given by

$$[\mathcal{L},\mathcal{L}] = [\partial_+ + J_+^\beta, \partial_- + J_-^\beta] = \partial_- J_+^\beta - \partial_+ J_-^\beta + [J_-^\beta, J_+^\beta] = 0 \quad . \tag{5.17a}$$

This is the integrability condition for the Lax pair[110]

$$\mathcal{L}_\mu M = 0 \quad , \quad \text{with} \quad \mathcal{L}_\mu = \partial_\mu - J_\mu^\beta \quad , \tag{5.17b}$$

where $J_\pm^\beta = J_0^\beta \pm J_1^\beta$ and M is the monodromy matrix. This is not a Lax pair as in the usual non-linear σ-models,[109] where J_μ^β is a conserved current, and where we obtain a conserved non-local charge from (5.17a), as well as higher local and non-local conservation laws, derived from an extension of (5.17a) in terms of an arbitrary spectral parameter.[110] However, to a certain extent, the situation is simpler in the present case, due to the rather unusual form of the currents (5.16), which permits us to write the commutator appearing in (5.17a) as a total derivative, in such a way that in terms of the current J_-^β we have

$$\partial_+ \left(4\pi\mu^2 J_-^\beta + \partial_+\partial_- J_-^\beta + [J_-^\beta, \partial_+ J_-^\beta] \right) = 0 \quad . \tag{5.18a}$$

Therefore the quantity

$$I_-^\beta(x^-) = 4\pi\mu^2 J_-^\beta(x^+,x^-) + \partial_+\partial_- J_-^\beta(x^+,x^-) + [J_-^\beta(x^+,x^-), \partial_+ J_-^\beta(x^+,x^-)] \tag{5.18b}$$

does not depend on x^+, and it is a simple matter to derive an infinite number of conservation laws from the above. These are non-local conservation laws, as is clear from (5.16).

This means that two-dimensional QCD is an integrable system![8,110,111] Moreover, it corresponds to an off-critical perturbation of the WZW action. If we write $\beta = \mathrm{e}^{i\phi} \sim 1 + i\phi$, we verify that the perturbing term corresponds to a mass term for ϕ. The next natural step is to obtain the algebra obeyed by (5.18b), and its representation. However, there is a difficulty presented by the non-locality of the perturbation. We now introduce a further auxiliary field defining a dual action, local in all fields, and representing the low-energy scales of the theory, and we later return to the problem of finding the algebra obeyed by (5.18b).

Consider the Δ-term of the action (5.14). We rewrite it introducing the integral over a Gaussian field C_- as

$$\mathrm{e}^{\frac{i}{2}\mu^2\Delta} = \int \mathcal{D}C_- \, \mathrm{e}^{i\int \mathrm{d}^2x \, \frac{1}{2}\mathrm{tr}\,(\partial_+ C_-)^2 - \mu\,\mathrm{tr}\int \mathrm{d}^2x \, C_-(\beta^{-1}\partial_+\beta)} \quad , \tag{5.19}$$

where the left-hand side is readily obtained by completing the square in the right-hand side.

Indeed, at this point we have two choices. We can proceed with the canonical quantization of the action (5.14) with the non-local term substituted in terms of the C_- field-dependent expression obtained in the exponent of the integrand of the right-hand side of eq. (5.19). Before that, motivated by the presence of the auxiliary vector field C_-, we again make a change of variables of the type

$$C_- = \frac{i}{4\pi\mu} W\partial_- W^{-1} \quad , \tag{5.20a}$$

$$\mathcal{D}C_- = \mathrm{e}^{-ic_V\Gamma[W]}\mathcal{D}W \quad , \tag{5.20b}$$

together with the now very frequently used identity (2.78) in order to find a dual action. We have for the β-partition function the expression

$$\mathcal{Z} = \int \mathcal{D}\beta\mathcal{D}W\, \mathrm{e}^{i\Gamma[\beta]-ic_V\Gamma[W]+\frac{i}{4\pi}\int \mathrm{d}^2x\, W\partial_- W^{-1}\beta^{-1}\partial_+\beta - i\int \mathrm{d}^2x\, \frac{1}{2(4\pi\mu)^2}[\partial_+(W\partial_- W^{-1})]^2} \quad , \tag{5.21}$$

from which we can separate the contribution $-\Gamma[\beta W] \equiv -\Gamma[\widetilde{\beta}]$; after such manœuvre we are left with

$$\mathcal{Z} = \int \mathcal{D}\widetilde{\beta}\, \mathrm{e}^{i\Gamma[\widetilde{\beta}]} \int \mathcal{D}W\, \mathrm{e}^{-i(c_V+1)\Gamma[W] - \frac{i}{2(4\pi\mu)^2}\mathrm{tr}\int \mathrm{d}^2z\,[\partial_+(W\partial_- W^{-1})]^2} \quad . \tag{5.22}$$

The dual action now has a coupling constant corresponding to the inverse of the initial charge. Therefore eq. (5.22) is appropriate to the study of a strongly coupled limit. Notice that the procedure is, in a sense, similar to the one used to obtain a dual action, where a non-dynamical field is introduced, and one eliminates the original fields by integration, leaving the so-called dual formulation. (See refs. [112-114] for further details on duality.) We separate a further WZW-conformal piece, and we are left with a local massive action for W. The drawback is the fact that now W itself has an action with a negative sign. Naïvely it also describes massive excitations, although a complete description of the spectrum can only be obtained after disentangling the non-linear relations and imposing the BRST conditions.

For the sources, we replace A_- (see eq. (2.108c)) by $\frac{i}{e}(U^{-1}W\widetilde{\beta}^{-1}\widetilde{\Sigma})\partial_-(\widetilde{\Sigma}^{-1}\widetilde{\beta}W^{-1}U)$. We also notice here that we have dual descriptions of QCD_2. In the first, valid in the perturbative region, for high energies, we find out a non-local perturbation of the WZW action. In terms of W the perturbation is local, but at the price of a negative sign in the naïve kinetic term in the W action, which is appropriate to describing the low energy (strong coupling) regime of the theory. In spite of such different complementary descriptions, both models are integrable. In the weak coupling regime we found the conservation laws (5.14–17). In the case of the W-theory, it is not difficult to find the equations of motion, and again derive similar relations for the quantity

$$I_-^W(x^-) = \frac{1}{4\pi}(c_V+1)J_-^W(x^+,x^-) + \frac{1}{(4\pi\mu)^2}\partial_+\partial_- J_-^W(x^+,x^-) + \frac{1}{(4\pi\mu)^2}[J_-^W, \partial J_-^W](x^+,x^-), \tag{5.23}$$

with $J_-^W = W\partial_- W^{-1}$ and $\partial_+ I_-^W = 0$, i.e. I_-^W does not depend on x^+. The conservation laws are local in this formulation.

Therefore, after finding isomorphic higher charges for both formulations, we are motivated to find their corresponding algebras, and later quantize them.

To obtain the algebra obeyed by the previously found conserved charges, it is easier to proceed with the canonical quantization,[115] obtaining first the Poisson algebra, and later the constraints and quantum commutators of the model. In fact, from the computation of the fermion determinant, we have an effective bosonic action that already takes into account some quantum corrections, namely the fermionic loops have been summed up. Therefore, the Poisson brackets already have quantum corrections arising from fermionic loops. This fact minimizes the possibilities of anomalies in the full quantum definition of the charges.[116] As a matter of fact, we shall see that quantum corrections are restricted to the introduction of renormalization constants.

We thus have to deal with the action

$$S=-(c_V+1)\Gamma[W]-\frac{1}{2(4\pi\mu)^2}\int \mathrm{d}^2x\ \left[\partial_+(W\partial_-W^{-1})\right]^2 \quad , \tag{5.24}$$

with the WZW functional given by

$$\Gamma[W]=\frac{1}{8\pi}\mathrm{tr}\int \mathrm{d}^2x\,\partial^\mu W^{-1}\partial_\mu W+\frac{1}{4\pi}\epsilon^{\mu\nu}\mathrm{tr}\int_0^1 \mathrm{d}r\int \mathrm{d}^2x\,\hat W^{-1}\dot{\hat W}\hat W^{-1}\partial_\mu\hat W\hat W^{-1}\partial_\nu\hat W \quad . \tag{5.25}$$

Due to the presence of higher derivatives in the above action, it is convenient to introduce an auxiliary field and rewrite it in the equivalent form

$$S=-(c_V+1)\Gamma[W]+\mathrm{tr}\,\frac{1}{2}\int \mathrm{d}^2x\,\left(-B^2+\frac{1}{2\pi\mu}\partial_+B\partial_-WW^{-1}\right) \quad , \tag{5.26}$$

where (4.1) is obtained by completing the square in the B-term in (5.26). The momentum canonically conjugated to the variable W is

$$\begin{aligned}\Pi^W_{ij}&=\frac{\partial S}{\partial\partial_0 W_{ij}}=-\frac{1}{4\pi}(c_V+1)\partial_0W^{-1}_{ji}-\frac{1}{4\pi}(c_V+1)A_{ji}+\frac{1}{4\pi\mu}(W^{-1}\partial_+B)_{ji}\\ &=\hat\Pi^W_{ij}-\frac{1}{4\pi}(c_V+1)A_{ji} \quad ,\end{aligned} \tag{5.27}$$

where the first term is obtained from the principal σ-model term in the WZW action, the second arises from the pure WZW term, and the third from the interaction with the auxiliary field. It is convenient to separate the WZW contribution A_{ij} to the momentum, since the new variable $\widehat\Pi^W$ is local in the original fields. The treatment of the WZ term, leading to A_{ij}, on the right-hand side above, follows closely ref. [115], see also ref. [8]. An explicit form for A_{ij} cannot be obtained in terms of local fields, but we need only its derivatives, which are not difficult to obtain, i.e.[8,115]

$$F_{ij;kl}=\frac{\delta A_{ij}}{\delta W_{lk}}-\frac{\delta A_{kl}}{\delta W_{ji}}=\partial_1W^{-1}_{il}W^{-1}_{kj}-W^{-1}_{il}\partial_1W^{-1}_{kj} \quad , \tag{5.28}$$

in terms of which we have the Poisson-bracket relation

$$\begin{aligned}\left\{\hat\Pi^W_{ij}(x),\hat\Pi^W_{kl}(y)\right\}&=-\frac{c_V+1}{4\pi}\left(\frac{\delta A_{lk}}{\delta W_{ij}}-\frac{\delta A_{ji}}{\delta W_{kl}}\right)\\ &=\frac{c_V+1}{4\pi}\left(\partial_1W^{-1}_{jk}W^{-1}_{li}-\partial_1W^{-1}_{li}W^{-1}_{jk}\right)\delta(x^1-y^1) \quad .\end{aligned} \tag{5.29}$$

The momentum associated with the B field is

$$\Pi^B_{ij} = -\frac{1}{4\pi\mu}(W\partial_- W^{-1})_{ji} \quad . \tag{5.30}$$

We can now list the relevant field operators appearing in the definition of the conservation law (5.23), that is

$$\begin{aligned} I^W_- &= \frac{1}{4\pi}(c_V+1)J^W_- + \frac{1}{(4\pi\mu)^2}\partial_+\partial_- J^W_- - \frac{1}{(4\pi\mu)^2}[J^W_-, \partial_+ J^W_-] \quad , \\ \partial_+ I^W_- &= 0 \quad . \end{aligned} \tag{5.31}$$

In terms of phase-space variables, they are

$$\begin{aligned} J^W_- &= W\partial_- W^{-1} = -4\pi\mu\widetilde{\Pi}_B \quad , \\ \partial_+ J^W_- &= -\,4\pi\mu\partial_+\widetilde{\Pi}_B = 4\pi\mu B \quad , \\ \partial_+\partial_- J^W_- &= (4\pi\mu)^2\left[W\widetilde{\hat{\Pi}}^W - (c_V+1)\mu\widetilde{\Pi}_B\right] \\ &\quad - (4\pi\mu)\mu(c_V+1)W'W^{-1} - 8\pi\mu B' \quad , \end{aligned} \tag{5.32}$$

where the tilde means a transposition of the matrix indices. It is straightforward to compute the Poisson algebra. We have

$$\left\{I^W_{ij}(t,x), I^W_{kl}(t,y)\right\} = \left[I^W_{kj}\delta_{il} - I^W_{il}\delta_{kj}\right]\delta(x^1-y^1) - \alpha\delta^{il}\delta^{kj}\delta'(x^1-y^1) \quad , \tag{5.33}$$

where $\alpha = \frac{1}{2\pi}(c_V+1)$. The affine algebra is thus realized while acting on the current operator, since

$$\begin{aligned} \left\{I^W_{ij}(t,x), J^W_{-kl}(t,y)\right\} &= (J^W_{-kj}\delta_{il} - J^W_{-il}\delta_{kj})\delta(x^1-y^1) \\ &\quad + 2\delta_{il}\delta_{kj}\delta'(x^1-y^1) \quad , \\ \left\{J^W_{ij}(t,x), J^W_{-kl}(t,y^1)\right\} &= 0 \quad . \end{aligned} \tag{5.34}$$

We thus obtain a current algebra for I^W_-, acting on J^W_- with a central extension. We shall return to this discussion later, after consideration of the quantization of the charge.

