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Large-x resummation of off-diagonal deep-inelastic parton scattering from d-dimensional refactorization

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Abstract: The off-diagonal parton-scattering channels $g+\gamma^*$ and $q+\phi^*$ in deep-inelastic scattering are power-suppressed near threshold $x \to 1$. We address the next-to-leading power (NLP) resummation of large double logarithms of $1-x$ to all orders in the strong coupling, which are present even in the off-diagonal DGLAP splitting kernels. The appearance of divergent convolutions prevents the application of factorization methods known from leading power resummation. Employing d-dimensional consistency relations from requiring $1/\epsilon$ pole cancellations in dimensional regularization between momentum regions, we show that the resummation of the off-diagonal parton-scattering channels at the leading logarithmic order can be bootstrapped from the recently conjectured exponentiation of NLP soft-quark Sudakov logarithms. In particular, we derive a result for the DGLAP kernel in terms of the series of Bernoulli numbers found previously by Vogt directly from algebraic all-order expressions. We identify the off-diagonal DGLAP splitting functions and soft-quark Sudakov logarithms as inherent two-scale quantities in the large-x limit. We use a refactorization of these scales and renormalization group methods inspired by soft-collinear effective theory to derive the conjectured soft-quark Sudakov exponentiation formula.

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Contents

1 Introduction

Resummations of logarithmically enhanced loop corrections are a powerful and often essential tool to enlarge the predictivity of QCD perturbation theory. Resummation is necessary when a ratio of kinematic invariants, λ, becomes small such that $\alpha_s \ln^k \lambda$, where α_s is the strong coupling and $k = 1$ or 2, is no longer a good expansion parameter. Recent interest in this subject has focused on understanding the structure of such logarithmic terms at next-to-leading power (NLP) in λ with the aim of summing them to all orders in α_s. This has been accomplished at the leading-logarithmic (LL) order in various contexts, covering final-state event shapes [1, 2], threshold resummation in Drell-Yan and Higgs production [3–5], and Higgs production or decay through light-quark loops [6–8]. A number of methods has been used, but it has become evident that a generalization to the next-to-leading-logarithmic (NLL) order is not straightforward. This is to be compared to the situation at leading power (LP), where resummation is often understood to any logarithmic order, even though one faces technical challenges of high-order loop calculations in practice.

The most natural framework to formulate resummation is through the factorization of scales and evolution equations. The all-order resummed expression is then obtained as the product or convolution of the factorized pieces. At NLP, one faces the new difficulty that these convolutions are divergent. While divergent convolutions are familiar from rapidity divergences, which are not regulated dimensionally and may occur already at LP, such as in transverse-momentum dependent factorization, the problem at NLP is of a different nature. The divergences can be regulated dimensionally and arise in convolutions of factors containing the physics at different virtualities. However, since factorization and resummation refer to the renormalized factors before convolution, the standard formalism fails to deal with this situation. An explicit example can be found in [9] for the case of next-to-leading logarithms near the $q\bar{q} \to \gamma^*$ Drell-Yan threshold.

In the present paper, we address these difficulties for the threshold of off-diagonal deep-inelastic parton scattering. The off-diagonal channels vanish at LP near threshold $x \to 1$, since they do not contain $1/[1-x]_+$ distributions at any order in α_s. However, the failure of standard resummation methods appears already at the LL order for the DGLAP splitting functions. Vogt and collaborators [10–12] found that the all-order quark-gluon splitting function with LL accuracy is given in moment space by

$$P_{gq}^{\rm LL}(N) = \frac{1}{N}\frac{\alpha_s C_F}{\pi}\,\mathcal{B}_0(a), \qquad a = \frac{\alpha_s}{\pi}(C_F - C_A)\ln^2 N\,, \tag{1.1}$$

where

$$\mathcal{B}_0(x) = \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n \tag{1.2}$$

is the Borel transform of the generating function of the Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, ...[1] The existence of an infinite series of double logarithmic terms shows

[1] Here and in the remainder of the paper α_s without argument denotes the strong coupling in the $\overline{\rm MS}$ scheme at the renormalization/factorization scale μ. With our definition (3.68) of the anomalous dimensions, the splitting kernel differs by a factor of two from [10].

that in the off-diagonal channels even the anomalous dimension is a two-scale quantity as $N \gg 1$, contrary to the diagonal anomalous dimension, a distinction that has not received as much attention as it deserves, except for [10–12]. This remarkable result was obtained from the structure of $1/\epsilon$ poles of the unfactorized parton-scattering cross sections in the exactly known $d = 4 - 2\epsilon$ dimensional low-order results, and their consistency with factorization. These structures were then extrapolated to all orders to find closed functional forms, including the reconstruction of the series of Bernoulli numbers. To our understanding (1.1) has not yet been proven by deriving it directly from algebraic all-order expressions. With this method, further results on the finite short-distance coefficients of the off-diagonal channels in deep-inelastic scattering and Drell-Yan production at large N were also obtained [10, 11].

What distinguishes the off-diagonal splitting functions from the diagonal ones in the $x \to 1$ / large-N limit is that the former describe the splitting into an energetic parton and a soft *quark*. We further notice that the double-logarithmic series involves the colour factor $C_F - C_A$. A connection between soft quarks and this colour factor of the large logarithms also appears for the Sudakov resummation of the $q\bar{q} \to \phi^*$ form factor, where ϕ^* denotes a Higgs boson, effectively coupled to two gluons, and q a light quark [13, 14]. The leading logarithms here originate from soft quark exchange. Further, the authors of [15] investigated the all-order structure of the $e^+e^- \to q\bar{q}g$ amplitude in the kinematic configuration where the quark and anti-quark are nearly collinear with small virtuality $s \ll Q^2$ and momentum fractions z and $\bar{z} = 1 - z$, respectively, recoiling against the energetic gluon. Keeping the leading double poles $1/\epsilon^2$ and logarithms of z or $\bar{z}$, as the quark or anti-quark become soft, they conjectured the exponentiation of the corresponding one-loop terms to all orders

$$\begin{aligned}\mathcal{P}_{q\bar{q}}(z) = \mathcal{P}_{q\bar{q}}^{\text{tree}}(z)\, \exp\Bigg[\frac{\alpha_s}{\pi\epsilon^2}\Bigg\{&\mathbf{T}_1\cdot\mathbf{T}_3\left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \mathbf{T}_2\cdot\mathbf{T}_3\left(\frac{\mu^2}{\bar{z}Q^2}\right)^{\epsilon} + \mathbf{T}_1\cdot\mathbf{T}_2\left(\frac{\mu^2}{s}\right)^{\epsilon}\\ &+\mathbf{T}_1\cdot\mathbf{T}_2\left(\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{z\bar{z}Q^2}\right)^{\epsilon}\right) - \mathbf{T}_1\cdot\mathbf{T}_2\left(\left(\frac{\mu^2}{s}\right)^{\epsilon} - \left(\frac{\mu^2}{z\bar{z}s}\right)^{\epsilon}\right)\Bigg\}\Bigg]. \end{aligned} \tag{1.3}$$

Here $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ are the colour operators of the quark, antiquark and gluon, respectively, such that $\mathbf{T}_1\cdot\mathbf{T}_2 = C_A/2 - C_F$, $\mathbf{T}_1\cdot\mathbf{T}_3 = \mathbf{T}_2\cdot\mathbf{T}_3 = -C_A/2$. If we now take $z \to 0$, which corresponds to the quark becoming soft, and focus on the terms involving Q^2, we see that the coefficient of $Q^{-2\epsilon}$ has the colour factor $\mathbf{T}_2\cdot\mathbf{T}_3 + \mathbf{T}_1\cdot\mathbf{T}_2 = -C_F$. The coefficient of $(zQ^2)^{-\epsilon}$, which involves the new scale $\sqrt{z}Q \ll Q$ in the soft-quark limit, however, is $\mathbf{T}_1\cdot\mathbf{T}_3 - \mathbf{T}_1\cdot\mathbf{T}_2 = C_F - C_A$. It is tempting to conjecture that in a splitting $1 \to 2+3$ with soft 3, the endpoint divergence, which occurs when integrating over z since $\mathcal{P}_{q\bar{q}}^{\text{tree}}(z) \propto 1/z$, and which requires extra resummation of the logarithms of z or $\bar{z}$ not captured by the usual formalism, is related to the difference of the Casimir charge of the energetic particles 1 and 2. Along this line, it was noted in [2, 15] that in supersymmetric QCD with quarks in the adjoint representation, the endpoint divergences and extra logarithms are absent. All three examples of the appearance of $C_F - C_A$ in front of double logarithms have in common that the resummed result was obtained without explicit factorization of the scales involved, using either d-dimensional arguments, diagrammatic arguments, or a conjecture.

In this paper, we establish a connection between some of these results in the context of NLP LL resummation for off-diagonal deep-inelastic parton scattering as $x \to 1$. To this end we adapt the soft-quark Sudakov exponentiation conjecture [15] from event shapes to deep-inelastic scattering (DIS). We then

- prove (1.1) for the resummed off-diagonal splitting function and the finite coefficient function from the soft-quark Sudakov exponentiation conjecture via d-dimensional consistency relations that follow from the requirement of pole cancellation between momentum regions. The adapted version of (1.3) plays the role of a "boundary condition" in the purely hard contribution to the process, from which the resummation of the full process follows in closed all-order form.
- derive the previously conjectured exponentiation formula through the refactorization of certain power-suppressed operators in soft-collinear effective theory (SCET) which have endpoint-singular matching coefficients. The renormalization group equations (RGEs) then exhibit the origin of the $C_F - C_A$ colour factor.

The outline of the paper is as follows. In section 2 we consider the $q + \phi^* \to q + g$ amplitude, define the light-cone momentum distribution for the qg-final state, and calculate its leading poles at the one-loop order. We then apply the exponentiation conjecture analogous to (1.3) to the soft-quark limit of this amplitude. Sections 3 and 4 contain the material related to the two bullet points above, respectively. We conclude in section 5. Appendices A and B collect SCET conventions and the field modes which appear in different parts of the paper, and some basic facts on DIS at large x. In appendices C and D we provide alternative derivations of a) the solution of the consistency relations at LP, and b) the resummation of the refactorized SCET operator, which confirms the result of section 4. The application of consistency relations to the thrust event shapes considered in [15] is presented in appendix E in order to note the similarities and differences between the two processes.

2 Off-diagonal DIS cross section and soft-quark Sudakov exponentiation

We begin by considering deep-inelastic scattering

$$q(p) + \phi^*(q) \to X(p_X) \tag{2.1}$$

of a quark off a Higgs boson, which couples to quarks and gluons through the gluonic interaction

$$\mathcal{L} = \kappa\, \phi\, \mathrm{tr}\left[G^{\mu\nu} G_{\mu\nu}\right] . \tag{2.2}$$

The coupling κ is related to the effective Higgs-gluon coupling in the infinite top-quark mass limit.[2] We are interested in the kinematic situation when the final state has small

[2]See (B.3) in appendix B. We refer to this appendix for a summary of the kinematics of DIS, the factorization of the hadronic structure function at leading power, and the relevant momentum regions at large x.

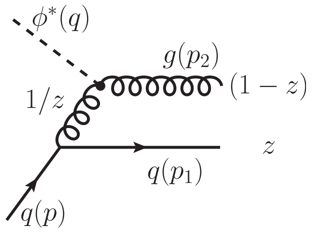

Figure 1. Scattering of a quark off a virtual Higgs boson at tree level.

invariant mass, $p_X^2 \ll Q^2 \equiv -q^2$, which corresponds to the limit $x \to 1$ for the Bjorken scaling variable $x \equiv Q^2/(2p \cdot q)$. The reason for considering this exotic process is that it is related to Drell-Yan production of a Higgs boson in the quark-gluon channel near the partonic threshold. The above DIS process allows us to extract the quark-gluon splitting kernel P_{gq} [10], which enters Higgs production. All these quantities are NLP, i.e. suppressed by one power of $(1-x)$ as $x \to 1$ relative to the leading diagonal gluon-gluon coupling. The off-diagonal channel $g + \gamma^* \to q + g$ for the more standard scattering on the vector current can be obtained by substitution of colour factors [10].

We consider the dimensionally regularized and unfactorized partonic DIS cross sections. Following [10], we introduce the partonic structure functions $W_{\phi,g}$, $W_{\phi,q}$ by defining

$$W_{\phi,i} = \frac{1}{8\pi Q^2} \int d^4x\, e^{iq\cdot x} \left\langle i(p) \middle| \left[G^A_{\mu\nu} G^{\mu\nu A}\right](x) \left[G^B_{\rho\sigma} G^{\rho\sigma B}\right](0) \middle| i(p) \right\rangle, \tag{2.3}$$

where $i = g$ or q. At the lowest non-vanishing order in α_s, to NLP in $(1-x)$, and neglecting $\mathcal{O}(\epsilon^0)$ terms not multiplied by logarithms of $(1-x)$ one has

$$W_{\phi,g} = \delta(1-x) + \mathcal{O}(\alpha_s), \qquad W_{\phi,q} = -\frac{1}{\epsilon}\frac{\alpha_s C_F}{2\pi}(1-x)^{-\epsilon} + \mathcal{O}(\alpha_s^2)\,. \tag{2.4}$$

The exact result is provided in (B.11) and (B.13).

2.1 Momentum distribution function and lowest-order result

At the lowest order the quark-scattering off the Higgs boson is realized by the process

$$q(p) + \phi^*(q) \to q(p_1) + g(p_2)\,. \tag{2.5}$$

The tree-level amplitude is shown in figure 1. We write the $d = 4 - 2\epsilon$ dimensional two-particle phase space as

$$\begin{aligned} d\Phi_2(p_X; p_1, p_2) &= \frac{\tilde{\mu}^{2\epsilon} d^d p_1}{(2\pi)^d}\, 2\pi\delta(p_1^2)\theta(p_1^0)\, 2\pi\delta\left[(p_X - p_1)^2\right]\theta(p_X^0 - p_1^0) \\ &= \frac{e^{\gamma_E \epsilon}}{8\pi\Gamma(1-\epsilon)}\, dz \left(\frac{\mu^2}{s_{qg} z\bar{z}}\right)^{\epsilon} \theta(z)\theta(\bar{z})\theta(s_{qg})\,, \end{aligned} \tag{2.6}$$

where $\tilde{\mu}^2 = \mu^2 e^{\gamma_E}/(4\pi)$, $s_{qg} = p_X^2$. We introduced the variable

$$z \equiv \frac{n_- p_1}{n_- p_1 + n_- p_2}\,, \qquad \bar{z} = 1 - z\,, \tag{2.7}$$

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which represents the distribution of the light-cone final-state momentum between the nearly collinear quark and gluon in the final state. The vector n_- represents a light-like vector which projects on the large momentum components of the final state particles.[3]

We can represent the matrix element squared of the process (2.5) integrated over the phase space in the form

$$W_{\phi,q}\big|_{q\phi^*\to qg} = \int_0^1 dz\,\left(\frac{\mu^2}{s_{qg}z\bar{z}}\right)^{\epsilon} \mathcal{P}_{qg}(s_{qg},z)\Big|_{s_{qg}=Q^2\frac{1-x}{x}}\,, \tag{2.8}$$

which holds to any order in perturbation theory in the strong coupling α_s. The momentum distribution function is defined as

$$\mathcal{P}_{qg}(s_{qg},z) \equiv \frac{e^{\gamma_E\epsilon}\,Q^2}{16\pi^2\Gamma(1-\epsilon)}\frac{|\mathcal{M}_{q\phi^*\to qg}|^2}{|\mathcal{M}_0|^2}\,, \tag{2.9}$$

where $|\mathcal{M}_0|^2$ denotes the tree-level matrix element squared, averaged (summed) over the spin and colour of the initial (final) state for the leading diagonal channel process $g+\phi^*\to g$. For $x\to 1$, we can expand $\mathcal{P}_{qg}(s_{qg},z)$ in s_{qg}/Q^2 or $\lambda\sim\sqrt{1-x}\ll 1$. For this purpose we note that

$$s_{qg} = (p+q)^2 = \frac{Q^2(1-x)}{x} = Q^2(1-x) + \mathcal{O}(\lambda^4)\,. \tag{2.10}$$

From (B.10) we have

$$|\mathcal{M}_0|^2 \equiv \left|\mathcal{M}_{g\phi^*\to g}\right|^2_{\text{tree}} = \frac{\kappa^2 Q^4}{x^2}\,(1-\epsilon) = \kappa^2 Q^4\,(1-\epsilon) + \mathcal{O}(\lambda^2)\,. \tag{2.11}$$

Similarly, the expansion of $|\mathcal{M}_{q\phi^*\to qg}|^2$, given in (B.12), which is itself a function of s_{qg} or x and of the momentum fraction z, gives

$$\left|\mathcal{M}^{(1)}_{q\phi^*\to qg}\right|^2 = 2\kappa^2\,g_s^2 C_F\,(1-\epsilon)\,Q^2\frac{\bar{z}^2}{z} + \mathcal{O}(\lambda^2) \tag{2.12}$$

at the lowest non-vanishing order in the coupling expansion, which implies

$$\mathcal{P}_{qg}(s_{qg},z)\big|_{\text{tree}} = \frac{\alpha_s C_F}{2\pi}\frac{\bar{z}^2}{z} + \mathcal{O}(\epsilon,\lambda^2)\,. \tag{2.13}$$

Integrating and neglecting $\mathcal{O}(\epsilon)$ corrections that are *not* multiplied by logarithms (i.e. counting $\epsilon\ll 1$ but $\epsilon\ln(1-x)\sim 1$, $(1-x)^{-\epsilon}\sim 1$ and $\epsilon\ln(\mu/Q)\sim 1$), gives

$$W_{\phi,q}\big|^{\text{NLP}}_{\mathcal{O}(\alpha_s),\,\text{leading pole}} = -\frac{1}{\epsilon}\,\frac{\alpha_s C_F}{2\pi}\left(\frac{\mu^2}{Q^2(1-x)}\right)^{\epsilon}, \tag{2.14}$$

in agreement with (2.4). $W_{\phi,q}\big|_{q\phi^*\to qg}$ represents the contribution to the *partonic* DIS structure function when only two partons are present in the final state. As such it is an infrared (IR) divergent quantity. In lowest order in α_s, the IR divergence is a single $1/\epsilon$ pole, which arises from the $z\to 0$ region of the integral (2.8) owing to the $1/z$ behaviour of the tree-level momentum distribution function. The $z\to 0$ limit corresponds to the kinematic configuration where the initial quark transfers all of its momentum to the final-state gluon, and the final-state quark becomes soft. It is therefore essential that the integration over z in (2.8) is done in d dimensions, a fact that will be of importance later on.

[3] See appendix A for more details on the definition of SCET reference vectors and of the power counting parameters relevant for DIS near threshold.

