From Veltman's conditions to Finite Unification

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Abstract

First we review Veltman's suggestion to attack the naturalness problem in the Standard model by requiring absence of quadratic divergences and the resulting mass formula. Then we emphasise the influence of Veltman's suggestion in strengthening the belief that supersymmetry is the natural playground for solving the problem of quadratic divergences. Going further, we recall few sporadic suggestions concerning the cancellation of the logarithmic divergences too, which in the framework of supersymmetry has led to the construction of all-loop Finite Theories with the use of the idea of reduction of couplings. Eventually, we concentrate on a specific Finite Unified Theory and its successful predictions for the top and Higgs mass, among others, and the prospects of its final justification in future collider searches.

1   Introduction

Tini Veltman was a great scientific personality who contributed continuously for decades in the most significant manner in establishing what everybody accepts today as the Standard Model (SM) of Elementary Particle Physics.[a]

His numerous important contributions in the field and in particular the breakthrough of Tini Veltman and Gerard 't Hooft on the renormalizability of the Standard Model (SM), one of the great moments in twentieth century physics, was awarded a Nobel Prize in Physics in 1999 "for elucidating the quantum structure of electroweak interactions in physics".

Here, we would like to present how one of Tini Veltman's ideas influenced the development of a particular direction of research, which eventually led to early successful predictions of the top quark and Higgs masses.

A concept that inspired Veltman in the direction that we would like to discuss and which is in the center of theoretical discussions after Veltman's work is the *naturalness* of a theory [1–3]. According to this idea, a theory is considered natural if at ordinary energies it is not too sensitive to the fundamental constants of nature. More specifically, a theory is considered unnatural if the radiative corrections to a physical observable have an intrinsic magnitude much greater than the observed value, so that a conspiracy among different orders in perturbation theory or a "fine tuning" is required. The naturalness criterion is particularly serious in the case of the SM since it belongs to the general category of renormalizable field theories with scalar masses which are known to suffer from quadratic divergences. Then quadratic divergences are indicative of the fact that the natural order of magnitude of the Higgs mass in the SM is $\mathcal{O}(f_L\Lambda)$, where f_L is a loop factor and Λ is the scale of new physics beyond the SM. Clearly then, absence of quadratic divergences is a necessary condition for the naturalness of the SM, which has to be modified in such a way so they are removed, and

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[a]A more personal, but also more detailed presentation of Veltman's contributions together with some biografical notes can be found at the Corfu Institute (EISA) homepage: http://eisa.institute .

that the mass scale of the modification should be in the TeV scale. This requirement is not sufficient, since such a theory might still suffer from the gauge hierarchy problem, i.e. could not provide the reason that there exist scales with huge differences in magnitude in nature, as for instance among the electroweak and the Planck scale.

With considerations along the above lines, Veltman was led to impose the condition of absence of the quadratic divergences in the SM in his famous paper published in Acta Physics Polonica [4]. It is a very important work since it was shown that this condition, known as the Veltman condition, is not just technical, but was leading to a relation among the masses of the SM, i.e. it has physical consequences, although after the discovery of the top and Higgs particles it appeared not to hold. Equally important is the fact that this work paved the way for supersymmetry (SUSY) [5] to be considered widely, and not only among the experts, as a theory with physical significance and consequences. Specifically, renormalizable supersymmetric theories are free of quadratic divergences to all orders in perturbation theory due to non-renormalization theorems. Of particular interest is the fact that such a property holds also in theories with softly broken supersymmetry (SSB) [6–9], such as the celebrated Minimal Supersymmetric Standard Model (MSSM) (for details see [10]), which has good chances to describe Physics beyond the SM. Finally, it should be stressed that for a very general class of theories with spontaneously broken supersymmetry, a mass formula was derived [11], which is very similar to the one resulting from Veltman's condition for cancellation of quadratic divergences.

The next question in this exciting avenue of development of ideas, starting from Veltman's fundamental work in [4], concerns the uniqueness of supersymmetry as a solution to the problem of cancellation of quadratic divergences in renormalizable field theories involving scalars. This was posed by two groups in [12] and [13,14] and was answered positively. Indeed, supersymmetry is the unique way to cancel the quadratic divergences in renormalizable field theories with scalars that can be examined perturbatively. Still, there is another very interesting way to avoid the problem by considering that the scalars are not fundamental but composite, i.e. a bound state of two fermions and was also mentioned in Veltman's paper in ref [4].

It should also be noted that Decker and Pestieau did, independently of Veltman, a similar analysis but they went a step further requiring that the lepton self-masses be *finite* [15], i.e. cancellation of the logarithmic divergences, too. As a result, new mass relations were found. Inspired by all the above ideas, we were searching for the construction of realistic Finite Theories with predictive power concerning some of the SM free parameters, the proliferation of which was always considered as another big obstacle of this theory. Quite naturally we were led to the framework of SSB supersymmetric theories where cancellation of quadratic divergences holds to all orders in perturbation theory and moreover to require absence of logarithmic divergences. It is remarkable that all-orders finite supersymmetric gauge theories can be constructed using the *reduction of couplings* scheme [15] and we consider ourselves lucky that we managed to construct the first realistic Finite Unified Theory [16,17]. Moreover, this model was predicting correctly the top quark mass one and a half year before its discovery; a prediction which survived for twelve years. Another version of the model [18] was predicting -in addition to the top quark mass- the Higgs boson mass, four and half years before the experimental discovery [19].

In the present paper we present in Sect. 2 briefly some details on Veltman's condition on the cancellation of quadratic divergences in SM with comments of other authors and similarly the works on the uniqueness of supersymmetry as a solution to the cancellation of quadratic divergences problem. Then we continue in Sect. 3 with the presentation of the scheme of reduction of couplings and Sect. 4 with the necessary conditions for finiteness. In Sect. 5 we review the above-mentioned all-loop finite $N = 1$ supersymmetric $SU(5)$ model and give its latest phenomenological analysis. Sect. 6 is dedicated to a few closing remarks.

2 Cancellation of quadratic divergences and Supersymmetry

2.1 Comments on Veltman's relation

Let us present few more details on Veltman's relation resulting from the requirement of cancellation of quadratic divergences in the SM at one loop. Veltman suggested that within the dimensional regularization [20], which does not catch the quadratic divergences, a suitable criterion of identifying such divergences is the occurence of poles in the complex dimensional plane of n less than four. Therefore quadratic divergences at the one loop level would correspond to poles for $n = 2$. Then, in the SM within the dimensional regularization scheme poles for n=2 occur in the vector boson and Higgs self energy and in the tadpole diagrams. However Veltman, inspired by the way dimensional regularisation has to be modified in order for the scheme to be suitable also for supersymmetric theories [21, 22], chose the dimension of the Dirac matrices to be four, independent of the space-time dimension. In other words, Veltman concluded that although conventional dimensional regularisation would suggest $n = 2$ as the dimension of the Dirac algebra, the appropriate choice is $n = 4$. This preserves the number of gauge degrees of freedom and hence respects supersymmetry, and corresponds to the use of regularisation by dimensional reduction [21, 22]. In any case, in ref. [23], the equivalence of dimensional reduction and dimensional regularisation was shown. With the above reasoning Veltman derived the following mass relation:

$$m_e^2 + m_\mu^2 + m_\tau^2 + 3(m_u^2 + m_d^2 + m_c^2 + m_s^2 + m_t^2 + m_b^2) = \frac{3}{2}m_W^2 + \frac{3}{4}m_Z^2 + \frac{3}{4}m_H^2 \ , \tag{1}$$

known as Veltman's mass relation. It is very interesting that the same formula was derived in ref. [24], based on the point-splitting regularization [25], which makes no reference to dimensions of space-time other than four. For discussions concerning the two-loop corrections we refer the reader to [26, 27]. Clearly now, given the measured values of the top and Higgs masses, the relation (1) does not hold. From the above discussion it is worth keeping the point that Veltman, although working within the SM had a vision that supersymmetric theories were the appropriate framework for the cancellation of quadratic divergences, in which the contributions of the bosons and the fermions have opposite sign. Equally important is the observation that the requirement of the cancellation of quadratic divergences was leading to very useful mass relations. Actually a strong support towards this direction was already provided by the work of Ferrara, Girardello and Palumbo [11], who derived in supersymmetric theories with spontaneously broken supersymmetry a very similar quadratic mass formula:

$$\sum_J (-1)^{2J}(2J+1)m_J^2 = 0. \tag{2}$$

Some further interesting comments on Veltman's relation were done by Kubo, Sibold and Zimmermann [28] using the reduction of couplings scheme, which will be discused in detail in the next section. Here we would like only to remind that based on this scheme the parameters of SM were related to α_s leading to predictions for the top and Higgs masses [29, 30] that do not hold. In ref. [28] it was analyzed whether it is possible to require in addition the absence of quadratical divergences using Veltman's relation. It has been shown first that postulating absence of quadratical divergences is a gauge and renormalization group invariant statement. Moreover the resulting constraint is compatible with reduction, at least with what they called 'the trivial one', meaning that the top mass was considered as another free parameter instead of been predicted by the dimensional reduction as in [29, 30].

2.2 Uniqueness of supersymmetry as solution of the quadratic divergences problem

The absence of quadratic divergences in supersymmetric theories was known due to the non-renormalization theorems, as is already mentioned in the Introduction. A further question was if this solution was unique, that is, if there are non-supersymmetric theories where also the quadratic divergences are absent.