The Hamiltonian density can also be computed, and we arrive at the phase-space expression

$$\begin{aligned} H_W &= \widetilde{\hat{\Pi}}^W W' + 4\pi\mu\widetilde{\hat{\Pi}}^W \widetilde{\Pi}^B W - \widetilde{\Pi}^B B' - 4\pi\mu^2(c_V+1)(\widetilde{\Pi}^B)^2 \\ &\quad - 2(c_V+1)\mu\widetilde{\Pi}^B W'W^{-1} + \frac{1}{4\pi}(c_V+1)(W'W^{-1})^2 + \frac{1}{2}B^2 \quad , \end{aligned} \tag{5.35}$$

where $B' = \partial_1 B$, $W' = \partial_1 W$; the above Hamiltonian can be rewritten in a quadratic form in terms of the currents, although in such a case we also have velocities, due to the appearance of the time derivatives:

$$H_W = \alpha\left(J^W_1\right)^2 - \frac{1}{(4\pi\mu)^2}\left[\partial^2_+ J^W_- J^W_+ + J^W_-\partial_-\partial_+ J^W_- - (\partial_+ J^W_-)^2\right] \quad , \tag{5.36}$$

where $J_1^W = \frac{1}{2}(J_+^W - J_-^W)$ and $J_+^W = W\partial_+ W^{-1}$. At this point we can compare the model with its β formulation. In this case we have the action

$$S = \Gamma[\beta] + i\mu \mathrm{tr} \int \mathrm{d}^2 x\, C_- \beta^{-1}\partial_+\beta + \frac{1}{2}\mathrm{tr}\int \mathrm{d}^2 x\,(\partial_+ C_-)^2 \quad . \tag{5.37}$$

The canonical quantization proceeds straightforwardly, and the relevant phase-space expressions are obtained for J_-^β in (5.16), which, due to the C_- equation of motion, read

$$J_-^\beta = -4\pi\mu^2\partial_+^{-2}(\beta^{-1}\partial_+\beta) = 4i\pi\mu C_- \quad , \tag{5.38a}$$

$$\Pi_- = \partial_+ C_- \quad , \tag{5.38b}$$

while the β-momentum is given by

$$\tilde{\hat{\Pi}}^\beta_{ji} = \frac{1}{4\pi}\partial_0\beta^{-1}_{ji} + \mu(C_-\beta^{-1})_{ji} \quad , \tag{5.39}$$

where the hat above Π^β means that we have neglected the WZW contribution as before,[115] and as a consequence

$$\left\{\tilde{\hat{\Pi}}^\beta_{ji}(t,x), \tilde{\hat{\Pi}}^\beta_{lk}(t,y)\right\} = -\frac{1}{4\pi}\left(\partial_1\beta^{-1}_{jk}\beta^{-1}_{li} - \partial_1\beta^{-1}_{li}\beta^{-1}_{jk}\right)\delta(x-y) \quad . \tag{5.40}$$

From the definition of the canonical momentum associated with C_- we have

$$\partial_+ J_-^\beta = 4i\pi\mu\Pi_- \quad . \tag{5.41}$$

The conserved charge is

$$\begin{aligned} I_-^\beta &= 4\pi\mu^2 J_-^\beta + \partial_+\partial_- J_-^\beta + [J_-^\beta, \partial_+ J_-^\beta] \quad , \\ \partial_+ I_-^\beta &= 0 \quad ; \end{aligned} \tag{5.42}$$

therefore the situation is analogous to the one we found previously by interchanging the (B, Π_B) phase-space variables with $(\Pi_-, -C_-)$ (noticing the exchanged order).

At this point the Hamiltonian might be computed. However, we will postpone that, since we will have to compute it in terms of more appropriate currents, making the problem easier to formulate in terms of the constraints that are hidden in the gauge-transformation properties.

We now come to the point where we should consider the quantization of the symmetry current (5.42). Let us consider the problem in the β language, since the short-distance expansion depends on the high-energy behaviour of the theory; since the only massive scale is the coupling constant, we have to consider the weak coupling limit. This limit is better described by the β action. In such a case, we need the short-distance expansion of the current $J_-^\beta = -4\pi\mu^2\partial_+^{-2}(\beta^{-1}\partial_+\beta)$ with itself. Since the short-distance expansion is compatible with the weak coupling limit, where the theory is conformally invariant, Wilson expansions can be dealt with in the usual way.

Due to the renormalization of the higher charge, we cannot give an interpretation of the field operator I^β_{ij} by itself, but only to an arbitrary linear combination involving the charge and the current. In any case, since I^β_{ij} is a right-moving field operator, it is natural to assume, in view of the Poisson algebra (5.33), that it obeys an algebra given by[117,118]

$$I^\beta_{ij}(x^-)I^\beta_{kl}(y^-) = (I^\beta_{kj}\delta_{il} - I^\beta_{il}\delta_{kj})(y^-)\frac{1}{x^- - y^-} - \alpha\frac{\delta^{il}\delta^{kj}}{(x^- - y^-)^2} \quad . \tag{5.43}$$

For J^β_{-ij} we are forced into a milder assumption. Indeed, the equation $\partial_+ J^\beta_{-ij} = 0$ would be too simple to realize the whole problem we are considering. In such a case we would be left with unequal time commutators for the second of eqs. (5.34). But in any case, since I^β_{ij} is a right-moving field operator, the equal-time requirement in the first of eqs. of (5.34) is also superfluous, and we get an operator-product algebra of the type

$$I^\beta_{-ij}(x^-)J^\beta_{-kl}(y^+,y^-) = (J^\beta_{-kj}\delta_{ij} - J^\beta_{-il}\delta_{kj})(y^+,y^-)\frac{1}{x^- - y^-} + 2\frac{\delta^{il}\delta^{kj}}{(x^- - y^-)^2} \quad . \tag{5.44}$$

The second equation in (5.34) cannot be taken at arbitrary times, since J^β_- depends on both x^+ *and* x^-. Moreover, if J^β_- were purely right-moving, the second equation would imply, for unequal times, that it is a trivial operator.

Some conclusions may be drawn for J^β_-. As we stressed above, $\partial_+ J^\beta_-$ cannot be zero[12)], in the full quantum theory; however, in view of (5.44), we conclude that left (−) derivatives of this current are primary fields,[118] since

$$I^\beta_{ij}\partial^n_+ J^\beta_{-kl} = \frac{1}{x^- - y^-}\left(\partial^n_+ J^\beta_{-il}\delta_{kj} - \partial^n_+ J^\beta_{-kl}\delta_{il}\right) \quad . \tag{5.45}$$

Therefore, we expect an affine Lie algebra for $I^\beta(x^+)$, and $\partial^n_+ J^\beta_-$ should be primary fields depending on parameters x^-.

Such an underlying structure is a rather unexpected result, since it arose out of a non-linear relation obeyed by the current, which can be traced back to an integrability condition of the model. Moreover, the theory has an explicit mass term – although free massive fermionic theories as well as some off-critical perturbations of conformally invariant theories in two dimensions may contain affine Lie symmetry algebras.

On the current itself there is now a realization of such an algebra in the right-moving sector.[119]

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12) In the case where J^β_- is left-moving, we expect further modifications of the commutators.

Reobtaining the $U(1)$ case

Again, as in the case of the β-Lagrangian, we may study the $U(1)$ limit by writing

$$W = \mathrm{e}^{2i\sqrt{\pi}\Xi} \quad , \tag{5.46}$$

to find

$$L = -\frac{1}{2}\left(\partial_\mu \Xi\right)^2 + \frac{\pi}{2e^2}\left(\partial^2 \Xi\right)^2 \quad . \tag{5.47}$$

The Ξ propagator is

$$D_\Xi = \frac{e^2}{\pi}\frac{1}{p^2\left(p^2 - \frac{e^2}{\pi}\right)} = -\frac{1}{p^2} + \frac{1}{p^2 - \frac{e^2}{\pi}} \quad , \tag{5.48}$$

which describes again a massive excitation corresponding to the previous Σ field (see eqs. (2.109–111)) and a massless negative metric excitation.

5.3 Constraint structure of the theory

Consider the effective action

$$S_{\mathrm{eff}} = \Gamma[\widetilde{g}] - (c_V + 1)\Gamma[\Sigma] + \Gamma[\beta] - \frac{1}{2}\mu^2 \int \mathrm{d}^2x \, [\partial_+^{-1}(\beta^{-1}\partial_+\beta)]^2 + S_{\mathrm{ghosts}} \quad . \tag{5.49}$$

Let us start by first coupling the fields $(\widetilde{g}, \Sigma, \text{ghosts})$ to external gauge fields

$$A_-^{\mathrm{ext}} = \frac{i}{e} V_{\mathrm{ext}} \partial_- V_{\mathrm{ext}}^{-1} \quad , \quad A_+^{\mathrm{ext}} = \frac{i}{e} U_{\mathrm{ext}}^{-1} \partial_+ U_{\mathrm{ext}} \quad . \tag{5.50}$$

Such a coupling may be obtained by substituting each WZW functional as prescribed in eq. (2.87). In the case of ghosts one has to perform a chiral rotation, as in the discussion following eq. (2.143). Therefore, after such a procedure and using again the invariance of the Haar measure to substitute $U_{\mathrm{ext}}(\widetilde{g})V_{\mathrm{ext}} \to (\widetilde{g}, \sigma)$, one finds the effective action

$$S_{\mathrm{eff}}(A) = \Gamma[\widetilde{g}] - (c_V + 1)\Gamma[\Sigma] + S_{\mathrm{ghosts}} + [1 - (c_V + 1) + c_V]\Gamma[U_{\mathrm{ext}} V_{\mathrm{ext}}] \quad . \tag{5.51}$$

Vanishing of the total central charge (i.e. the vanishing coefficient of the last term above) tells us that the action does not depend on the external gauge fields. Nevertheless, from minimal coupling the effective action can also be written as

$$\begin{aligned} S_{\mathrm{eff}}(A) = S_{\mathrm{eff}}(0) &- \frac{1}{4\pi} A_+^{\mathrm{ext}} \left[ie\widetilde{g}\partial_-\widetilde{g}^{-1} - ie(c_V+1)\Sigma\partial_-\Sigma^{-1} + J_-(\text{ghost})\right] \\ &- \frac{1}{4\pi} A_-^{\mathrm{ext}} \left[ie\widetilde{g}^{-1}\partial_+\widetilde{g} - ie(c_V+1)\Sigma^{-1}\partial_+\Sigma + J_+(\text{ghost})\right] + \mathcal{O}(A^2)\,. \end{aligned} \tag{5.52}$$

Functionally differentiating the partition function once with respect to $A_+^{\rm ext}$ and separately with respect to $A_-^{\rm ext}$, and putting $A_\pm^{\rm ext} = 0$ we find the constraints[46,53]

$$i\widetilde{g}\partial_-\widetilde{g}^{-1} - i(c_V+1)\Sigma\partial_-\Sigma^{-1} + J_-(\text{ghosts}) \sim 0 \quad , \tag{5.53}$$
$$i\widetilde{g}^{-1}\partial_+\widetilde{g} - i(c_V+1)\Sigma^{-1}\partial_+\Sigma + J_+(\text{ghosts}) \sim 0 \quad , \tag{5.54}$$

leading to two BRST charges $Q^{(\pm)}$ as discussed in ref. [53], which are nilpotent. Therefore we find two first-class constraints.

The field $A_+^{\rm ext}$ can also be coupled to the field β instead of $\widetilde{g}$, since the system $(\beta, \Sigma, \text{ghosts})$ has vanishing central charge too. In such a case we have to disentangle the non-local interaction considering instead of the third and fourth terms in (5.49), the β action

$$S(\beta) = \Gamma[\beta] + \int \mathrm{d}^2x\, \frac{1}{2}(\partial_+C_-)^2 + i\int \mathrm{d}^2x\, \mu\, C_-\beta^{-1}\partial_+\beta \quad . \tag{5.55}$$

We make the minimal substitution $\partial_+ \to \partial_+ - ieA_+^{\rm ext}$, repeating the previous arguments for the $(\beta, \Sigma, \text{ghosts})$ system, and we now arrive at the constraint (the minus gauging is not an invariance if one includes the β system):

$$\beta\partial_-\beta^{-1} + 4i\pi\mu\beta C_-\beta^{-1} - i(c_V+1)\Sigma\partial_-\Sigma^{-1} + J_-(\text{ghost}) \sim 0 \quad . \tag{5.56}$$

One could naïvely expect that, by repeating the previous arguments, one obtains a system having a new set of first-class constraints. But if we consider instead the equivalent system of constraints defined by the first set (5.53), together with the difference of the $(-)$ currents, i.e. (5.54) and (5.56) as given by

$$\Omega_{ij} = (\beta\partial_-\beta^{-1})_{ij} + 4i\pi\mu(\beta C_-\beta^{-1})_{ij} - (\widetilde{g}\partial_-\widetilde{g}^{-1})_{ij} \quad , \tag{5.57}$$

one readily verifies that the latter cannot lead to a nilpotent BRST charge due to the absence of ghosts. Therefore, it must be treated as a second-class constraint.[55] The Poisson algebra obeyed by Ω_{ij} is

$$\{\Omega_{ij}(t,x), \Omega_{kl}(t,y)\} = (\widetilde{\Omega}_{il}\delta_{kj} - \widetilde{\Omega}_{kj}\delta_{il})(t,x)\delta(x-y) + 2\delta_{il}\delta_{kj}\delta'(x-y) \quad , \tag{5.58}$$
$$\widetilde{\Omega} = \widetilde{g}\partial_-\widetilde{g}^{-1} + \beta\partial_-\beta^{-1} + 4i\pi\mu\beta C_-\beta^{-1} \quad . \tag{5.59}$$

(Notice the change of sign in $\widetilde{\Omega}$.) Using the above, we can thus define the undetermined velocities, and no further constraint is generated.

The fact that the theory possesses second-class constraints is very annoying, since these cannot be realized by the usual cohomology construction. Therefore, instead of building a convenient Hilbert space, one has to modify the dynamics, since the usual relation between Poisson brackets and commutators is replaced by the relation between Dirac brackets and commutators.

Nevertheless, as we will see, several nice structures unravelled so far remain untouched after such a harsh mutilation. Indeed, we shall see that there is a rather deep separation between the "right" currents, obeying equations analogous to those written so far, and the "left" currents, which will obey a modified dynamics, due to the second-class constraints.