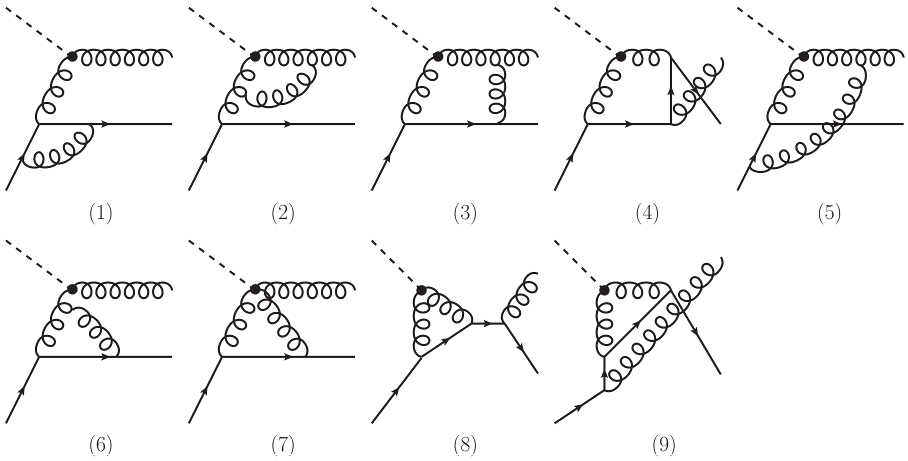

Figure 2. One-loop corrections to the scattering of a quark off a virtual Higgs boson. Only the triangle and box diagrams are shown.

2.2 One-loop momentum distribution function

In this subsection, we calculate the one-loop (virtual) correction to the $2 \to 2$ scattering process $q\phi^* \to qg$ and obtain the corresponding momentum distribution function (2.9). We are interested in the leading double poles and logarithms. We therefore need the leading pole $1/\epsilon^2$ without expanding its coefficient in a series of ϵ. The relevant Feynman diagrams are shown in figure 2. We find that only the first five diagrams give non-vanishing leading poles. Calculating the interference of these one-loop diagrams with the tree diagram, we obtain for the leading-pole terms

$$\begin{aligned}\mathcal{P}_{qg}(s_{qg},t,u)|_{\text{1-loop}} = \mathcal{P}_{qg}(s_{qg},t,u)|_{\text{tree}} \frac{\alpha_s}{\pi}\frac{\mu^{2\epsilon}}{\epsilon^2} & \\ \Big\{ \big[\mathbf{T}_1\cdot\mathbf{T}_2\,(-s_{qg})^{-\epsilon} + \mathbf{T}_1\cdot\mathbf{T}_0\,(-t)^{-\epsilon} + \mathbf{T}_2\cdot\mathbf{T}_0\,(-u)^{-\epsilon}\big] & \\ + \mathbf{T}_1\cdot\mathbf{T}_2\left((Q^2)^{-\epsilon} - (-t)^{-\epsilon} - (-s_{qg})^{-\epsilon} + \frac{(-s_{qg})^{-\epsilon}(-t)^{-\epsilon}}{(Q^2)^{-\epsilon}}\right) & \\ + \mathbf{T}_1\cdot\mathbf{T}_0\left((Q^2)^{-\epsilon} - (-s_{qg})^{-\epsilon} - (-u)^{-\epsilon} + \frac{(-s_{qg})^{-\epsilon}(-u)^{-\epsilon}}{(Q^2)^{-\epsilon}}\right) & \\ + \mathbf{T}_2\cdot\mathbf{T}_0\left((Q^2)^{-\epsilon} - (-t)^{-\epsilon} - (-u)^{-\epsilon} + \frac{(-t)^{-\epsilon}(-u)^{-\epsilon}}{(Q^2)^{-\epsilon}}\right)\Big\} & \qquad (2.15)\end{aligned}$$

with $t = (p_1 - p)^2$, $u = (p_2 - p)^2$, and

$$\mathcal{P}_{qg}(s_{qg},t,u)|_{\text{tree}} = \frac{\alpha_s C_F}{2\pi}\frac{s_{qg}^2 + u^2}{-tQ^2} + \mathcal{O}(\epsilon)\,, \qquad (2.16)$$

employing colour operator notation, $\mathbf{T}_i$, for particle i ($i = 0$ for the incoming quark). Eq. (2.15) is valid for general values of the Mandelstam variables s_{qg}, t, u. The first line in the curly bracket is the familiar double-pole structure when all s_{qg}, t, u are of order of the hard scale Q^2. The last three lines represent additional contributions, which are suppressed by ϵ^2 compared to the first line. Although technically finite as $\epsilon \to 0$, these cannot be simply omitted because after integration over the phase space to obtain $W_{\phi,q}$, they may generate poles and logarithms of equal order as the pole terms from the first line, as will be shown below.

When the final-state quark and gluon become collinear, $s_{qg} \ll Q$. With $s_{qg} + t + u = -Q^2$, we can parameterize $t = -zQ^2 - s_{qg}/2, u = -\bar{z}Q^2 - s_{qg}/2$. The collinear expansion amounts to expanding in $s_{gq}/Q^2 \sim (1-x) \sim \lambda^2$ at fixed z, which yields

$$
\begin{aligned}
\mathcal{P}_{qg}(s_{qg},z)|_{\text{1-loop}} = \mathcal{P}_{qg}(s_{qg},z)|_{\text{tree}} \frac{\alpha_s}{\pi}\frac{\mu^{2\epsilon}}{\epsilon^2} \\
\Big\{ \left[\mathbf{T}_1\cdot\mathbf{T}_2\,(-s_{qg})^{-\epsilon} + \mathbf{T}_1\cdot\mathbf{T}_0\,(zQ^2)^{-\epsilon} + \mathbf{T}_2\cdot\mathbf{T}_0\,(\bar{z}Q^2)^{-\epsilon}\right] \\
+ \mathbf{T}_1\cdot\mathbf{T}_2\left[(Q^2)^{-\epsilon} - (zQ^2)^{-\epsilon} - (-s_{qg})^{-\epsilon} + (-zs_{qg})^{-\epsilon}\right] \\
+ \mathbf{T}_1\cdot\mathbf{T}_0\left[(Q^2)^{-\epsilon} - (\bar{z}Q^2)^{-\epsilon} - (-s_{qg})^{-\epsilon} + (-\bar{z}s_{qg})^{-\epsilon}\right] \\
+ \mathbf{T}_2\cdot\mathbf{T}_0\left[(Q^2)^{-\epsilon} - (zQ^2)^{-\epsilon} - (\bar{z}Q^2)^{-\epsilon} + (z\bar{z}Q^2)^{-\epsilon}\right] \Big\}.
\end{aligned}
\tag{2.17}
$$

Given that the leading order result (2.12) becomes singular only at the end point $z = 0$, we can safely expand the $\bar{z}^{-\epsilon}$ terms in a series of ϵ in the above equation. Therefore, keeping only terms contributing to the leading poles after integration over the phase space, we have

$$
\begin{aligned}
\mathcal{P}_{qg}(s_{qg},z)|_{\text{1-loop}} = \mathcal{P}_{qg}(s_{qg},z)|_{\text{tree}} \frac{\alpha_s}{\pi}\frac{1}{\epsilon^2}\Bigg(\mathbf{T}_1\cdot\mathbf{T}_0\left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \mathbf{T}_2\cdot\mathbf{T}_0\left(\frac{\mu^2}{\bar{z}Q^2}\right)^{\epsilon} \\
+ \mathbf{T}_1\cdot\mathbf{T}_2\left[\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \left(\frac{\mu^2}{zs_{qg}}\right)^{\epsilon}\right]\Bigg).
\end{aligned}
\tag{2.18}
$$

In the framework of SCET this result exhibits a profound problem. The tree amplitude represented in figure 1 corresponds to a J^{B1} SCET operator (see appendix A) with a quark field in the collinear direction, and a quark and a gluon field in the anti-collinear direction with light-cone momentum fractions z and $\bar{z}$, respectively. The tree-level matching coefficient of this operator from the diagram in figure 1 is proportional to $1/z$, which gives the $1/z$ behaviour of $\mathcal{P}_{qg}(s_{qg},z)|_{\text{tree}}$ after squaring the amplitude and accounting for a factor of z from the sum of the final-state quark spin. From the general formula for the anomalous dimension of subleading power operators [16, 17], we get the double pole terms with $\mathbf{T}_1\cdot\mathbf{T}_0$ and $\mathbf{T}_2\cdot\mathbf{T}_0$ from the standard cusp anomalous dimension terms. However, one cannot obtain a cusp term for the two fields within the same collinear sector, i.e. the $\mathbf{T}_1\cdot\mathbf{T}_2$ term. In this part, there are three terms involving three different scales. The third contains the scale zs_{qg}. We may disregard it here because the dependence on s_{qg} identifies it as a term related to the final-state jet function, rather than the renormalization of the J^{B1} operator at the hard DIS vertex. The first two terms, however, contain the hard scales Q^2 and zQ^2, and they are supposed to be predicted by the corresponding anomalous dimension. However, the anomalous dimension given in [16, 17] applies when the convolution of the coefficient function with the anomalous dimension is convergent, which is not the case here as discussed next.

The difference between these two terms in the coefficient of $\mathbf{T}_1 \cdot \mathbf{T}_2$ is $\mathcal{O}(\epsilon)$ and hence does not contribute to the double pole. Instead, the expansion in ϵ produces $1/\epsilon \times \ln z$. The important point is that the $1/z$ singularity of the matching coefficient promotes this term to the same leading-pole order $1/\epsilon^3$ as the standard double pole terms after integration over z as in (2.8). Moreover, the integral over z must itself be regularized due to the singularity at $z = 0$, and the correct result is obtained by *not* expanding (2.18) before integration. This can easily be seen by comparing (no expansion before integration)

$$\frac{1}{\epsilon^2}\int_0^1 dz\, \frac{1}{z^{1+\epsilon}}\left(1 - z^{-\epsilon}\right) = -\frac{1}{2\epsilon^3} \tag{2.19}$$

to (expansion of (2.18) before integration)

$$\frac{1}{\epsilon^2}\int_0^1 dz\, \frac{1}{z^{1+\epsilon}}\left(\epsilon \ln z - \frac{\epsilon^2}{2!}\ln^2 z + \frac{\epsilon^2}{3!}\ln^3 z + \cdots\right) = -\frac{1}{\epsilon^3} + \frac{1}{\epsilon^3} - \frac{1}{\epsilon^3} + \cdots . \tag{2.20}$$

If only the pole part of the integrand were kept, the result would be incomplete. This explains why it was necessary to keep the exact d-dimensional coefficient of the double pole terms in the one-loop momentum distribution function.

To summarize, when we attempt to interpret (2.18) in the SCET framework, we discover two problems with the standard treatment of factorization in SCET. First, the renormalization and logarithmic terms of some SCET operators with singular matching coefficients are not obtained correctly. Second, the convolution integrals of the hard matching coefficients with the jet functions — the integral over z above — diverge and must themselves be regularized, for instance dimensionally. These obstacles appear first at NLP.

2.3 Exponentiation conjecture

We shall pursue the SCET interpretation further in section 4. Here, we observe that crossing symmetry relates $q\phi^* \to qg$ to $H \to q\bar{q}g$ discussed in [15], and we follow [15] by *conjecturing* that the leading poles are given correctly to all orders in α_s by exponentiating the one-loop expression (2.18):

$$\begin{aligned}\mathcal{P}_{qg}(s_{qg}, z) = \mathcal{P}_{qg}(s_{qg}, z)|_{\text{tree}} \exp\Bigg[\frac{\alpha_s}{\pi}\frac{1}{\epsilon^2}\Bigg(&\mathbf{T}_1 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \mathbf{T}_2 \cdot \mathbf{T}_0 \left(\frac{\mu^2}{\bar{z}Q^2}\right)^{\epsilon} \\ &+ \mathbf{T}_1 \cdot \mathbf{T}_2 \left(\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{zQ^2}\right)^{\epsilon} + \left(\frac{\mu^2}{zs_{qg}}\right)^{\epsilon}\right)\Bigg) + \mathcal{O}\left(\frac{1}{\epsilon}\right)\Bigg],\end{aligned} \tag{2.21}$$

where, for the qg case, $\mathbf{T}_1 \cdot \mathbf{T}_0 = C_A/2 - C_F$, $\mathbf{T}_2 \cdot \mathbf{T}_0 = \mathbf{T}_1 \cdot \mathbf{T}_2 = -C_A/2$, but the colour-operator notation is used to emphasize the generality of the conjecture. The exponentiation refers to the d-dimensional expression, since it must be integrated in d dimensions over z as discussed above.

In the subsequent section, we employ consistency relations from the cancellation of poles between all relevant momentum regions to infer the structure of the DIS structure function from the contribution with only hard loops and a single anti-hardcollinear loop, as shown in the left diagram of figure 4 below. The term involving s_{qg} in the exponent in (2.21) arises from the exponentiation of the one-loop anti-hardcollinear leading pole to all orders, and should therefore be dropped for this consideration. Also the tree-level momentum distribution function is non-singular as $z \to 1$, hence for the leading poles after the z-integration, we may replace $\bar{z} \to 1$. We can therefore simplify (2.21) to

$$\mathcal{P}_{qg,\text{hard}}(s_{qg}, z) = \frac{\alpha_s C_F}{2\pi} \frac{1}{z} \exp\left[\frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left((\mathbf{T}_2 \cdot \mathbf{T}_0 + \mathbf{T}_1 \cdot \mathbf{T}_2) \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right.\right.$$
$$\left.\left. + (\mathbf{T}_1 \cdot \mathbf{T}_0 - \mathbf{T}_1 \cdot \mathbf{T}_2) \left(\frac{\mu^2}{zQ^2} \right)^\epsilon \right) + \mathcal{O}\left(\frac{1}{\epsilon} \right) \right], \tag{2.22}$$

where, for the qg case, $\mathbf{T}_2 \cdot \mathbf{T}_0 + \mathbf{T}_1 \cdot \mathbf{T}_2 = -C_A$ and $\mathbf{T}_1 \cdot \mathbf{T}_0 - \mathbf{T}_1 \cdot \mathbf{T}_2 = C_A - C_F$. In SCET we interpret this as the resummation of the matching coefficient of a non-standard B1 operator, squared and convoluted with the tree-level jet function. We shall come back to this in section 4, where we provide a derivation of this result with factorization methods. Notice that the above expression has a homogeneous power counting even when $z \ll 1$, since in a d-dimensional treatment we count $z^{-\epsilon}$ as an $\mathcal{O}(1)$ quantity.

Integrating (2.22) over z yields the contribution to the off-diagonal quark DIS structure function from any number of hard loops and a single anti-hardcollinear loop from the two-particle phase-space, which corresponds to the integral over the tree-level final state jet function. We obtain

$$\begin{aligned} W_{\phi,q}\Big|^{\text{hard}}_{q\phi^* \to qg} &= \int_0^1 dz \left(\frac{\mu^2}{s_{qg} z} \right)^\epsilon \mathcal{P}_{qg,\text{hard}}(s_{qg}, z)\Big|_{s_{qg} = Q^2(1-x)} \\ &= \frac{\alpha_s C_F}{2\pi} \left(-\frac{1}{\epsilon} \right) \left(\frac{\mu^2}{Q^2(1-x)} \right)^\epsilon \exp\left[-\frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right] \\ &\quad \times \frac{\exp\left[\frac{\alpha_s (C_A - C_F)}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right] - 1}{\frac{\alpha_s (C_A - C_F)}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^\epsilon} \end{aligned} \tag{2.23}$$

in the leading-pole approximation. We note that the expression contains the standard Sudakov factor with colour factor C_A and a non-standard factor involving the colour factor $C_F - C_A$. The latter arises from the integral

$$\int_0^1 dz \, \frac{\exp[a z^{-\epsilon}]}{z^{1+\epsilon}} = -\frac{1}{\epsilon} \frac{e^a - 1}{a} \,. \tag{2.24}$$

3 Resummed off-diagonal partonic cross section and qg splitting function from consistency relations

The theory of deep-inelastic scattering and of resummation for $x \to 1$ is based on factorization formulas for the scattering of hadrons in terms of partonic quantities. The latter are usually IR and ultraviolet (UV) divergent, but can be defined in terms of a renormalization prescription. Consistency relations follow from the requirement that an observable must be finite as $\epsilon \to 0$ and allow one to deduce the expansion in ϵ of unrenormalized partonic quantities based on partial information.

The LL resummation of the quark-gluon splitting function (1.1) was obtained in [10] from the requirement that the DIS cross section is an observable and hence must be finite, together with additional assumptions on the all-order colour structure as well as an exponentiation ansatz for the full partonic cross section. A stronger form of consistency relations from pole cancellations can be obtained when the regions of virtuality relevant to the observable are known. The different scaling of every region with the dimensionless parameters of the problem implies a larger number of consistency relations [18]. For example, the LL resummation of the thrust event shapes at NLP was derived in [1] from the contributions with a single collinear and an arbitrary number of hard loops alone and invoking pole cancellations between all regions.

In this section we use consistency relations to derive the NLP LL resummation of the off-diagonal DGLAP kernel (1.1) and short-distance coefficient function from the exponentiation conjecture (2.23). In this way, we infer the resummation of the short-distance functions in the DIS factorization formula from the resummation in a single momentum region. In this section we work in moment space, following [10], to avoid dealing with convolutions. Moments of functions $g(x)$ of the Bjorken scaling variable x are taken with the standard definition $g(N) \equiv \int_0^1 dx\, x^{N-1} g(x)$. The $x \to 1$ limit corresponds to $N \to \infty$ in moment space.

3.1 Consistency relations for DIS

In this section, we consider the *hadronic* DIS process $p + \phi^* \to X$. From standard factorization theorems at leading twist in Λ/Q, where Λ denotes the QCD scale, we can write the hadronic tensor as

$$W = \sum_i W_{\phi,i}\, f_i\,, \tag{3.1}$$

where i sums over all partonic scattering channels,[4] and f_i denotes the unfactorized (unrenormalized) parton distribution function (PDF) of i in the proton p. Thus f_i contains dimensionally regulated UV divergences. The finite, $\overline{\text{MS}}$ subtracted parton distributions and short-distance coefficients (partonic cross sections) are related to $W_{\phi,i}$, f_i by

$$\tilde{f}_k = Z_{ki} f_i, \qquad W_{\phi,i} = \tilde{C}_{\phi,k} Z_{ki}\,, \tag{3.2}$$

such that

$$W_{\phi,i} f_i = \tilde{C}_{\phi,k} \tilde{f}_k\,. \tag{3.3}$$

Note that the short-distance coefficients $\tilde{C}_{\phi,k}$ have finite limits for $\epsilon \to 0$, but are still d-dimensional. The splitting kernels are given by

$$P_{ij} = -\gamma_{ij} = \frac{dZ_{ik}}{d\ln\mu}(Z^{-1})_{kj}\,. \tag{3.4}$$

[4] In the following, we imply the summation convention over repeated partonic channel indices and often leave out the sum symbol.

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For generic N the leading-twist DIS factorization formula involves hard and collinear physics related to the scales Q and Λ. The latter is non-perturbative and factorized into the PDFs. For large N, the small invariant mass of the final state (see also appendix B) introduces a new scale into the problem, which is also the source of the large logarithms that we wish to sum. The four relevant virtualities are:

- hard, $p^2 = Q^2$
- anti-hardcollinear, $p^2 = Q^2\lambda^2 = Q^2/N$
- collinear, $p^2 = \Lambda^2$
- softcollinear, $p^2 = \Lambda^2\lambda^2 = \Lambda^2/N$

The anti-hardcollinear virtuality arises from the requirement of a small-mass final state X. In the adopted large-momentum frame, its large momentum is in the opposite direction of the incoming proton, hence "anti-hardcollinear". We also need a softcollinear virtuality $\Lambda/N \ll \Lambda$, which accounts for the anomalously small momentum of the target remnant as $x \to 1$ [19]. Note, however, that there is no soft region in DIS.