In order to address this problem, the inverse question was posed, given the absence of quadratic divergences, what kind of solutions does it imply.

The first development in this direction, following the spirit of [4], was done in [12]. In order to show whether the absence of quadratic divergences implies supersymmetry, they studied the case of one Majorana field and an arbitrary number of scalar and pseudoscalar fields systematically. By carrying out the loop expansion to two-loop order, and requiring that the quadratic divergences cancel order by order, they derived relationships among the dimensionless couplings. They found no solutions with only one spin 0 field, either scalar or pseudoscalar. In case of either a pair of scalars or pseudoscalar fields, the only solutions are the trivial ones. The combination of one scalar and one pseudoscalar gives a non-trivial solution to the cancellation of quadratic divergences, which corresponds to the massive Wess-Zumino model with soft supersymmetry breaking terms.

They concluded that the necessary and sufficient conditions for the absence of quadratic divergences to two-loop order in cases of two or less bosonic fields lead uniquely to the softly broken supersymmetric theories. The statement can be extended to all-loops, since in theories with soft breaking terms there will appear no quadratic divergences.

Shortly later, in [13], the requirement of absence of quadratic divergences in a quantum field theory with one or more scalar bosons was studied. The analysis was done at one-loop, but it was further required that the constraints resulting from eliminating the quadratic divergences should be renormalization group invariant (RGI), in order to have a physical meaning. More specifically, each scalar boson has associated a quadratic divergence and demanding that these are eliminated leads to parametric conditions, which then were required to be preserved under a change of the renormalization scale. In general the resulting systems are severely overconstrained. With a procedure very close to the reduction of couplings (see next section) they determined the independent couplings and the relations among them. In all cases considered in that paper, the only solutions found were supersymmetric.

This work was extended and detailed in [14], where they considered general classes of theories with scalars, Abelian and non-Abelian, with quartic and Yukawa interactions. By requiring renormalization group invariance at one-loop, besides the cancellation of quadratic divergences, they found that the only solutions possible are supersymmetric. As a notable exception they found that in a chiral supersymmetric $U(1)$ model the quadratic divergence associated with a radiatively induced Fayet-Iliopoulos D-term, do not cancel. It seems in their analysis they missed that the Veltman relation is RGI according to ref. [28].

Thus, the absence of quadratic divergences induced by scalar couplings, leads in general to a supersymmetric solution. Given that SSB terms are by construction free of quadratic divergences, the necessity to add them in any supersymmetric model in the prospect to become realistic is very welcome without any cost.

3   Theoretical Basis of Reduction of Couplings

The idea of reduction of couplings was introduced in [31] and evolved over the next two decades. It aims to express the parameters of a theory -that are considered independent- in terms of one basic parameter, which is called primary coupling. This is achieved by searching for Renormalization Group Invariant (RGI) relations among couplings and using them to reduce the -seemingly- free parameters. In this section we will outline the procedure, first applied to parameters without mass dimension, and then it will be extended to parameters of dimension one or two, i.e. the parameters of the soft breaking sector of an $N = 1$ SUSY theory.

3.1 Reduction of Dimensionless Parameters

Any RGI relation among couplings $g_1, ..., g_A$ of a renormalizable theory can be written in the form $\Phi(g_1, \cdots, g_A) = $ const., which has to satisfy the partial differential equation

$$\mu \frac{d\Phi}{d\mu} = \vec{\nabla}\Phi \cdot \vec{\beta} = \sum_{a=1}^{A} \beta_a \frac{\partial\Phi}{\partial g_a} = 0 , \tag{3}$$

where β_a is the β-function of g_a. Solving this partial differential equation is equivalent to solving a set of ordinary differential equations, known as reduction equations (REs) [31–33],

$$\beta_g \frac{dg_a}{dg} = \beta_a , \ a = 1, \cdots, A , \tag{4}$$

where g and β_g are the primary coupling and its β-function, respectively, while the counting on a does not include g. Since the Φ_a's can impose a maximum of $(A-1)$ independent RGI "constraints" in the A-dimensional space of parameters, one could express them all in terms of a single coupling g. However, the general solutions of Eqs. (4) contain as many integration constants as the number of equations. Thus, we have just traded an integration constant for each renormalized coupling and such general solutions cannot be considered "reduced ones". The crucial requirement is to demand power series solutions to the REs which preserve perturbative renormalizability,

$$g_a = \sum_n \rho_a^{(n)} g^{2n+1} , \tag{5}$$

This ansatz fixes the integration constant in each of the REs and chooses a special solution. Remarkably, the uniqueness of these power series solutions can be decided already at one-loop level [31–33]. As an illustration, we assume β-functions of the form

$$\beta_a = \frac{1}{16\pi^2} \left[\sum_{b,c,d \neq g} \beta_a^{(1)\,bcd} g_b g_c g_d + \sum_{b \neq g} \beta_a^{(1)\,b} g_b g^2 \right] + \cdots , \tag{6}$$

$$\beta_g = \frac{1}{16\pi^2} \beta_g^{(1)} g^3 + \cdots .$$

$\cdots$ stands for higher-order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b, c, d. We will assume that the $\rho_a^{(n)}$'s with $n \leq r$ are uniquely determined. To obtain $\rho_a^{(r+1)}$'s we insert the power series (5) into the REs (4) and collect terms of $\mathcal{O}(g^{2r+3})$:

$$\sum_{d \neq g} M(r)_a^d \rho_d^{(r+1)} = \text{lower order quantities} ,$$

where the right-hand side is known by assumption and

$$M(r)_a^d = 3 \sum_{b,c \neq g} \beta_a^{(1)\,bcd} \rho_b^{(1)} \rho_c^{(1)} + \beta_a^{(1)\,d} - (2r+1) \beta_g^{(1)} \delta_a^d , \tag{7}$$

$$0 = \sum_{b,c,d \neq g} \beta_a^{(1)\,bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d \neq g} \beta_a^{(1)\,d} \rho_d^{(1)} - \beta_g^{(1)} \rho_a^{(1)} . \tag{8}$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s are uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

The couplings in SUSY theories have the same asymptotic behaviour. Thus, it is natural to search for such a power series solution to the REs. The prospect of coupling unification described in this section is very attractive, as the "completely reduced" theory contains only one independent coupling,

with primary examples the FUTs [16–18]. However, since it is often unrealistic, one usually imposes fewer RGI constraints, achieving "partial reduction" [29, 30].

All the above hint (recall also [28]) towards an underlying connection among reduction of couplings and supersymmetry. As an example, consider an $SU(N)$ gauge theory with $\phi^i(\mathbf{N})$ and $\hat{\phi}_i(\overline{\mathbf{N}})$ complex scalars, $\psi^i(\mathbf{N})$ and $\hat{\psi}_i(\overline{\mathbf{N}})$ left-handed Weyl spinors and λ^a $(a = 1, \ldots, N^2 - 1)$ right-handed Weyl spinors in the adjoint representation of $SU(N)$, i.e. a model with the field content of a supersymmetric theory, but not with the corresponding couplings. The Lagrangian then includes

$$\mathcal{L} \supset i\sqrt{2}\{ \ g_Y \overline{\psi} \lambda^a T^a \phi - \hat{g}_Y \overline{\hat{\psi}} \lambda^a T^a \hat{\phi} + \text{h.c.} \ \} - V(\phi, \overline{\phi}), \tag{9}$$

where

$$V(\phi, \overline{\phi}) = \frac{1}{4}\lambda_1(\phi^i \phi_i^*)^2 + \frac{1}{4}\lambda_2(\hat{\phi}_i \hat{\phi}^{* \ i})^2 + \lambda_3(\phi^i \phi_i^*)(\hat{\phi}_j \hat{\phi}^{* \ j}) + \lambda_4(\phi^i \phi_j^*)(\hat{\phi}_i \hat{\phi}^{* \ j}) \ . \tag{10}$$

This is the most general renormalizable form in 4D. Searching for a solution like those in Eq. (5) for the REs, one finds among the many possible solutions in lowest order:

$$g_Y = \hat{g}_Y = g \ ,$$
$$\lambda_1 = \lambda_2 = \frac{N-1}{N}g^2 \ , \tag{11}$$
$$\lambda_3 = \frac{1}{2N}g^2 \ , \ \lambda_4 = -\frac{1}{2}g^2 \ ,$$

which corresponds to a $N = 1$ SUSY gauge theory. While the above do not provide an answer about the relation of reduction of couplings and SUSY, they indeed point to further study in that direction.