As a consequence of the definition of the canonical momenta, eq. (5.39), the constraints have a simpler phase-space formulation, and are given by

$$\Omega_{ij} = 4\pi(\beta\hat{\widetilde{\Pi}}^{\beta})_{ij} + \partial_1\beta\beta^{-1} - 4\pi(\widetilde{g}\hat{\widetilde{\Pi}}^{\tilde{g}})_{ij} - \partial_1\widetilde{g}\widetilde{g}^{-1} \quad , \tag{5.60}$$

which has actually been used to compute (5.58). Notice that the structure of the right-hand side of the phase-space expression is rather simple. Indeed, the C_- field just redefines the momentum associated with β, and the above constraints are analogous to those appearing in the description of non-Abelian chiral bosons,[119] i.e. WZW theory with a constraint on a chiral current. It follows that the Poisson algebra is very simple. Indeed, one obtains[119]

$$\begin{aligned}\{\Omega_{ij}(x), \Omega_{kl}(y)\} &= 16\pi\delta_{il}\delta_{kj}\delta'(x^1 - y^1) + 4\pi\Big[(4\pi\beta\hat{\widetilde{\Pi}}^{\beta} + \beta'\beta^{-1} + 4\pi\widetilde{g}\hat{\widetilde{\Pi}}^{\tilde{g}} + \widetilde{g}'g^{-1})_{kj}\delta_{il} \\ &\quad - (4\pi\beta\hat{\widetilde{\Pi}}^{\beta} + \beta'\beta^{-1} + 4\pi\widetilde{g}\hat{\widetilde{\Pi}}^{\tilde{g}} + \widetilde{g}'g^{-1})_{il}\delta_{kj}\Big]\delta(x^1 - y^1) \\ &= 16\pi\delta_{il}\delta_{kj}\delta'(x^1 - y^1) + 8\pi[j_{-kj}\delta_{il} - j_{-il}\delta_{kj}]\delta(x^1 - y^1) \quad ,\end{aligned} \tag{5.61}$$

where $j_- = 4\pi\beta\hat{\widetilde{\Pi}}^{\beta} + \beta'\beta^{-1}$ satisfies the Poisson algebra

$$\{j_{-ij}, j_{-kl}\} = 8\pi\delta_{il}\delta_{kj}\delta'(x-y) + 4\pi(j_{-kj}\delta_{il} - j_{-il}\delta_{kj})\delta(x-y) \quad . \tag{5.62}$$

The above expression also defines the Q matrix

$$Q_{ij;kl} = \{\Omega_{ij}(x), \Omega_{kl}(y)\}\Big|_{\text{equal time}} \quad , \tag{5.63}$$

which is not a combination of constraints, and therefore no further constraint is generated by the Dirac algorithm. The inverse of the Dirac matrix is not difficult to compute and we have the expression[119]

$$\begin{aligned}(Q^{-1})_{ij;kl} &= \frac{1}{32\pi}\delta_{il}\delta_{kj}\varepsilon(x) + \\ &+ \frac{1}{64\pi}(\delta_{il}j_{-jk} - \delta_{jk}j_{-li})|x| + \\ &+ \frac{1}{128\pi}(\delta_{ia}j_{-jb} - \delta_{jb}j_{-ai})(\delta_{al}j_{-bk} - \delta_{bk}j_{-la})\frac{1}{2}x^2\varepsilon(x) + \\ &+ \frac{1}{256\pi}(\delta_{ia}j_{-jb} - \delta_{jb}j_{-ai})(\delta_{ca}j_{-bd} - \delta_{bd}j_{-ac})(\delta_{lc}j_{-dk} - \delta_{dk}j_{-cl})\frac{1}{3}x^3\varepsilon(x) + \cdots,\end{aligned} \tag{5.64}$$

where x is the space component of x^μ.

The next step consists in replacing the Poisson brackets by Dirac brackets. Thus we have to compute the Poisson brackets of the relevant quantities with the constraints. We use

$$\{A, B\}_{DB} = \{A, B\}_{PB} - \{A, \Omega_\alpha\}_{PB}Q^{-1}_{\alpha\beta}\{\Omega_\beta, B\}_{PB} \quad . \tag{5.65}$$

We will see that functions of

$$J_+^\beta = \beta^{-1}\partial_+\beta = -4\pi\widetilde{\Pi}\beta + \beta^{-1}\beta' + 4i\pi\mu C_- \tag{5.66}$$

commute with Ω_α, and that their Dirac brackets coincide with their Poisson brackets.

Canonical quantization through the Dirac formulation of the β sector is achieved by the formulae (5.38a, b), (5.39) and (5.41), from which we obtain the phase space expression

$$\frac{1}{4\pi}\beta^{-1}\partial_\pm\beta = -\widetilde{\hat{\Pi}}^\beta \beta \pm \frac{1}{4\pi}\beta^{-1}\beta' + i\mu C_- \quad . \tag{5.67}$$

It is useful, in view of (5.58), to consider the combination

$$\frac{1}{4\pi}\partial_-\beta\beta^{-1} = -\beta\widetilde{\hat{\Pi}}^\beta - \frac{1}{4\pi}\beta'\beta^{-1} + i\mu\beta C_-\beta^{-1} \quad ; \tag{5.68}$$

or also, aiming at the expression of the constraint (5.58), which contains the C_- field, we have

$$\beta\partial_-\beta^{-1} + 4i\pi\mu C_-\beta^{-1} = -4\pi\beta\widetilde{\hat{\Pi}}^\beta - \beta'\beta^{-1} \quad . \tag{5.69}$$

Thus, in terms of phase-space variables the constraint is given by (5.60). Using the above phase-space expressions we find that

$$\left\{J_-^\beta, \Omega\right\} = 0 \quad , \tag{5.70}$$

$$\left\{[J_-^\beta, \partial_+ J_-^\beta], \Omega\right\} = \{[C_-, \Pi_-], \Omega\} = 0 \quad . \tag{5.71}$$

For $\left\{\partial_+\partial_- J_-^\beta, \Omega\right\}$ we first have to compute

$$\partial_+\partial_- J_-^\beta = \partial_+^2 J_-^\beta - 2(\partial_+ J_-^\beta)' \quad , \tag{5.72}$$

$$= 4\pi\mu^2\beta^{-1}\partial_+\beta - 2(\Pi_-)' \quad . \tag{5.73}$$

We use the fact that $\{\Pi'_-, \Omega\} = 0$ and we are left with

$$\beta^{-1}\partial_+\beta = -4\pi\widetilde{\hat{\Pi}}^\beta \beta + \beta^{-1}\beta' + 4i\pi\mu C_- \quad . \tag{5.74}$$

Using now $\{C_-, \Omega\} = 0$, we just have to consider

$$j_{+ij} = \left(-4\pi\widetilde{\hat{\Pi}}^\beta \beta + \beta^{-1}\beta'\right)_{ij} \quad . \tag{5.75}$$

However, since $\{j_+, j_-\} = 0$ we have $\{j_+, \Omega\} = 0$! As a conclusion, for the objects relevant to us, the Dirac algebra is the same as the Poisson algebra! This is a non-trivial result, because it holds even though, due to (5.60), the Dirac algebra obeyed by $\hat{\Pi}^\beta$ and β changes drastically, especially if we take into account the expression of the inverse Dirac matrix (5.64), which is non-local and has an infinite number of terms!

In the duality transformation relating the β and the W fields, we also find interesting relations arising out of the constraint structure of the theory. First let us perform a more detailed analysis of the ghost structure. Going back to the transformations defined by (5.20) we have the factor $(\det \partial_+ \ \det \partial_-)^{c_V}$ left out, which contributes as

$$\mathcal{Z}^{\mathrm{gh}'} = \int \mathcal{D}b'_{++}\mathcal{D}b'_{--}\mathcal{D}c'_{+}\mathcal{D}c'_{-}\ \mathrm{e}^{-\mathrm{tr}\int \mathrm{d}^2x\,(b'_{++}\partial_- c'_- + b'_{--}\partial_+ c'_-)} \quad . \tag{5.76}$$

The coupling of a subset of fields to an external gauge potential written in the form (5.50), can be made as in the usual way. If such a set has a vanishing total central charge, the partition function does not depend on the gauge potential, and we are led to constraints again. With the partition function written in the W language as in (5.26), and taking into account all appropriate ghosts, we have various self-commuting constraints. Some of them, such as

$$J_{\tilde g} - (c_V+1)J_\Sigma + J_{\mathrm{ghost}} \sim 0 \quad , \tag{5.77}$$

$$J_{\tilde \beta} - (c_V+1)J_\Sigma + J_{\mathrm{ghost}} \sim 0 \quad , \tag{5.78}$$

are the same as before, with the advantage that now $\widetilde\beta$ is a pure WZW field, so that it can be simply identified with $\widetilde g$, without further consequences. However, further constraints involving also the W field arise, such as

$$J_+^{\tilde g} - (c_V+1)J_+^W + J_{+\mathrm{ghost}} \sim 0 \quad , \tag{5.79}$$

so that we have, as a consequence, the non-trivial second-class constraint

$$J_+^\Sigma - J_+^W \sim 0 \quad , \tag{5.80}$$

or, more explicitly,

$$(c_V+1)\Sigma^{-1}\partial_+\Sigma - (c_V+1)W^{-1}\partial_+W + \frac{1}{\mu}W^{-1}\partial_+BW = 0 \quad . \tag{5.81}$$

We have proceeded as in the β formulation, but with the interaction of the A_-^{ext} field with the W, while in the (dual) β case we considered A_+^{ext}.

The phase-space expression is given in the formula

$$\Omega^{W,\Sigma} = -\tilde{\hat\Pi}^W W + \frac{1}{4\pi}W^{-1}W' + \tilde{\hat\Pi}^\Sigma \Sigma - \frac{1}{4\pi}\Sigma^{-1}\Sigma' \sim 0 \quad , \tag{5.82}$$

and resembles the β formulation (see (5.60)). However, if we now substitute the B field from the constraint (5.80) back into the action we find a non-local term. This means that while in the β formulation, which is non-local at the beginning, we end up with a local action after inserting the constraint back, in the W formulation, which is local at the beginning, we end up with a non-local action; another feature of duality in both formulations.

Keeping the Dirac algebra in mind, we substitute back the configuration-space constraints into the action, maintaining the phase-space structure. In such a case, using (5.57) and (2.78), we redefine $\beta g \equiv P$, $\beta = Pg^{-1}$, and find the effective action

$$\begin{aligned} S = &\Gamma[P] - \frac{1}{2\pi} g^{-1}\partial_+ g g^{-1}\partial_- g - \frac{1}{4\pi} P^{-1}\partial_+ P P^{-1}\partial_- P + \frac{1}{4\pi} P^{-1}\partial_+ P g^{-1}\partial_- g \\ &+ \frac{1}{4\pi} P^{-1}\partial_- P g^{-1}\partial_+ g + \frac{1}{4\pi}\partial_- g g^{-1}\partial_+ P P^{-1} - \frac{1}{2\pi}\partial_- g g^{-1} P g^{-1}\partial_+ g g^{-1} P \\ &+ \frac{1}{2(4\pi\mu)^2}\left[\partial_+\left\{ g P^{-1} g \partial_-(g^{-1} P g^{-1})\right\}\right]^2 \quad . \end{aligned} \tag{5.83}$$

The equation of motion/conservation law (5.18a) still holds, as previously proved. From action (5.83) we can find the equations of motion. Notice that the final action is a WZW theory off the critical point, a principal σ-model, and current–current-type interactions between them.

For the dual formulation a further interesting structure arises. The constraint is now

$$\partial_+ B = -\mu(c_V + 1) W\Sigma^{-1}\partial_+(\Sigma W^{-1}) \quad . \tag{5.84}$$

As with to the above, we use (5.84) and (2.6) to introduce $S = W\Sigma$, replacing the W field. It is interesting enough to note that it is now the dual formulation that is non-local due to the presence of the B field. We again arrive at the WZW theory for S, a principal σ-model term for Σ, current–current-type interactions, and principal σ-model terms for S. The latter are such that the (wrong) sign of the principal σ term in $\Gamma[S]$ changes, and we arrive at the WZW model with a relative minus sign, or $\Gamma[S^{-1}]$!

However, the standard procedure for dealing with the constraints is to substitute the phase-space expressions in the Hamiltonian. But in such a case, the constraint (5.60) does not depend upon C_-, and leads just to a connection between the right-moving current of the g sector, the left-moving current being untouched by such a relation! Therefore, still in the present case, where we have witnessed the appearance of second-class constraints, their main role was to ensure the positive metric requirement, as we have seen by means of the change of sign of the WZW action in the dual formulation. A rich algebraic structure arises in both (β and W) formulations of the theory[47].

5.4 Spectrum and comparison with the $1/N$ expansion

Having recognized the role played by the β action, we pass to a discussion of the spectrum of the theory. The first step towards understanding the model was taken by 'tHooft, proving that the bound states form a Regge trajectory, valid for the weak coupling case (heavy quarks). Later, Steinhardt[22] studied the strong coupling case (light quarks) finding the baryon as the soliton of a generalized sine-Gordon interaction.

Here we do not intend to provide a definite answer to such a complex question, but some directions may be outlined from the computations performed. Indeed, we have an appropriate formulation to deal separately with the two regimes: the weak-coupling regime described by the β action may be discussed perturbatively. We will see that in the large-N limit the relevant mass parameter is the one defined by 'tHooft, and we arrive at a possibility of computing the exact mass spectrum, once the complicated constraint structure is disentangled.