The calculation of the DIS process is imagined to be strictly factorized into contributions from the different virtualities. A multi-loop diagram is considered as a sum of terms, in which every loop momentum has one of the above virtualities, in the spirit of the strategy of expanding by regions [20]. Each loop is then associated with a factor $(\mu^2/p^2)^\epsilon$ times a function of ϵ, which will usually be singular. The consistency relations follow from the requirement that the sum of all terms is non-singular as $\epsilon \to 0$. We note that dimensional regularization only factorizes regions with different virtualities. It is not sensitive to the scaling of the momentum components separately. However, for the present problem, this will be sufficient to obtain non-trivial consistency constraints.

3.1.1 Leading power

The resummation of leading large-N logarithms at leading power is simple and well-known. We rederive it here from consistency relations and the RGE for the hard function to illustrate the method.

At LP there is no mixing of partonic channels. Only the gluon channel contributes to the DIS cross section. We expand the diagonal gluon channel in α_s according to

$$W_{\phi,g}\, f_g = f_g(\Lambda) \times \sum_n \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{2n}} \sum_{k=0}^{n}\sum_{j=0}^{n} b_{kj}^{(n)}(\epsilon) \left(\frac{\mu^{2n} N^j}{Q^{2k}\Lambda^{2(n-k)}}\right)^{\epsilon} + \mathcal{O}\left(\frac{1}{N}\right). \tag{3.5}$$

Here k denotes the number of hard plus anti-hardcollinear loops, which determines the dependence $Q^{-2k\epsilon}$ on Q, j is the number of anti-hardcollinear and softcollinear loops, which determines the number of times the factor N^ϵ appears. $n-k$ is then the number of collinear plus softcollinear loops related to the PDF. At $\mathcal{O}(\alpha_s^n)$, the leading singularity is $1/\epsilon^{2n}$. This factor has been extracted, such that the coefficients $b_{kj}^{(n)}(\epsilon)$ can be expanded in non-negative powers of ϵ. For the LL resummation, we focus on the leading poles, and need only the constant part $b_{kj}^{(n)} \equiv b_{kj}^{(n)}(0)$. At LP, we also drop the $\mathcal{O}(1/N)$ correction. The above expansion holds when expressed in terms of the dimensionless bare coupling $\mu^{-2\epsilon}\alpha_{s0}$, since some of the poles are related to coupling renormalization. However, since the relation between the bare and renormalized coupling involves at most a single pole per loop, we can identify the expansion parameter with the renormalized $\overline{\text{MS}}$ coupling $\alpha_s \equiv \alpha_s(\mu)$ to leading-pole accuracy.

We regard f_g on the left-hand side as the unrenormalized gluon PDF at the factorization scale μ. To make the dependence on the collinear and softcollinear scale explicit, we relate it to a non-perturbative reference PDF via

$$f_g(\mu) = U_{gg}(\mu) f_g(\Lambda) = f_g(\Lambda)\left[1 + \mathcal{O}(\alpha_s)\right], \tag{3.6}$$

where $U_{gg}(\mu)$, defined by this equation, contains the evolution from the scale Λ to μ. Another way of reading (3.5) is that it represents DIS on a gluon with IR singularities regulated non-dimensionally rather than DIS on a hadron. It is only important that the left-hand side is finite as $\epsilon \to 0$, so all poles on the right-hand side originate from factorization, and have to cancel.

The requirement of pole cancellation implies not only the obvious consistency condition

$$\sum_{k=0}^{n}\sum_{j=0}^{n} b_{kj}^{(n)} = 0\,, \tag{3.7}$$

from the vanishing of the coefficient of the $1/\epsilon^{2n}$ pole. In addition, the coefficients of all terms of the form

$$(\ln N)^r \left(\ln\frac{\Lambda}{Q}\right)^s \times \frac{1}{\epsilon^{2n-r-s}}\,, \tag{3.8}$$

must vanish for $s+r<2n$, $r,s\geq 0$, since other subleading pole terms from the ϵ expansion of $b_{kj}^{(n)}(\epsilon)$ cannot produce the same logarithmically enhanced coefficients. This gives the conditions

$$\sum_{k=0}^{n}\sum_{j=0}^{n} j^r (n-k)^s\, b_{kj}^{(n)} = 0\,. \tag{3.9}$$

Using the binomial expansion of $(n-k)^s$, they are equivalent to the consistency relations

$$\sum_{k=0}^{n}\sum_{j=0}^{n} j^r k^s\, b_{kj}^{(n)} = 0 \qquad \text{for } s+r<2n,\ \ r,s\geq 0\,. \tag{3.10}$$

At order $\mathcal{O}(\alpha_s^n)$ this provides $2n^2+n$ equations, but only n^2+2n of them are linearly independent. There are $(n+1)^2$ coefficients $b_{kj}^{(n)}$, hence at any order in perturbation theory we can determine all $b_{kj}^{(n)}$ through consistency equations in terms of a single remaining one.

A particularly convenient choice is $b_{n0}^{(n)}$, which corresponds to n-loop diagrams with only hard loops. We show below that $b_{n0}^{(n)}$ can be determined from the one-loop hard coefficient $b_{10}^{(1)}$ by solving a RGE, such that the purely hard-loop contribution is given to all orders by

$$W_{\phi,g}^{\text{LP,LL}}\Big|_{\text{hard loops}} = \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right], \tag{3.11}$$

which captures the leading-pole part, denoted by "LL", and implies

$$b_{n0}^{(n)} = (-4C_A)^n \,. \tag{3.12}$$

This provides the single condition required at LP to fix all of the $b_{kj}^{(n)}$. We make the ansatz

$$(W_{\phi,g}\, f_g)^{\mathrm{LP,LL}} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} - \left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}(N^\epsilon - 1)\right] f_g(\Lambda)\,, \tag{3.13}$$

which is finite for $\epsilon \to 0$ and contains only products of factors of the form $(\mu^2/p^2)^\epsilon$, with p^2 any of the four relevant virtualities. This corresponds to

$$\begin{aligned} b_{kj}^{(n)} &= (4C_A)^n \sum_{m_1,m_2,m_3,m_4\geq 0} \frac{(-1)^{m_2}(-1)^{m_3}}{m_1!m_2!m_3!m_4!}\delta_{m_1+m_2,k}\delta_{m_3+m_4,n-k}\delta_{m_1+m_3,j} \\ &= (-1)^{k+j}\,(4C_A)^n \sum_{m=m_0}^{\min(j,k)} \frac{1}{m!(k-m)!(j-m)!(n-k-j+m)!}\,, \end{aligned} \tag{3.14}$$

where $m_0 = \max(0, k+j-n)$. We checked (up to $n = 10$) that this ansatz indeed satisfies the consistency conditions (3.10). Since the system is fully constrained (equal number of free coefficients and linearly independent consistency conditions), this solution is the unique solution of the consistency relations given (3.11).

Clearly, (3.13) factorizes into

$$W_{\phi,g}^{\mathrm{LP,LL}} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^\epsilon - 1)\right], \tag{3.15}$$

$$f_g^{\mathrm{LP,LL}} = \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}(N^\epsilon - 1)\right] f_g(\Lambda)\,, \tag{3.16}$$

where the first expression is the unfactorized partonic cross section, and the second the unfactorized PDF. The latter shows that the gluon PDF in the $x \to 1$ limit must be considered as a two-scale object already at LP, since $f_g^{\mathrm{LP,LL}}$ depends on the softcollinear in addition to the collinear virtuality. Recall that in the derivation of these expressions N^ϵ is treated as an $\mathcal{O}(1)$ quantity that must not be expanded in ϵ. However, by definition of the $\overline{\mathrm{MS}}$ scheme, to define the $\overline{\mathrm{MS}}$ renormalization constants, the pole part is extracted by expanding in ϵ at fixed (large) N. From (3.16) and the requirement that $\tilde{f}_g$ in (3.2) be finite, we obtain

$$Z_{gg}^{\mathrm{LP,LL}} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{\ln N}{\epsilon}\right], \tag{3.17}$$

$$\tilde{C}_{\phi,g} = \exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^\epsilon - 1) - \epsilon \ln N\right)\right]. \tag{3.18}$$

The anomalous dimension in the gluon channel is obtained from

$$\gamma_{gg}(N) = -\left(\frac{d}{d\ln\mu}Z_{gg}\right)Z_{gg}^{-1}\,. \tag{3.19}$$

In the leading (double) pole approximation, the evolution of d-dimensional $\overline{\text{MS}}$ coupling is given by

$$\frac{d\alpha_s}{d\ln\mu} = -2\epsilon\alpha_s\,, \tag{3.20}$$

hence

$$\gamma_{gg}^{\text{LP,LL}}(N) = \frac{\alpha_s C_A}{\pi}\,2\ln N\,, \tag{3.21}$$

with no $(\alpha_s \ln^2 N)^k$ corrections. This corresponds to the well-known fact that the DGLAP kernel for $x \to 1$ at LP is

$$P_{gg}(x) = \frac{2\Gamma_{\text{cusp}}(\alpha_s)}{[1-x]_+} + 2\gamma^g(\alpha_s)\delta(1-x) + \mathcal{O}((1-x)^0)\,, \tag{3.22}$$

with no $\ln^n(1-x)/[1-x]_+$ corrections to the $1/[1-x]_+$ term to any order in α_s.

3.1.2 Derivation of the resummed hard function (3.11)

In the $x \to 1$ limit the QCD part of the Higgs-gluon interaction is closely related to the Sudakov form factor for gluon scattering. In SCET notation (see appendix A), it matches to the operator

$$J^{\text{A0}} = 2g_{\mu\nu}\, n_-\partial\mathcal{A}^{\mu A}_{\perp hc}(sn_-)n_+\partial\mathcal{A}^{\nu A}_{\perp c}(tn_+)\,, \tag{3.23}$$

with a collinear gauge-invariant transverse gluon field in the collinear direction of the initial-state gluon and an anti-hardcollinear one for the outgoing gluon. The square of the hard matching coefficient C^{A0} of this operator contains the large logarithms at LP as $x \to 1$ in the DIS structure functions, which we associated with $W_{\phi,g}^{\text{LP,LL}}|_{\text{hard loops}}$ above.[5] Here we need the resummation of the pole part of the bare coefficient rather than of the large logarithms in the renormalized coefficient.

The anomalous dimension of J^{A0} takes the general form

$$\Gamma(\alpha_s,\mu) \equiv -\frac{dZ}{d\ln\mu}Z^{-1} = \Gamma_{\text{cusp}}(\alpha_s)\ln\frac{Q^2}{\mu^2} + \gamma(\alpha_s)\,. \tag{3.24}$$

With

$$\frac{d\alpha_s}{d\ln\mu} \equiv -2\epsilon\alpha_s + \beta(\alpha_s)\,, \tag{3.25}$$

we can solve (3.24) for [21]

$$\ln Z(\mu) = \int_0^{\alpha_s(\mu)} \frac{d\alpha}{\alpha}\frac{1}{2\epsilon-\beta(\alpha)/\alpha}\left(\Gamma(\alpha,\mu) - \int_0^{\alpha}\frac{d\alpha'}{\alpha'}\frac{2\Gamma_{\text{cusp}}(\alpha')}{2\epsilon-\beta(\alpha')/\alpha'}\right)\,. \tag{3.26}$$

The bare coefficient function is given by

$$C^{\text{A0}}_{\text{bare}} = Z(\mu)C^{\text{A0}}(\mu)\,, \tag{3.27}$$

where $C^{\text{A0}}(\mu)$ is free of poles. The bare coefficient does not depend on μ and is a function of the dimensionless quantities $\alpha_{s0}/Q^{2\epsilon}$ and ϵ. The resummation of the pole part is obtained

[5] See appendix B for a very brief summary of factorization for $x \to 1$ at LP.

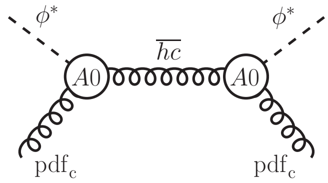

Figure 3. Diagrammatic SCET representation of LP resummation. The Wilson lines attached to SCET operators are set to 1 in this graph.

most easily by choosing $\mu = Q$, in which case $C^{\mathrm{A0}}(Q)$ contains no large logarithms, and by expressing $\alpha_s(Q)$ in terms of the bare coupling α_{s0}.

The cusp anomalous dimension is responsible for the double logarithms. To sum the leading poles, the one-loop approximation suffices. We therefore set

$$\Gamma_{\mathrm{cusp}}(\alpha_s) = \frac{\alpha_s C_A}{\pi}\,, \qquad \gamma(\alpha_s) = 0\,, \qquad \beta(\alpha_s) = 0\,, \tag{3.28}$$

and obtain

$$\ln Z^{\mathrm{LL}}(Q) = -\frac{\alpha_s(Q) C_A}{2\pi}\frac{1}{\epsilon^2} = -\frac{\alpha_{s0} C_A}{2\pi}\frac{1}{\epsilon^2}\frac{1}{Q^{2\epsilon}} = -\frac{\alpha_s(\mu) C_A}{2\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}. \tag{3.29}$$

It is sufficient to use the tree-level coefficient $C^{\mathrm{A0}}(Q) = 1$ to obtain the leading poles. We then find

$$W_{\phi,g}^{\mathrm{LP,LL}}\Big|_{\text{hard loops}} = |C^{\mathrm{A0}}_{\mathrm{bare}}|^2_{\mathrm{LL}} = \exp\left(2\ln Z^{\mathrm{LL}}(Q)\right) = \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right], \tag{3.30}$$

which proves (3.11). The above method can be used to include running coupling and higher-order effects. However, we restrict ourselves to the leading double logarithms here.

3.1.3 SCET intepretation

The SCET interpretation of DIS at LP as $x \to 1$ is sketched in figure 3. A collinear gluon from the PDF is converted into an anti-hardcollinear gluon by the A0 current (3.23), which sources the final-state jet. The figure shows the cross section $W_{\phi,g}\, f_g$ with hard vertices and lines corresponding to the (anti-hard) collinear fields. It does not show the Wilson lines attached to these fields. Since the softcollinear PDF modes enter at LP only through Wilson lines, they do not appear in the graph, despite the fact that the result (3.16) for the resummed PDF shows that they are necessary at LP to achieve pole cancellation. The picture is consistent with the explicit SCET computations [19]. The appearance of softcollinear modes only in Wilson lines is also the reason why they do not appear explicitly in the LP factorization theorem [19], once the standard PDFs are introduced. Their presence in (3.16) leads us to suspect that this will no longer hold at NLP. There is some similarity of this with *collinear functions* in the factorization of the Drell-Yan process near the kinematic threshold. These do not appear in the well-known LP factorization theorem, because soft modes appear only through Wilson lines. However, this no longer holds beyond LP, and collinear functions do appear at NLP [3, 9].

3.2 Next-to-leading power

Having introduced the method for the well-understood case of LP large-N resummation, we proceed to the main subject of this paper, the NLP suppressed off-diagonal quark-Higgs scattering channel. All partonic channels are relevant at NLP, hence we consider the expansion of $\sum_i (W_{\phi,i} f_i)$ in powers of $1/N$. The LP term $W_{\phi,g}^{\rm LP} f_g^{\rm LP}$ was considered before in (3.5). The NLP term in the hadronic cross section consists of

$$\sum_i (W_{\phi,i} f_i)^{\rm NLP} = W_{\phi,q}^{\rm NLP} f_q^{\rm LP} + W_{\phi,\bar{q}}^{\rm NLP} f_{\bar{q}}^{\rm LP} + W_{\phi,g}^{\rm NLP} f_g^{\rm LP} + W_{\phi,g}^{\rm LP} f_g^{\rm NLP} . \tag{3.31}$$

The evolution factors that express the unrenormalized PDF at the scale μ in terms of the PDFs at the initial scale Λ must also be expanded. We generalize (3.6) to

$$f_i(\mu) = U_{ij}(\mu) f_j(\Lambda) , \tag{3.32}$$

and find

$$\begin{aligned} f_g^{\rm LP}(\mu) &= U_{gg}^{\rm LP}(\mu) f_g(\Lambda) , \\ f_q^{\rm LP}(\mu) &= U_{qq}^{\rm LP}(\mu) f_q(\Lambda) \qquad \text{(similarly for } \bar{q}) , \\ f_g^{\rm NLP}(\mu) &= U_{gg}^{\rm NLP}(\mu) f_g(\Lambda) + U_{gq}^{\rm NLP}(\mu) \left(f_q(\Lambda) + f_{\bar{q}}(\Lambda) \right) . \end{aligned} \tag{3.33}$$

The LP leading-pole resummed factor

$$U_{gg}^{\rm LP,LL}(\mu) = \exp \left[-\frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{\Lambda^2} \right)^{\epsilon} (N^{\epsilon} - 1) \right] , \tag{3.34}$$

can be inferred from (3.16). $U_{qq}^{\rm LP,LL}(\mu)$ and $U_{\bar{q}\bar{q}}^{\rm LP,LL}(\mu)$ are obtained by replacing $C_A \to C_F$. The hadronic cross section should be finite for any choice of non-perturbative initial conditions $f_g(\Lambda)$, $f_q(\Lambda)$ and $f_{\bar{q}}(\Lambda)$. For the off-diagonal quark-gluon channel we focus on the terms proportional to $f_q(\Lambda)$, given by

$$\sum_i (W_{\phi,i} f_i)^{\rm NLP} \Big|_{\propto f_q(\Lambda)} = \left(W_{\phi,q}^{\rm NLP} U_{qq}^{\rm LP} + W_{\phi,g}^{\rm LP} U_{gq}^{\rm NLP} \right) f_q(\Lambda) . \tag{3.35}$$

3.2.1 Consistency relations

Assuming that the same hard, anti-hardcollinear, collinear and softcollinear virtualities describe the physics of large-x DIS at NLP, we expand the hadronic cross section as

$$\begin{aligned} \sum_i (W_{\phi,i} f_i)^{\rm NLP} = \; & f_q(\Lambda) \times \frac{1}{N} \sum_{n=1} \left(\frac{\alpha_s}{4\pi} \right)^n \frac{1}{\epsilon^{2n-1}} \sum_{k=0}^{n} \sum_{j=0}^{n} c_{kj}^{(n)}(\epsilon) \left(\frac{\mu^{2n} N^j}{Q^{2k} \Lambda^{2(n-k)}} \right)^{\epsilon} \\ & + f_{\bar{q}}(\Lambda),\, f_g(\Lambda) \quad \text{terms} . \end{aligned} \tag{3.36}$$

Compared to the previous LP expansion formula (3.36), we note the overall NLP factor $1/N$, the power $2n-1$ rather than $2n$ for the leading pole and the absence of a tree term $n = 0$. This follows from the fact that a quark must be radiated into the final state in the off-diagonal quark-gluon channel. This first emission brings a factor of α_s but produces only a single $1/\epsilon$ pole. As mentioned above, the poles must cancel in all channels separately, and we can therefore disregard the $f_{\bar{q}}(\Lambda)$, $f_g(\Lambda)$ terms.[6] In the following we will only be interested in the leading pole at any order, in which case we can replace $c_{kj}^{(n)}(\epsilon)$ by their four-dimensional values $c_{kj}^{(n)} \equiv c_{kj}^{(n)}(0)$.