3.2 Reduction of Couplings in $N = 1$ SUSY Gauge Theories - Partial Reduction

Let us consider a chiral, $N = 1$ supersymmetric gauge theory with group G and gauge coupling g. The superpotential of the theory can be written:

$$W \ = \ \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} C_{ijk} \phi_i \phi_j \phi_k \ , \tag{12}$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the chiral superfield ϕ_i belongs to the irreducible representation R_i of the gauge group. The renormalization constants associated with the superpotential, for preserved SUSY, are:

$$\phi_i^0 = \left(Z_i^j\right)^{(1/2)} \phi_j \ , \tag{13}$$

$$m_{ij}^0 = Z_{ij}^{i'j'} m_{i'j'} \ , \tag{14}$$

$$C_{ijk}^0 = Z_{ijk}^{i'j'k'} C_{i'j'k'} \ . \tag{15}$$

By virtue of the $N = 1$ non-renormalization theorem [5, 6, 34, 35] there are no mass and cubic interaction term infinities:

$$Z_{ij}^{i'j'} \left(Z_{i'}^{i''}\right)^{(1/2)} \left(Z_{j'}^{j''}\right)^{(1/2)} = \delta_{(i}^{i''} \delta_{j)}^{j''} \ ,$$
$$Z_{ijk}^{i'j'k'} \left(Z_{i'}^{i''}\right)^{(1/2)} \left(Z_{j'}^{j''}\right)^{(1/2)} \left(Z_{k'}^{k''}\right)^{(1/2)} = \delta_{(i}^{i''} \delta_j^{j''} \delta_{k)}^{k''} \ . \tag{16}$$

Therefore, the only surviving infinities are the wave function renormalization constants Z_i^j, so just on infinity per field. The one-loop β-function of g is given by [36–40]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\left[\sum_i T(R_i) - 3\,C_2(G)\right] ,\tag{17}$$

where $C_2(G)$ is the quadratic Casimir operator of the adjoint representation of the gauge group G and $\mathrm{Tr}[T^a T^b] = T(R)\delta^{ab}$, where T^a are the group generators in the appropriate representation. Due to the non-renormalization theorem [5, 6, 35] the β-functions of C_{ijk} are related to the anomalous dimension matrices γ_{ij} of the matter fields as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl}\,\gamma_k^l + C_{ikl}\,\gamma_j^l + C_{jkl}\,\gamma_i^l .\tag{18}$$

The one-loop γ_j^i is given by [36]:

$$\gamma^{(1)\,i}_{\;\;\;j} = \frac{1}{32\pi^2}\,[\,C^{ikl}\,C_{jkl} - 2\,g^2\,C_2(R_i)\delta_j^i\,],\tag{19}$$

where $C^{ijk} = C_{ijk}^*$.

We take C_{ijk} to be real so that C_{ijk}^2 are always positive. The squares of the couplings are convenient to work with, and the C_{ijk} can be covered by a single index i $(i = 1, \cdots, n)$:

$$\alpha = \frac{g^2}{4\pi}\ ,\ \alpha_i = \frac{g_i^2}{4\pi}\ .\tag{20}$$

Then the evolution of α's in perturbation theory will take the form

$$\begin{aligned}
\frac{d\alpha}{dt} &= \beta = -\beta^{(1)}\alpha^2 + \cdots ,\\
\frac{d\alpha_i}{dt} &= \beta_i = -\beta_i^{(1)}\,\alpha_i\,\alpha + \sum_{j,k}\beta_{i,jk}^{(1)}\,\alpha_j\,\alpha_k + \cdots ,
\end{aligned}\tag{21}$$

Here, $\cdots$ denotes higher-order contributions and $\beta_{i,jk}^{(1)} = \beta_{i,kj}^{(1)}$. For the evolution equations (21), following ref [41] we investigate the asymptotic properties. First, we define [31, 33, 42–44]

$$\tilde\alpha_i \equiv \frac{\alpha_i}{\alpha}\ ,\ i = 1, \cdots, n ,\tag{22}$$

and derive from Eq. (21)

$$\begin{aligned}
\alpha\frac{d\tilde\alpha_i}{d\alpha} &= -\tilde\alpha_i + \frac{\beta_i}{\beta} = \left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\tilde\alpha_i\\
&\quad - \sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\,\tilde\alpha_j\,\tilde\alpha_k + \sum_{r=2}\left(\frac{\alpha}{\pi}\right)^{r-1}\tilde\beta_i^{(r)}(\tilde\alpha) ,
\end{aligned}\tag{23}$$

where $\tilde\beta_i^{(r)}(\tilde\alpha)$ $(r = 2, \cdots)$ are power series of $\tilde\alpha$'s and can be computed from the rth-loop β-functions. We then search for fixed points ρ_i of Eq. (22) at $\alpha = 0$. We have to solve the equation

$$\left(-1 + \frac{\beta_i^{(1)}}{\beta^{(1)}}\right)\rho_i - \sum_{j,k}\frac{\beta_{i,jk}^{(1)}}{\beta^{(1)}}\,\rho_j\,\rho_k = 0 ,\tag{24}$$

assuming fixed points of the form

$$\rho_i = 0 \text{ for } i = 1, \cdots, n'\ ;\ \rho_i > 0 \text{ for } i = n'+1, \cdots, n .\tag{25}$$

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Next, we treat $\tilde{\alpha}_i$ with $i \leq n'$ as small perturbations to the undisturbed system (defined by setting $\tilde{\alpha}_i$ with $i \leq n'$ equal to zero). It is possible to verify the existence of the unique power series solution of the reduction equations (23) to all orders already at one-loop level [31–33, 42]:

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \rho_i^{(r)} \alpha^{r-1} \ , \ i = n' + 1, \cdots, n \ . \tag{26}$$

These are RGI relations among parameters, and preserve formally perturbative renormalizability. So, in the undisturbed system there is only one independent parameter, the primary coupling α.

The non-vanishing $\tilde{\alpha}_i$ with $i \leq n'$ cause small perturbations that enter in a way that the reduced couplings ($\tilde{\alpha}_i$ with $i > n'$) become functions both of α and $\tilde{\alpha}_i$ with $i \leq n'$. Investigating such systems with partial reduction is very convenient to work with the following PDEs:

$$\left\{ \tilde{\beta} \, \frac{\partial}{\partial \alpha} + \sum_{a=1}^{n'} \tilde{\beta}_a \, \frac{\partial}{\partial \tilde{\alpha}_a} \right\} \ \tilde{\alpha}_i(\alpha, \tilde{\alpha}) = \tilde{\beta}_i(\alpha, \tilde{\alpha}) \ ,$$

$$\tilde{\beta}_{i(a)} \ = \ \frac{\beta_{i(a)}}{\alpha^2} - \frac{\beta}{\alpha^2} \, \tilde{\alpha}_{i(a)}, \qquad \tilde{\beta} \ \equiv \ \frac{\beta}{\alpha} \ . \tag{27}$$

These equations are equivalent to the REs (23), where, in order to avoid any confusion, we let a, b run from 1 to n' and i, j from $n' + 1$ to n. Then, we search for solutions of the form

$$\tilde{\alpha}_i = \rho_i + \sum_{r=2} \left(\frac{\alpha}{\pi} \right)^{r-1} f_i^{(r)}(\tilde{\alpha}_a) \ , \ i = n' + 1, \cdots, n \ , \tag{28}$$

where $f_i^{(r)}(\tilde{\alpha}_a)$ are power series of $\tilde{\alpha}_a$. The requirement that in the limit of vanishing perturbations we obtain the undisturbed solutions (26) [30, 45] suggests this type of solutions. Once more, one can obtain the conditions for uniqueness of $f_i^{(r)}$ in terms of the lowest order coefficients.

3.3 Reduction of Dimension-1 and -2 Parameters

The extension of the reduction of couplings method to massive parameters is not straightforward, since the technique was originally aimed at massless theories on the basis of the Callan-Symanzik equation [31, 32]. Many requirements have to be met, such as the normalization conditions imposed on irreducible Green's functions [46], etc. significant progress has been made towards this goal, starting from [47], where, as an assumption, a mass-independent renormalization scheme renders all RG functions only trivially dependent on dimensional parameters. Mass parameters can then be introduced similarly to couplings.

This was justified later [48, 49], where it was demonstrated that, apart from dimensionless parameters, pole masses and gauge couplings, the model can also include couplings carrying a dimension and masses. To simplify the analysis, we follow Ref. [47] and use a mass-independent renormalization scheme as well.

Consider a renormalizable theory that contains $(N + 1)$ dimension-0 couplings, $(\hat{g}_0, \hat{g}_1, ..., \hat{g}_N)$, L parameters with mass dimension-1, $\left(\hat{h}_1, ..., \hat{h}_L \right)$, and M parameters with mass dimension-2, $\left(\hat{m}_1^2, ..., \hat{m}_M^2 \right)$. The renormalized irreducible vertex function Γ satisfies the RGE

$$\mathcal{D}\Gamma \left[\Phi's; \hat{g}_0, \hat{g}_1, ..., \hat{g}_N; \hat{h}_1, ..., \hat{h}_L; \hat{m}_1^2, ..., \hat{m}_M^2; \mu \right] = 0 \ , \tag{29}$$

with

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \sum_{i=0}^{N} \beta_i \frac{\partial}{\partial \hat{g}_i} + \sum_{a=1}^{L} \gamma_a^h \frac{\partial}{\partial \hat{h}_a} + \sum_{\alpha=1}^{M} \gamma_\alpha^{m^2} \frac{\partial}{\partial \hat{m}_\alpha^2} + \sum_J \Phi_I \gamma^{\phi I}{}_J \frac{\delta}{\delta \Phi_J} \ , \tag{30}$$

where β_i are the β-functions of the dimensionless couplings g_i and Φ_I are the matter fields. The mass, trilinear coupling and wave function anomalous dimensions, respectively, are denoted by $\gamma_\alpha^{m^2}$, γ_a^h and $\gamma^{\phi I}{}_J$ and μ denotes the energy scale. For a mass-independent renormalization scheme, the γ's are given by

$$\gamma_a^h = \sum_{b=1}^{L} \gamma_a^{h,b}(g_0, g_1, ..., g_N)\hat{h}_b,$$

$$\gamma_\alpha^{m^2} = \sum_{\beta=1}^{M} \gamma_\alpha^{m^2,\beta}(g_0, g_1, ..., g_N)\hat{m}_\beta^2 + \sum_{a,b=1}^{L} \gamma_\alpha^{m^2,ab}(g_0, g_1, ..., g_N)\hat{h}_a\hat{h}_b \ . \tag{31}$$

The $\gamma_a^{h,b}$, $\gamma_\alpha^{m^2,\beta}$ and $\gamma_\alpha^{m^2,ab}$ are power series of the (dimensionless) g's.