In order to understand the question concerning the spectrum, we first have to know which is the mass of the simplest excitation, or the mass parameter characterizing the theory. We thus consider the action

$$S[\beta] = \Gamma[\beta] + \frac{1}{2}\mu^2 \int \mathrm{d}^2x \left[\partial_+^{-1}(\beta^{-1}\partial_+\beta)\right]^2 \quad , \tag{5.85}$$

and write a background quantum splitting for the β field as

$$\beta = \beta_0 \mathrm{e}^{i\xi} \quad , \tag{5.86}$$

after which we have the background quantum splitting of the action up to second order in the quantum field ξ. However, we have to be careful since, in the large-N limit, the second term is the zeroth-order Lagrangian, from which we suppose that the ξ field acquires a lowest mass m_1 to be computed. The WZW term splits as

$$\Gamma[\beta] = \Gamma[\beta_0] + \frac{1}{2}\int \mathrm{d}^2x\ \beta_0^{-1}\partial_\mu\beta_0\, \xi\overleftrightarrow{\partial_\nu}\xi\,(g^{\mu\nu} + \epsilon^{\mu\nu}) \quad . \tag{5.87}$$

Using the fact that $\Gamma[\beta]$ is at the critical point, it is not difficult to compute the $\beta_0^{-1}\partial_\mu\beta_0$ two-point function at the one-loop order. We have the zeroth-order contribution from the second term of (5.85), and the one-loop contribution, which leads to the result

$$\beta^{-1}\partial_+\beta\frac{\mu^2}{p_+^2}\beta^{-1}\partial_+\beta - N\frac{p_\mu p_\nu}{p^2}(g^{\mu\rho}+\epsilon^{\mu\rho})(g^{\nu\sigma}+\epsilon^{\nu\sigma})F(p)\beta^{-1}\partial_\rho\beta\beta^{-1}\partial_\sigma\beta \quad , \tag{5.88}$$

where

$$F(p) = \frac{1}{2\pi}\sqrt{\frac{p^2-4m_1^2}{p^2}}\ln\frac{\sqrt{-p^2+4m_1^2}+\sqrt{-p^2}}{\sqrt{-p^2+4m_1^2}-\sqrt{-p^2}} - \frac{1}{\pi} \quad . \tag{5.89}$$

For $p^2 = m_1^2$, we find that

$$\beta_0^{-1}\partial_+\beta_0\,\beta_0^{-1}\partial_+\beta_0\frac{1}{p_+^2}\left[\mu^2 - 4Nm_1^2F(m_1^2)\right] \quad . \tag{5.90}$$

The zero of the two-point function contribution to the action is at

$$m_1^2 = fe^2N = f\alpha \quad , \tag{5.91}$$

where f is a numerical constant, in accordance with 'tHooft's results.[14]

That the second term of (5.85) has an extra factor N arises from the fact that the fermion loops are suppressed by a factor $1/N$. Since the fermion loops contribute with a WZW functional, while the μ term stems from the gauge-field self-interaction (see eqs. (2.108)) the factors of N are correct. Moreover, it is exactly the given assignment that is compatible with the planar expansion. Finally, we have to quote the fact that 'tHooft's analysis for the bound state $\overline{\psi}\gamma_+\psi$ leads to a Bethe–Salpeter equation compatible with the previous results, the methods following closely his analysis.

More detailed information about the spectrum of the theory can be obtained from the Hamiltonian formulation. From the action

$$S = \Gamma[\beta] + \int d^2x \frac{1}{2}(\partial_+ C_-)^2 + i\int d^2x \mu C_- \beta^{-1}\partial_+\beta \quad , \tag{5.92}$$

we obtain the canonical momenta

$$\tilde{\hat{\Pi}}^{\beta} = \frac{1}{4\pi}\partial_0\beta^{-1} + i\mu C_-\beta^{-1} \quad , \tag{5.93}$$

$$\Pi_- = \partial_+ C_- \quad , \tag{5.94}$$

and the Hamiltonian density

$$\begin{aligned} H = &\frac{1}{2}\Pi_-(\Pi_- - 2C') - 2\pi(\tilde{\hat{\Pi}}^{\beta}\beta)^2 - \frac{1}{8\pi}(\beta^{-1}\beta')^2 \\ &+ 4\pi\mu\tilde{\hat{\Pi}}^{\beta}\beta C_- - 2\pi\mu^2 C_-^2 - \mu\beta^{-1}\beta' C_- \quad . \end{aligned} \tag{5.95}$$

The important currents are

$$J_+^{\beta} = \beta^{-1}\partial_+\beta = -4\pi\tilde{\hat{\Pi}}^{\beta}\beta + \beta^{-1}\beta' + 4\pi\mu C_- \quad , \tag{5.96}$$

$$j_-^{\beta} = \beta\partial_-\beta^{-1} + 4\pi\mu\beta C_-\beta^{-1} = 4\pi\beta\tilde{\hat{\Pi}}^{\beta} + \beta'\beta^{-1} \quad , \tag{5.97}$$

in terms of which the Hamiltonian density reads (notice that j_-^{β} is not related to J_-^{β}, eqs. (5.16) and (5.38a)):

$$H_{\beta} = -\frac{1}{16\pi}\left[\left(J_+^{\beta}\right)^2 + \left(j_-^{\beta}\right)^2\right] - \frac{1}{2}\mu J_+^{\beta} C_- + \pi\mu^2 C_-^2 + \frac{1}{2}\Pi\left(\Pi_- - 2C'\right) \quad . \tag{5.98}$$

From the previously discussed constraint structure (5.60), the current j_- is related to the free right-moving fermion current $j_-^g = g\partial_- g^{-1}$, and we will drop it in the discussion of the spectrum for β. Moreover, from the Sugawara construction of the Virasoro algebra, in terms of the affine algebra generators, we know that the Sugawara piece $H_+ = \frac{-1}{16\pi}(J_+^{\beta})^2$ acquires a factor $(c_V+1)^{-1} \overset{SU(N)}{=} (N+1)^{-1}$ in the quantum theory. The C_-^2 terms are not known, since the C_- equation of motion is not easily solvable. Nevertheless, in terms of C_- and its conjugate momentum, the Hamiltonian is quadratic. If we take it for granted that the zero-mode term is just the squared momentum, moreover neglecting the $C_- J_+$ interaction, the Hamiltonian eigenstates have masses m_n obeying the Regge behaviour

$$m_n^2 \sim n\, m_1^2 \quad . \tag{5.99}$$

Corrections to this equation can be obtained using a large-N expansion for the field C_-, a procedure that is at least possible upon considering the large-N limit of (5.92). There are in fact further eigenstates, and for a choice of the normalization of the fields one finds asymptotically a low mass eigenstate, compatible also with Steinhardt's baryon.[22]

5.5 Integrability conditions and Calogero-type model

The issue of integrability has been discussed for a long time in the literature. It started almost a century ago, with the observations of Korteweg and de Vries[120], and today it is an area of research in itself, including several applications.

In two-dimensional space-time the situation is simple. In fact, the Coleman–Mandula theorem[8,121] prevents the existence of higher conservation laws in more than two dimensions. The theorem states that for dimensions higher than two, the most general invariance group of a non-trivial field theory is the product of the Poincaré group and an internal symmetry group. Allowing anticommutators, we can at most have a supersymmetry algebra. But in two dimensions there are integrable systems, with an infinite number of conservation laws.[13)]

Non-linear σ models are two-dimensional counterparts of four-dimensional Yang–Mills theories, sharing several desired properties.[8] When such models are defined on a symmetric space they are classically integrable — as are their supersymmetric extensions. Such a fact, when not spoiled by quantum anomalies, leads to a solution of the theory at the S-matrix level.

It is natural to ask whether the two-dimensional counterpart of the Yang–Mills theory is also integrable or not, and whether the $(3+1)$-dimensional Yang–Mills theory itself displays, in some limit, such a property. The answer to both questions seems to be positive, although most details should still be worked out. In the case of two-dimensional Yang–Mills theory such an information can be derived in two ways; first the pure-gauge model is equivalent to a system of non-relativistic fermions, or to the $c = 1$ matrix model, a property which seems to hold true where there is fermionic matter to begin with. Moreover, the bosonization of the model reveals the existence of a Lax pair.

As far as the second question is concerned, namely concerning the $(3+1)$-dimensional Yang–Mills theory, one has to consider the high-energy scattering amplitudes. Such a

[13)] It is not yet established how one might rephrase the property of integrability for real higher-dimensional systems. In fact, due to the severe constraints imposed it should not be generally true as higher conservations laws. However, several simplified models, important for physical understanding, are integrable in a weak sense. In higher dimensions, we can quote the self-dual Yang–Mills theory, as well as the $N = 4$ supersymmetric Yang–Mills theory in four dimensions, and gravity in ten dimensions[122]. In such cases, however, the integrability properties, although implying constraints, do not lead to a solution of the theory, or imply the existence of higher conserved charges.

problem was studied by several authors, and the outcome is an effective two-dimensional theory describing the transverse components of the momentum transfer. Verlinde and Verlinde[123] arrived at an effective two-dimensional model of two matrices, and also proved that the effective interaction is described by the Lipatov vertex.[124] Such a vertex, on the other hand, leads to amplitudes obeying Hamiltonian equations which seem to factorize into simple Hamiltonian systems,[125–127] being in principle integrable and solvable by the Bethe ansatz technique.[128]

Although all such indications have to be further analyzed, and several details must be understood, the possibilities of studying such models thus have a wide avenue ahead, and high-energy scattering might prove to be equivalent to an exactly soluble model, around which perturbative expansions can be performed.

We have seen that pure QCD_2 in the Coulomb gauge ($A_0 = 0$), on a cylinder whose cross section has length L, has a Hamiltonian given by eq. (3.102), and that the Gauss law, corresponding to the A_0 equation of motion, was written in eq. (3.105). We also defined the Wilson line in eq. (3.104), and the "dressed" electric field in eq. (3.103). Due to the Gauss law, the dressed electric field $V(x)$ is a constant, ($\partial_1 V(x) = 0$), therefore $V(x) = V(0)$. We defined also the Wilson loop around the cylinder as $W_{\rm cyl} = W[0, L]$ (now we specify $W_{\rm cyl}$), and we immediately verified, writing $V(0) = V(L)$, that

$$[W_{\rm cyl}, \dot{A}_1(0)] = 0 \quad . \tag{5.100}$$

The time derivative of the cylindric Wilson loop $\dot{W}_{\rm cyl}$ can also be computed from the known formula for the derivative of the exponential of an algebra-valued object, that is

$$\dot{W}_{\rm cyl} = ie \int {\rm d}x\, W[0, x]\dot{A}_1(x)W[x, L] = ie \int {\rm d}x\, V(x) = ieLV(0) \quad , \tag{5.101}$$

or in terms of $W_{\rm cyl}$ and $\dot{A}_1$:

$$\dot{W}_{\rm cyl} = ieLW_{\rm cyl}\dot{A}_1(0) = ieL\dot{A}_1(0)W_{\rm cyl} \quad , \tag{5.102}$$

which together with (5.100) implies $[W_{\rm cyl}, \dot{W}_{\rm cyl}] = 0$.

Rewriting $\dot{A}_1$ in terms of the cylindric Wilson loop, one obtains the Hamiltonian (3.108), that is a one-dimensional unitary matrix model.

Moreover, we notice also that as matrices, $W_{\rm cyl}$ and $\dot{W}_{\rm cyl}$ commute, therefore the space of states only contains singlets! This means that we have to deal only with the eigenvalues of the cylindric Wilson variable $W_{\rm cyl}$.

On the other hand, if we consider the Hamiltonian[88]

$$H = {\rm tr} \left[-\frac{1}{2} \frac{\delta^2}{\delta\phi_{ij}^2} + U(\phi) \right] \quad , \tag{5.103}$$

where $U(\phi)$ depends only on the eigenvalues, that is it does not depend on angular variables, we obtain eigenfunctions of the type

$$\psi = \frac{\chi}{\Delta(\lambda)} \quad , \tag{5.104}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_i < \lambda_j)$ is the Vandermonde determinant. In such a case the expectation of the Hamiltonian is

$$\int \prod_i \mathrm{d}\lambda_i \, \mathcal{D}\Omega \sum \left(\frac{\partial\psi}{\partial\phi_{ij}}\right) \Delta^2(\lambda_i) = \text{volume} \int \prod_i \mathrm{d}\lambda_i \sum \left(\frac{\partial\psi}{\partial\lambda_i}\right)^2 \Delta^2(\lambda_i) \quad , \qquad (5.105)$$

where we used $\mathcal{D}\phi = \mathcal{D}\Omega\mathcal{D}\lambda\Delta^2(\lambda)$. Writing ψ in terms of χ we find $\int \prod \mathcal{D}\lambda_i \sum \left(\frac{\partial\chi}{\partial\lambda_i}\right)^2$ and therefore may use as Hamiltonian

$$H = \sum_i \left[-\frac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} + U(\lambda_i)\right] \quad , \qquad (5.106)$$

for wave functions χ, or else for ψ

$$H = -\frac{1}{2\Delta(\lambda)} \sum \frac{\partial^2}{\partial\lambda_i^2}\Delta(\lambda) + U(\lambda_i) + \sum \frac{\pi_{ij}^2 + \overline{\pi}_{ij}^2}{(\lambda_i - \lambda_j)^2} \quad , \qquad (5.107)$$

where π_{ij} and $\overline{\pi}_{ij}$ are the generators of (left and right) rotations. Due to the presence of the Vandermonde determinant, which is antisymmetric under the exchange of two "particles", the problem boils down to the theory of N fermions in the non-relativistic limit.