The similarity of (3.5) and (3.36) implies that the consistency relations from pole cancellation take the same form:

$$\sum_{k=0}^{n}\sum_{j=0}^{n} j^r k^s c_{kj}^{(n)} = 0 \qquad \text{for } s+r < 2n-1,\ r,s \geq 0\,. \tag{3.37}$$

However, the absence of the $1/\epsilon^{2n}$ pole now leads to the condition $s+r<2n-1$ as compared to $s+r<2n$ at LP, (3.10). There are still $(n+1)^2$ coefficients $c_{kj}^{(n)}$ at order n, but (3.37) provides only $2n^2-n$ equations. Moreover, not all of them are linearly independent. We can write $c_{kj}^{(n)}$ as a $(n+1)^2$ dimensional vector $c^{(n)}$ in the compound index $[kj]$ with ordering $00, 01, \ldots 0n, 10, \ldots nn$, and regard $j^r k^s$ as the entries of a $(n+1)^2 \times (2n^2-n)$ matrix $M^{(n)}$ with indices $[kj]$ and $[sr]$ (these ordered as $00, 10, 01, 20, 11, 02, \ldots$). Then (3.37) is expressed as $M^{(n)} c^{(n)} = 0$. For example, for $n=2$, the $2n^2-n=6$ consistency conditions read in matrix form

$$\begin{pmatrix} 1&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&2&2&2 \\ 0&1&2&0&1&2&0&1&2 \\ 0&0&0&1&1&1&4&4&4 \\ 0&0&0&0&1&2&0&2&4 \\ 0&1&4&0&1&4&0&1&4 \end{pmatrix} \cdot \begin{pmatrix} c_{00}^{(2)} \\ c_{01}^{(2)} \\ c_{02}^{(2)} \\ c_{10}^{(2)} \\ c_{11}^{(2)} \\ c_{12}^{(2)} \\ c_{20}^{(2)} \\ c_{21}^{(2)} \\ c_{22}^{(2)} \end{pmatrix} = 0\,. \tag{3.38}$$

The number of linearly independent consistency relations is related to the rank of matrix $M^{(n)}$, which is $(n+1)^2-3$. Hence, the consistency relations allow us to determine all $(n+1)^2$ coefficients $c_{kj}^{(n)}$ in terms of three unknowns at every order n.

3.2.2 Solution

Two of the three "initial conditions" at every n for solving the consistency relations can be fixed trivially. In the absence of collinear and softcollinear loops ($k=n$), there must be at least one anti-hardcollinear loop, since the final state cannot be made up of hard modes for $x \to 1$. This implies

$$c_{n0}^{(n)} = 0\,, \tag{3.39}$$

[6]The $f_g(\Lambda)$ terms can be used to formulate consistency relations for the NLP resummation of the gluon channel. The antiquark scattering terms are completely analogous to the quark terms and need not be considered separately.

for all n. Similarly, without any hard or anti-hardcollinear loops ($k = 0$), the necessary off-diagonal $q \to qg$ splitting always produces a softcollinear quark. Thus there must be at least one softcollinear loop, such that

$$c_{00}^{(n)} = 0 \qquad \text{for all } n\,. \tag{3.40}$$

The third "initial condition" is provided by the Sudakov exponentiation conjecture (2.23). Recall that this refers to the all-hard loop corrections to the square of the $q\phi^* \to qg$ amplitude integrated over the anti-hardcollinear two-particle phase space, which gives the series of terms $c_{n1}^{(n)}$ in present notation. In moment space, we replace $(1-x) \to 1/N$ in (2.23). Expanding in α_s, we obtain

$$W_{\phi,q}\Big|_{q\phi^*\to qg}^{\text{hard}} = \sum_{n=1} \left(\frac{\alpha_s}{4\pi}\right)^n c_{n1}^{(n)} \frac{1}{\epsilon^{2n-1}} \left(\frac{\mu^{2n} N}{Q^{2n}}\right)^{\epsilon}, \tag{3.41}$$

with

$$c_{n1}^{(n)} = \frac{1}{2}(-4)^n \frac{C_F}{C_F - C_A} \frac{C_F^n - C_A^n}{n!} = \frac{(-4)^n}{2n!} C_F \left(C_F^{n-1} + C_F^{n-2} C_A + \cdots + C_A^{n-1}\right). \tag{3.42}$$

This particular finite series of C_F and C_A terms has already been seen in [10, 15].

The consistency equations can now be solved. For example, at $n = 2$, we can rewrite (3.38) as equations for the unknown coefficients as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 1 & 4 & 0 & 1 & 4 & 4 \end{pmatrix} \cdot \begin{pmatrix} c_{01}^{(2)} \\ c_{02}^{(2)} \\ c_{10}^{(2)} \\ c_{11}^{(2)} \\ c_{12}^{(2)} \\ c_{22}^{(2)} \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \\ 2 \\ 1 \end{pmatrix} c_{21}^{(2)}, \tag{3.43}$$

where we used $c_{00}^{(2)} = c_{20}^{(2)} = 0$, and $c_{21}^{(2)}$ is given by (3.42) for $n = 2$. All unknown coefficients are uniquely determined, since the matrix has full rank (equal to 6) and is therefore invertible. At general n we can proceed analogously, and write the $n(2n-1)$ consistency conditions in the form

$$\sum_{k=0}^{n} \sum_{\substack{j=0 \\ (jk)\neq(00),(n0),(n1)}}^{n} j^r k^s c_{kj}^{(n)} = -n^s c_{n1}^{(n)} \qquad \text{for } s + r < 2n-1,\; r, s \geq 0\,, \tag{3.44}$$

where the right-hand side is fixed by (3.42) and the left-hand side can be written as a matrix-vector multiplication with a matrix of dimension $((n+1)^2 - 3) \times n(2n-1)$. This matrix is quadratic only for $n = 1$ and $n = 2$, and has more rows than columns for $n \geq 3$. This means the free coefficients are over-constrained. Nevertheless, as expected from the previous discussion, not all consistency conditions are linearly independent, and the rank of the matrix is such that there is a solution, which is then the unique solution.

Rather than attempting a direct solution of these linear systems, we will guess a suitable ansatz. From (3.35), (3.36), we deduce that we must match

$$\begin{aligned}
&W^{\rm NLP}_{\phi,q}U^{\rm LP}_{qq}+W^{\rm LP}_{\phi,g}U^{\rm NLP}_{gq}\\
&\overset{(3.15),\,(3.34)}{=} W^{\rm NLP}_{\phi,q}\exp\left[-\frac{\alpha_s C_F}{\pi\epsilon^2}\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}(N^\epsilon-1)\right]+\exp\left[\frac{\alpha_s C_A}{\pi\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(N^\epsilon-1)\right]U^{\rm NLP}_{gq}\\
&\overset{!}{=}\frac{1}{N}\sum_{n=1}\left(\frac{\alpha_s}{4\pi}\right)^n\frac{1}{\epsilon^{2n-1}}\sum_{k=0}^{n}\sum_{j=0}^{n}c^{(n)}_{kj}\left(\frac{\mu^{2n}N^j}{Q^{2k}\Lambda^{2(n-k)}}\right)^{\epsilon},
\end{aligned}\tag{3.45}$$

while satisfying (3.42). The form of the LP leading-pole solution (3.13) together with the fact that

$$\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}-\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}(N^\epsilon-1)\tag{3.46}$$

appears to be the unique finite combination of all four regions in the leading-pole approximation, suggests the ansatz

$$\begin{aligned}
W^{\rm NLP}_{\phi,q}U^{\rm LP}_{qq}+W^{\rm LP}_{\phi,g}U^{\rm NLP}_{gq}=n(\epsilon)\times\Bigg\{&\exp\left[\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}-\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}(N^\epsilon-1)\right]\\
&-\exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}-\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}(N^\epsilon-1)\right]\Bigg\},
\end{aligned}\tag{3.47}$$

where $n(\epsilon)$ is a normalization factor yet to be determined.[7] At $\mathcal{O}(\alpha_s)$, (3.37) implies $c^{(1)}_{01}+c^{(1)}_{11}=0$, given that $c^{(1)}_{00}=c^{(1)}_{10}=0$. Expanding (3.47) to $\mathcal{O}(\alpha_s)$ and matching it to (3.45) gives

$$n(\epsilon)\,\frac{\alpha_s}{\pi}\frac{C_F-C_A}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}-\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}(N^\epsilon-1)\overset{!}{=}\frac{1}{N}\frac{\alpha_s}{4\pi}\frac{1}{\epsilon}c^{(1)}_{11}\left\{\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}-\left(\frac{\mu^2}{\Lambda^2}\right)^{\epsilon}\right\}N^\epsilon,\tag{3.48}$$

which yields

$$n(\epsilon)=\frac{c^{(1)}_{11}}{4N}\frac{1}{C_F-C_A}\frac{\epsilon N^\epsilon}{N^\epsilon-1}\overset{(3.42)}{=}-\frac{1}{2N}\frac{C_F}{C_F-C_A}\frac{\epsilon N^\epsilon}{N^\epsilon-1}\,.\tag{3.49}$$

With the normalization determined, (3.47) reproduces all $c^{(n)}_{n1}$ or (2.23). Since (3.47) is finite as $\epsilon\to 0$, and since the content of consistency relations is the finiteness of the physical cross section assuming (2.23), (3.47) provides the unique solution.

[7]The relative factor between the two terms follows, because there is no $\mathcal{O}(\alpha_s^0)$ term in the off-diagonal channel.

Given that $W_{\phi,q}^{\rm NLP}$ must not depend on (μ^2/Λ^2), while $U_{gq}^{\rm NLP}$ must not depend on (μ^2/Q^2), the solution (3.47) implies

$$\begin{aligned} W_{\phi,q}^{\rm NLP,LL} &= -\frac{1}{2N}\frac{C_F}{C_F-C_A}\frac{\epsilon N^\epsilon}{N^\epsilon-1}\left(\exp\left[\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon (N^\epsilon-1)\right]\right. \\ &\qquad \left. -\exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon (N^\epsilon-1)\right]\right), \end{aligned} \tag{3.50}$$

$$\begin{aligned} U_{gq}^{\rm NLP,LL} &= -\frac{1}{2N}\frac{C_F}{C_F-C_A}\frac{\epsilon N^\epsilon}{N^\epsilon-1}\left(\exp\left[-\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{\Lambda^2}\right)^\epsilon (N^\epsilon-1)\right]\right. \\ &\qquad \left. -\exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{\Lambda^2}\right)^\epsilon (N^\epsilon-1)\right]\right). \end{aligned} \tag{3.51}$$

The first of these equations reproduces eq. (17) in [10] for $\mu = Q$ and $C_A = 0$ as assumed there, and therefore proves and generalizes the conjectured all-order structure of the full partonic cross section. In addition, the dependence of $W_{\phi,q}^{\rm NLP,LL}$ on C_F and C_A is consistent with the colour structure conjectured in eq. (13) of [10].

It is remarkable that in the leading-pole approximation, the full result follows from the exponentiation conjecture for the hard-only amplitude (2.23) by a simple substitution. Let us define

$$A \equiv \frac{\alpha_s(C_F-C_A)}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon, \qquad S \equiv \frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon. \tag{3.52}$$

Then the solution of the consistency equations in terms of the hard-only amplitude (2.23) can be summarized in moment space as

$$\begin{aligned} W_{\phi,q}\Big|^{\rm hard}_{q\phi^*\to qg} &= \frac{1}{N}\frac{\alpha_s C_F}{2\pi\epsilon}\left(\frac{\mu^2 N}{Q^2}\right)^\epsilon \exp\left[-S\right] \times \frac{\exp(-A)-1}{A} \\ \longrightarrow \quad W_{\phi,q}^{\rm NLP,LL} &= -\frac{1}{N}\frac{\alpha_s C_F}{2\pi\epsilon}\left(\frac{\mu^2 N}{Q^2}\right)^\epsilon \exp\left[S\,(N^\epsilon-1)\right] \\ &\qquad \times \frac{\exp(A\,(N^\epsilon-1))-1}{A\,(N^\epsilon-1)}, \end{aligned} \tag{3.53}$$

i.e. $A\to A\,(1-N^\epsilon)$, $S\to S\,(1-N^\epsilon)$. The appearance of the factor $(N^\epsilon-1)$ is characteristic of the leading-pole solution. The prefactor of the Sudakov factors accounts for the anti-hardcollinear $\mathcal{O}(\alpha_s)$ contribution that must always be present at NLP.

3.2.3 DGLAP kernel and coefficient function

To determine the resummed off-diagonal splitting function and the $\overline{\rm MS}$-subtracted short-distance partonic cross section, we decompose the unfactorized partonic cross section $W_{\phi,q}^{\rm NLP,LL}$ into its finite and divergent parts. From (3.2) we deduce

$$W_{\phi,q}^{\rm NLP} = \tilde{C}_{\phi,q}^{\rm NLP}\, Z_{qq}^{\rm LP} + \tilde{C}_{\phi,g}^{\rm LP} Z_{gq}^{\rm NLP}, \tag{3.54}$$

where $Z_{qq}^{\rm LP}$ and $C_{\phi,g}^{\rm LP}$ are known from (3.17) (replacing C_A by C_F) and (3.18), respectively, and the NLP off-diagonal factors $Z_{gq}^{\rm NLP}$, $\tilde{C}_{\phi,q}^{\rm NLP}$ are to be determined.

From the structure of the LP expressions (3.17), (3.18) it is apparent that the split into pole and finite part in the exponents must be done according to

$$\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon (N^\epsilon-1) = \frac{\ln N}{\epsilon} + \frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^\epsilon (N^\epsilon-1) - \epsilon\ln N\right\}. \tag{3.55}$$

It will be convenient to introduce the abbreviations

$$w \equiv -\epsilon\ln N, \qquad a = \frac{\alpha_s}{\pi}(C_F-C_A)\ln^2 N\,, \tag{3.56}$$

$$\widehat{S}_i = \frac{\alpha_s C_i}{\pi}\frac{1}{\epsilon^2}\left\{\left(\frac{\mu^2}{Q^2}\right)^\epsilon (N^\epsilon-1) - \epsilon\ln N\right\}, \qquad i = A, F\,, \tag{3.57}$$

which allow us to write (3.50) as

$$W_{\phi,q}^{\rm NLP,LL} = \frac{1}{2N\ln N}\frac{C_F}{C_F-C_A}\exp\left[\frac{\alpha_s C_F}{\pi}\frac{\ln N}{\epsilon}\right]\frac{w}{e^w-1}\left(e^{a/w}e^{\widehat{S}_A} - e^{\widehat{S}_F}\right). \tag{3.58}$$

Next we note that $w/(e^w-1)$, $e^{\widehat{S}_A}$ and $e^{\widehat{S}_F}$ do not have poles in $1/\epsilon$, while $\exp\left[\frac{\alpha_s C_F}{\pi}\frac{\ln N}{\epsilon}\right]$ matches $Z_{qq}^{\rm LP,LL}$, hence to obtain $\tilde{C}_{\phi,q}^{\rm NLP}$ in (3.54), we must separate

$$F(w,a) \equiv \frac{we^{a/w}}{e^w-1} \tag{3.59}$$

into its pole and finite part. Using

$$F(w,0) = \frac{w}{e^w-1} = \sum_{n=0}^\infty \frac{B_n}{n!}w^n\,, \tag{3.60}$$

where

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \ldots \tag{3.61}$$

are the Bernoulli numbers, we obtain by expanding $e^{a/w}$ that

$$F_{\rm pole}(w,a) = \sum_{k\geq 1}\frac{1}{w^k}\sum_{n\geq 0}\frac{B_n}{n!(n+k)!}a^{n+k}\,, \tag{3.62}$$

$$F_{\rm fin}(w,a) = \sum_{k\geq 0}w^k\sum_{n\geq k}\frac{B_n}{n!(n-k)!}a^{n-k}\,, \tag{3.63}$$

where the sums over n on the right-hand side can be regarded as a generalization of Bernoulli polynomials. Inserting this decomposition into (3.58) and matching the resulting expression to (3.54), we identify the splitting kernel and short-distance coefficient as

$$Z_{gq}^{\rm NLP,LL} = \frac{1}{2N\ln N}\frac{C_F}{C_F-C_A}\exp\left[\frac{\alpha_s C_F}{\pi}\frac{\ln N}{\epsilon}\right]F_{\rm pole}(w,a)\,, \tag{3.64}$$

$$\begin{aligned}\tilde{C}_{\phi,q}^{\rm NLP,LL} = \frac{1}{2N\ln N}\frac{C_F}{C_F-C_A}&\Bigg(F_{\rm fin}(w,a)\exp\left[\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{Q^2}\right)^\epsilon(N^\epsilon-1)-\epsilon\ln N\right)\right]\\ &-\frac{w}{e^w-1}\exp\left[\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{Q^2}\right)^\epsilon(N^\epsilon-1)-\epsilon\ln N\right)\right]\Bigg).\end{aligned} \tag{3.65}$$

Note that by construction $Z_{gq}^{\rm NLP,LL}$ is a pure pole term, hence corresponds to the $\overline{\rm MS}$ PDF renormalization factor, while the short-distance coefficient is finite as it should be. Indeed, for $\epsilon \to 0$ at fixed N, which implies $w \to 0$, we find

$$\tilde{C}_{\phi,q}^{\rm NLP,LL}\Big|_{\epsilon\to 0} = \frac{1}{2N\ln N}\frac{C_F}{C_F - C_A}\Bigg(\mathcal{B}_0(a)\exp\left[C_A\frac{\alpha_s}{\pi}\left(\frac{1}{2}\ln^2 N + \ln N \ln\frac{\mu^2}{Q^2}\right)\right] \\ - \exp\left[\frac{\alpha_s C_F}{\pi}\left(\frac{1}{2}\ln^2 N + \ln N\ln\frac{\mu^2}{Q^2}\right)\right]\Bigg)\,, \tag{3.66}$$

which agrees eq. (29) of [10] for $\mu = Q$, and generalizes it to $\mu \neq Q$. Here

$$\mathcal{B}_0(a) = F_{\rm fin}(0,a) \tag{3.67}$$

is the Borel transform of the generating function $F(w,0)$ of the Bernoulli numbers, already defined in (1.2).

The anomalous dimension in the $q \to gq$ splitting channel is obtained from

$$\gamma_{gq}(N) = -\sum_{k=g,q,\bar{q}}\left(\frac{dZ_{gk}}{d\ln\mu}\right)(Z^{-1})_{kq} = \gamma_{gg}^{\rm LP}\,\frac{-Z_{gq}^{\rm NLP}}{Z_{qq}^{\rm LP}} - \left(\frac{dZ_{gq}^{\rm NLP}}{d\ln\mu}\right)(Z^{-1})_{qq}^{\rm LP}\,, \tag{3.68}$$

where the second equality holds to NLP accuracy and uses the vanishing of the off-diagonal terms at LP. Inserting the leading-pole resummed results for the Z-factors and the LP leading-pole anomalous dimension (3.21) in the gluon-gluon channel, we obtain after a short calculation

$$\gamma_{gq}^{\rm NLP,LL}(N) = \frac{1}{N}\frac{\alpha_s C_F}{\pi}\left[F_{\rm pole}(w,a) - w\,\frac{d}{da}F_{\rm pole}(w,a)\right] = -\frac{1}{N}\frac{\alpha_s C_F}{\pi}\,\mathcal{B}_0(a)\,, \tag{3.69}$$

which has no poles as it must be and proves (1.1) first given in [10]. We close this derivation with three observations:

- Comparison of the summed large-N anomalous dimensions (3.21), (3.69) shows that there is an infinite series of (double) logarithmic terms only for the off-diagonal channel. This implies that not only the short-distance coefficient, but also the anomalous dimension is a two-scale object in the off-diagonal channel. The double logarithms are associated with the colour charge change $C_F - C_A$ of the partons that carry large momentum. The absence of large logarithms in the diagonal channel seems related to the fact that the energetic particles have the same colour charge.
- The Sudakov exponentiation conjecture was originally proposed and explored in [15] for the case of e^+e^- or Higgs-decay event-shape distributions in the two-jet limit, when the final state includes a soft quark. There are interesting differences and similarities between the DIS and event-shape case, which we elaborate on in appendix E. The solution of the consistency relations takes a form similar to (3.50). The Bernoulli series does not arise for event shapes. Event shapes are infrared finite, and the Bernoulli numbers arise in DIS from the need to factorize the pole part of (3.50) to obtain the renormalized short-distance coefficient and parton distribution, as seen above.