We search for a reduced theory where

$$g \equiv g_0, \qquad h_a \equiv \hat{h}_a \quad \text{for } 1 \le a \le P, \qquad m_\alpha^2 \equiv \hat{m}_\alpha^2 \quad \text{for } 1 \le \alpha \le Q$$

are independent parameters. The reduction of the rest of the parameters, namely

$$\hat{g}_i = \hat{g}_i(g), \qquad (i = 1, ..., N),$$

$$\hat{h}_a = \sum_{b=1}^{P} f_a^b(g)h_b, \qquad (a = P+1, ..., L),$$

$$\hat{m}_\alpha^2 = \sum_{\beta=1}^{Q} e_\alpha^\beta(g)m_\beta^2 + \sum_{a,b=1}^{P} k_\alpha^{ab}(g)h_a h_b, \qquad (\alpha = Q+1, ..., M) \tag{32}$$

is consistent with the RGEs (29) and (30). The following relations should be satisfied

$$\beta_g \frac{\partial \hat{g}_i}{\partial g} = \beta_i, \qquad (i = 1, ..., N),$$

$$\beta_g \frac{\partial \hat{h}_a}{\partial g} + \sum_{b=1}^{P} \gamma_b^h \frac{\partial \hat{h}_a}{\partial h_b} = \gamma_a^h, \qquad (a = P+1, ..., L),$$

$$\beta_g \frac{\partial \hat{m}_\alpha^2}{\partial g} + \sum_{a=1}^{P} \gamma_a^h \frac{\partial \hat{m}_\alpha^2}{\partial h_a} + \sum_{\beta=1}^{Q} \gamma_\beta^{m^2} \frac{\partial \hat{m}_\alpha^2}{\partial m_\beta^2} = \gamma_\alpha^{m^2}, \qquad (\alpha = Q+1, ..., M). \tag{33}$$

Using Eqs. (31) and (32), they reduce to

$$\beta_g \frac{df_a^b}{dg} + \sum_{c=1}^{P} f_a^c \left[\gamma_c^{h,b} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^b \right] - \gamma_a^{h,b} - \sum_{d=P+1}^{L} \gamma_a^{h,d} f_d^b = 0,$$

$$(a = P+1, ..., L; \, b = 1, ..., P),$$

$$\beta_g \frac{de_\alpha^\beta}{dg} + \sum_{\gamma=1}^{Q} e_\alpha^\gamma \left[\gamma_\gamma^{m^2,\beta} + \sum_{\delta=Q+1}^{M} \gamma_\gamma^{m^2,\delta} e_\delta^\beta \right] - \gamma_\alpha^{m^2,\beta} - \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,d} e_\delta^\beta = 0,$$

$$(\alpha = Q+1, ..., M; \, \beta = 1, ..., Q),$$

$$\beta_g \frac{dk_\alpha^{ab}}{dg} + 2 \sum_{c=1}^{P} \left(\gamma_c^{h,a} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^a \right) k_\alpha^{cb} + \sum_{\beta=1}^{Q} e_\alpha^\beta \left[\gamma_\beta^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\beta^{m^2,cd} f_c^a f_d^b \right. \tag{34}$$

$$+ 2 \sum_{c=P+1}^{L} \gamma_\beta^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\beta^{m^2,d} k_\delta^{ab} \bigg] - \left[\gamma_\alpha^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\alpha^{m^2,cd} f_c^a f_d^b \right.$$

$$+ 2 \sum_{c=P+1}^{L} \gamma_\alpha^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,\delta} k_\delta^{ab} \bigg] = 0,$$

$$(\alpha = Q+1, ..., M; \, a, b = 1, ..., P) \, .$$

The above relations ensure that the irreducible vertex function of the reduced theory

$$\Gamma_R \left[\Phi\text{'s}; g; h_1, ..., h_P; m_1^2, ..., m_Q^2; \mu \right] \equiv$$

$$\Gamma \left[\Phi\text{'s}; g, \hat{g}_1(g)..., \hat{g}_N(g); h_1, ..., h_P, \hat{h}_{P+1}(g,h), ..., \hat{h}_L(g,h); \right. \tag{35}$$

$$\left. m_1^2, ..., m_Q^2, \hat{m}_{Q+1}^2(g,h,m^2), ..., \hat{m}_M^2(g,h,m^2); \mu \right]$$

has the same renormalization group flow as the original one.

Assuming a perturbatively renormalizable reduced theory, the functions $\hat{g}_i$, f_a^b, e_α^β and k_α^{ab} are expressed as power series in the primary coupling:

$$\hat{g}_i = g \sum_{n=0}^{\infty} \rho_i^{(n)} g^n, \qquad f_a^b = g \sum_{n=0}^{\infty} \eta_a^{b(n)} g^n,$$

$$e_\alpha^\beta = \sum_{n=0}^{\infty} \xi_\alpha^{\beta(n)} g^n, \qquad k_\alpha^{ab} = \sum_{n=0}^{\infty} \chi_\alpha^{ab(n)} g^n. \tag{36}$$

These expansion coefficients are found by inserting the above power series into Eqs. (33), (34) and requiring the equations to be satisfied at each order of g. It is not trivial to have a unique power series solution; it depends both on the theory and the choice of independent couplings.

If there are no independent dimension-1 parameters $(\hat{h})$, their reduction becomes

$$\hat{h}_a = \sum_{b=1}^{L} f_a^b(g) M,$$

where M is a dimension-1 parameter (i.e. a gaugino mass, corresponding to the independent gauge coupling). If there are no independent dimension-2 parameters $(\hat{m}^2)$, their reduction takes the form

$$\hat{m}_a^2 = \sum_{b=1}^{M} e_a^b(g) M^2.$$

3.4 Reduction of Couplings of Soft Breaking Terms in $N = 1$ SUSY Theories

The reduction of dimensionless couplings was extended [47, 50] to the SSB dimensionful parameters of $N = 1$ supersymmetric theories. It was also found [18, 51] that soft scalar masses satisfy a universal sum rule.

Consider the superpotential (12)

$$W = \frac{1}{2} \mu^{ij} \Phi_i \Phi_j + \frac{1}{6} C^{ijk} \Phi_i \Phi_j \Phi_k , \tag{37}$$

and the SSB Lagrangian

$$- \mathcal{L}_{\text{SSB}} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)^j_i \phi^{*i} \phi_j + \frac{1}{2} M \lambda_i \lambda_i + \text{h.c.} \tag{38}$$

The ϕ_i's are the scalar parts of chiral superfields Φ_i, λ are gauginos and M the unified gaugino mass.

The one-loop gauge and Yukawa beta-functions are given by (17) and (18), respectively, and the one-loop anomalous dimensions by (19). We make the assumption that the REs admit power series solutions:

$$C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^{2n} . \tag{39}$$

Since we want to obtain higher-loop results instead of knowledge of explicit β-functions, we require relations among β-functions. The spurion technique [9, 35, 52–54] gives all-loop relations among SSB β-functions [55–62]:

$$\beta_M = 2 \mathcal{O} \left(\frac{\beta_g}{g} \right) , \tag{40}$$

$$\beta_h^{ijk} = \gamma^i_l h^{ljk} + \gamma^j_l h^{ilk} + \gamma^k_l h^{ijl}$$
$$- 2 (\gamma_1)^i_l C^{ljk} - 2 (\gamma_1)^j_l C^{ilk} - 2 (\gamma_1)^k_l C^{ijl} , \tag{41}$$

$$(\beta_{m^2})^i_j = \left[\Delta + X \frac{\partial}{\partial g} \right] \gamma^i_j , \tag{42}$$

where

$$\mathcal{O} = \left(M g^2 \frac{\partial}{\partial g^2} - h^{lmn} \frac{\partial}{\partial C^{lmn}} \right) , \tag{43}$$

$$\Delta = 2 \mathcal{O} \mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} + \tilde{C}_{lmn} \frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn} \frac{\partial}{\partial C^{lmn}} , \tag{44}$$

$$(\gamma_1)^i_j = \mathcal{O} \gamma^i_j, \tag{45}$$

$$\tilde{C}^{ijk} = (m^2)^i_l C^{ljk} + (m^2)^j_l C^{ilk} + (m^2)^k_l C^{ijl} . \tag{46}$$

Assuming (following [57]) that the relation among couplings

$$h^{ijk} = -M(C^{ijk})' \equiv -M \frac{dC^{ijk}(g)}{d \ln g} , \tag{47}$$

is RGI to all orders and the use of the all-loop gauge β-function of [63–65]

$$\beta_g^{\text{NSVZ}} = \frac{g^3}{16\pi^2} \left[\frac{\sum_l T(R_l)(1 - \gamma_l/2) - 3C_2(G)}{1 - g^2 C_2(G)/8\pi^2} \right] , \tag{48}$$

we are led to an all-loop RGI sum rule [66] (assuming $(m^2)^i_j = m^2_j \delta^i_j$),

$$m^2_i + m^2_j + m^2_k = |M|^2 \left\{ \frac{1}{1 - g^2 C_2(G)/(8\pi^2)} \frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2} \frac{d^2 \ln C^{ijk}}{d(\ln g)^2} \right\}$$
$$+ \sum_l \frac{m^2_l T(R_l)}{C_2(G) - 8\pi^2/g^2} \frac{d\ln C^{ijk}}{d\ln g} . \tag{49}$$

It is worth noting that the all-loop result of Eq. (49) coincides with the superstring result for the finite case in a certain class of orbifold models [18, 67, 68] if $\frac{d\ln C^{ijk}}{d\ln g} = 1$ [17].