The introduction of a Wilson line along the time at $x = 0$ is rather instructive.[88] It can be seen as a static source at $x = 0$, and in that case we consider the Euclidian action

$$S_E = \frac{1}{4}\int \mathrm{d}^2x \, \mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} + \int \mathrm{d}t \, \overline{\psi}(i\partial_t - eA_0^a(x=0)T^a + M)\psi \quad . \qquad (5.108)$$

The partition function as computed in ref. [88] in a series of e^{-TM} is obtained by noticing that there are $(\dim R)$ independent fermions with energy $M - \frac{i}{T}w_n$, where e^{iw_n} are the eigenvalues of the Wilson loops in the representation R. We are interested in the Hamiltonian, which in the gauge $A_0 = 0$ reads

$$H = \frac{1}{2}\int_0^L \mathrm{d}x \, \mathrm{tr}\, \dot{A}_1^2 + M\overline{\psi}\psi \quad , \qquad (5.109)$$

and the Gauss law has a source term, that is

$$\nabla_1 F_{10} = \partial_1\dot{A}_1 + ie[A_1, \dot{A}_1] = e\overline{\psi}T^a\psi\tau^a\delta(x - L + \epsilon) \quad ; \qquad (5.110)$$

we again find it useful to introduce the field

$$V(x) = W[0,x]\dot{A}_1(x)W[x,L] \quad , \qquad (5.111)$$

whose derivative is concentrated at $x = L - \epsilon$,

$$\partial_1 V = eW[0, L-\epsilon]\overline{\psi}T^a\psi\tau^a W[L-\epsilon, L]\delta(x - L + \epsilon) \quad , \qquad (5.112)$$

which implies that $V(x)$ is almost always constant, with a singularity at $x = L - \epsilon$, such that

$$V(L) = V(0) + eW(\overline{\psi}\psi) \quad . \tag{5.113}$$

The last term implies that the commutator of the Wilson loop W with the electric field no longer vanishes,

$$[W, \dot{A}_1(0)] = eW\overline{\psi}\psi \quad . \tag{5.114}$$

The time derivative of the Wilson line is easily obtained from the derivative of the ordered exponential, and reads

$$\dot{W} = ie \int_0^L \mathrm{d}x\, V(x) \quad , \tag{5.115}$$

which can be integrated due to the constancy of $V(x)$ (up to the δ-function in eq. (5.112)). Upon use of (5.111) we find

$$\dot{W} = ieL\dot{A}_1(0)W \quad , \tag{5.116}$$

which implies, together with (5.114), that

$$[\dot{W}, W^{-1}] = ie^2 L\overline{\psi}\psi \quad . \tag{5.117}$$

Using again the diagonalized form of the matrix W

$$W = U\Lambda U^\dagger \quad , \tag{5.118}$$

the constraint leads to the equation

$$2U^\dagger\dot{U} - \Lambda U^\dagger \dot{U}\Lambda^\dagger - \Lambda^\dagger U^\dagger \dot{U}\Lambda = -ie^2 LJ \quad , \tag{5.119}$$

where $J = U^\dagger\overline{\psi}\psi U$. Defining $\Omega = U^\dagger\dot{U}$, one rewrites (5.119) as

$$\Omega_{ij}\left(2 - \mathrm{e}^{i(\theta_i-\theta_j)} - \mathrm{e}^{-i(\theta_i-\theta_j)}\right) = -ie^2 LJ_{ij} \quad , \tag{5.120}$$

which enables us to compute Ω_{ij}; moreover the Hamiltonian

$$H = -\frac{1}{2e^2 L}\mathrm{tr}\left(W^{-1}\dot{W}\right)^2 + M\overline{\psi}\psi \tag{5.121}$$

turns into

$$H = \frac{1}{2e^2 L}\sum \dot{\theta}_i^2 + \frac{1}{2}e^2 L \sum_{i\neq j} \frac{J_{ij}J_{ji}}{4\sin^2(\theta_i - \theta_j)/2} \quad , \tag{5.122}$$

which is a theory of non-relativistic fermions interacting via a two-body potential analogous to that of the Calogero–Sutherland model.[129]

Such results are in close analogy with those obtained in ref. [111] relating Yang–Mills theory with sources and the Calogero–Sutherland model. The main issue is the fact that the Yang–Mills Hamiltonian is quadratic in momentum, generating the motion of a free system on the cotangent bundle of the Lie algebra; after diagonalization, such a matrix model describes fermions interacting with a well-defined Sutherland-type potential. The study of ref. [104] points also to the same direction, relating QCD_2 on a cylinder to a one-dimensional matrix model of the type introduced by Kazakov and Migdal.[130]

5.6 QCD at high energies and two-dimensional field theory

The fact that high-energy scattering amplitudes have a corresponding description in terms of two-dimensional field theory is remarkable, and deserves further study. A deeper insight into such a problem was obtained in the work of Verlinde and Verlinde,[123] who starting from $(3+1)$-dimensional Yang–Mills theory, studied the limit where the incoming energy squared s is much larger than the exchanged momentum squared t. This will be related to the so-called leading logarithmic approximation (LLA).[124–126,131–133] By means of a scaling argument, they found a two-dimensional theory and the corresponding correction.

The problem is understood by considering the splitting of the $(3+1)$-dimensional coordinates into "fast coordinates" $x^\alpha = (x^+, x^-) = (x+t, x-t)$, whose Fourier counterparts are large, and "slow coordinates" $z^i = (y, z)$, which describe large-distance physics.

The crucial observation made by the authors is that due to the Lorentz contraction in the direction of the motion of the fast particles, the field strength will be of the form of a shock wave, non-vanishing only on a hyper-plane passing through the trajectory of the particle; in this case, the wave function of a test particle is submitted to a gauge rotation, which depends on the transverse distance (z^i above), explaining the two-dimensional nature of the effective interaction, and providing a Lagrangian mechanism first discovered in terms of the Feynman diagrammatic expansion.[132]

One first redefines the "fast coordinates" as

$$x^\alpha \to \lambda x^\alpha \quad , \tag{5.123}$$

which scales the incoming energy as

$$s' = \lambda^2 s \quad . \tag{5.124}$$

In such a case, one chooses $\lambda \sim 1/\sqrt{s}$, which tends to zero in the desired limit (short distance in the fast coordinates) and keeps s' fixed. In such a case, the four-dimensional Yang–Mills action

$$S = -\frac{1}{4}\int \mathrm{d}^4x \, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} \tag{5.125}$$

transforms into

$$\begin{aligned} S' &= \mathrm{tr}\int \mathrm{d}^4x' \left[-\frac{1}{4\lambda^2}F^2_{\alpha\beta} - \frac{1}{2}\left(F_{\alpha i}\right)^2 - \frac{\lambda^2}{4}\left(F_{ij}\right)^2 \right] \\ &= \int \mathrm{d}^4x' \, \mathrm{tr}\left[\frac{1}{2}E^{\alpha\beta}F_{\alpha\beta} + \frac{1}{2}\left(F_{\alpha i}\right)^2 + \frac{\lambda^2}{4}\left(E_{\alpha\beta}\right)^2 + \frac{\lambda^2}{4}\left(F_{ij}\right)^2 \right] \quad , \end{aligned} \tag{5.126}$$

where one uses the fact that the gauge potential transforms as

$$A_i \to A_i \quad , \quad A_\alpha \to \lambda^{-1}A_\alpha \quad . \tag{5.127}$$

An auxiliary antisymmetric field $E_{\alpha\beta}$ has been introduced above.

The $\lambda \to 0$ limit may be singular and terms having λ as a coefficient may be necessary at a later stage. For the time being, we just consider the zeroth-order approximation, namely neglecting the terms containing positive powers of λ.

One may also include matter fields, writing the action

$$S = \int \mathrm{d}^4 x \left[-\frac{1}{2} \mathrm{tr} \left(E^{\alpha\beta} F_{\alpha\beta} + F^{\alpha i} F_{\alpha i} \right) + \overline{\psi} \gamma^\alpha \left(\partial_\alpha + ie A_\alpha \right) \psi \right] \quad , \tag{5.128}$$

where the quark-mass term also disappears in such a limit due to the factor λ^2 from the measure (notice that $\psi \to \lambda^{-1/2} \psi$). Due to the absence of the mass term, the fermion action factorizes into two chiral terms. In the above action, the auxiliary field ($E_{\alpha\beta}$) acts as a Lagrange multiplier for the constraint $F_{\alpha\beta} = 0$, or in light-cone components $F_{+-} = 0$, which is equivalent to the previous observation that the main contribution follows from transverse configurations in the gauge field.

The chirality conservation of the action (5.128) implies that the current satisfies

$$\nabla_+ j_- = \nabla_- j_+ = 0 \quad . \tag{5.129}$$

The gauge-field equation of motion can be written perturbatively in terms of the parameter λ and from its perturbative solution the Lipatov vertex may be obtained. Indeed, keeping λ as a parameter, one develops the gauge field and current in a coupling perturbative expansion as

$$A_{i,\alpha} = \sum_{n \geq 0} A_{i,\alpha}^{(n)} e^n \quad , \tag{5.130a}$$

$$j_\alpha = \sum_{n \geq 0} j_\alpha^{(n)} e^n \quad , \tag{5.130b}$$

where the current j_α is chirally conserved in the high-energy limit. Plugging the above eqs. (5.130) back into the equation of motion

$$\nabla_\mu F^{\mu\nu} = j^\nu \quad , \tag{5.131}$$

one obtains for the first few perturbative equations, in the Lorentz gauge

$$A_i^{(0)} = 0 \quad , \tag{5.132a}$$

$$\partial_i^2 A_\alpha^{(0)} = -\, j_\alpha^{(0)} \quad , \tag{5.132b}$$

$$\partial^2 A_i^{(1)} = [A_\alpha^{(0)}, \partial_i A_\alpha^{(0)}] \quad , \tag{5.132c}$$

$$\partial^2 A_\alpha^{(1)} = j_\alpha^{(1)} + \lambda^{-2} [A_\beta^{(0)}, \partial_\beta A_\alpha^{(0)}] \quad , \tag{5.132d}$$

where $\partial^2 = \lambda^{-2} \partial^\alpha \partial_\alpha - \partial_i^2$. Recall that in the Lorentz gauge $\partial_+ A_-^{(0)} = 0 = \partial_- A_+^{(0)}$. One can use the chiral conservation law of the currents in order to eliminate them order by order in perturbation theory. From the first-order result,

$$\partial^2 A_-^{(1)} = - \left[\frac{1}{\partial_+} A_+^{(0)}, \partial_i^2 A_-^{(0)} \right] + \lambda^{-2} \left[A_+^{(0)}, \partial_- A_-^{(0)} \right] \quad , \tag{5.133}$$

we write for the diagram of Fig. 8 the Lipatov vertex[124]

Fig. 8: The Lipatov vertex

$$C_\mu = -k^{\perp}_{i_\mu} - k'^{\perp}_{i_\mu} + P_{A_\mu}\left(\frac{2k_i^2}{\alpha_i s} + \beta_i\right) - P_{B_\mu}\left(\frac{2k_i'^2}{\beta_i s} + \alpha_i\right) \quad , \tag{5.134}$$

where α and β correspond to the Sudakov decomposition of the momenta of the gluons

$$k_i = \alpha_i P_A + \beta_i P_B + k_i^{\perp} \quad , \tag{5.135}$$

so that $k_i^{\perp} \cdot P_A = 0 = k_i^{\perp} P_B$, that is $k_i^{\perp}$ is the momentum in the impact parameter space. In such a case, the curvature tensor is proportional to λ^2:

$$\begin{aligned} F_{+-}^{(1)} &= \partial_+ A_-^{(1)} - \partial_- A_+^{(1)} - ie\left[A_+^{(0)}, A_-^{(0)}\right] \\ &= 2\lambda^2 \frac{1}{\partial^\alpha \partial_\alpha}\left[\partial_i A_+^{(0)}, \partial_i A_-^{(0)}\right] \quad . \end{aligned} \tag{5.136}$$

In the effective action (5.126), the term $\lambda^{-2} F_{+-}^2$ is replaced by the lagrange multiplier, that is $E^{(1)} = \lambda^{-2} F_{+-}$. Therefore, the perturbation theory result is equivalent to (5.126) at $\lambda = 0$. This permits a good simplification of the theory, since A_α is now a pure gauge field. The theory is now described by two Wilson lines summarized by

$$W_\pm|z| = \mathrm{tr}\, P \exp\left(ie \int_{-\infty}^{\infty} \mathrm{d}x^\pm\, A_\pm(z)\right) \quad . \tag{5.137}$$

The two-point function of two right-moving quarks is given by

$$\begin{aligned} &\langle T\overline{\psi}(x_\alpha, z_i)\psi(x'_\alpha, z'_i)\rangle \\ &= -\delta^{(2)}(z-z')\left(\delta(x^- - x'^-)\theta(x^+ - x'^+) + \frac{1}{x^- - x'^- + i\epsilon}\right) P \mathrm{e}^{ie\int_x^{x'} \mathrm{d}x^+ A_+(z)} . \end{aligned} \tag{5.138}$$

The fact that the gauge field $A_\alpha(z)$ has pure gauge content, permits an effective simplification of the theory. Indeed, we have

$$A_\alpha = \frac{i}{e}\partial_\alpha U U^{-1} \quad , \tag{5.139}$$

and the action reads

$$S[U, A_i] = \frac{1}{2e^2}\int \mathrm{d}^4x \,\mathrm{tr}\,[\partial_\alpha(U^{-1}D_iU)]^2 \quad , \tag{5.140}$$

where $D_i = \partial_i - ieA_i$. If one redefines the field $A_i \to \widetilde{A}_i = \frac{i}{e}U^{-1}D_iU$, every local interaction disappears, and only the Wilson-line description survives. One can use for them the definition

$$P\mathrm{e}^{ie\int \mathrm{d}x^+ A_+(z)} = g_2(z)g_1^{-1}(z) \quad , \tag{5.141}$$

where the variables $g_A(z)$ (resp. $h_A(z)$ for A_-), $A = 1, 2$ are dynamical variables.