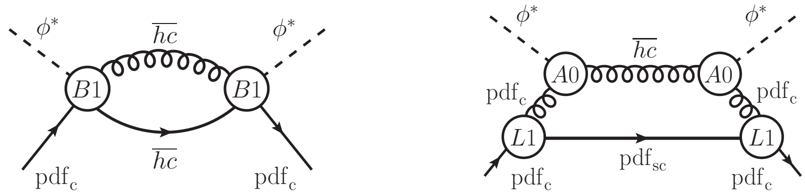

Figure 4. SCET representation of the content of (3.35) for quark-Higgs scattering at NLP as $x \to 1$. Wilson lines are set to 1.

- Let us comment on similarities and differences compared to the derivation of (3.69) presented in [10]. Both derivations use finiteness and pole cancellation. In addition, [10] conjectures a specific form of the full unfactorized partonic cross section (including hard and hardcollinear contributions to all orders) as well as a particular assumption for the colour structure (as stated in eqs. (13) and (14) in [10]). We require (2.22) as an input for the derivation, which involves a single region only (specifically the hard region), that we consider as a weaker assumption compared to those used in [10]. In addition, the exponentiation conjecture (2.22) lends itself to a derivation based on RGE methods, that we turn to in section 4. Finally, we obtain the Bernoulli series in (3.69) by an algebraic derivation in a closed form.

3.2.4 SCET interpretation

The SCET interpretation of DIS at NLP as $x \to 1$ in the off-diagonal channel is sketched in figure 4. The figure shows

$$\left(W^{\mathrm{NLP}}_{\phi,q} U^{\mathrm{LP}}_{qq} + W^{\mathrm{LP}}_{\phi,g} U^{\mathrm{NLP}}_{gq}\right) f_q(\Lambda) \tag{3.70}$$

from (3.35), the external quark line representing the quark PDF $f_q(\Lambda)$. As before, Wilson lines of whatever fields are set to 1.

The left diagram in figure 4 represents the first term in this expression and describes the hard scattering of a quark off the Higgs boson. The corresponding hard vertex is of the B1 type with field content $\mathcal{A}_{\overline{hc}\perp}\bar{\chi}_{\overline{hc}}\chi_c$ (see appendix A), and the presence of two anti-hardcollinear fields provides the power suppression. The circled operator vertex labelled "B1" in the graph represents the hard subgraph, which corresponds to figure 1 at tree level, and to the hard region of the diagrams shown in figure 2 at one-loop order, respectively. At this vertex the incoming quark is converted into the anti-hardcollinear quark and gluon in the final state. The right diagram describes the second term in (3.70). Here the hard scattering occurs through the LP gluon-Higgs scattering vertex of the A0 type with field content $\mathcal{A}_{\overline{hc}\perp}\mathcal{A}_{c\perp}$. In the $x \to 1$ limit, the gluon in the $q \to gq$ splitting carries almost the entire momentum of the initial quark, leaving a remnant softcollinear q. The interaction that couples soft(collinear) quarks to collinear modes is power-suppressed, and part of the NLP $\mathcal{L}^{(1)}$ SCET Lagrangian [22, 23]. In (3.70) this process is part of U^{NLP}_{qg}, the off-diagonal evolution of the PDF in the $x \to 1$ limit. Now there is a softcollinear mode in the final state, which does not arise from a Wilson line, which is shown explicitly in the right diagram in figure 4.

It is evident that both diagrams in the figure are related. If the gluon propagators labelled pdf_c in the right diagram became hard, the diagram would turn into the left one. What prevents standard SCET factorization methods to be applied to this situation, is that the convolution of the short-distance coefficient of the B1 operator with the final-state jet function is divergent after SCET renormalization of the hard and jet functions. The above treatment through consistency relations circumvents this problem, since it is d-dimensional to the end. The divergent convolutions do not appear, but they are implicit, and done in d dimensions, where they exist. In the LL NLP resummation of the diagonal quark-quark and gluon-gluon channels for Drell-Yan production near threshold with SCET methods [3, 4] these complications did not appear, but they do at the next-to-leading logarithmic order [9].

The appearance of an endpoint divergence and the breakdown of standard SCET factorization points to the emergence of a new scale in the problem, which requires a refactorization of the B1-type SCET operator. In the following, we show how this idea can be implemented. The resummation of logarithms from the new scale will allow us to derive the exponentiation conjecture that was used above as a boundary condition to solve the consistency equations for the NLP LL resummation of the quark-gluon channel.

4 Derivation of the soft-quark Sudakov factor

In the previous sections, we have seen that the SCET interpretation of NLP off-diagonal DIS involves a B1-type current, i.e. an operator constructed from one collinear and two anti-hardcollinear SCET fields, where the latter two are light-like separated. The operator creates the anti-hardcollinear final-state quark and gluon carrying momentum fractions z and $1-z$, respectively, of the total anti-hardcollinear final-state momentum. The peculiarity of our problem manifests itself in the fact that the convolution of tree-level matching coefficient and the anomalous dimension is not well defined, because the integral exhibits an endpoint divergence as $z \to 0$, i.e. when the quark becomes soft. This prevents the standard application of the RGE to the summation of the large logarithms. Instead, we must first consider the limit $z \to 0$ and sum $\ln z$ terms, which become large in the singular region, to all orders, while still working in d dimensions. This goes beyond the standard paradigm of SCET, where the large component of collinear momenta are assumed to be of the order of the hard scale, hence the momentum fraction z appearing in a B-type current is treated as an order one parameter $z = \mathcal{O}(1)$. z is nevertheless integrated from 0 to 1. This is justified since the contribution to the integral from an interval $[0, \delta]$ can be made arbitrarily small as $\delta \to 0$, as long as the matching coefficient is less singular than $1/z$.

A similar problem has recently been discussed in [24], where it was noted that the singular part of the hard matching coefficient can be included into the definition of the operator. This leads to the concept of singular and regular B1 operators. In that case, the singular B1 operator was related by reparameterization invariance (RPI) to the leading-power A0 operator. The singular B1 operator can then be combined with the time-ordered product of the leading-power A0 operator and the NLP Lagrangian to obtain a well-defined operator, whose evolution is governed by the standard RGE. The analogous construction does not work for DIS considered here, as there is no RPI relation of the NLP B1 operator to the LP one. This fact is easily understood from the observation that the LP and NLP operators relevant to the dicussion of off-diagonal DIS have different fermion numbers in the collinear *and* anti-collinear sectors, unlike in [24], where the NLP current was obtained by adding a gluon building block to the LP current.

From figure 1 it is evident that the intermediate gluon propagator is proportional to $1/z$ as $z \to 0$. When the momentum fraction of the outgoing quark is $\mathcal{O}(1)$, the intermediate gluon propagator is hard, and it has to be integrated out, which gives the matching coefficient of the B1 operator that behaves as $1/z$ for $z \to 0$. In this limit, however, the virtuality of the intermediate propagator approaches zero, and the intermediate gluon should still be present as a dynamical mode in the effective theory (EFT) rather than having been integrated out. This causes a breakdown of the standard application of SCET to this problem, and its failure to reproduce the IR singularity structure of full QCD correctly. We are therefore forced to revise the structure of the modes in the presence of an endpoint-singular matching coefficient.

4.1 Scales relevant for the endpoint-singular B1 operator

To cure the lack of proper scale separation due to the endpoint singularity, we must identify the intermediate scales and modes relevant to the $z \to 0$ limit, which goes beyond the $\mathrm{SCET_I}$ paradigm. Only then, the soft-quark Sudakov expoentiation conjecture (2.22) can be derived with EFT methods.

To understand the scales relevant for our problem, we perform a method-of-region analysis [20] of the integrals appearing in the one-loop diagrams shown in figure 2. We assume that the external momenta are slightly off-shell so that we can identify all modes which contribute to the loop integrals when there is a dimensionful infrared scale. As elsewhere in this paper, we focus on the terms giving rise to the leading logarithms. This means that we want to capture terms which diverge in the limit when the quark momentum fraction goes to zero, $z \to 0$, and we focus on the double poles in ϵ or single ϵ poles multiplied by $\ln z$. This allows us to drop many terms and perform simplifications such that the final expression for the leading one-loop correction takes the form

$$\mathcal{A} = \mathcal{M}^{(1)}_{q\phi^* \to qg} \times 2i\, g_s^2 \left(\mathbf{T}_1 \cdot \mathbf{T}_0\, I_1 + \mathbf{T}_2 \cdot \mathbf{T}_0\, I_2 + \mathbf{T}_1 \cdot \mathbf{T}_2\, I_{12} \right), \tag{4.1}$$

with $\mathcal{M}^{(1)}_{q\phi^* \to qg}$ the Born amplitude for the process $q(p) + \phi^*(q) \to q(p_1) + g(p_2)$. The result can be expressed in terms of the three master integrals

$$I_1 = 2p_1 \cdot p\, \tilde{\mu}^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k-p_1)^2} \frac{1}{(k-p)^2}\,, \tag{4.2}$$

$$I_2 = 2p_2 \cdot p\, \tilde{\mu}^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k-p_2)^2} \frac{1}{(k-p)^2}\,, \tag{4.3}$$

$$I_{12} = 2p_2 \cdot p\, \tilde{\mu}^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k-p_2)^2} \frac{1}{(k-p+p_1)^2}\,. \tag{4.4}$$

The integral I_1 corresponds to diagram (1) of figure 2, while the sum of diagrams (2) and (5) can be expressed in terms of the integrals I_2 and I_{12}. The remaining diagrams in figure 2 do not contribute to the terms that we consider here. The modes relevant to the endpoint problem can be identified by expanding these integrals. To perform the expansion by regions we introduce the variable z as defined in (2.7) as a new power counting parameter and take the limits $1 \gg z \gg \lambda$. We work to leading order in the z and λ expansion. We assume the following scaling for the external momenta p_1, p_2, p, respectively:

$$z\text{-anti-softcollinear} \qquad p_1 \sim Q(\lambda^2, \sqrt{z}\lambda, z) \qquad p_1^2 \sim z\lambda^2 Q^2\,, \tag{4.5}$$

$$\text{anti-hardcollinear} \qquad p_2 \sim Q(\lambda^2, \lambda, 1) \qquad p_2^2 \sim \lambda^2 Q^2\,, \tag{4.6}$$

$$\text{hardcollinear} \qquad p \sim Q(1, \lambda, \lambda^2) \qquad p^2 \sim \lambda^2 Q^2\,, \tag{4.7}$$

with component notation $l \sim (n_+l, l_\perp, n_-l)$. With off-shell external momenta all three integrals are IR and UV finite. When the off-shell regulator is set to zero, this computation corresponds to standard one-loop matching of the B1 operator in SCET$_\mathrm{I}$. Note that we choose the momentum p to have hardcollinear rather than collinear virtuality here to facilitate the interpretation of the result as a SCET$_\mathrm{I}$ matching computation — only subsequently the hardcollinear modes shall be matched on the collinear modes corresponding to the external initial-state momentum in DIS as discussed before. In the following, we focus only on the pole parts of contributing regions.

We begin with the I_2 integral. This is a standard vertex integral exhibiting a double logarithmic enhancement, which can be decomposed into the following loop momentum regions:

- hard $k \sim (n_+k, k_\perp, n_-k) \sim Q\,(1,1,1)$

$$I_2|_h = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}\right] + \mathcal{O}\left(\epsilon^0\right) \tag{4.8}$$

- hardcollinear $k \sim Q\left(1, \lambda, \lambda^2\right)$

$$I_2|_{hc} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{-p^2}\right] + \mathcal{O}(\epsilon^0) \tag{4.9}$$

- anti-hardcollinear $k \sim Q\left(\lambda^2, \lambda, 1\right)$

$$I_2|_{\overline{hc}} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{-p_2^2}\right] + \mathcal{O}(\epsilon^0) \tag{4.10}$$

- soft $k \sim Q\left(\lambda^2, \lambda^2, \lambda^2\right)$

$$I_2|_s = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2\mu^2}{\left(-p_2^2\right)\left(-p^2\right)}\right] + \mathcal{O}(\epsilon^0) \tag{4.11}$$

As expected, $I_2|_h + I_2|_{hc} + I_2|_{\overline{hc}} + I_2|_s = \mathcal{O}(\epsilon^0)$, and this integral is reproduced by the standard SCET$_\mathrm{I}$ modes. The soft modes appear here because we assumed that momentum

p has hardcollinear virtuality. The complete EFT description must take into account that the external momentum p has collinear virtuality, and in this case, the soft modes ought to be replaced by the softcollinear modes.

The integral I_1 has a similar mode structure, but in this case the hard mode results in a scaleless expression and gives a vanishing contribution. Instead, a new z-hardcollinear mode appears. It is obtained by combining z-anti-softcollinear (4.5) and hardcollinear (4.7) momenta, in analogy with the hard mode being a sum of hardcollinear (4.7) and anti-hardcollinear (4.6) momenta. We find following decomposition of the I_1 integral into modes:

- z-hardcollinear $k \sim Q\left(1, z^{1/2}, z\right)$

$$I_1|_{z\text{-}hc} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{zQ^2}{\mu^2}\right] + \mathcal{O}(\epsilon^0) \tag{4.12}$$

- hardcollinear $k \sim Q\left(1, \lambda, \lambda^2\right)$

$$I_1|_{hc} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{-p^2}\right] + \mathcal{O}(\epsilon^0) \tag{4.13}$$

- z-anti-softcollinear $k \sim Q\left(\lambda^2, z^{1/2}\lambda, z\right)$

$$I_1|_{z\text{-}\overline{sc}} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{-p_1^2}\right] + \mathcal{O}(\epsilon^0) \tag{4.14}$$

- soft $k \sim Q\left(\lambda^2, \lambda^2, \lambda^2\right)$

$$I_1|_s = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{zQ^2\mu^2}{(-p_1^2)\,(-p^2)}\right] + \mathcal{O}(\epsilon^0) \tag{4.15}$$

These results show the emergence of the new scale $\sqrt{z}Q$ related to the endpoint singularity in the z-integral (2.8). The new scale is not directly related to the scales present in the factorization of the DIS process at LP. Rather, it is generated dynamically at NLP, due to the breakdown of naïve factorization for the B1 current with a singular matching coefficient. As the momentum fraction of a collinear quark becomes parametrically small, the scalar product $p_1 \cdot p$ cannot be treated as being of the same order as $p_2 \cdot p$.

The presence of the new scale manifests itself in a particularly subtle manner for the I_{12} integral. Before we proceed to consider the relevant regions for I_{12}, let us look at the on-shell result. For I_1 and I_2, the on-shell results are equal to the hard and z-hardcollinear contributions, respectively. Thus, they are both single scale integrals, though the scale for I_1 is zQ^2, while for I_2 it is Q^2. On the contrary, the on-shell result for I_{12} contains a large logarithm

$$I_{12}|_{\text{on-shell}} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\ln z\right] + \mathcal{O}(\epsilon^0), \tag{4.16}$$

which cannot be removed by any choice of μ. This observation is troublesome as the on-shell integral is already a two-scale object, and it needs to be factorized to achieve a proper EFT interpretation of the result. Returning to the case of off-shell external momenta, we find contributions from the following integration regions:

- hard $k \sim Q\,(1,1,1)$

$$I_{12}|_h = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}\right]+\mathcal{O}(\epsilon^0) \tag{4.17}$$

- z-hardcollinear $k \sim Q\left(1, z^{1/2}, z\right)$

$$I_{12}|_{z\text{-}hc} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{zQ^2}{\mu^2}\right]+\mathcal{O}(\epsilon^0) \tag{4.18}$$

- anti-hardcollinear $k \sim Q\left(\lambda^2, \lambda, 1\right)$

$$I_{12}|_{\overline{hc}} = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2}{-p_2^2}\right]+\mathcal{O}(\epsilon^0) \tag{4.19}$$

- z-anti-softcollinear $k \sim Q\left(\lambda^2, z^{1/2}\lambda, z\right)$

$$I_{12}|_{z\text{-}\overline{sc}} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{-zp_2^2}\right]+\mathcal{O}(\epsilon^0) \tag{4.20}$$

We observe that $I_{12}|_{\text{on-shell}} = I_{12}|_h + I_{12}|_{z\text{-}hc}$, and the large logarithm appears as the result of a cancellation between the hard and z-hardcollinear contributions. When considering the limit $z \to 0$ it is thus crucial to factorize (2.22) into these two contributions. The cusp anomalous dimension governing the LL resummation should not itself contain large logarithms, which cannot be removed by *some* choice of the scale μ. If that is not the case, then the anomalous dimension should itself be resummed or factorized. In the following, we perform a refactorization such that the logarithms from the hard scale may be resummed independently of the logarithms whose origin is the z-hardcollinear scale. Such refactorization is necessary to correctly sum *all* the logarithms of z.

4.2 Resummation

Having understood the scales relevant to the one-loop result, we attempt to construct the EFT framework to derive (2.22). At this point, we restrict ourselves to LL accuracy and do not claim that the construction presented here could be used to perform resummation at NLL accuracy or beyond. As the analysis of regions showed, we must distinguish the hard scale Q^2 and the z-hardcollinear scale zQ^2. This suggests that the matching of QCD to SCET_I should be separated into two steps: first we integrate out the hard modes, then we the remove z-hardcollinear modes to obtain the EFT at the hardcollinear scale $\lambda^2 Q^2$.

The intermediate gluon propagator in the tree diagram shown in figure 1 has z-hardcollinear virtuality and thus it must be associated with the dynamical degrees of freedom at the z-hardcollinear scale. The matching equation at the hard scale reads

$$F^{\mu\nu}_A F^A_{\mu\nu} = C^{\text{A}0}(Q^2,\mu^2)\, J^{\text{A}0}\,, \tag{4.21}$$

where the LP SCET_I current

$$J^{\text{A}0} = 2g^{\mu\nu} n_-\partial\mathcal{A}^{A,z\text{-}\overline{hc}}_{\perp\mu}\; n_+\partial\mathcal{A}^{A,z\text{-}hc}_{\perp\nu} \tag{4.22}$$

represents the point-like coupling of the Higgs boson to two gluons in SCET, and $C^{\mathrm{A0}}(Q^2,\mu^2)=1$ at tree level. In this theory, the diagram in figure 1 is represented by the matrix element of the time-ordered product

$$C^{\mathrm{A0}}\left(Q^2,\mu^2\right)\langle q(p_1)g(p_2)|\int d^dx\, T\left\{J^{\mathrm{A0}},\mathcal{L}^{(1)}_{\xi q_{z\text{-}\overline{sc}}}(x)\right\}|q(p)\rangle\,. \tag{4.23}$$

The Lagrangian mediating the z-anti-softcollinear quark interaction with z-hardcollinear modes is

$$\mathcal{L}^{(1)}_{\xi q_{z\text{-}\overline{sc}}}(x)=\overline{q}_{z\text{-}\overline{sc}}(x_-)\,W^\dagger_{z\text{-}hc}i\slashed{D}_{\perp,z\text{-}hc}\xi_{z\text{-}hc}+\text{h.c.} \tag{4.24}$$

Let us note a peculiarity which distinguishes this problem from the one discussed in [24]. The z-anti-softcollinear quark is generated by a subleading-power Lagrangian insertion in the collinear sector. This observation suggests that the endpoint-singular contribution in a collinear sector should be combined with a time-ordered product in the corresponding anti-collinear sector, while in [24], the time-ordered product and the singular current belong to the same direction. The hard matching coefficient C^{A0} of the operator (4.22) satisfies a standard RGE (3.24), which at LL accuracy reads

$$\frac{d}{d\ln\mu}C^{\mathrm{A0}}(Q^2,\mu^2)=\Gamma_{\mathrm{A0}}\,C^{\mathrm{A0}}(Q^2,\mu^2)=\frac{\alpha_sC_A}{\pi}\ln\frac{Q^2}{\mu^2}\,C^{\mathrm{A0}}(Q^2,\mu^2)\,. \tag{4.25}$$

After this first matching, the SCET with z-hardcollinear modes is defined at the zQ^2 scale. To describe DIS factorization in the limit $x\to 1$, we need an EFT at the scale λ^2Q^2 where $\lambda^2\sim 1-x$. This EFT should contain only modes with hardcollinear virtuality or lower, as is the case for SCET factorization of DIS at LP. Besides, we need to include z-anti-softcollinear modes as separate entities represented in the EFT by their own set of fields. The detailed construction of this EFT is left for future work. Fortunately, it is not needed to perform LL resummation, since we already possess all the essential ingredients. We discuss the resummation in the following, and then provide some partly speculative comments on the SCET with z-anti-softcollinear modes in the following subsection.