As mentioned above, the all-loop results on the SSB β-functions, Eqs.(40)-(46), lead to all-loop RGI relations. We assume:

(a) the existence of an RGI surface on which $C = C(g)$, or equivalently that the expression

$$\frac{dC^{ijk}}{dg} = \frac{\beta^{ijk}_C}{\beta_g} \tag{50}$$

holds (i.e. reduction of couplings is possible)

(b) the existence of a RGI surface on which

$$h^{ijk} = -M \frac{dC(g)^{ijk}}{d\ln g} \tag{51}$$

holds to all orders.

Then it can be proven [69–71] that the relations that follow are all-loop RGI (note that in both assumptions we do not rely on specific solutions of these equations)

$$M = M_0 \frac{\beta_g}{g}, \tag{52}$$

$$h^{ijk} = -M_0 \beta^{ijk}_C, \tag{53}$$

$$b^{ij} = -M_0 \beta^{ij}_\mu, \tag{54}$$

$$(m^2)^i_j = \frac{1}{2} |M_0|^2 \mu \frac{d\gamma^i{}_j}{d\mu}, \tag{55}$$

where M_0 is an arbitrary reference mass scale to be specified shortly. Assuming

$$C_a \frac{\partial}{\partial C_a} = C^*_a \frac{\partial}{\partial C^*_a} \tag{56}$$

for an RGI surface $F(g, C^{ijk}, C^{*ijk})$ we are led to

$$\frac{d}{dg} = \left(\frac{\partial}{\partial g} + 2 \frac{\partial}{\partial C} \frac{dC}{dg} \right) = \left(\frac{\partial}{\partial g} + 2 \frac{\beta_C}{\beta_g} \frac{\partial}{\partial C} \right) , \tag{57}$$

where Eq. (50) was used. Let us now consider the partial differential operator $\mathcal{O}$ in Eq. (43) which (assuming Eq. (47)), becomes

$$\mathcal{O} = \frac{1}{2} M \frac{d}{d\ln g} \tag{58}$$

and β_M, given in Eq. (40), becomes

$$\beta_M = M \frac{d}{d\ln g} \left(\frac{\beta_g}{g} \right) , \tag{59}$$

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which by integration provides us [62, 69] with the generalized, i.e. including Yukawa couplings, all-loop RGI Hisano - Shifman relation [58]

$$M = \frac{\beta_g}{g} M_0 \ . \tag{60}$$

M_0 is the integration constant and can be associated to the unified gaugino mass M (of an assumed covering GUT), or to the gravitino mass $m_{3/2}$ in a supergravity framework. Therefore, Eq. (52) becomes the all-loop RGI Eq. (52). β_M, using Eqs.(59) and (52) can be written as follows:

$$\beta_M = M_0 \frac{d}{dt}(\beta_g/g) \ . \tag{61}$$

Similarly

$$(\gamma_1)^i_j = \mathcal{O}\gamma^i_j = \frac{1}{2} \ M_0 \ \frac{d\gamma^i_j}{dt} \ . \tag{62}$$

Next, from Eq.(47) and Eq.(52) we get

$$h^{ijk} = -M_0 \ \beta_C^{ijk} \ , \tag{63}$$

while β_h^{ijk}, using Eq.(62), becomes [69]

$$\beta_h^{ijk} = -M_0 \ \frac{d}{dt}\beta_C^{ijk}, \tag{64}$$

which shows that Eq. (63) is RGI to all loops. Eq. (54) can similarly be shown to be all-loop RGI as well.

It should be noted concerning the β-functions of the SBB parameters, as in Eqs. (61) and (64), that the vanishing of the dimensionless β-functions, even to all-orders, as will be discussed in the next section, is transferred to the dimensionful SSB sector of the theory.

4   Finiteness

A natural development of the ideas started with Veltman's work on the cancellation of quadratic divergences in renormalizable field theories with scalars, which found an excellent realisation in supersymmetric theories with soft supersymmetry breaking terms, as we have already discussed, led to the search of constructing renormalizable field theories free also of logarithmic divergences, i.e. completely *Finite Theories*.

The finiteness that will be discussed here is a consequence of the reduction of couplings, presented in the previous section, and is based on the fact that in supersymmetric theories is possible to find RGI relations among couplings that keep finiteness in perturbation theory, even to all orders. Accepting finiteness as a guiding principle in constructing realistic theories of EPP, the first thing that comes to mind is to look for an $N = 4$ supersymmetric unified gauge theory, since any ultraviolet (UV) divergences are absent in these theories. However nobody has managed so far to produce realistic models in the framework of $N = 4$ SUSY. In the best case one could try to do a drastic truncation of the theory like the orbifold projection of refs. [72,73], but this is already a different theory than the original one. The next possibility is to consider an $N = 2$ supersymmetric gauge theory, whose β-function receives corrections only at one loop. Then it is not hard to select a spectrum to make the theory all-loop finite. However a serious obstacle in these theories is their mirror spectrum, which in the absence of a mechanism to make it heavy, does not permit the construction of realistic models. Therefore, one is naturally led to consider $N = 1$ supersymmetric gauge theories, which can be chiral and in principle realistic.

It should be noted that in the approach followed here (UV) finiteness means the vanishing of all the β-functions, i.e. the non-renormalization of the coupling constants, in contrast to a complete (UV) finiteness where even field amplitude renormalization is absent. Before the work of several members of our group, the studies on $N = 1$ finite theories were following two directions: (i) construction of finite theories up to two loops examining various possibilities to make them phenomenologically viable, (ii) construction of all-loop finite models without particular emphasis on the phenomenological consequences. The success of the work of our group started in refs [16, 17] with the construction of the first realistic all-loop finite model, based on the theorem presented below, realising in this way an old theoretical dream of field theorists.

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Finiteness in N=1 Supersymmetric Gauge Theories

Let us, once more, consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by (see Eq. (12))

$$W = \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} C_{ijk} \phi_i \phi_j \phi_k . \tag{65}$$

The $N = 1$ non-renormalization theorem, ensuring the absence of mass and cubic-interaction-term infinities, leads to wave-function infinities only; one for each superfield. As one can see from Eqs. (17) and (19), all the one-loop β-functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma^{(1)i}_j$ vanish, i.e.

$$\sum_i T(R_i) = 3C_2(G) , \tag{66}$$

$$C^{ikl} C_{jkl} = 2\delta_j^i g^2 C_2(R_i) . \tag{67}$$

The conditions for finiteness for $N = 1$ field theories with $SU(N)$ gauge symmetry are discussed in [74], and the analysis of the anomaly-free and no-charge renormalization requirements for these theories can be found in [75]. A very interesting result is that the conditions (66) and (67) are necessary and sufficient for finiteness at the two-loop level [36–40].

In case SUSY is broken by soft terms, the requirement of finiteness in the one-loop soft breaking terms imposes further constraints among them [76]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of the theory two-loop finite [77].

The one- and two-loop finiteness conditions of Eqs. (66) and (67) restrict considerably the possible choices of the irreducible representations (irreps) R_i for a given group G, as well as the Yukawa couplings in the superpotential (65). Note in particular that the finiteness conditions cannot be applied to the MSSM, since the presence of a $U(1)$ gauge group is incompatible with the condition (66), due to $C_2[U(1)] = 0$. This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the MSSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that SUSY (most probably) can only be broken due to the soft breaking terms. Indeed, due to the unacceptability of gauge singlets, F-type spontaneous symmetry breaking [78] terms are incompatible with finiteness, as well as D-type [79] spontaneous breaking which requires the existence of a $U(1)$ gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [15, 80] which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As we have seen (see Eq. (50)), the necessary and also sufficient, condition for this to happen is to require that such relations are solutions to the REs

$$\beta_g \frac{dC_{ijk}}{dg} = \beta_{ijk} \tag{68}$$

and hold at all orders. Remarkably, the existence of all-order power series solutions to (68) can be decided at one-loop level, as already mentioned.

Let us now turn to the all-order finiteness theorem [15,80], which states under which conditions an $N = 1$ supersymmetric gauge theory can become finite to all orders in perturbation theory, that is attain physical scale invariance. It is based on (a) the structure of the supercurrent in $N = 1$ supersymmetric gauge theory [81–83], and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [15, 80, 84–86]. Details of the proof can be found in refs. [15, 80] and further discussion in Refs. [84–88]. Here, following mostly Ref. [88] we present a comprehensible sketch of the proof.

Consider an $N = 1$ supersymmetric gauge theory, with simple Lie group G. The content of this theory is given at the classical level by the matter supermultiplets S_i, which contain a scalar field ϕ_i and a Weyl spinor ψ_{ia}, and the vector supermultiplet V_a, which contains a gauge vector field A_μ^a and a gaugino Weyl spinor λ_α^a.