Due to the U equations of motion, the integration over $\mathrm{d}x^+$ and $\mathrm{d}x^-$ can be performed in the action for classical configurations, leaving only the value of the fields at boundaries, corresponding to the above-defined (effectively two-dimensional) fields g_A, h_A. Indeed

$$\begin{aligned} S[U, A] =& \frac{1}{2e^2}\int \mathrm{d}^2z \,\mathrm{tr} \int \mathrm{d}x^+\,\partial_+\left(U^{-1}D_iU\right)\int \mathrm{d}x^-\,\partial_-\left(U^{-1}D_-U\right) \\ =& \frac{1}{2e^2}\int \mathrm{d}^2z\, M^{AB}\mathrm{tr}\,\left(g_A^{-1}D_i^+g_A\,h_B^{-1}D_i^-h_B\right) \quad , \end{aligned} \tag{5.142}$$

where $M^{11} = M^{22} = 1 = -M^{12} = -M^{21}$, and $D_i^\pm$ contains the boundary fields $a_i^\pm(z)$.

High-energy scattering thus has a very simplified description as compared to the full difficulty of general scattering. The simplification comes from a general issue connected with an effective dimensional reduction occurring in such a limit, as exemplified in the simple case of one-loop high-energy fermionic scattering in QED, as described[132,133] in

Fig. 9

Fig. 9: Fermionic high-energy scattering in four-dimensional QED.

The upper line of the diagram is given respectively by

$$\frac{\gamma_\nu(\not{p}_1 + \not{k} + m)\gamma_\mu}{(p_1+k)^2 - m^2 + i\epsilon} \qquad \text{and} \qquad \frac{\gamma_\mu(\not{p}_2 - \not{k} + m)\gamma_\nu}{(p_2+k)^2 - m^2 + i\epsilon} \quad . \tag{5.143}$$

In the high-energy limit, we use the external fermions equation of motion $\overline{u}_2\gamma_\mu u_1 = \frac{p_{1_\mu}}{m}\chi_2^\dagger\chi_1$, where χ are bispinors. Moreover, we work on shell, and define $k_\pm = k_0 \pm k_3$, where the z-axis is chosen in the centre-of-mass system as the direction of $\vec{p}_1$. Adding the two contributions in (5.143), with all above ingredients we arrive at

$$\frac{p_{1_\mu}p_{1_\nu}}{mw}\left(\frac{1}{q_- + i\epsilon} + \frac{1}{-q_- + i\epsilon}\right) = -2\pi i \frac{p_{1_\mu}p_{1_\nu}}{mw}\delta(q_-) \quad , \tag{5.144}$$

where w is the centre-of-mass energy. Notice already the presence of the first δ-function in one of the non-transverse components of the momentum. From the remaining part of the diagram, one obtains a term proportional to $\frac{1}{-q_+ + i\epsilon}$, whose principal part vanishes due to symmetry, and one remains with a second delta term ($\delta(q_+)$). Therefore the scattering is described, in the loop variables, by an effective two-dimensional theory.

Such an idea as applied to the scattering of high-energy hadrons in QCD leads to the possibility of summing an infinite series of diagrams, and leads to the so-called Reggeization of the gluons, which up to an effective Regge form factor given by

$$s^{\alpha(t)-1} \tag{5.145}$$

(s and t are the Mandelstam variables, to be appropriately defined below), are described by the Born approximation of the perturbative expansion. This is the leading logarithmic approximation (LLA)[131]. Such a study of high-energy scattering has been performed by several authors, both in QED[133] and QCD.[124,131,132]

There is a surprising regularity in the QCD results,[131] where in higher orders the only new issue, in the LLA, namely $\alpha_s \ln \frac{s}{M^2} \approx 1$, are the logarithmic factors $\left(\alpha_s \ln \frac{s}{M^2}\right)^J$.

Taken separately, the diagrams violate the Froissart bound, leading to non-unitary amplitudes. A previous knowledge of Reggeon field theory leads to a solution of the problem. Indeed, higher orders are necessary, and as a matter of fact, the above-mentioned regularity of higher-order computations in the LLA can be summarized by the Reggeon factor, given by eq. (5.145). A non-perturbative solution should obey unitarity, which relates elastic and non-elastic scattering through the equation

$$\frac{1}{s}\mathrm{Im}\, T_{2\to 2} = \sum_{n=0}^{\infty} \int \mathrm{d}\Omega_n\, T_{2\to 2+n}(s+i\epsilon) T_{2\to 2+n}(s-i\epsilon) \quad , \tag{5.146}$$

where "n" is the number of new particles produced and $\{\mathrm{d}\Omega_n\}$ is the integration over the phase space of the intermediate states, to be defined in detail in eq. (5.162). This means that we have to supplement the LLA with an infinite subclass of diagrams, in such a way that unitarity holds.

The amplitude resulting from the LLA, and matching the lowest-order computations in perturbation theory, leads to the Reggeized amplitude

$$T_{2\to 2} = -\frac{\mathrm{e}^{-i\pi\alpha(t)} - 1}{t - M^2} s^{\alpha(t)} g t^a_{bc} g t^{a'}_{cb'} \quad , \tag{5.147}$$

where $\alpha(t) = 1 + (t - M^2)\beta(t)$ and

$$\beta(p^2) = e^2 N \int \frac{\mathrm{d}^2 k}{(2\pi)^3} \frac{1}{(k_\perp^2 + M^2)[(k-p)_\perp^2 + M^2]} \quad , \tag{5.148}$$

where $p_\perp$ is the component of momentum perpendicular to the incoming direction, or equivalently the impact parameter space. Also $s = (P_A + P_B)^2$ and $t = (P_A - P'_A)^2$.

In the high-energy limit the transferred momentum has mostly transverse components, $q = (P_A - P'_A) \simeq (0, 0, q_\perp)$ and $s >> s_1 \sim s_2 \sim \cdots \sim s_{n+1} >> q_{1_\perp}^2 q_{n+1_\perp}^2 \sim M^2$, where we define, according to Lipatov[124]

$$
\begin{aligned}
& s=(P_A+P_B)^2 \simeq 2P_AP_B \quad , \\
& s_i=(k_i+k_{i+1})^2 \simeq 2k_ik_{i+1} \quad , \\
& k_i=q_i-q_{i+1} \quad , \\
& k_0 \equiv P'_A=P_A-q_0 \, , \; k_{n+1} \equiv P'_B=P_B+q_{n+1} \quad ,
\end{aligned} \tag{5.149}
$$

and momentum conservation requires $P_A+P_B=\sum_{i=0}^{n+1} k_i$.

As we have already mentioned, order-by-order computations, in the high-energy limit have been performed by several authors, and the common feature in non-Abelian gauge theories is the fact that corrections exponentiate, leading to the so-called Reggeization of the gluon. This means that in the LLA, previously defined, one can just compute the Born amplitude, substituting the vertex by Lipatov's vertex, which includes the energy-dependent correction $s^{\alpha(t)}$. Thus we write for the $2 \to 2+n$ amplitude, in the LLA approximation, the result

$$
\begin{aligned}
& A_{2\to 2+n}(q_i) \\
& \quad = 2se\frac{(s_{01}/M^2)^{w(q_1)}}{q_1^2} et^{a_1}_{i_1,i_2}(C(q_1,q_2)\cdot\varepsilon(k_1))\frac{(s_{12}/M^2)^{w(q_2)}}{q_2^2} et^{a_2}_{i_2,i_3}(C(q_2,q_3)\cdot\varepsilon(k_2)) \\
& \qquad \times \cdots et^{a_n}_{i_n,i_{n+1}}(C(q_n,q_{n+1})\cdot\varepsilon(k_n))\frac{(s_{n,n+1}/M^2)^{w(q_{n+1})}}{q_{n+1}^2} \quad ,
\end{aligned} \tag{5.150}
$$

where $\varepsilon_\mu(k)$ is the polariazation vetor of the Reggeon. The trajectory of the Reggeized gluon is

$$
w(q)=-\frac{e^2N}{2(2\pi)^3}\int\frac{\mathrm{d}^2k\, q^2}{k^2(q-k)^2} \quad , \tag{5.151}
$$

and the result (5.150) can be summarized by

$$
A^{\mathrm{LLA}}=A^{\mathrm{tree}}s_1^{w(q_1)}\cdots s_{n+1}^{w(q_{n+1})} \quad , \tag{5.152}
$$

which corresponds to the leading logarithmic approximation (LLA). The form of the vertex for effective gluon production as computed by Lipatov[124]

$$
C^\mu(q_i,q_{i+1})=-q_i^{\mu\,\perp}-q_{i+1}^{\mu\;\;\perp}+P_A^\mu\left(\frac{2q_i^2}{\alpha_i s}+\beta_i\right)-P_B^\mu\left(\frac{2q_{i+1}^2}{\beta_i s}+\alpha_i\right) \quad , \tag{5.153}
$$

has been discussed previously in the comparison with the Lagrangian methods.

The fact that unitarity has to be satisfied in the s-channel for all subenergy variables has been emphasized by Bartels.[131] This is the moment where it is crucial to use unitarity equations to compute the discontinuity in the elastic-scattering amplitude (5.146). In order to use the unitarity equation, we have to rewrite the kinematical variables in a suitable form.

Since details are generally not available in the literature, we present the whole derivation of the proof of integrability of the high energy scattering amplitude.

We have to use again Sudakov's decomposition (5.135) for the outgoing gluons on shell in the energy limit ($k_i^2 = s\alpha_i\beta_i - (k_i^\perp)^2 = 0$), we find

$$\alpha_i\beta_i = \frac{(k_i^\perp)^2}{s} \quad , \tag{5.154}$$

and

$$s_{i,i+1} = (\alpha_{i+1}\beta_i + \alpha_i\beta_{i+1})s - 2k_i^\perp \cdot k_{i+1}^\perp \quad , \tag{5.155}$$

where, in general, the first term dominates, and $\alpha_i\beta_{i+1} + \alpha_{i+1}\beta_i \sim 1$. Moreover, substituting $\alpha_i\beta_i = \frac{(k_i^\perp)^2}{s}$, we find that in the multi-Regge limit $\left(\frac{\alpha_{i+1}}{\alpha_i} + \frac{\beta_{i+1}}{\beta_i}\right)$ must be large, and since $\alpha_0 \sim 1$, and $\beta_{n+1} \sim 1$, we find the order

$$1 \sim \alpha_0 >> \alpha_1 >> \cdots >> \alpha_{n+1} \sim q_{n+1}^2/s \quad , \tag{5.156}$$

and

$$q_0^2/s \sim \beta_0 << \beta_1 \cdots << \beta_{n+1} \sim 1 \quad . \tag{5.157}$$

Thus we obtain for the partial energy variables the results

$$\begin{aligned} s_{i,i+1} &= \alpha_i\beta_{i+1}s \quad , \\ \prod_{i=0}^{n} s_{i,i+1} &= \alpha_0\beta_{n+1}s\prod_{i=1}^{n}(k_i^\perp)^2 \quad . \end{aligned} \tag{5.158}$$

The phase-space integral is obtained from

$$\begin{aligned} \frac{\mathrm{d}^4k}{(2\pi)^3}\delta(k^2) &= s\mathrm{d}\alpha\mathrm{d}\beta\frac{\mathrm{d}^2k^\perp}{2(2\pi)^3}\delta(s\alpha\beta - k_\perp^2) \quad , \\ &= \frac{\mathrm{d}\alpha}{\alpha}\frac{\mathrm{d}^2k^\perp}{2(2\pi)^3} \quad , \end{aligned} \tag{5.159}$$

leading to the physical momentum volume

$$\prod_{i=0}^{n+1}\frac{\mathrm{d}^4k_i}{(2\pi)^3}\delta(k_i^2) = \prod_{i=0}^{n}\frac{\mathrm{d}s_{i,i+1}}{s_{i,i+1}}\prod_{i=0}^{n+1}\frac{\mathrm{d}^2k_i^\perp}{2(2\pi)^3}\frac{\mathrm{d}\alpha_0}{\alpha_0} \quad . \tag{5.160}$$

From this expression we find the total phase-space volume which reads

$$\begin{aligned} \mathrm{d}\Omega_n &= (2\pi)^4\delta^4\left(P_A + P_B - \sum_{i=0}^{n+1}k_i\right)\prod_{i=0}^{n+1}\frac{\mathrm{d}^4k_i}{(2\pi)^3}\delta(k_i^2) \\ &= \frac{\pi}{s}\prod_{i=0}^{n+1}\frac{\mathrm{d}\alpha_i}{\alpha_i}\prod_{i=1}^{n+1}\frac{\mathrm{d}^2k_i^\perp}{(2\pi)^3}\delta(1-\alpha_0)\delta(1-\beta_{n+1}) \quad . \end{aligned} \tag{5.161}$$

We can now integrate over $s_{i,i+1}$ instead of α_i and use eq. (5.158) as well as integrate over $k_0^\perp$ to arrive at

$$\mathrm{d}\Omega_n = \frac{\pi}{s}\left[\prod_{i=0}^{n}\left(\mathrm{d}s_{i,i+1}\frac{\mathrm{d}^2 k_{i+1}^\perp}{(2\pi)^3}\right)\right]\delta\left(\prod s_{i,i+1} - s\prod (k_i^\perp)^2\right) \quad . \tag{5.162}$$

The Mellin transformation is now easily performed since the s variable enters in a simple way as an overall factor, as well as in the argument of the delta function. Thus we have

$$\int\int_0^\infty \frac{\mathrm{d}s}{M^2}\left(\frac{s}{M^2}\right)^{-w-1} s\mathcal{F}(s)\mathrm{d}\Omega_n = \pi\int\frac{\mathrm{d}s_{0,1}}{M^2}\left(\prod_{i=1}^{n}\frac{\mathrm{d}s_{i,i+1}}{(k_i^\perp)^2}\right)\left(\frac{s^{\mathrm{tot}}}{M^2}\right)^{-w-1}$$
$$\times\prod_{i=1}^{n+1}\frac{\mathrm{d}^2 k_i^\perp}{(2\pi)^3}\mathcal{F}(s^{\mathrm{tot}}) \quad , \tag{5.163}$$

where $s^{\mathrm{tot}} = s_{0,1}\prod_{i=0}^{n}\frac{s_{i,i+1}}{(k_i^\perp)^2}$.