We assume the existence of an operator J^{B1}, such that we can match the time-ordered product (4.23)

$$\int d^dx\, T\left\{J^{\mathrm{A0}},\mathcal{L}^{(1)}_{\xi q_{z\text{-}\overline{sc}}}(x)\right\}=D^{\mathrm{B1}}(zQ^2,\mu^2)\,J^{\mathrm{B1}} \tag{4.26}$$

to this J^{B1}, which must be composed of a hardcollinear quark, a z-anti-softcollinear quark and an anti-hardcollinear gluon field. While the matching of the z-hardcollinear quark field to the hardcollinear quark field is trivial, the non-trivial content of this equation is the reinterpretation of the anti-collinear sector, where zQ^2 now appears as the large scale on which the matching coefficient and anomalous dimension of the operator can depend. As this operator is supposed to reproduce the IR poles of the QCD amplitude, its renormalization factor can be deduced from our previous computation of the amplitude (4.1) in the on-shell limit. Using (4.8), (4.12) and (4.16) we find that the LL UV divergence of this operator can be removed by the counterterm

$$Z_{\mathrm{B1,B1}}=1-\frac{\alpha_s}{2\pi}\left[\left(C_F-\frac{C_A}{2}\right)\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{zQ^2}{\mu^2}\right]+\frac{C_A}{2}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}\right]+\frac{1}{\epsilon}\frac{C_A}{2}\ln z\right]. \tag{4.27}$$

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We implicitly assumed that there is no operator mixing at the LL level involved in the matching equation (4.26), i.e. the complete IR pole in QCD is reproduced in the EFT by $Z_{\mathrm{B1,B1}}$. Using

$$\frac{d}{d\ln\mu}C^{\mathrm{A0}}\left(Q^2,\mu^2\right)D^{\mathrm{B1}}\left(zQ^2,\mu^2\right)J^{\mathrm{B1}}=0\,, \tag{4.28}$$

we find that the matching coefficient obeys at LL accuracy the RGE

$$\begin{aligned}\frac{d}{d\ln\mu}D^{\mathrm{B1}}\left(zQ^2,\mu^2\right)&=-\left[\left(\frac{d}{d\ln\mu}Z_{\mathrm{B1,B1}}\right)Z^{-1}_{\mathrm{B1,B1}}+\Gamma_{\mathrm{A0}}\right]D^{\mathrm{B1}}\left(zQ^2,\mu^2\right)\\&=\frac{\alpha_s}{\pi}\left(C_F-C_A\right)\ln\frac{zQ^2}{\mu^2}\,D^{\mathrm{B1}}\left(zQ^2,\mu^2\right)\equiv\Gamma_{\mathrm{B1}}D^{\mathrm{B1}}\left(zQ^2,\mu^2\right).\end{aligned} \tag{4.29}$$

It is pivotal that Γ_{B1} depends only on the scale zQ^2. The fact that the other logarithms dropped out serves as a consistency check of this construction. The anomalous dimension of the D^{B1} coefficient is proportional to C_F-C_A, reflecting earlier observations that double logarithmic enhancement [25] and endpoint divergence [15] vanish in $\mathcal{N}=1$ supersymmetric QCD.

It is straightforward to solve (4.25) and (4.29). Recalling that we work with bare, d-dimensional objects for the boundary terms for the d-dimensional consistency relations, we solve for the bare coefficients, and find

$$\begin{aligned}\left[C^{\mathrm{A0}}\left(Q^2,\mu^2\right)\right]_{\mathrm{bare}}&=C^{\mathrm{A0}}\left(Q^2,Q^2\right)\exp\left[-\frac{\alpha_sC_A}{2\pi}\frac{1}{\epsilon^2}\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\right],\\\left[D^{\mathrm{B1}}\left(zQ^2,\mu^2\right)\right]_{\mathrm{bare}}&=D^{\mathrm{B1}}\left(zQ^2,zQ^2\right)\exp\left[-\frac{\alpha_s}{2\pi}\left(C_F-C_A\right)\frac{1}{\epsilon^2}\left(\frac{zQ^2}{\mu^2}\right)^{-\epsilon}\right].\end{aligned} \tag{4.30}$$

The product of the square of these two coefficients gives (2.22), proving the exponentiation of soft-quark Sudakov logarithms conjectured in [15].

4.3 Tentative EFT interpretation

While it was not essential to know the exact form of the operator J^{B1} to achieve LL resummation, we nonetheless comment on the possible structure of SCET with z-antisoftcollinear modes. We expect that the operator J^{B1} has the form[8]

$$J^{\mathrm{B1}}=\overline{\chi}_{hc}\gamma_\mu\left[in_-\partial_{\overline{hc}}\mathcal{A}^\mu_{\perp\overline{hc}}\right]\left[\frac{1}{in_-\partial_{z\text{-}\overline{sc}}}\chi_{z\text{-}\overline{sc}}\right]. \tag{4.31}$$

In this theory, we must decompose the large component of the momentum in the anticollinear sector into a hardcollinear part, which is of the order of Q, and a residual momentum of the order of zQ. Above we accordingly decomposed the anti-hardcollinear derivative as $in_-\partial=in_-\partial_{\overline{hc}}+in_-\partial_{z\text{-}\overline{sc}}$. Unlike in $\mathrm{SCET_{II}}$, where the soft modes do not

[8]For aesthetic reasons we write the Hermitian conjugate operators corresponding to antiquark scattering here and in appendix D.

interact with the collinear modes, here the z-anti-softcollinear modes can still interact with the anti-hardcollinear modes. This interaction is responsible for the part of the anomalous dimension proportional to $\mathbf{T}_1 \cdot \mathbf{T}_2$.

To compute the anomalous dimension of (4.31), one would need to derive the Lagrangian for this z-SCET, which we leave for future work. Instead, extending the notion of singular $\mathrm{SCET_I}$ operators in [24], we consider the family of $\mathrm{SCET_I}$ operators

$$J_{\mathrm{B1}}^{(n)} = \overline{\chi}_{hc}(0)\gamma_\mu \left[(in_-\partial)^{1+n\epsilon}\mathcal{A}^\mu_{\perp\overline{hc}}\left[\left(\frac{1}{in_-\partial}\right)^{1+n\epsilon}\chi_{\overline{hc}}\right]\right](0), \tag{4.32}$$

which are defined at a single point $x = 0$, but contain the singular part of the matching coefficient in its definition through the inverse derivative. Computing the mixing of these operators into themselves together with the assumption that the relevant scale is zQ^2, we find again the product of the resummed coefficient functions in (4.30). We provide this alternative derivation in appendix D. In addition, we recover the renormalization factor (4.27) as the standard $\overline{\mathrm{MS}}$ renormalization constant of $J_{\mathrm{B1}}^{(0)}$, if we expand in ϵ, in which case all $J_{\mathrm{B1}}^{(n)}$ collapse to $J_{\mathrm{B1}}^{(0)}$. This gives us confidence that a SCET operator whose anomalous dimension is equal to the QCD IR poles can in principle be constructed.

The next step of the EFT construction involves integrating out the hardcollinear scale and matching hardcollinear fields onto the collinear fields which describe the modes inside the PDF. The hierarchy of scales is such that soft modes are also integrated out, and instead softcollinear modes appear. Similar to the leading power, the anti-hardcollinear fields in the operator (4.31) give rise to the so-called jet function at the level of the amplitude squared. Besides, the cross-section contains a contribution due to the time-ordered product of the LP current and subleading-power soft-collinear Lagrangian, as shown in figure 4.

5 Conclusion

Contrary to the expectation that the resummation of large logarithms in $1 - x$ in the expansion of the off-diagonal parton scattering channels in deep-inelastic scattering or Drell-Yan production near threshold ought to be simpler than for the diagonal channels, since the former vanish at leading power in the expansion in $1 - x$, the converse is true. This can already be seen from the fact that even the DGLAP splitting functions contain an infinite series of double logarithms for quark-gluon or gluon-quark transitions, for which the formula (1.1) was found [10] a decade ago, but a method for systematic improvements beyond leading logarithms is still missing. The difficulty appears to be related to the emission of a soft quark rather than gluon in the parton splitting, or more generally to the change of colour charge of the energetic partons in the splitting.

The present work was motivated by the desire to understand (1.1) from the perspective of scale separation and effective field theory as a necessary step towards a general resummation formalism at next-to-leading power. In the first part of the paper, we showed that given that the relevant modes are hard, collinear, softcollinear and anti-hardcollinear, the leading-logarithmic resummation of off-diagonal deep-inelastic parton scattering as $x \to 1$ follows from the resummation of the purely hard virtual contribution to the process. The condition that all $1/\epsilon$ poles in dimensional regularization cancel between the various regions is sufficient to bootstrap the full solution. For the resummed purely hard contribution, which acts as a "boundary condition" to solve these consistency conditions, we adapted the "soft-quark Sudakov" exponentiation conjecture [15] from event shapes in e^+e^- collisions to DIS. In this way we derived the expression for the resummed off-diagonal DGLAP kernel in terms of the series of Bernoulli numbers found previously [10] directly from algebraic all-order expressions, that is, without extrapolating the structure of an iteratively generated finite series of terms.

The second part of the paper is concerned with the derivation of the "soft-quark Sudakov" exponentiation of the hard function. The hard function can be alternatively interpreted either as the light-cone momentum distribution amplitude of the final-state qg pair in the off-diagonal $2 \to 2$ scattering process, or the matching coefficient of a B1-type collinear operator in SCET. The crucial feature is that the amplitude is singular as $1/z$ when the quark momentum fraction $z \to 0$. The failure of standard Sudakov resummation or SCET factorization is caused by the divergence of the convolution of the hard amplitude with the final-state jet function and the emergence of the new scale $\sqrt{z}Q$. Based on this observation, we derive the previously conjectured exponentiation formula through the refactorization of certain power-suppressed operators in SCET which have endpoint-singular matching coefficients. The renormalization group equations then exhibit the origin of the peculiar $C_F - C_A$ colour factor through an additional exponent related to the scale $\sqrt{z}Q$.

We cannot offer a precise effective field theory formulation for this refactorization at this point, and, furthermore must note that in this treatment, the problem of endpoint-divergent convolutions is side-stepped by effectively regulating them dimensionally, since the consistency relations and the "boundary condition" for their solution are formulated in d dimensions for unrenormalized objects. A truly satisfactory solution would express the result as finite convolutions of properly renormalized functions. Nevertheless, we believe that the connection between various ideas made manifest here for the first time should provide useful insight on NLP resummations. In particular, the formalization of the refactorization of SCET operators appears to be a promising avenue for the systematic understanding of resummation at next-to-leading power.

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A SCET conventions and relevant modes

- We define light-like reference vectors $n_\pm$ with $n_\pm^2 = 0$, $n_+ \cdot n_- = 2$. Any four-momentum can be decomposed as

$$p^\mu = \frac{1}{2} n_+ p \, n_-^\mu + \frac{1}{2} n_- p \, n_+^\mu + p_\perp^\mu \,. \tag{A.1}$$

  Collinear modes have large $n_+ p$, anti-collinear modes large $n_- p$.

- SCET operators are conveniently constructed from collinear quark and gluon fields, which are invariant under collinear gauge transformations

$$\chi_c(x) = (W_c^\dagger \xi_c)(x), \qquad \mathcal{A}_{\perp c}^\mu(x) = W_c^\dagger(x) \left[iD_{\perp c}^\mu(x) \, W_c(x) \right], \tag{A.2}$$

  where

$$W_c(x) = \mathbf{P} \exp \left[i g_s \int_{-\infty}^0 ds \, n_+ A_c \, (x + s n_+) \right] \tag{A.3}$$

  is the collinear Wilson line. Similar definitions apply to hardcollinear fields. For anticollinear fields n_+ and n_- are interchanged.

- In SCET operators, J_i^{Xn} denotes the product of collinear fields for direction i, here collinear or anticollinear. $X = A, B, \ldots$ refers to the number $1, 2, \ldots$ of collinear fields from (A.2), and n to the power suppression relative to the leading term consisting of a single field without derivative [16, 22]. In this paper, when we refer to an A0 or B1 operator, we refer to field operators involving a product of collinear fields in directions n_+ and n_- with field content $\mathcal{A}_{\perp \overline{hc}}^\nu \mathcal{A}_{\perp c}^{\mu A}$, for which we employ the short-hand notation $J^{\rm A0}$, and $[\mathcal{A}_{\perp \overline{hc}}^\nu \bar{\chi}_{\overline{hc}}] \chi_c$, which we refer to as $J^{\rm B1}$. Both arise in the matching of the Higgs-gluon coupling (2.2) to SCET. The explicit forms are

$$J^{\rm A0}(t, \bar{t}\,) = 2 g_{\mu\nu} \, n_- \partial \mathcal{A}_{\overline{hc}\perp}^{\nu A}(\bar{t} n_-) \, n_+ \partial \mathcal{A}_{c\perp}^{\mu A}(t n_+) \,, \tag{A.4}$$

$$J^{\rm B1}(t, \bar{t}_1, \bar{t}_2) = \frac{g_{\mu\nu}}{2} \left[n_- \partial \mathcal{A}_{\overline{hc}\perp}^{\nu A}(\bar{t}_1 n_-) \right] \left[\bar{\chi}_{\overline{hc}}(\bar{t}_2 n_-) \frac{\overleftarrow{1}}{i n_- \partial} \right] i g_s T^A \gamma_\perp^\mu \, \chi_c(t n_+) \,. \tag{A.5}$$

  The B1 operator as given is the one that appears in section 3. The one in section 4 looks similar but its precise mode content is different and the two B1 operators must be carefully distinguished. See table 1 below for a summary of modes and their abbreviations.

- The following scaling and power counting variables are used in this work: $\lambda \sim \sqrt{1-x} \ll 1$ related to factorization DIS at large x; $\eta \sim \Lambda/Q \ll \lambda$ related to the twist expansion. The QCD scale Λ appears in modes for the non-perturbative PDFs. We consider large-x factorization at NLP, but always work at LP in the twist expansion parameter η. For the refactorization of the B1 operator in section 4, we also need to consider $z \equiv n_- p_1 / n_-(p_1 + p_2) \ll 1$, where p_1 is the momentum of the quark, which becomes soft.

  The scalings assigned to the momentum modes used throughout the paper are summarized in table 1.

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Name	$(n_+l, l_\perp, n_-l)$	virtuality l^2
hard $[h]$	$Q(1,1,1)$	Q^2
z-hardcollinear $[z\text{-}hc]$	$Q(1,\sqrt{z},z)$	$z\,Q^2$
z-anti-hardcollinear $[z\text{-}\overline{hc}]$	$Q(z,\sqrt{z},1)$	$z\,Q^2$
z-soft $[z\text{-}s]$	$Q(z,z,z)$	$z^2\,Q^2$
z-anti-softcollinear $[z\text{-}\overline{sc}]$	$Q(\lambda^2,\sqrt{z}\,\lambda,z)$	$z\,\lambda^2\,Q^2$
hardcollinear $[hc]$	$Q(1,\lambda,\lambda^2)$	$\lambda^2\,Q^2$
anti-hardcollinear $[\overline{hc}]$	$Q(\lambda^2,\lambda,1)$	$\lambda^2\,Q^2$
soft $[s]$	$Q(\lambda^2,\lambda^2,\lambda^2)$	$\lambda^4\,Q^2$
collinear $[c]$	$Q(1,\eta,\eta^2)$	$\eta^2\,Q^2$
softcollinear $[sc]$	$Q(\lambda^2,\lambda\,\eta,\eta^2)$	$\lambda^2\,\eta^2\,Q^2$

Table 1. Scaling of the momentum modes relevant for DIS.

B DIS at $x \to 1$

We briefly summarize some results and definitions for DIS off a scalar particle and factorization at large x at LP here.

B.1 DIS off a virtual scalar

We consider DIS of a particle N off a Higgs boson,

$$\phi^*(q) + N(P) \to X(P')\,, \tag{B.1}$$

as represented in figure 5. The large momentum transfer Q and the Bjorken scaling variable x are defined by

$$Q^2 = -q^2, \qquad x = \frac{Q^2}{2P\cdot q}. \tag{B.2}$$

Partons in particle N carry momentum fraction ξ, such that $p = \xi P$, $0 < \xi < 1$. DIS mediated by the exchange of a scalar particle (dubbed Higgs boson) occurs via the effective gluon-gluon-scalar coupling (2.2), where the coupling κ for an actual Higgs boson would be given by

$$\kappa(m_t,\mu) = \frac{\alpha_s(\mu)}{6\pi v} C_t(m_t,\mu) \tag{B.3}$$

with

$$C_t(m_t,\mu) = 1 + \frac{\alpha_s(\mu)}{4\pi}\big(5C_A - 3C_F\big) + \mathcal{O}(\alpha_s^2). \tag{B.4}$$

The DIS structure function W_ϕ is defined as[9]

$$W_\phi = \frac{1}{8\pi Q^2}\int d^4x\, e^{iq\cdot x}\,\langle N(P)\big|\big[G^A_{\mu\nu}G^{\mu\nu A}\big](x)\big[G^B_{\rho\sigma}G^{\rho\sigma B}\big](0)\big|N(P)\rangle\,. \tag{B.5}$$

[9]We define W_ϕ with an additional factor of $1/Q^2$ compared to the more common definition for DIS of an off-shell photon, to compensate for the dimensionful coupling $\kappa \sim 1/v$. An average over the spin and colour of the state $N(P)$ is implicitly understood when taking the matrix element.

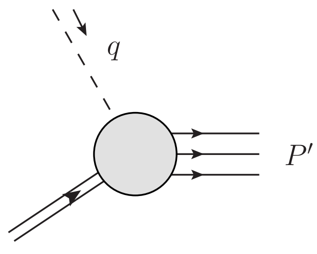

Figure 5. DIS process mediated by a scalar boson, as in (B.1).