Let us first recall certain facts about the theory:
(1) A massless $N = 1$ supersymmetric theory is invariant under a $U(1)$ chiral transformation R under which the various fields transform as follows

$$
\begin{aligned}
A'_\mu &= A_\mu, \quad \lambda'_\alpha = \exp(-i\theta)\lambda_\alpha \\
\phi' &= \exp(-i\tfrac{2}{3}\theta)\phi, \quad \psi'_\alpha = \exp(-i\tfrac{1}{3}\theta)\psi_\alpha, \quad \cdots
\end{aligned}
\tag{69}
$$

The corresponding axial Noether current $J_R^\mu(x)$ is

$$
J_R^\mu(x) = \bar{\lambda}\gamma^\mu\gamma^5\lambda + \cdots
\tag{70}
$$

is conserved classically, while in the quantum case is violated by the axial anomaly

$$
\partial_\mu J_R^\mu = r\left(\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} + \cdots\right).
\tag{71}
$$

From its known topological origin in ordinary gauge theories [89–91], one would expect the axial vector current J_R^μ to satisfy the Adler-Bardeen theorem and receive corrections only at the one-loop level. Indeed it has been shown that the same non-renormalization theorem holds also in supersymmetric theories [84–86]. Therefore

$$
r = \hbar\beta_g^{(1)}.
\tag{72}
$$

(2) The massless theory we consider is scale invariant at the classical level and, in general, there is a scale anomaly due to radiative corrections. The scale anomaly appears in the trace of the energy momentum tensor $T_{\mu\nu}$, which is traceless classically. It has the form

$$
T_\mu^\mu = \beta_g F^{\mu\nu}F_{\mu\nu} + \cdots
\tag{73}
$$

(3) Massless, $N = 1$ supersymmetric gauge theories are classically invariant under the supersymmetric extension of the conformal group – the superconformal group. Examining the superconformal algebra, it can be seen that the subset of superconformal transformations consisting of translations, SUSY transformations, and axial R transformations is closed under SUSY, i.e. these transformations form a representation of SUSY. It follows that the conserved currents corresponding to these transformations make up a supermultiplet represented by an axial vector superfield called the supercurrent J,

$$
J \equiv \left\{J_R'^\mu, \ Q_\alpha^\mu, \ T_\nu^\mu, ...\right\},
\tag{74}
$$

where $J_R'^\mu$ is the current associated to R invariance, Q_α^μ is the one associated to SUSY invariance, and T_ν^μ the one associated to translational invariance (energy-momentum tensor).
The anomalies of the R current $J_R'^\mu$, the trace anomalies of the SUSY current, and the energy-momentum tensor, form also a second supermultiplet, called the supertrace anomaly

$$
S = \{Re\ S,\ Im\ S,\ S_\alpha\} = \left\{T_\mu^\mu,\ \partial_\mu J_R'^\mu,\ \sigma_{\alpha\dot{\beta}}^\mu \bar{Q}_\mu^{\dot{\beta}} + \cdots\right\}
$$

where T_μ^μ is given in Eq.(73) and

$$
\begin{aligned}
\partial_\mu J_R'^\mu &= \beta_g \epsilon^{\mu\nu\sigma\rho}F_{\mu\nu}F_{\sigma\rho} + \cdots \tag{75} \\
\sigma_{\alpha\dot{\beta}}^\mu \bar{Q}_\mu^{\dot{\beta}} &= \beta_g \lambda^\beta \sigma_{\alpha\beta}^{\mu\nu}F_{\mu\nu} + \cdots \tag{76}
\end{aligned}
$$

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(4) It is very important to note that the Noether current defined in (70) is not the same as the current associated to R invariance that appears in the supercurrent J in (74), but they coincide in the tree approximation. So starting from a unique classical Noether current $J^\mu_{R(class)}$, the Noether current J^μ_R is defined as the quantum extension of $J^\mu_{R(class)}$ which allows for the validity of the non-renormalization theorem. On the other hand, J'^μ_R, is defined to belong to the supercurrent J, together with the energy-momentum tensor. The two requirements cannot be fulfilled by a single current operator at the same time.

Although the Noether current J^μ_R which obeys (71) and the current J'^μ_R belonging to the supercurrent multiplet J are not the same, there is a relation [15, 80] between quantities associated with them

$$r = \beta_g(1 + x_g) + \beta_{ijk}x^{ijk} - \gamma_A r^A \tag{77}$$

where r was given in Eq. (72). The r^A are the non-renormalized coefficients of the anomalies of the Noether currents associated to the chiral invariances of the superpotential, and –like r– are strictly one-loop quantities. The γ_A's are linear combinations of the anomalous dimensions of the matter fields, and x_g, and x^{ijk} are radiative correction quantities. The structure of Eq. (77) is independent of the renormalization scheme.

One-loop finiteness, i.e. vanishing of the β-functions at one-loop, implies that the Yukawa couplings λ_{ijk} must be functions of the gauge coupling g. To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in g, which is solution of the REs (68). We can now state the theorem for all-order vanishing β-functions [15].

Theorem:
Consider an $N = 1$ supersymmetric Yang-Mills theory, with simple gauge group. If the following conditions are satisfied

1. There is no gauge anomaly.

2. The gauge β-function vanishes at one-loop
$$\beta^{(1)}_g = 0 = \sum_i T(R_i) - 3\,C_2(G). \tag{78}$$

3. There exist solutions of the form
$$C_{ijk} = \rho_{ijk}g, \qquad \rho_{ijk} \in \mathbb{C} \tag{79}$$
   to the conditions of vanishing one-loop matter fields anomalous dimensions
$$\gamma^{(1)i}_{\ \ j} \;=\; 0 = \frac{1}{32\pi^2}\,[\,C^{ikl}\,C_{jkl} - 2\,g^2\,C_2(R)\delta^i_j]. \tag{80}$$

4. These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa β-functions:
$$\beta_{ijk} = 0. \tag{81}$$

Then, each of the solutions (79) can be uniquely extended to a formal power series in g, and the associated super Yang-Mills models depend on the single coupling constant g with a β-function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a unique formal power series solution to the reduction equations. The vanishing of the gauge β-function at one-loop, $\beta^{(1)}_g$, is equivalent to the vanishing of the R current anomaly (71). The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings β-functions at that order. It also implies the vanishing of the chiral anomaly coefficients r^A. This last property is a necessary condition for having β-functions vanishing at all orders.[b]

---

[b]There is an alternative way to find finite theories [92–95].

Proof:

Insert β_{ijk} as given by the REs into the relationship (77). Since these chiral anomalies vanish, we get for β_g an homogeneous equation of the form

$$0 = \beta_g(1 + O(\hbar)). \tag{82}$$

The solution of this equation in the sense of a formal power series in $\hbar$ is $\beta_g = 0$, order by order. Therefore, due to the REs (68), $\beta_{ijk} = 0$ too.

Thus, we see that finiteness and reduction of couplings are intimately related. Since an equation like eq. (77) is lacking in non-supersymmetric theories, one cannot extend the validity of a similar theorem in such theories.

A very interesting development was done in ref [56]. Based on the all-loop relations among the β-functions of the soft supersymmetry breaking terms and those of the rigid supersymmetric theory with the help of the differential operators, discussed in Sections 3.3 and 3.4, it was shown that certain RGI surfaces can be chosen, so as to reach all-loop finiteness of the full theory. More specifically it was shown that on certain RGI surfaces the partial differential operators appearing in Eqs. (40,41) acting on the β- and γ- functions of the rigid theory can be transformed to total derivatives. Then the all-loop finiteness of the β- and γ-functions of the rigid theory can be transferred to the β-functions of the soft supersymmetry breaking terms. Therefore a totally all-loop finite $N = 1$ SUSY gauge theory can be constructed, including the soft supersymmetry breaking terms.

5   Successful Finite Unification

Below we briefly review the basic properties of a phenomenologically successful SUSY model with reduced couplings, which can be made finite to all-loops in perturbation theory. Its predictions for the top and bottom quark masses, the SM Higgs boson mass, as well as the supersymmetric and the other Higgs spectra are discussed in 5.3, while experimental constraints considered are listed in 5.2. A few comments on Cold Dark Matter (CDM) are mentioned too. Other models with reduced couplings that were analyzed in [96] and [97] (see also [98] and [99]) are the Reduced Minimal $N = 1$ $SU(5)$ [41], the two-loop Finite $N = 1$ $SU(3)^3$ [100–102] and the Reduced Minimal Supersymmetric Standard Model [103,104].

5.1   The Finite $N = 1$ Supersymmetric $SU(5)$ Model

The model under review is a finite to all-orders $SU(5)$ $N = 1$ SUSY GUT (also referred to as **FUTB**), where the finiteness conditions, resulting from the application of the reduction of couplings method and the requirement of vanishing one-loop β-functions, have been applied.

The particle content of the model, resulting from applying condition (78), consists of three $(\overline{\mathbf{5}} + \mathbf{10})$ supermultiplets, where the quarks and leptons are accomodated, while in the Higgs sector there are four supermultiplets $(\overline{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$.