Notice now that for $i \leq n$, $q_i \sim f(k_i^\perp, \alpha_i; q_{i+1}) \sim h(k_i^\perp, s_{i-1,i}; q_i)$. Thus all $s_{i,i+1}$ integrations, but for $i = m$, are non-trivial.

The denominators $q_i^2 \simeq -(\vec{q}_i)^2$ are factored to the integration over the perpendicular momenta. The partial energy integration (over $s_{i,i+1}$) is not too complicated now. From $A_{2\to 2+n}$ and its conjugate we find terms of the type

$$\left[\frac{s_{i,i+1}}{(k_i^\perp)^2}\right]^{-w-1}\left[\frac{s_{i,i+1}}{M^2}\right]^{w(q_i)+w(q-q_i)} \quad . \tag{5.164}$$

From the LLA $s_{i,i+1} = s\alpha_i\beta_{i+1} >> s\alpha_i\beta_i = (k_i^\perp)^2$; therefore, using the latter as a lower bound for integration, we find for the contribution (5.164) for the elastic scattering

$$\int_{(k_i^\perp)^2}^\infty \frac{\mathrm{d}s_{i,i+1}}{(k_i^\perp)^2}\left[\frac{s_{i,i+1}}{(k_i^\perp)^2}\right]^{-w-1}\left[\frac{s_{i,i+1}}{M^2}\right]^{w(q_i)+w(q-q_i)} = \frac{1}{w - w(q_i) - w(q-q_i)} \quad . \tag{5.165}$$

The sum over polarizations may be performed as usual, and we find

$$\sum_a C^\mu(q_i, q_{i+1})C^\nu(q-q_i, q-q_{i+1})\epsilon_\mu^{(a)}(k_i)\epsilon_\nu^{(a)}(k_i) = C_\mu(q_i, q_{i+1})C^\mu(q-q_i, q-q_{i+1})$$
$$= K(q_i, q_{i+1}|q) \quad , \tag{5.166}$$

defining a kernel that will be used in the derivation of the Bethe–Salpeter equation, and which can be computed from the expression of the Lipatov vertex (5.153). One first defines the function $F(q,q')$ by

$$A(w,t) = \int \mathrm{d}^2 q' F(q,q')\phi(q,q')\prod_{i=2}^{n+1}\int \mathrm{d}^2 q_i \frac{K(q_{i-1}, q_i|q)}{q_i^2(q-q_i)^2[w - w(q_i) - w(q-q_i)]} \quad , \tag{5.167}$$

where

$$F(q,q_1)=\sum_{n=1}^{\infty}\left(\frac{e^2N}{(2\pi)^3}\right)^n\frac{1}{q_1^2(q-q_1)^2[w-w(q_1)-w(q-q_1)]}\quad . \tag{5.168}$$

The vertex function $\phi(q,q')$ has to be used in order to define the hadronic state. The Bethe–Salpeter equation obeyed by $F(q,q_1)$ can be simply read from the above series representation:

$$q_1^2(q-q_1)^2[w-w(q_1)-w(q-q_1)]F(qq_1)=\frac{e^2N}{(2\pi)^3}\left[1+\int \mathrm{d}^2q_1' K(q_1',q-1|q)F(q,q_1')\right]\quad . \tag{5.169}$$

The resulting amplitude is essentially two-dimensional[134–136], and it is useful to define the Fourier transformation in the impact parameter space as

$$\delta^2(q-q')f_w(k,k';q)=\frac{1}{(2\pi)^8}\int\prod_1^2\mathrm{d}^2\rho_r\prod_1^2\mathrm{d}^2\rho_{r'}f_w(\rho_1,\rho_2,\rho_{1'},\rho_{2'})$$
$$\times\,\mathrm{e}^{ik\rho_1+i(q-k)\rho_2-ik'\rho_{1'}-i(q'-k')\rho_{2'}}\quad . \tag{5.170}$$

Lipatov rewrote the Bethe–Salpeter equation in such a way that conformal invariance in two-dimensional space can be verified, that is

$$\begin{aligned}
&w\frac{\partial^2}{\partial\rho_1^2}\frac{\partial^2}{\partial\rho_2^2}f_w(\rho_1,\rho_2)\\
&=(2\pi)^4\delta^2(\rho_1-\rho_{1'})\delta^2(\rho_2-\rho_{2'})+\frac{e^2N}{(2\pi)^3}\left\{(2\pi)^2\delta^2(\rho_1-\rho_2)\left(\frac{\partial}{\partial\rho_1}+\frac{\partial}{\partial\rho_2}\right)^2f_w(\rho_1,\rho_2)\right.\\
&+\frac{\partial^2}{\partial\rho_1^2}\int\frac{\mathrm{d}^2\rho_0}{|\rho_{01}|^2}\left[\frac{\partial^2}{\partial\rho_2^2}f_w(\rho_0,\rho_2)-\frac{|\rho_{12}|^2}{|\rho_{01}|^2+|\rho_{02}|^2}\frac{\partial^2}{\partial\rho_2^2}f_w(\rho_1,\rho_2)\right]\\
&\left.+\frac{\partial}{\partial\rho_2^2}\int\frac{\mathrm{d}^2\rho_0}{|\rho_{02}|^2}\left[\frac{\partial^2}{\partial\rho_1^2}f_w(\rho_1,\rho_0)-\frac{|\rho_{12}|^2}{|\rho_{01}|^2+|\rho_{02}|^2}\frac{\partial^2}{\partial\rho_1^2}f_w(\rho_1,\rho_2)\right]\right\}\quad ,
\end{aligned}\tag{5.171}$$

where $\rho_{ij}=\rho_i-\rho_j$. In terms of $f_w(\rho_1,\rho_2,\rho_{1'},\rho_{2'})$, the Bethe–Salpeter equation is an integro-differential equation, with quartic derivatives with respect to ρ_i. The problem, although appearing far from the solubility from the point of view of such a difficult differential equation, is facilitated from the fact that it is conformally invariant in the impact parameter space. In two dimensions, conformal invariance means factorization in terms of holomorphic and anti-holomorphic functions. Neglecting the contact terms, the remaining integrals are not difficult to compute. Consider

$$\int\frac{\mathrm{d}^2\rho_0}{|\rho_{01}|^2}\frac{\partial^2}{\partial\rho_2^2}f(\rho_0,\rho_2)=\int\frac{\mathrm{d}^2\rho_0}{|\rho_{01}|^2}\mathrm{e}^{-iP\rho_{01}}\frac{\partial^2}{\partial\rho_2^2}f(\rho_1,\rho_2)\quad , \tag{5.172}$$

where P is the generator of translations. The above integral is in general infrared-divergent, but as proved by Lipatov, considering all contributions the infrared divergence will cancel (see also next term in (5.171)). Neglecting this divergence, the integral corresponds to the massless boson propagator in two dimensions with the phase-space variables interchanged, and one obtains[8] the result $2\pi \ln P^2$. For the remaining term we have

$$\begin{aligned}\int \frac{\mathrm{d}^2\rho_0}{|\rho_{01}|^2} \frac{|\rho_{12}|^2}{|\rho_{01}|^2 + |\rho_{02}|^2} \frac{\partial^2}{\partial \rho_2^2} f(\rho_1, \rho_2) &= \int \mathrm{d}^2\rho_0 \left[\frac{1}{|\rho_{01}|^2} - \frac{2}{|\rho_{01}|^2 + |\rho_{02}|^2} \right] \frac{\partial^2}{\partial \rho_2^2} f(\rho_1, \rho_2) \\ &= 2\pi \ln |\rho_{12}|^2 \frac{\partial^2}{\partial \rho_2^2} f(\rho_1, \rho_2) \quad . \end{aligned} \tag{5.173}$$

Writing the derivative terms in momentum space, and separating the holomorphic and anti-holomorphic pieces we obtain the pair Hamiltonian

$$H_{ik} = P_i^{-1} \ln \rho_{ik} P_i + P_k^{-1} \ln \rho_{ik} P_k + \ln P_i P_k \quad . \tag{5.174}$$

The final situation is outlined in ref. [125], where Lipatov states that the Bethe–Salpeter equation, for n reggeized gluons for a large number (N) of colours, is described in terms of a wave function $f_w(\vec{\rho}_1, \cdots, \vec{\rho}_n; \vec{\rho}_0)$ satisfying factorization

$$f_w(\vec{\rho}_1, \cdots, \vec{\rho}_n; \vec{\rho}_0) = \sum f^2(\rho_{10}, \cdots, \rho_{n0}) \overline{f}^2(\overline{\rho}_{10}, \cdots, \overline{\rho}_{n0}) \quad , \tag{5.175}$$

where $\rho_{ij} = \rho_i - \rho_j$ and $\overline{\rho}_{ij} = \overline{\rho}_i - \overline{\rho}_j$ are the coordinates of the gluons, and these functions satisfy independent Schrödinger equations

$$H f^2 = \epsilon f^2 \quad , \tag{5.176a}$$

$$\overline{H}\, \overline{f}^2 = \overline{\epsilon} \overline{f}^2 \quad . \tag{5.176b}$$

From the previous analysis, an extremely interesting physics describing the high-energy scattering of QCD arises. Thus, recapitulating, gluons interact in such a way that they build up collective states, the Reggeons, interacting almost as free particles, but conveniently dressed. Unitarity imposes strong conditions. In particular it is possible to compute the partial-wave scattering amplitude via unitarity condition, since the plain LLA leads to amplitudes that do not satisfy the Froissart bound. The unitary partial-wave scattering amplitude can be proved to satisfy a Bethe–Salpeter equation. The solutions to such an equation describe the singlet composite states built out of Reggeons, the first state with vacuum quantum numbers being the Pomeron.[137] The Bethe–Salpeter equation is a Hamiltonian equation describing an effective interaction for pairs of gluons, being of the form

$$\mathcal{H} = -\frac{e^2}{2\pi} \sum H_{ij}\, t_i^a t_j^a \quad , \tag{5.177}$$

where (i, j) are the pairs of Reggeons. Such two-particle Hamiltonians $\mathcal{H}_{ij}$, as presented in (5.174), are effectively two-dimensional, and may be interpreted as describing a lattice model with pair interactions, the size of the lattice being the number of external Reggeons.

A big simplification arises in the large-N (number of colours) limit, where the theory is planar, and thus the external Reggeons interact only with their nearest neighbours. In such a case one substitutes in eq. (5.177), $t_i^a t_j^a \to -\frac{N}{2}\delta_{i,j+1}$. Faddeev and Korchemsky[127] proved that the Hamiltonian thus obtained corresponds to the XXX Heisenberg chain with spin zero.

The form of such an identification arises from the computation of the Heisenberg Hamiltonian in terms of the solution of the Yang–Baxter equation. The solution is given by[138]

$$R_{ij}(\lambda) = \frac{\Gamma(i\lambda - 2s)\Gamma(i\lambda + 2s + 1)}{\Gamma(i\lambda - J_{ij})\Gamma(i\lambda + J_{ij} + 1)} \quad , \tag{5.178}$$

where s is the spin and the operator J_{ij} acts on the quantum space $h_i \otimes h_j$, h_i corresponding to the i^{th} site of the Heisenberg lattice, and satisfies

$$J_{ij}(J_{ij}+1) = (\vec{s}_i + \vec{s}_j)^2 = 2\vec{s}_i \cdot \vec{s}_j + 2s(s+1) \quad . \tag{5.179}$$

The form of the Hamiltonian is

$$H = \sum H_{jj+1} \quad , \tag{5.180}$$

$$H_{jj+1} = -i\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln R_{jj+1}(\lambda)\Big|_{\lambda=0} \quad . \tag{5.181}$$

Astonishingly enough, Faddeev and Korchemsky[127] computed this Hamiltonian and verified that for $s = 0$ it is given by eq. (5.174)! Therefore one can use the Bethe ansatz to obtain the solution of high-energy QCD.[128]

6. Conclusions

After twenty years of development, QCD_2 stays in an outstanding position in the way towards non-perturbative comprehension of strong interactions. The large-N limit of the theory revealed a desirable structure for the mesonic spectrum, whose higher levels display a Regge behaviour. These properties were later generalized for fermions in the adjoint representation, an important step towards understanding the theory in higher dimensions, since in such a case adjoint matter substitutes the lack of the transverse degrees of freedom of the gauge field in two dimensions. Further properties of the perturbative theory are also in accordance with expectations for strong interactions, and it proves therefore to have certain advantages over the usual non-linear σ-models in the description of strong interactions by means of simplified models.

The central issue of the computation of the non-Abelian fermionic determinant is the key to understanding the theory, bypassing the severe question of confinement since it provides an effective theory for the description of the mesonic bound states, opening the possibility of understanding baryons as solitons of the effective interactions. The full QCD problem can be dealt with using these methods. There are paralels with the QED developments, but even well known facts as the role played by negative metric states[140] are further complicated in the present case, where the coset construction needs to be advocated.

The string interpretation of pure Yang–Mills theory, as well as its Landau–Ginzburg-type generalizations connected the previous picture to that of non-critical string theory. The relevance of these developments is found in the basis they form for a deeper understanding of the role of non-critical string theory in the realm of strong interactions. Although far from being realized, such is aparently the correct way to understand strong interactions at intermediate energies. A very general formulation of QCD_2 has been recently studied including many features discussed here.[141]

Finally, the question of high energy scattering in strong interactions is linked with integrable models, a property shared in full by two-dimensional QCD. Thus one sees the important role of higher symmetries algebras, spectrum generating algebras, and integrability conditions, which might give a clue to the full solution of the theory.