The QCD factorization theorem relates the structure function W_ϕ to the partonic short-distance coefficients $\tilde{C}_{\phi,i}(x)$ by means of the convolution

$$W_\phi(x) = \int_x^1 \frac{d\xi}{\xi} \left\{ \tilde{f}_g\left(\frac{x}{\xi}\right) \tilde{C}_{\phi,g}(\xi) + \sum_q \tilde{f}_q\left(\frac{x}{\xi}\right) \tilde{C}_{\phi,q}(\xi) \right\} \tag{B.6}$$

with renormalized PDFs $\tilde{f}_i$ of parton i in N. The notation follows (3.1), (3.2) of the main text. There we often refer to unfactorized (unrenormalized) partonic structure functions $W_{\phi,i}(x)$ and PDFs f_i, related to the above by (3.2). Eq. (B.6) also holds with $\tilde{f}_i \otimes \tilde{C}_{\phi,i} \to f_i \otimes W_{\phi,i}$. The equation also applies to the case when the particle N is itself a quark or gluon, but then the left-hand side is IR divergent for massless, on-shell partons. When the IR divergences are regulated non-dimensionally with a regulator introducing the scale Λ, the consistency arguments based on pole cancellations used in the main text apply to this partonic scattering. If dimensional regularization is employed, the unfactorized partonic structure functions $W_{\phi,i}(x)$ do not change but the unrenormalized PDFs $f_i(x)$ become trivial, because the loop integrals are scaleless, and the left-hand side of (B.6) is simply $W_{\phi,i}$.

The hadronic structure function is related to the phase-space integrated, initial-state spin- and colour-averaged and final-state spin- and colour-summed scattering amplitude as

$$\int d\Phi_X \, |\mathcal{M}_{N\phi^* \to X}|^2 = 2\pi\kappa^2 Q^2 \, W_\phi \,. \tag{B.7}$$

The same relation applies to the partonic structure functions for the scattering of gluons and quarks. The respective lowest order contributions in powers of the strong coupling are obtained from

$$\int d\Phi_1 \big|\mathcal{M}_{\phi^* g \to g}\big|^2_{\text{tree}} = 2\pi\kappa^2 Q^2 \, W_{\phi,g}\big|_{\mathcal{O}(\alpha_s^0)}, \tag{B.8}$$

$$\int d\Phi_2 \big|\mathcal{M}_{\phi^* q \to qg}\big|^2_{\text{tree}} = 2\pi\kappa^2 Q^2 \, W_{\phi,q}\big|_{\mathcal{O}(\alpha_s)} \,. \tag{B.9}$$

The two-particle phase space $d\Phi_2$ has been defined in (2.6), and $d\Phi_1$ denotes the d-dimensional one-particle phase space. The tree-level contribution (see diagram (a) in figure 6) for gluon scattering is

$$\big|\mathcal{M}_{\phi^* g \to g}\big|^2_{\text{tree}} = 4\kappa^2 (1-\epsilon) \, (p \cdot q)^2 \,, \tag{B.10}$$

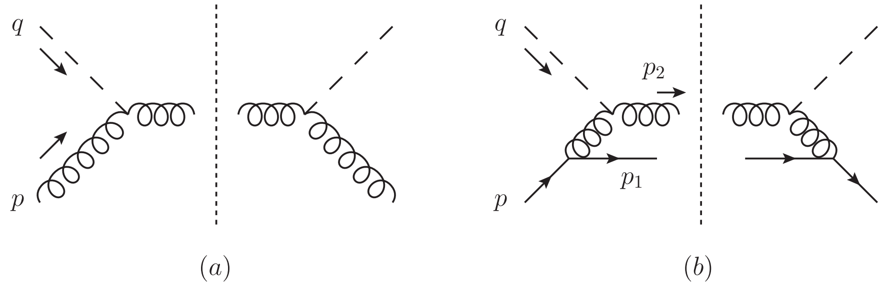

Figure 6. Partonic contribution to DIS with scalar exchange: (a) LO diagram; (b) NLO contribution in the quark-gluon channel.

which inserted in (B.8) gives

$$W_{\phi,g}(x) = \frac{1-\epsilon}{x}\,\delta(1-x) + \mathcal{O}(\alpha_s)\,. \tag{B.11}$$

The quark-scattering channel starts contributing at $\mathcal{O}(\alpha_s)$ with diagram (b) in figure 6. The spin- and colour-averaged/summed matrix element squared expressed in terms of the variable z defined in (2.7) reads

$$\left|\mathcal{M}_{\phi^* q\to qg}\right|^2 = 4\kappa^2 g_s^2 C_F\,\frac{1-\epsilon}{2}\,\frac{Q^2}{x}\,\frac{\bar{z}^2}{z} + \mathcal{O}(\alpha_s^2)\,. \tag{B.12}$$

Inserting this expression into (B.9), we get

$$\begin{aligned} W_{\phi,q}(x)\big|_{\mathcal{O}(\alpha_s)} &= -\frac{\alpha_s C_F}{2\pi}\,\frac{1}{x}\left(\frac{\mu^2}{s_{qg}}\right)^{\epsilon}\frac{(1-\epsilon)\left(1-\frac{\epsilon}{2}\right)}{\epsilon(1-2\epsilon)}\,\frac{e^{\gamma_E\epsilon}\,\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &= \frac{\alpha_s C_F}{2\pi}\left[-\frac{1}{\epsilon} - \frac{1}{2} - \ln\left(\frac{\mu^2}{Q^2(1-x)}\right) + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\lambda^2)\,, \end{aligned} \tag{B.13}$$

with s_{qg} as defined in (2.10).

B.2 Large x

In this paper we focus on the threshold region $x \to 1$. This region is characterized by the fact that the scattering leaves a soft target nucleon (or parton, depending on whether we consider the hadronic or partonic threshold) and a jet-like final state X with parametrically small invariant mass squared $p_X^2 = Q^2(1-x)/x \ll Q^2$. The existence of two scales for $x \to 1$ is the basis of the factorization of the partonic structure functions $W_{\phi,i}(x)$ into a hard and jet function [31–33]. For the structure function W_ϕ, focusing on the leading gluon scattering channel, such factorization takes the form

$$W_\phi(x) = H(Q^2,\mu)\int_x^1 \frac{d\xi}{\xi}\,J\!\left(Q^2\frac{1-\xi}{\xi},\mu\right)\frac{x}{\xi}f_g\!\left(\frac{x}{\xi},\mu\right), \tag{B.14}$$

valid to leading power in $\lambda \sim (1-x)$ and in $\eta \sim \Lambda/Q$. We assume $\lambda \gg \eta$, in which case the hard and jet functions in (B.14) can be calculated in perturbation theory. A similar

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equation holds for the *partonic* structure functions $W_{\phi,i}(x)$ themselves, as discussed above, if we interpret f_g as the distribution of gluons in parton i.

The factorization theorem (B.14) can also be derived within SCET [19]. Near threshold the parton undergoing the hard scattering has collinear momentum scaling $p = \xi P \sim Q(1, 0, \eta^2)$ and carries away almost all momentum of the initial state $N(p)$. The target remnant is then made out of partons which must have softcollinear momenta, i.e. $p_{\text{remnant}} \sim Q(\lambda^2, \lambda\eta, \eta^2)$. This explains the need for softcollinear modes in the PDFs, as listed in table 1. The energetic parton scattering off the Higgs boson (or virtual photon in standard DIS) is converted into a jet of partons with anti-hardcollinear momentum, $p_{\overline{hc}} \sim Q(\lambda^2, \lambda, 1)$. The factorization theorem (B.14) is derived by constructing the sequence

$$\text{QCD} \quad \to \quad \text{SCET}(\overline{hc}, c, sc) \quad \to \quad \text{SCET}(c, sc) \tag{B.15}$$

of effective theories, where in the first matching step the hard modes are integrated out, and the QCD gluon-gluon-scalar interaction is matched onto A0, B1 etc. SCET$(\overline{hc}, c, sc)$ operators built from the collinear gauge-invariant building blocks (see appendix A). In the second step, the jet function emerges as the matching coefficient containing the anti-hardcollinear final-state, leaving a parton distribution made up of collinear modes and the softcollinear target remnant modes. At LP, only an A0-type current is required, and the hard function is given by the square of its short-distance coefficient,

$$H(Q^2, \mu) = |C^{\text{A0}}(Q^2, \mu)|^2 \,. \tag{B.16}$$

More important for the present work is the observation [19] that the softcollinear mode appears only through a Wilson line in the definition of the PDF for $x \to 1$. It is then possible to identify this PDF with the standard PDF. The two-step matching scheme (B.15) should be expected to hold beyond the LP. However, when (B.14) is naively generalized to NLP, the convolutions of generalized renormalized hard and jet functions diverge. As discussed in the main text, this requires, at least for the present, a partly d-dimensional treatment and a refactorization within SCET$(\overline{hc}, c, sc)$ to generate the correct large-x logarithms that would otherwise be missed.

C Alternative derivation of the LP solution (3.13)

There is a simpler way to obtain the leading-power leading-pole expression (3.13) for $(W_{\phi,g}\, f_g)^{\text{LP,LL}}$, which bypasses the combinatorially involved solution for the coefficients $b_{kj}^{(n)}$ in the consistency relation.

At any N the DIS factorization theorem implies multiplicative factorization. At LP in $1/N$, only the gluon channel contributes. We can therefore write the expansion for the logarithm of the DIS cross section (3.5) as

$$\begin{aligned} \ln\left(W_{\phi,g}\, f_g\right) = &\sum_{n=1} \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{n+1}} \sum_{j=0}^{n} \left[t_j^{(n)}(\epsilon) \left(\frac{\mu^{2n} N^j}{Q^{2n}}\right)^\epsilon + f_j^{(n)}(\epsilon) \left(\frac{\mu^{2n} N^j}{\Lambda^{2n}}\right)^\epsilon \right] \\ &+ \ln f_g(\Lambda) + \mathcal{O}\left(\frac{1}{N}\right) , \end{aligned} \tag{C.1}$$

using that $W_{\phi,g}$ cannot depend on Λ, and f_g cannot depend on Q. The form of this expansion contains the non-trivial statement that for the logarithm of the cross section the highest pole at $\mathcal{O}(\alpha_s^n)$ is $1/\epsilon^{n+1}$.

For the leading poles, we can drop the ϵ-dependence of the coefficients $t_j^{(n)}(\epsilon)$, $f_j^{(n)}(\epsilon)$. Instead of the $(n+1)^2$ coefficients $b_{kj}^{(n)}$ at $\mathcal{O}(\alpha_s^n)$, we now have only $2(n+1)$. There are $2n+1$ consistency relations from the requirement of pole cancellation,

$$\sum_{j=0} j^r (t_j^{(n)} + f_j^{(n)}) = 0 \qquad \text{for} \qquad r = 0, \ldots, n\,, \tag{C.2}$$

$$\sum_{j=0} j^r f_j^{(n)} = 0 \qquad \text{for} \qquad r = 0, \ldots, n-1\,, \tag{C.3}$$

leaving one undetermined coefficient per order n, which we choose to be $t_0^{(n)}$. The solution to the consistency relations is

$$t_j^{(n)} = (-1)^j \frac{n!}{j!(n-j)!}\, t_0^{(n)}, \qquad f_j^{(n)} = -t_j^{(n)}\,, \tag{C.4}$$

resulting in

$$\ln W_{\phi,g}^{\mathrm{LP,LL}} = \sum_{n=1} \left(\frac{\alpha_s}{4\pi}\right)^n \frac{1}{\epsilon^{n+1}} (-1)^n \left(\frac{\mu^2}{Q^2}\right)^{n\epsilon} (N^\epsilon - 1)^n\, t_0^{(n)}\,, \tag{C.5}$$

and a corresponding expression for $\ln f_g^{\mathrm{LP,LL}}$ with $Q \to \Lambda$ and an overall minus sign. Comparison with the resummed hard-only expression (3.11) implies that the logarithm of $W_{\phi,g}^{\mathrm{LP,LL}}$ is one-loop exact, that is $t_0^{(1)} = -4C_A$ and $t_0^{(n)} = 0$, $n > 1$. This conclusion can also be reached directly, without the explicit resummed result for $W_{\phi,g}^{\mathrm{LP,LL}}$, from the requirement that $\gamma_{gg}^{\mathrm{LP}}(N)$ should have at most a single power of $\ln N$ at any order in α_s, and the observation that $\gamma_{gg}^{\mathrm{LP}}(N)$ is related to the coefficient of the single pole in (C.5). We then recover the previous results (3.15), (3.16).

D Alternative derivation of the resummed singular B1 current

In this appendix we present an alternative derivation of the exponentiation conjecture (2.22) for the momentum distribution $\mathcal{P}_{qg,\mathrm{hard}}(s_{qg}, z)$ in the limit $z \to 0$ that is complementary to the presentation in section 4. Both rely on the observation made in the first part of section 4 concerning the relevant regions for $z \to 0$. However, instead of considering a "refactorization" into hard and z-hardcollinear regions as in section 4, the derivation presented here solely relies on the SCET$_{\mathrm{I}}$ description of DIS for large x involving collinear, anti-hardcollinear and softcollinear modes.

We start from the SCET$_{\mathrm{I}}$ description of the $q\phi^* \to qg$ scattering process depicted in figure 1 assuming that the incoming q has collinear scaling, and the outgoing q and g are both anti-hardcollinear. The relevant SCET B1 current has field content $\bar{\chi}_c\, \mathcal{A}_{\overline{hc},\perp} \chi_{\overline{hc}}$, and its tree-level matching coefficient diverges as $1/z$ for $z \to 0$, where z is the momentum fraction of the anti-hardcollinear quark. The momentum distribution $\mathcal{P}_{qg,\mathrm{hard}}(s_{qg}, z)$ is related to the square of the B1 matching coefficient (note that one factor of $1/z$ cancels when computing the matrix element squared for $q\phi^* \to qg$).

As a first attempt one may naively apply RG evolution to the B1 matching coefficient. Keeping only the cusp part of its anomalous dimension (that is diagonal in Lorentz and spinor indices as well as with respect to the momentum fraction)

$$\Gamma^{\rm cusp}_{\rm B1,non\text{-}singular} = \frac{\alpha_s}{\pi}\left(\mathbf{T}_1\cdot\mathbf{T}_0 \ln\frac{\mu^2}{zQ^2} + \mathbf{T}_2\cdot\mathbf{T}_0 \ln\frac{\mu^2}{\bar{z}Q^2}\right), \tag{D.1}$$

and performing a d-dimensional RG evolution analogously as described in section 3.1.2 yields precisely the exponential factors involving $\mathbf{T}_1\cdot\mathbf{T}_0$ and $\mathbf{T}_2\cdot\mathbf{T}_0$ in (2.22) (for $z\to 0$, i.e. setting $\bar{z}\to 1$), but misses the contribution involving $\mathbf{T}_1\cdot\mathbf{T}_2$. The reason is that the one-loop anomalous dimension for the B1 operator with generic momentum fractions of the anti-hardcollinear quark and gluon [16] contains contributions that would diverge in four dimensions, when convoluted with a matching coefficient that goes like $1/z$.

To avoid the problem of divergent convolutions, we retain a d-dimensional description, and define a singular B1 current analogous to [24], that absorbs the $1/z$ factor

$$J^{(0)}_{\rm B1} = \overline{\chi}_c(0)\gamma_\mu \left[in_-\partial \mathcal{A}^\mu_{\perp\overline{hc}}\left[\frac{1}{in_-\partial}\chi_{\overline{hc}}\right]\right](0)\,. \tag{D.2}$$

This operator essentially agrees with the one considered in section 4. However, in the present discussion, we do not consider a further refactorization. Also note that, in contrast to the singular B1 current considered in [24], the inverse derivative acts on the quark building block instead of the gluon building block. For the latter case, the matching coefficient of the singular B1 current was linked to the leading power coefficient by reparameterization invariance, such that its anomalous dimension coincides with the one of the corresponding LP current. This relation does not exist in the present case.

In order to find the anomalous dimension of $J^{(0)}_{\rm B1}$, we compute its off-shell regulated one-loop matrix element in an external state with momentum fraction z of the anti-hardcollinear quark. We are interested in the double-pole part in the limit $z\to 0$, when counting factors of z^ϵ as order one. Apart from the expected cusp part in accordance with (D.1), one finds an additional piece involving a factor $\alpha_s\,\mathbf{T}_1\cdot\mathbf{T}_2\,(z^{-\epsilon}-1)/z$. When expanding for $\epsilon\to 0$ this gives a factor $1/z\times\ln z$ that cannot be interpreted as a renormalization of the singular current $J^{(0)}_{\rm B1}$ due to the additional factor of $\ln z$. Instead, we may interpret this result as an operator mixing, requiring us to introduce an additional singular operator. At higher orders we expect contributions of the form $1/z\times\ln^n z$ and a d-dimensional z-dependence of the form $z^{-1-n\epsilon}$. This prompts us to consider a tower of singular operators, defined in d dimensions as

$$J^{(n)}_{\rm B1} = \overline{\chi}_c\,(0)\,\gamma_\mu\left[(in_-\partial)^{1+n\epsilon}\mathcal{A}^\mu{}_{\perp\overline{hc}}\left[\left(\frac{1}{in_-\partial}\right)^{1+n\epsilon}\chi_{\overline{hc}}\right]\right](0)\,. \tag{D.3}$$

The off-shell regulated one-loop matrix element, including the sum of collinear, anti-hardcollinear and softcollinear loops is given by

$$
\begin{aligned}
&\langle \bar{q}_{\overline{hc}}(p_1) g_{\overline{hc}}(p_2)| J_{\rm B1}^{(n)} |\bar{q}_c(p)\rangle_{\text{1-loop}} \\
&= \frac{\alpha_s}{2\pi}\frac{1}{\epsilon^2}\Bigg\{ \mathbf{T}_1\cdot\mathbf{T}_0 \left[\left(\frac{\mu^2 z Q^2}{(-p_1^2)(-p^2)}\right)^\epsilon - \left(\frac{\mu^2}{-p_1^2}\right)^\epsilon - \left(\frac{\mu^2}{-p^2}\right)^\epsilon \right] \\
&\quad + \mathbf{T}_2\cdot\mathbf{T}_0 \left[\left(\frac{\mu^2 \bar{z} Q^2}{(-p_2^2)(-p^2)}\right)^\epsilon - \left(\frac{\mu^2}{-p_2^2}\right)^\epsilon - \left(\frac{\mu^2}{-p^2}\right)^\epsilon \right] \\
&\quad + \mathbf{T}_1\cdot\mathbf{T}_2 \left(\frac{\mu^2}{-p_2^2}\right)^\epsilon \left[\frac{1}{z^\epsilon} - 1\right] \Bigg\} \frac{1}{z^{1+n\epsilon}}\, g_s \bar{v}_c(p) \not{\mathcal{A}}_{a\perp\overline{hc}} t^a v_{\overline{hc}}(p_1) + \mathcal{O}\left(z^0\right) \,, \qquad \text{(D.4)}
\end{aligned}
$$

where we used the colour neutrality relation $\sum \mathbf{T}_i = 0$ to rearrange terms and kept only the leading double poles. The dependence on the off-shell regulator cancels in the coefficient of the logarithmically enhanced single-pole part when expanding in ϵ, as expected. In addition, we find a term involving $\mathbf{T}_1 \cdot \mathbf{T}_2$. From this point we could proceed to derive a Z-factor for the set of singular currents. However, since all $J_{\rm B1}^{(n)}$ coincide for $\epsilon \to 0$ the mixing of these currents cannot be determined unambigously in this way. Therefore, we instead consider the product of the d-dimensional *bare* Wilson coefficient and the bare currents, and use that the sum $\sum_n C_n J_n$ has no UV poles for any IR regulated matrix element. For the present case this leads to the condition

$$
\sum_n \left\{ C^{(n)}_{{\rm B1},\mathcal{O}(\alpha_s)} \langle J_{\rm B1}^{(n)}\rangle_{\rm tree} + C^{(n)}_{\rm B1,tree} \langle J_{\rm B1}^{(n)}\rangle_{\text{1-loop}} \right\}_{1/\epsilon^2 \text{ and } \ln(X)/\epsilon \text{ poles}} = 0 + \mathcal{O}\left(z^0\right) \,. \qquad \text{(D.5)}
$$

When matching to QCD, the first summand captures the hard one-loop contribution. The analysis at the beginning of section 4 implies that the relevant regions contributing for $z \to 0$ have virtuality Q^2 or zQ^2. We therefore make the *ansatz*

$$
\sum_n C^{(n)}_{{\rm B1},\mathcal{O}(\alpha_s)} \langle J_{\rm B1}^{(n)}\rangle_{\rm tree} = \frac{\alpha_s}{2\pi}\frac{1}{\epsilon^2} \sum_n \left[c_n \left(\frac{\mu^2}{Q^2}\right)^\epsilon + d_n \left(\frac{\mu^2}{zQ^2}\right)^\epsilon \right] C^{(n)}_{\rm B1,tree} \langle J_{\rm B1}^{(n)}\rangle_{\rm tree} + \mathcal{O}\left(\frac{1}{\epsilon}\right) . \qquad \text{(D.6)}
$$

Inserting this ansatz along with (D.4) into (D.5) yields $c_n = \mathbf{T}_2 \cdot \mathbf{T}_0 + \mathbf{T}_1 \cdot \mathbf{T}_2 = -C_A$ and $d_n = \mathbf{T}_1 \cdot \mathbf{T}_0 - \mathbf{T}_1 \cdot \mathbf{T}_2 = C_A - C_F$.