The most general $SU(5)$ invariant, cubic superpotential, where the R-parity that forbids fast proton decay has been imposed, and that is also consistent with the above particle content, is given by

$$
\begin{aligned}
W & = H_a \left[\, f_{ab} \, \overline{H}_b \mathbf{24} + h_{ia} \, \overline{\mathbf{5}}_i \mathbf{24} + \overline{g}_{ija} \, \mathbf{10}_i \overline{\mathbf{5}}_j \right] + p \, (\mathbf{24})^3 \\
& + \frac{1}{2} \mathbf{10}_i \left[\, g_{ija} \, \mathbf{10}_j H_a + \hat{g}_{iab} \, \overline{H}_a \overline{H}_b + g'_{ijk} \, \overline{\mathbf{5}}_j \overline{\mathbf{5}}_k \, \right] ,
\end{aligned} \tag{83}
$$

where $i, j, k = 1, 2, 3$ and $a, b = 1, \cdots, 4$, and we sum over all indices in W (notice that the $SU(5)$ indices are suppressed). The $\mathbf{10}_i$'s and $\overline{\mathbf{5}}_i$'s are the usual three generations, and the four $(\mathbf{5} + \overline{\mathbf{5}})$ Higgses are denoted by H_a , $\overline{H}_a$. As further restrictions, to make the model viable, the anomalous dimensions have been assumed

diagonal, and couplings between the fermions and the **24** in the adjoint are not allowed. To achieve all-loop finiteness the conditions 3 and 4 from the all-loop finiteness theorem have to be satisfied. These require the existence of isolated and non-degenerate solutions to the vanishing of the anomalous dimensions, and thus the vanishing of the Yukawa β-functions. One can check that this is indeed the case. As explained in the previous section, these conditions guarantee a unique solution to the reduction equations.

The existence of these solutions implies an enhanced symmetry of the superpotential, which can be found e.g. in refs. [18, 105], and is given by :

$$W = \sum_{i=1}^{3} \big[\ \frac{1}{2} g_i^u \ \mathbf{10}_i \mathbf{10}_i H_i + g_i^d \ \mathbf{10}_i \overline{\mathbf{5}}_i \ \overline{H}_i \ \big] + g_{23}^u \ \mathbf{10}_2 \mathbf{10}_3 H_4 \tag{84}$$

$$+ \ g_{23}^d \ \mathbf{10}_2 \overline{\mathbf{5}}_3 \ \overline{H}_4 + g_{32}^d \ \mathbf{10}_3 \overline{\mathbf{5}}_2 \ \overline{H}_4 + g_2^f \ H_2 \, \mathbf{24} \, \overline{H}_2 + g_3^f \ H_3 \, \mathbf{24} \, \overline{H}_3 + \frac{g^\lambda}{3} \, (\mathbf{24})^3 \ ,$$

while the solutions to the reduction equations, which ensure the vanishing of $\gamma_i^{(1)}$, and are *non-degenerate* and *isolated* as:

$$(g_1^u)^2 = \frac{8}{5} \ g^2 \ , \ (g_1^d)^2 = \frac{6}{5} \ g^2 \ , \ (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5} \ g^2 \ ,$$

$$(g_2^d)^2 = (g_3^d)^2 = \frac{3}{5} \ g^2 \ , \ (g_{23}^u)^2 = \frac{4}{5} \ g^2 \ , \ (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5} \ g^2 \ , \tag{85}$$

$$(g^\lambda)^2 = \frac{15}{7} g^2 \ , \ (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2} \ g^2 \ , \ (g_1^f)^2 = 0 \ , \ (g_4^f)^2 = 0 \ .$$

Regarding the SSB sector of the model, assuming the existence of a RGI surface on which Eq. (51) holds, we obtain at one-loop the generic relation $h = -MC$, while the sum rule leads to:

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2 \ , \ m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3} \ , \ m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3} \ . \tag{86}$$

As a result there exist two free parameters in the dimensionful sector, $m_{\mathbf{10}}$ and M.

After the $SU(5)$ breaking, it is required that the resulting model is the MSSM. To achieve this, it is necessary to perform a rotation of the Higgs sector, so that the MSSM Higgs doublets are mostly composed from the 5 and $\bar{5}$ that couple to the third generation. At the same time, the usual doublet-triplet mechanism has to be implemented to ensure there is no fast proton decay [16, 17, 106–109]. The solutions to the vanishing of the anomalous dimensions (85) and the sum rule (86) for the third generation are thus boundary conditions for the MSSM at the GUT scale. The other two generations are minimally coupled to the MSSM Higgs doublets and are therefore taken to zero in this analysis. The model is discussed in more detail in [16, 17, 41, 105].

5.2 Phenomenological Constraints

Before the analysis of the above-mentioned model, we will review the experimental constraints applied.[c]

We have consider the pole mass of the top quark while the bottom quark mass is evaluated at the M_Z scale, in order to avoid pole mass uncertainties. The experimental values [110] are:

$$m_t^{\mathrm{exp}} = 173.1 \pm 0.9 \ \mathrm{GeV} \qquad , \qquad m_b(M_Z) = 2.83 \pm 0.10 \ \mathrm{GeV} \ . \tag{87}$$

The Higgs-like particle discovered in July 2012 by ATLAS and CMS [111, 112] is interpreted as the light CP-even Higgs boson of the MSSM [113–115]. Its experimental average mass is [110]

$$M_h^{\mathrm{exp}} = 125.10 \pm 0.14 \ \mathrm{GeV} \ . \tag{88}$$

---

[c]The used values do not correspond to the latest experimental results, which, however, has a negligible impact on our analysis.

However, it is the theoretical uncertainty [116, 117] that dominates the total uncertainty, since it is much larger than the experimental one. For the prediction of the Higgs mass we used the version 2.16.0 of the `FeynHiggs` code [116–124]. This version gives a $\mathcal{O}(2 \text{ GeV})$ downward shift on the Higgs mass M_h (for large SUSY masses). More importantly, it gives a reliable point-by-point evaluation of the uncertainty [125]. The theoretical uncertainty calculated is added linearly to the experimental error of Eq. (88).

Furthermore, recent ATLAS experiment results [126] limit the neutral Higgs boson masses with respect to $\tan\beta$. For our case $\tan\beta \sim 45 - 55$ the lowest limit for the physical neutral Higgs masses is

$$M_{A,H} \gtrsim 1900 \text{ GeV}. \tag{89}$$

For the calculation of the heavy Higgs sector and the full supersymmetric spectrum a `SARAH` [127] generated, custom module for `SPheno` [128, 129] was used. The cross sections for their particle productions at the HL-LHC and FCC-hh were calculated with `MadGraph5_aMC@NLO` [130].

We also considered the following flavour observables. For $\text{BR}(b \to s\gamma)$ we take a value from [131–136], while for $\text{BR}(B_s \to \mu^+\mu^-)$ we use a combination of [137–142]:

$$\frac{\text{BR}(b \to s\gamma)^{\text{exp}}}{\text{BR}(b \to s\gamma)^{\text{SM}}} = 1.089 \pm 0.27 \quad , \quad \text{BR}(B_s \to \mu^+\mu^-) = (2.9 \pm 1.4) \times 10^{-9} . \tag{90}$$

For the B_u decay to $\tau\nu$ we use [136, 143–145] and for ΔM_{B_s} we use [146, 147]:

$$\frac{\text{BR}(B_u \to \tau\nu)^{\text{exp}}}{\text{BR}(B_u \to \tau\nu)^{\text{SM}}} = 1.39 \pm 0.69 \quad , \quad \frac{\Delta M_{B_s}^{\text{exp}}}{\Delta M_{B_s}^{\text{SM}}} = 0.97 \pm 0.2 . \tag{91}$$

Finally, we consider Cold Dark Matter (CDM) relic density constraints. Since the Lightest SUSY Particle (LSP), which in our case is the lightest neutralino, could be a promising CDM candidate [148, 149], we examine if the model is within the CDM relic density experimental limits. The current bound on the CDM relic density at $2\,\sigma$ level is given by [150]

$$\Omega_{\text{CDM}} h^2 = 0.1120 \pm 0.0112 . \tag{92}$$

For the calculation of the CDM relic density the `MicrOMEGAs 5.0` code [151–153] was used.

5.3   Numerical Analysis of the Finite $SU(5)$

We continue with the analysis of the predicted spectrum of the model. Below the GUT scale we get the MSSM, where the third generation is given by the finiteness conditions (the first two remain unrestricted). However, these conditions do not restrict the low-energy renormalization properties, so the above relations between gauge, Yukawa and the various dimensionful parameters serve as boundary conditions at M_{GUT}. The third generation quark masses $m_b(M_Z)$ and m_t are predicted within 3σ and 2σ uncertainties, respectively, of their experimental values (see the complete analysis in [96, 154]), as shown in Fig. 1. The tau lepton mass is used as an input. μ turns out to be negative, as shown in [96, 99, 155–161].
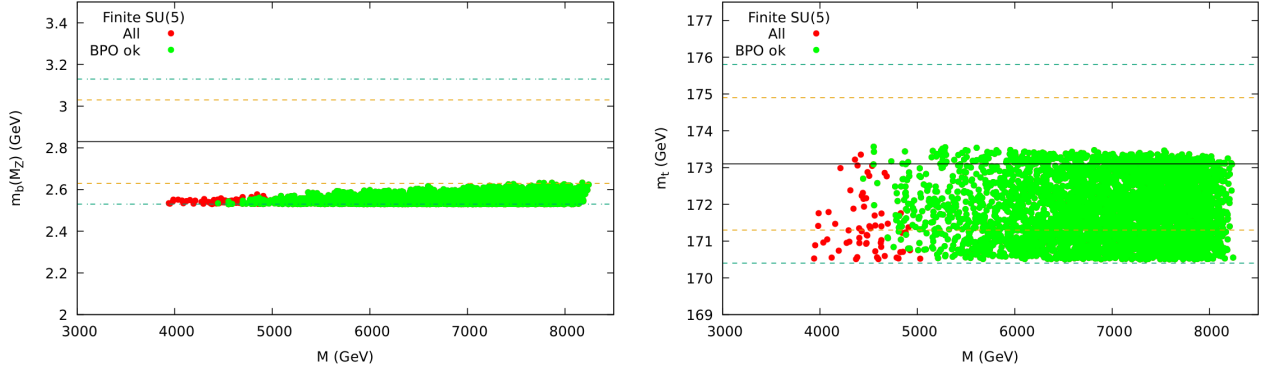
Figure 1: $m_b(M_Z)$ (left) and m_t (right) as a function of M for the Finite $N = 1$ $SU(5)$. The green points are the ones that satisfy the B-physics constraints. The orange (blue) dashed lines denote the 2σ (3σ) experimental uncertainties.