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Appendix A

In this appendix we summarize our notation and conventions. In most of the review we work in Minkowski two-dimensional space, but we give here the necessary dictionary for translating results to Euclidian space. For the metric and ϵ tensor we use

$$g^{00} = -g^{11} = 1 \quad , \quad \epsilon_{01} = -\epsilon^{01} = 1 \quad ; \tag{A.1}$$

the gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma_\mu \gamma_5 = \epsilon_{\mu\nu}\gamma^\nu \quad , \tag{A.2}$$

in such a way that $\gamma_\pm = \gamma_0 \pm \gamma_1$, $\gamma_+\gamma_- = 2$ and we use also

$$\epsilon^{\mu\nu}\epsilon_{\rho\sigma} = \delta^\mu_\sigma \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\sigma \quad . \tag{A.3}$$

The definition of the light-cone variables

$$J_+ = J_0 + J_1 \quad , \quad A_+ = A_0 + A_1 \tag{A.4}$$

leads to extra factors of 2, as for example in eq. (2.62). The tilde is generically reserved for the definitions of pseudo-vectors

$$\widetilde{A}_\mu = \epsilon_{\mu\nu} A^\nu \quad , \quad \text{or} \quad \widetilde{D}_\mu = \epsilon_{\mu\nu} D^\nu \quad . \tag{A.5}$$

Notice that $\partial_\mu \widetilde{\partial}_\nu - \partial_\nu \widetilde{\partial}_\mu = \epsilon_{\mu\nu}\Box$, where $\Box$ is the d'Alembertian. The massless propagator obeying $\Box D(x) = \delta(x)$ is $D(x) = -\frac{i}{4\pi}\ln(-x^2 + i\epsilon)$. In Minkowski space,

$$x^\mu = (x^0, x^1) \quad , \quad \partial_\pm = \partial_0 \pm \partial_1 \quad , \quad x^\pm = x^0 \pm x^1 \quad . \tag{A.6}$$

On the other hand, in Euclidian space:

$$x_\mu = (x_1, x_2)\, , \, \overline{\partial} = \partial_1 - i\partial_2 \equiv \partial_-^E \, , \, \partial = \partial_1 + i\partial_2 \equiv \partial_+^E \, , \, z = x_1 - ix_2 \, , \, \overline{z} = x_1 + ix_2 \, . \tag{A.7}$$

In order to translate from one space to the other, we have $x_2 = ix_0$, implying (notice the important $(-)$ sign!)

$$\overline{\partial} \longleftrightarrow -\partial_- \quad , \quad \partial \longleftrightarrow \partial_+ \quad . \tag{A.8}$$

Notice also that

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\overline{\partial} \quad , \quad \frac{\partial}{\partial z} = \frac{1}{2}\partial \quad . \tag{A.9}$$

With these conventions,

$$F_{\mu\nu}F_{\mu\nu} = 2F_{12}^2 = -\frac{1}{2}F_{z\bar{z}}^2 \quad \text{and} \quad F_{z\bar{z}} = -iF_{12} + iF_{21} = -2iF_{12} \quad . \tag{A.10}$$

Path integrals are always performed in Euclidian space, while in the canonical quantization we use the Minkowski version:

$$\frac{\partial}{\partial x^\pm} = \frac{1}{2}\partial_\pm \quad , \quad \mathrm{d}x^+\,\mathrm{d}x^- = \left\| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\| \mathrm{d}^2x \, = 2\mathrm{d}^2x \quad . \tag{A.11}$$

The gauge field and the covariant derivative, in the direct and adjoint representations, and the τ matrices are

$$A_\mu = \sum_a \tau^a A_\mu^a \equiv \tau^a A_\mu^a \quad , \tag{A.12a}$$

$$D_\mu = \partial_\mu - ieA_\mu \quad , \tag{A12b}$$

$$\nabla_\mu^{ab} = \partial_\mu \delta^{ab} + ef^{acb}A_\mu^c \quad , \tag{A.12c}$$

$$[\tau^a, \tau^b] = if^{abc}\tau^c \quad , \quad \mathrm{tr}\,\tau^a\tau^b = \delta^{ab} \quad . \tag{A.12d}$$

The gauge field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu] \quad , \tag{A.13a}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ef^{abc}A_\mu^b A_\nu^c \quad . \tag{A.13b}$$

The covariant derivative on fermions in the fundamental representation, ψ^i, are defined by (A.12a)

$$(D_\mu\psi)_i = \partial_\mu\psi_i - ie(A_\mu\psi)_i \quad , \tag{A.14}$$

while for fermions in the adjoint representation, ψ^a, we have

$$(\nabla_\mu\psi)^a = \partial_\mu\psi^a + ef^{acb}A_\mu^c\psi^b = \partial_\mu\psi^a - ie[A_\mu,\psi]^a \quad . \tag{A.15}$$

In two dimensions, it is true that

$$F^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu}\epsilon_{\rho\sigma}F^{\rho\sigma} \quad . \tag{A.16}$$

Appendix B

Considering maps of a closed orientable two-dimensional surface $\mathcal{M}_g$ of genus g, onto another two-dimensional surface $\mathcal{M}_G$ of genus G, a relation between the genera G, g and the winding number (n) of the mapping can be obtained. In the case of smooth maps, this relation is given by Kneser's formula[139]

$$2(g-1) \geq 2n(G-1) \quad . \tag{B.1}$$

To understand this bound one uses the idea of covering maps, which in case of smooth maps do not have branch points or collapsed handles. It can be proved that such smooth maps can be continuously deformed into the covering maps. To construct them, first consider a closed orientable two-dimensional surface $\mathcal{M}_G$ of genus $G > 1$, as in Fig. 10a. Then cut this surface along a cycle, as in Fig. 10b, leading to a surface of genus $(G-1)$. This surface is topologically equivalent to the surface of Fig. 10c. Now add n copies of the same topologically equivalent surface, as in Fig. 10d, obtaining a surface of genus $n(G-1)$. Finally close the covering gluing, the circles a and b forming a surface with an additional handle, ending up with an n-fold covering of $\mathcal{M}_G$ by $\mathcal{M}_g$, where the genera and the winding number are related by

$$g = n(G-1) + 1 \quad . \tag{B.2}$$

Fig. 10: Covering space to obtain the Kneser formula.

This is not the whole history, because there are no smooth maps of a genus $g > 1$ surface that wind around it more than once. This means that we have to include branch points or collapsed tubes, modifying the above relation to

$$2(g-1) = 2n(G-1) + B \quad , \tag{B.3}$$

where B is the total branching number. The above relation is known as the Riemann–Hurwitz theorem. (See ref. [83] for more details.)

In fact the relation (B.2) pictures only the surfaces that saturate the bound given by Kneser's formula; adding extra handles to $\mathcal{M}_g$, the Euler characteristic can only increase.

Appendix C

A character is the trace of the matrix in a given representation; that is, given a representation R

$$R\colon g \to T(g) \quad , \tag{C.1}$$

the character is

$$\chi[g] = \mathrm{tr}\; T(g) \quad . \tag{C.2}$$

Its utility lies upon the fact that it is invariant under a unitarity transformation, and the representation is fully decomposed in terms of uniquely characterized irreducible representations.

In a given representation, characters may be computed in terms of the eigenvalues of the given element. In the case of $U(N)$, the eigenvalues are phases $\epsilon_j = e^{i\varphi_j}$, and the irreducible representations are labelled by N integers n_i, ordered as $n_1 \geq n_2 \geq \cdots \geq n_N$, and it has been proved that the character of U is a ratio of the determinants (see refs. [76, 77] for details)

$$\chi_{\{\lambda_i\}}[U] = \frac{\det\, \epsilon_i^{n_j+N-j}}{\det\, \epsilon_i^{N-j}} = \frac{\Delta_f(\epsilon)}{\Delta_0(\epsilon)} \quad . \tag{C.3}$$

The dimension of the representation is simply obtained as the character computed on the unit element, that is

$$\dim R = \mathrm{tr}_{\,R}\, 1 = \chi_R[1] \quad . \tag{C.4}$$

Consider a manifold M on which the group acts. It is clear that tensors also provide representations for the group action. The irreducible representations are classified by means of the symmetry of the tensor with respect to indices exchange, in a Young tableau as in Fig. 11; there, the i$^{\mathrm{th}}$ row has n_i elements, $n_i \geq n_{i+1}$, and the tensor is constructed in such a way as to be symmetric in the indices in a same row, and later antisymmetrized with respect to indices in the same column:

Fig. 11: A general Young tableau.

The Young frame is then given in terms of the properties under permutations. The length of the rows, $n_1 \geq n_2 \geq \cdots \geq n_k > 0$ defines the representation. They are the analogue of the total angular momentum. A standard tableau is defined as the one where the integers $1, 2, \cdots, N$, are distributed in non-decreasing order from the left in every row and in increasing order from the top in every column. The n_i's above represent the highest-weight vectors.

The best property of characters, however, is the fact that they obey orthogonality and completeness relations, once an invariant metric is defined. This means that functions of the group can be decomposed in a "Fourier series", where the "Fourier elements" are the characters.

Let us denote by $\mathcal{D}^R_{\alpha\beta}(g)$ a d_R-dimensional representation in terms of a matrix. The integral

$$\int \mathcal{D}g\, \mathcal{D}^R_{\alpha\beta}(g) \mathcal{D}^S_{\gamma\delta}(g^{-1}) \quad , \tag{C.5}$$

where $\mathcal{D}g$ is the Haar measure, cannot be non-vanishing unless $R = S$, otherwise one would be able to construct a homomorphism from R to S. On the other hand, when they are equal, the result must be proportional to $\delta_{\alpha\delta}\delta_{\beta\gamma}$, and one finds

$$\int \mathcal{D}g\, \mathcal{D}^R_{\alpha\beta}(g) \mathcal{D}^S_{\delta\gamma}(g^{-1}) = \frac{1}{\dim R}\, \delta_{RS}\delta_{\alpha\gamma}\delta_{\beta\delta} \quad . \tag{C.6}$$

Along the same lines one also finds (for proofs, we refer to the mathematical literature; see ref. [76])

$$\sum_{R,\alpha,\beta} (\dim R)\, \mathcal{D}^R_{\alpha\beta}(g) \mathcal{D}^R_{\beta\alpha}(h^{-1}) = \delta(g,h) \quad . \tag{C.7}$$

These relations imply, for characters, the results

$$\int \mathcal{D}g\, \chi_R[g]\, \chi_S[g] = \delta_{RS} \quad , \tag{C.8a}$$

$$\sum_R (\dim R)\, \chi_R[gh^{-1}] = \delta(g,h) \quad . \tag{C.8b}$$

Moreover

$$\int \mathcal{D}g\, \chi_R[g]\, \chi_S[g^{-1}h] = \frac{1}{\dim R} \delta_{RS} \chi_R[h] \quad , \tag{C.9}$$

which upon use of Haar measure invariance implies

$$\int \mathcal{D}g\, \chi_R[gh]\, \chi_S[g^{-1}m] = \frac{1}{\dim R} \delta_{RS} \chi_R[hm] \quad . \tag{C.10}$$

Using (C.6), we can also show that

$$\int \mathcal{D}g\, \chi_R[ghg^{-1}m] = \frac{1}{\dim R} \chi_R[h] \chi_R[m] \quad , \tag{C.11}$$

In view of the orthogonality relation of the characters:

$$\int \mathrm{d}V \, \chi_{R_1}[U_{p_1} V] \, \chi_{R_2}[V^+ U_{p_2}] = \frac{1}{\dim R} \, \delta_{R_1, R_2} \, \chi[U_1 U_2] \quad , \tag{C.12}$$

and finally we write down the combination formula for characters

$$\sum_{\sigma \in S_n} d_R^{-1} \chi_R[\sigma] \chi_R[\varphi] = \sum_{\sigma} \chi_R[\sigma\varphi] \quad , \tag{C.13}$$

Casimir and dimensions

Besides the characters, important concepts in the theory use of representations are the Casimir operators, and the dimension. The latter can be simply obtained from the character computed at the unit element, namely

$$\dim R = \chi_R[1] \quad . \tag{C.13}$$

The Casimirs are invariants under the group transformation, and the order-"p" Casimir is given by the trace of the product of all possible p elements of the group, that is

$$C_p = \sum_{\{i_i\}} A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_p i_1} \quad . \tag{C.14}$$

We will need here the quadratic Casimir

$$C_2 = \mathrm{tr} \ A^2 \quad , \tag{C.15}$$

Given the Young tableau, which is equivalent to giving their representation, the computation of Casimirs and dimensions is a well-defined problem in group theory, and the result is known. The actual procedure is however too long to be discussed in full generality. For $U(N)$ or $SU(N)$, the results are rather simple. The quadratic Casimir can be computed acting with

$$X_2 = E_{-\alpha} E_\alpha + H^2 \quad , \tag{C.16}$$

on a highest-weight vector. One finds, for the eigenvalue:

$$C_2(R) = m^2 + \sum_\alpha \vec{\alpha} \cdot \vec{m} = m^2 + 2\vec{m} \cdot \vec{r} \quad . \tag{C.17}$$

The values of r_i can be found in ref. [76], $2r_i = N + 1 - 2i$, and one finds

$$C_2(R) = \sum_{i=1}^{N} n_i(n_i + 1 - 2i) + Nn \quad , \tag{C.18}$$

for $U(N)$, with $\sum n_i = n$. For $SU(N)$ the last term must be dropped, and one has to use $\tilde{n}_i = n_i - \frac{1}{N}\sum n_i$, and after some simple algebra one finds

$$C_2^{SU(N)}(R) = \sum n_i(n_i + 1 - 2i) + Nn - \frac{n^2}{N} \quad . \tag{C.19}$$

The dimension is given by Weyl's formula and we have, using $h_i = n_i + N - i$,

$$\dim R = \frac{\prod_{i<j}(h_i - h_j)}{\prod_{i<j}(n_i - n_j + j - i)} \quad . \tag{C.20}$$

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