Since the building blocks in $J_{\rm B1}^{(n)}$ are all evaluated at the same space-time position, the current operator itself cannot depend on z, and therefore the same applies to the Wilson coefficients. Nevertheless, when evaluated in the matrix element, $\langle J_{\rm B1}^{(n)}\rangle_{\rm tree} \propto 1/z^{1+n\epsilon}$, where z is the momentum fraction of the external anti-hardcollinear quark. Therefore, the term involving the coefficient d_n has to be interpreted as an operator mixing $J_{\rm B1}^{(n)} \to J_{\rm B1}^{(n+1)}$, i.e.

$$
C^{(n+1)}_{{\rm B1},\mathcal{O}(\alpha_s)} = \frac{\alpha_s}{2\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^\epsilon \left[c_{n+1} C^{(n+1)}_{\rm B1,tree} + d_n C^{(n)}_{\rm B1,tree} \right] + \mathcal{O}\left(\frac{1}{\epsilon}\right) , \qquad \text{(D.7)}
$$

implying for the $\mathcal{O}\left(\alpha_s\right)$ cusp part of the anomalous dimension matrix

$$
\Gamma^{B1,\rm cusp}_{nm} = \frac{\alpha_s}{2\pi} \ln\frac{\mu^2}{Q^2} \times \begin{cases} -C_A & n = m\,, \\ C_A - C_F & n = m+1\,, \\ 0 & \text{else}\,. \end{cases} \qquad \text{(D.8)}
$$

Solving the d-dimensional RG evolution (see section 3.1.2) for the corresponding Z-factor,

$$\frac{d}{d\ln\mu^2}\mathbf{Z} = -\mathbf{Z}\mathbf{\Gamma}\,, \tag{D.9}$$

yields the bare Wilson coefficients $C^{(n)}_{\rm B1} = Z_{nm}(Q)C^{(m)}_{\rm B1,ren}(Q)$ with

$$Z_{nm}(Q) = \exp\left[-C_A\frac{\alpha_s(\mu)}{2\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right]\sum_{j\geq 0}\frac{1}{j!}\left((C_A-C_F)\frac{\alpha_s}{2\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}\right)^{j}\delta_{n,m+j}\,, \tag{D.10}$$

where $\delta_{n,m+j}$ is the Kronecker symbol. At LL accuracy it is sufficient to evaluate the renormalized Wilson coefficients $C^{(m)}_{\rm B1,ren}(Q)$ at tree-level, such that all of them are zero, except for $m=0$, $C^{(0)}_{\rm B1,ren}(Q) = C^{(0)}_{\rm B1,tree} = \kappa$. In the previous equation, we can therefore set $j=n$ and drop the sum over j.

We are interested in the matrix element for $q\phi^* \to qg$, with leading poles arising from *hard loops only*, which is given by the hard matching coefficients multiplied with the tree-level SCET matrix element,

$$\begin{aligned}\mathcal{M}_{q\phi^*\to qg}\Big|_{\text{hard loops only}} &= \sum_n C^{(n)}_{\rm B1}\langle J^{(n)}_{\rm B1}\rangle_{\rm tree} + \mathcal{O}(z^0)\\ &= C^{(0)}_{\rm B1,tree}\exp\left[-C_A\frac{\alpha_s}{2\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon} + (C_A-C_F)\frac{\alpha_s}{2\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{zQ^2}\right)^{\epsilon}\right]\\ &\quad\times\frac{1}{z}\,g_s\bar{v}_c(p)\not{\epsilon}_{a\perp\overline{hc}}t^a v_{\overline{hc}}(p_1) + \mathcal{O}(z^0)\,.\end{aligned} \tag{D.11}$$

Inserting this result into (2.9) precisely yields the exponentiation conjecture (2.22).

E Relation between DIS at large x and event shapes in the two-jet limit

In this appendix we discuss the relation between NLP contributions to DIS for large x and the thrust distribution in $e^+e^- \to \gamma^*(Q) \to$ jets [15]. In particular, we consider the power expansion in the two-jet limit $\tau = 1-T\to 0$, where T is the thrust event-shape variable, such that τ plays the role of $1-x$ (or $1/N$ in Mellin space) in DIS. The leading logarithmic corrections to the differential cross section at NLP have the form [18]

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}\bigg|^{\rm NLP,LL} = \sum_n\left(\frac{\alpha_s(Q)}{4\pi}\right)^n c^{(n)}_{\rm LL}\,\ln^{2n-1}\tau\,. \tag{E.1}$$

The relevant regions are hard, (anti-)hardcollinear and soft, with virtualities Q^2, τQ^2 and $\tau^2 Q^2$, respectively. The leading poles can therefore be expanded in the form

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}\bigg|^{\rm NLP,LL} = \sum_n\left(\frac{\alpha_s}{4\pi}\right)^n\frac{1}{\epsilon^{2n-1}}\sum_{j=1}^{2n}c^{(n)}_j\left(\frac{\mu^{2n}}{Q^{2n}\tau^j}\right)^{\epsilon}\,. \tag{E.2}$$

Hard loops contribute a factor $\alpha_s\times(\mu^2/Q^2)^\epsilon$, (anti-)hardcollinear loops $\alpha_s\times(\mu^2/Q^2\tau)^\epsilon$ and soft loops $\alpha_s\times(\mu^2/Q^2\tau^2)^\epsilon$. At NLP at least one (anti-)hardcollinear or soft loop is required, such that the expansion starts at $j = 1$. Compared to DIS, the virtualities are composed of only two independent scales instead of three, and therefore the coefficients $c_j^{(n)}$ depend only on a single index j at each order in α_s. This implies that each coefficient can receive contributions from different combinations of regions, for example $c_2^{(2)}$ from one hard and one soft loop or two hardcollinear loops. Pole cancellation yields $2n-1$ conditions

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 3 & \dots & 2n \\ 1 & 4 & 9 & \dots & (2n)^2 \\ \vdots & & & & \\ 1 & 2^{2n-2} & 3^{2n-2} & \dots & (2n)^{2n-2} \end{pmatrix} \cdot \begin{pmatrix} c_1^{(n)} \\ c_2^{(n)} \\ \vdots \\ c_{2n}^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{E.3}$$

that are all linearly independent. This means only one of the $2n$ coefficients $c_j^{(n)}$ is free. As noticed in [18], the finite part of (E.2) is completely determined by a single coefficient, specifically

$$c_{\mathrm{LL}}^{(n)} = -\frac{1}{(2n-1)!} \sum_{j=1}^{2n} j^{2n-1}\, c_j^{(n)} = c_1^{(n)}\,, \tag{E.4}$$

where the last equality follows from solving (E.3). The only possible combination of regions contributing to $c_1^{(n)}$ are $n-1$ hard loops and one (anti-)hardcollinear loop. More precisely, the (anti-)hardcollinear loop arises from a phase-space integration with a three-particle final state $\gamma^* \to q\bar{q}g$, with two of them being either both hardcollinear or both anti-hardcollinear. We note that, for the thrust distribution, the hardcollinear and anti-hardcollinear directions have equal virtuality, and both refer to particles in the *final* state. Therefore, we are free to choose a convention for the light-cone basis such that $c_1^{(n)}$ receives contributions from $n-1$ hard loops and one *anti*-hardcollinear loop. This choice is made to make the analogy to DIS as close as possible, see below.

In the following we focus exclusively on those contributions to the NLP cross section for which no analog at LP exists (termed category II in [18]), in analogy to the off-diagonal DIS process. Category II requires either an anti-hardcollinear $q\bar{q}$ pair (IIc) or a soft q or $\bar{q}$ (IIs), respectively. In the following it is understood that $c_j^{(n)}$ refers to category II only, assuming that poles cancel separately in each category. Then $c_1^{(n)}$ receives only contributions from virtual hard loop corrections to $\gamma^* \to [q\bar{q}]g$, where the square bracket denotes the anti-hardcollinear particles. Such contributions are given by

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}\bigg|_{\gamma^*\to[q\bar{q}]g} = \int_0^1 dz \left(\frac{\mu^2}{s_{q\bar{q}}z\bar{z}}\right)^{\epsilon} \mathcal{P}_{q\bar{q}}(s_{q\bar{q}}, z)\bigg|_{s_{q\bar{q}}=Q^2\tau} + \mathcal{O}(\lambda^2)\,, \tag{E.5}$$

where z and $\bar{z}$ are the collinear momentum fractions, $s_{q\bar{q}}$ is the virtuality of the $q\bar{q}$ pair, and

$$\mathcal{P}_{q\bar{q}}(s_{q\bar{q}}, z) \equiv \frac{e^{\gamma_E\epsilon}Q^2}{16\pi^2\Gamma(1-\epsilon)} \frac{|\mathcal{M}_{\gamma^*\to[q\bar{q}]g}|^2}{|\mathcal{M}_0|^2}\,, \tag{E.6}$$

where $|\mathcal{M}_0|^2$ is the LO matrix element squared for $\gamma^* \to q\bar{q}$, and $|\mathcal{M}_{\gamma^*\to[q\bar{q}]g}|^2$ involves an arbitrary number of hard loop corrections. This expression can be compared to (2.8)

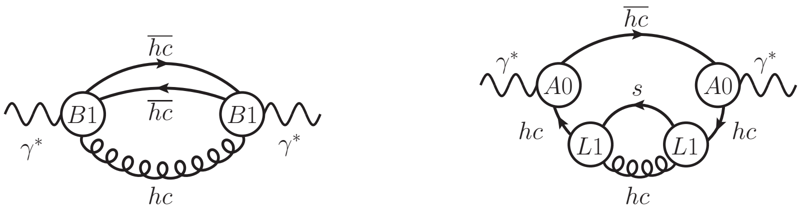

Figure 7. SCET representation of the content of (E.8) for the thrust distribution in $e^+e^- \to \gamma^*(Q) \to$ jets at NLP as $\tau \to 0$. Wilson lines are set to 1.

for DIS, which has a similar structure, except that here we consider a $1 \to 3$ instead of a $2 \to 2$ process, and the anti-hardcollinear particles are a quark and an antiquark instead of a quark and a gluon. More importantly, DIS involves an additional scale related to the PDF, that is absent for the thrust distribution. Nevertheless, as for DIS, the tree-level momentum distribution

$$\mathcal{P}_{q\bar{q}}(s_{q\bar{q}}, z)|_{\text{tree}} = \frac{\alpha_s C_F}{2\pi} \frac{e^{\gamma_E \epsilon}(1-\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\bar{z}}{z} + \frac{z}{\bar{z}} \right) , \tag{E.7}$$

leads to endpoint divergences in (E.5) for $z \to 0, 1$. Using the conjecture from [15] for the all-order expression for $\mathcal{P}_{q\bar{q}}$ allows one to perform the z-integration in d dimensions, and, after expanding in α_s, one can read off the coefficients $c_1^{(n)}$. Remarkably, the result coincides with (3.42) multiplied by a factor of two. Using $c_{\text{LL}}^{(n)} = c_1^{(n)}$ one directly obtains the LL contributions to the NLP thrust distribution from this result, which coincides with the "soft quark Sudakov" factor given in [15].

At this point one may wonder why, despite of the similarities, the LL resummed off-diagonal DGLAP kernel (3.69) obtained from the DIS process is considerably more complex than the thrust distribution. To understand this difference, it is useful to separately consider contributions with an anti-hardcollinear $q\bar{q}$ pair (denoted by IIc) and those with a soft quark or antiquark (denoted by IIs). The tentative SCET interpretation given in [15] suggests that IIc is represented by diagrams involving a B1 current operator with anti-hardcollinear $q\bar{q}$ building blocks, and IIs by diagrams with an insertion of a time-ordered product operator involving the LP current and $\mathcal{L}_{\xi q}^{(1)}$ (see figure 7). This motivates the following ansatz for the (partial) factorization of hard, (anti-)hardcollinear and soft loop contributions to IIs and IIc,

$$\frac{1}{\sigma_0} \frac{d\sigma}{d\tau} \bigg|_{\text{II}}^{\text{NLP,LL}} \equiv H_{\text{IIs}}^{\text{LP}} \cdot [J \times S]_{\text{IIs}}^{\text{NLP}} + [H \times J]_{\text{IIc}}^{\text{NLP}} \cdot S_{\text{IIc}}^{\text{LP}} , \tag{E.8}$$

with factorized hard and soft functions for IIs and IIc, respectively. They are governed by the usual LP cusp anomalous dimension, with leading poles given by

$$\begin{aligned} H_{\text{IIs}}^{\text{LP}} &\equiv \exp \left[-\frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^\epsilon \right] , \\ S_{\text{IIc}}^{\text{LP}} &\equiv \exp \left[-\frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau^2} \right)^\epsilon \right] . \end{aligned} \tag{E.9}$$

The non-trivial information resides in the combined jet and hard function for IIc, involving a convolution in momentum fractions of the B1 operator, as well as in the combined soft and jet function for IIs, involving convolutions related to the spatial separation of the A0 current and the Lagrangian insertion. Following the discussion above, we expect these convolutions to feature endpoint divergences in four dimensions. The decomposition is analogous to (3.35) in DIS, with $[H \times J]^{\rm NLP}_{\rm IIc}$ corresponding to the bare NLP partonic cross section $W^{\rm NLP}_{\phi,q}$ and $[J \times S]^{\rm NLP}_{\rm IIs}$ to the bare NLP PDF evolution factor $U^{\rm NLP}_{gq}$.

Here we point out that, using the results from above allows one to bootstrap the resummed leading poles of $[H \times J]_{\rm IIc}$ and $[J \times S]_{\rm IIs}$ in d dimensions. To see this, we note that the consistency conditions (E.3) determine all coefficients $c^{(n)}_j$ for $1 \leq j \leq 2n$ given the result for $c^{(n)}_1$. In addition, we use that the leading poles can be expanded in the form

$$
\begin{aligned}
[H \times J]_{\rm IIc} &= \epsilon \sum_{n_c \geq 1} \left(-\frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau} \right)^{\epsilon} \right)^{n_c} \sum_{n_h \geq 0} \left(-\frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \right)^{n_h} c_{\rm IIc}(n_c, n_h) \,, \\
[J \times S]_{\rm IIs} &= \epsilon \sum_{n_c \geq 0} \left(-\frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau} \right)^{\epsilon} \right)^{n_c} \sum_{n_s \geq 1} \left(-\frac{\alpha_s}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau^2} \right)^{\epsilon} \right)^{n_s} c_{\rm IIs}(n_c, n_s) \,,
\end{aligned}
\tag{E.10}
$$

where $n_{s,c,h}$ denote the number of soft, (anti-)hardcollinear and hard loops. Note that $n_c > 0$ for IIc and $n_s > 0$ for IIs. Inserting this expansion into (E.8) and requiring that the sum of IIs and IIc contributions has to reproduce (E.2) with known coefficients $c^{(n)}_j$ allows one to uniquely determine the coefficients $c_{\rm IIc}(n_c, n_h)$ and $c_{\rm IIs}(n_c, n_s)$. We find

$$
\begin{aligned}
[H \times J]_{\rm IIc} = {} & \frac{C_F}{C_F - C_A} \frac{\epsilon \tau^{-\epsilon}}{\tau^{-\epsilon} - 1} \\
& \left\{ \exp\left[\frac{2\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau} \right)^{\epsilon} - \frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \right] \right. \\
& \left. - \exp\left[\frac{\alpha_s (C_F + C_A)}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau} \right)^{\epsilon} - \frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2} \right)^{\epsilon} \right] \right\},
\end{aligned}
\tag{E.11}
$$

$$
\begin{aligned}
[J \times S]_{\rm IIs} = {} & \frac{C_F}{C_F - C_A} \frac{\epsilon \tau^{-\epsilon}}{\tau^{-\epsilon} - 1} \\
& \left\{ - \exp\left[\frac{2\alpha_s C_F}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau} \right)^{\epsilon} - \frac{\alpha_s C_F}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau^2} \right)^{\epsilon} \right] \right. \\
& \left. + \exp\left[\frac{\alpha_s (C_F + C_A)}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau} \right)^{\epsilon} - \frac{\alpha_s C_A}{\pi} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{Q^2 \tau^2} \right)^{\epsilon} \right] \right\}.
\end{aligned}
\tag{E.12}
$$

These expressions can be compared to (3.50) for $W^{\rm NLP,LL}_{\phi,q}$ and (3.51) for $U^{\rm NLP,LL}_{gq}$, respectively. In particular, the last lines in each expression would lead to the appearance of Bernoulli functions when expanding the thrust distribution in ϵ. Remarkably, however, these terms precisely cancel when adding the IIc and IIs pieces in (E.8). The remaining terms combine to exponential factors that are finite by themselves for $\epsilon \to 0$, giving

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau}\Big|_{\rm II}^{\rm NLP,leading\ poles} = \frac{C_F}{C_F - C_A}\frac{\epsilon\tau^{-\epsilon}}{\tau^{-\epsilon}-1}\Bigg\{\exp\left[-\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(1-\tau^{-\epsilon})^2\right] - \exp\left[-\frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon^2}\left(\frac{\mu^2}{Q^2}\right)^{\epsilon}(1-\tau^{-\epsilon})^2\right]\Bigg\}. \tag{E.13}$$

This expression indeed has no poles in $1/\epsilon$, and approaches a finite limit for $\epsilon \to 0$, that precisely agrees with the LL resummed "soft quark Sudakov" formula given in [15].

While it is reassuring to recover the result for the LL resummed NLP thrust distribution given in [15], the main purpose of this appendix is to point out the form of the two individual contributions (E.11), (E.12) and the formal analogy as well as difference to the DIS process.

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