The plot of the light Higgs mass satisfies all experimental constraints considered in 5.2 (including B-physics constraints) for a unified gaugino mass $M \sim 4500 - 7500$ GeV, while its point-by-point theoretical uncertainty [125] drops significantly (w.r.t. the previous analysis) to $0.65 - 0.70$ GeV. This can be found in Fig. 2. The improved evaluation of M_h and its uncertainty prefer a heavier (Higgs) spectrum (compared to previous analyses [96, 99, 105, 155–160, 162–165]), and thus allows only a heavy supersymmetric spectrum, which is in agreement with all existing experimental data. Very heavy coloured supersymmetric particles are favoured, in agreement with the non-observation of such particles at the LHC [166, 167].
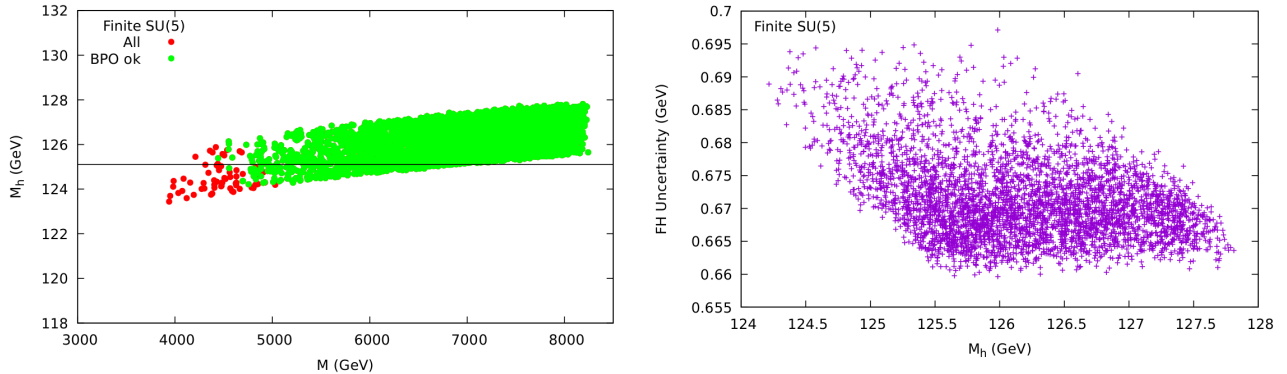

Figure 2: Left: M_h as a function of M. Green points comply with B-physics constraints. Right: The lightest Higgs mass theoretical uncertainty calculated with FeynHiggs 2.16.0 [125].

Concerning CDM, although no point fulfills the strict bound of Eq. (92), since we have overproduction of CDM in the early universe (for the original analysis see [98]), we can extend the model by considering bilinear R-parity violating terms (that preserve finiteness) and thus introduce neutrino masses [168, 169]. R-parity violation [170–173] would remove the CDM bound of Eq. (92) completely.

As explained in more detail in [97], the three benchmarks chosen (for the purposes of collider phenomenology) feature the LSP above 2100 GeV, 2400 GeV and 2900 GeV, respectively. The resulting masses that

are relevant to our analysis were generated by `SPheno` 4.0.4 [128, 129] and are listed in Table 1 for each benchmark (with the corresponding $\tan\beta$). The two first masses refer to the heavy Higgs bosons. The gluino mass is $M_{\tilde{g}}$, the neutralinos and the charginos are denoted as $M_{\tilde{\chi}_i^0}$ and $M_{\tilde{\chi}_i^\pm}$, while the slepton and sneutrino masses for all three generations are given as $M_{\tilde{e}_{1,2,3}}$, $M_{\tilde{\nu}_{1,2,3}}$. Similarly, the squarks are denoted as $M_{\tilde{d}_{1,2}}$ and $M_{\tilde{u}_{1,2}}$ for the first two generations. The third generation masses are given by $M_{\tilde{t}_{1,2}}$ for stops and $M_{\tilde{b}_{1,2}}$ for sbottoms.

	$tan\beta$	$M_{A,H}$	$M_{H^\pm}$	$M_{\tilde{g}}$	$M_{\tilde{\chi}_1^0}$	$M_{\tilde{\chi}_2^0}$	$M_{\tilde{\chi}_3^0}$	$M_{\tilde{\chi}_4^0}$	$M_{\tilde{\chi}_1^\pm}$	$M_{\tilde{\chi}_2^\pm}$
FUTSU5-1	49.9	5.688	5.688	8.966	2.103	3.917	4.829	4.832	3.917	4.833
FUTSU5-2	50.1	7.039	7.086	10.380	2.476	4.592	5.515	5.518	4.592	5.519
FUTSU5-3	49.9	16.382	16.401	12.210	2.972	5.484	6.688	6.691	5.484	6.691
	$M_{\tilde{e}_{1,2}}$	$M_{\tilde{\nu}_{1,2}}$	$M_{\tilde{\tau}}$	$M_{\tilde{\nu}_\tau}$	$M_{\tilde{d}_{1,2}}$	$M_{\tilde{u}_{1,2}}$	$M_{\tilde{b}_1}$	$M_{\tilde{b}_2}$	$M_{\tilde{t}_1}$	$M_{\tilde{t}_2}$
FUTSU5-1	3.102	3.907	2.205	3.137	7.839	7.888	6.102	6.817	6.099	6.821
FUTSU5-2	3.623	4.566	2.517	3.768	9.059	9.119	7.113	7.877	7.032	7.881
FUTSU5-3	4.334	5.418	3.426	3.834	10.635	10.699	8.000	9.387	8.401	9.390

Table 1: Masses for each of the three benchmarks of the Finite $N = 1$ $SU(5)$ (in TeV) [97].

At 14 TeV HL-LHC none of the Finite $SU(5)$ scenarios listed above has a SUSY production cross section above 0.01 fb, and thus will most probably remain unobservable [174]. The discovery prospects for the heavy Higgs-boson spectrum is significantly better at the FCC-hh [175]. Theoretical analyses [175, 176] have shown that for large $\tan\beta$ heavy Higgs mass scales up to ~ 8 TeV could be accessible. Since in this model we have $\tan\beta \sim 50$, the first two benchmark points are well within the reach of the FCC-hh (as explained in [97]). The third point, however, where $M_A \sim 16$ TeV, will be far outside the reach of the collider. At 100 TeV we have in principle production of SUSY particles in pairs, although their production cross section is at the few fb level . This is a result of the heavy spectrum of the model. Comparing our benchmark predictions with the simplified model limits of [177], we have found that the lighter stop might be accessible in FUTSU5-1 (see [97]). For the squarks of the first two generations there are better prospects. All benchmarks could be tested at the $2\,\sigma$ level, but no discovery at the $5\,\sigma$ can be expected and the same holds for the gluino.

6   Conclusions

Veltman's contributions to the field of Particle Physics have a huge impact in the development of the field. Here we have presented only one of the roads that Veltman opened in particle physics, published in his celebrated paper in Acta Physica Polonica [4]. This work of Veltman was guided by the current at that time notion of naturality which appeared to be of fundamental importance in physics discussions and guiding principle in searches of new physics up to now. Veltman required the absence of quadratic divergences in the SM, which led to a quadratic mass relation among the SM particles. The fact that Veltman's relation eventually does not hold can be taken as sign that the SM, despite its phenomenological successes cannot be considered as a complete theory. Moreover, the whole discussion on the cancellation of quadratic divergences in renormalizable field theories with scalars was uniquelly pointing to the supersymmetric ones, where naturally do not appear such divergences to all-orders of perturbation theory, as the arena of searching for a more complete theory. The possibility to achieve unification of the gauge couplings of the SM in the supersymmetric framework with supersymmetry broken in the TeV scale gave a huge push in the research in such theories and in particular in MSSM for many years.

Still, the problem of the several independent parameters of the SM is much more severe even in the minimal version of MSSM, when the supersymmetry breaking sector is taken into account. Correspondingly the

necessity of *reduction of couplings* in the SM becomes substantially stronger in the MSSM. The application of the method of searching for RGI relations as a way to reduce the independent parameters of the SM failed, as the Veltman relation, when it was confronted with the experimental discoveries of the top quark and Higgs particles and the determination of their masses. Now after several years of research it seems that so far supersymmetric unified schemes such as the Finite $SU(5)$, the minimal $SU(5)$ and the $SU(3)^3$ with reduced couplings (i.e. satifying RGI relations) can be realistic. Among them clearly the most interesting is the $SU(5)$ FUT, since beyond the unification scale a complete reduction of couplings in favour of the gauge coupling can be achieved, and it is furthermore finite to all-orders in perturbation theory. In the latter clearly even the logarithmic divergences are absent, fulfilling an old dream of theoretical physicists who were seriously disturbed by the presence of divergences in field theories. Moreover, it is a realistic theory with the great successes of predicting successfully the top and Higgs masses well before their experimental discoveries and passing successfully all experimental tests so far and having chances to be tested further at FCC-hh.

Acknowledgements

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