

# Topology and Strings: Topics in $N = 2^*$

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*Or là, où il n'y a point de parties, il n'y a ni étendue, ni figure, ni divisibilité possible.  
Et ces Monades sont les véritables Atomes de la Nature et en un mot les Elements des  
Choses.*

*G.W. Leibnitz (La Monadologie-3)*

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# 1 $N = 2$ Algebra and Topological Field Theory

## 1.1 $N = 2$ Algebra and BRST-cohomology

Given the  $N = 2$  supersymmetric algebra

$$\begin{aligned}(Q^\pm)^2 &= 0 \\ \{Q^+, Q^-\} &= H\end{aligned}\tag{1}$$

a topological field theory (TFT) can be defined by declaring one of the two SUSY generators, let us say  $Q^+$ , to be a BRST-charge. The physical Hilbert space  $\mathcal{H}$  of the TFT is defined as the BRST cohomology and the physical observables  $\phi_i$  are constrained by the symmetry requirement

$$[Q^+, \phi_i] = 0\tag{2}$$

We can provide the Hilbert space  $\mathcal{H}$  with an inner product  $\langle \ , \ \rangle$  such that the adjoint of  $Q^+$  is  $Q^-$ . This allow us to associate with each cohomology class a "Hodge-representative"<sup>1</sup> satisfying

$$Q^+|i\rangle = Q^-|i\rangle = 0\tag{3}$$

From (1.b) we observe that this basis is one to one related to the vacuum states

$$H|i\rangle = 0\tag{4}$$

In these lectures we will mostly reduce our study to two dimensional topological field theories [1], [2]. Physically, topological invariance means that the only space-time dependence of correlation functions will be on its topology, which in two dimensions is simply given by the genus. Topological invariance is certainly a much larger symmetry than the more familiar conformal invariance, this however does not mean that all topological field theories are massless or, equivalently, with a traceless energy-momentum tensor. As we will see it is possible to write down lagrangians which are manifestly independent of the metric, and in this sense topological, possessing dimensionful coupling constants. The

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<sup>1</sup>Hodge's theorem for compact manifolds without boundary establish that any p-form can be uniquely decomposed as a sum of exact, co-exact and a harmonic form. The harmonic form (3) is the Hodge-representative.

renormalization group can be directly applied to these topological theories. The critical points of the renormalization group flow will define topological conformal field theories, which will be characterized by two chiral,  $Q^\pm$  and  $\bar{Q}^\pm$ , N=2 algebras.

## 1.2 Two Dimensional TFT's: Operator Formalism

In two dimensions, a TFT can be nicely described using the operator formalism [3]. Let  $\mathcal{H}$  be the physical Hilbert space and let us choose as a basis the Hodge-representatives defined by equation (3). Given now a generic Riemann surface  $\Sigma_g$  of genus  $g$  with  $n$  punctures  $p_1, \dots, p_n$ , the operator formalism definition of the corresponding TFT will consist in associating with these geometrical data a quantum state  $|\Sigma_g; p_1, \dots, p_n\rangle$  satisfying

$$\begin{aligned} Q^+ |\Sigma_g; p_1, \dots, p_n\rangle &= 0 \\ \delta |\Sigma_g; p_1, \dots, p_n\rangle &= Q^+ |\eta\rangle \end{aligned} \tag{5}$$

where by  $\delta$  we mean any change of the metric and the positions of the punctures. Condition (5.1) implies that  $|\Sigma_g; p_1, \dots, p_n\rangle \in \otimes^n \mathcal{H}$ , and condition (5.2) reflects the topological nature of the theory, namely, any geometrical change is represented by  $Q^+$ -exact forms and therefore all the geometrical dependence of the state  $|\Sigma_g; p_1, \dots, p_n\rangle$  can be mapped into the same BRST-cohomology class. Hence we can associate with any genus  $g$  and any number of punctures  $n$  a Hodge-representative state  $|g, n\rangle$  as follows

$$|g, n\rangle = \sum_{i_1, \dots, i_n} C_g^{i_1 \dots i_n} |i_1\rangle \otimes \dots \otimes |i_n\rangle \tag{6}$$

where we sum over the basis (3) of the physical Hilbert space  $\mathcal{H}$ , and with the constants  $C_g^{i_1 \dots i_n}$  depending only on the topological data, namely the genus and the number of punctures. To define the theory reduces now to fix these constants. In order to do it we will impose, as usual, consistency with sewing.

A topological sewing can be defined by two operations  $*$  and  $\hat{*}$  such that

$$\begin{aligned} |g, n\rangle &= |g_1, n_1\rangle * |g_2, n_2\rangle, & n_1 + n_2 &= n + 2 \\ |g, n\rangle &= \hat{*}|g-1, n+2\rangle \end{aligned} \tag{7}$$

Using (6), we can define the  $*$ -operation as follows

$$|g, n\rangle = \sum_i \left( \sum_j C_{g_1}^{i_1 \dots i_{n_1-1}} C_{g_2}^{j i_{n_1+2} \dots i_{n_1+n_2}} \right) |i_1\rangle \otimes \dots \otimes |i_{n_1+n_2}\rangle \quad (8)$$

with

$$C_g^{i_1 \dots i_n} \equiv C_g^{i_1 \dots i_n l} \eta_{lj} \quad (9)$$

where we have introduced a "sewing metric"  $\eta_{ij}$ . The  $*$ -operation can be analogously defined as follows

$$|g, n\rangle = \sum_i \sum_j C_{g-1}^{i_1 \dots i_n j} |i_1\rangle \otimes \dots \otimes |i_n\rangle \quad (10)$$

Using (8) and (10), we get the following type of sewing equations

$$C_g^{i_1 \dots i_n} = C_{g_1}^{i_1 \dots i_k} C_{g_2}^{j i_{k+1} \dots i_n} = \sum_j C_{g-1}^{i_1 \dots i_n j} \quad (11)$$

An immediate consequence of sewing is that all constants  $C_g^{i_1 \dots i_n}$  can be written as products of the elementary three point functions  $C_0^{ijk}$ . The sewing equations (11) will be automatically fulfilled if the elementary three point constants satisfy the associativity condition

$$\sum_m C_0^{ij}{}_m C_0^{mkl} = \sum_m C_0^{ik}{}_m C_0^{mjl} \quad (12)$$

The net result of the sewing construction is that a TFT is completely determined by a set of constants  $C_0^{ijk}$  and the sewing metric  $\eta_{ij}$ . In the previous discussion, we have not considered the dependence of  $C_0^{ijk}$  on the coupling constants of the theory. Before entering into that problem, we will like to use the previous formalism for the explicit construction of correlation functions.

### 1.3 Observables and Hodge-representatives

Let us consider a physical observable  $\phi_i$  satisfying condition (2). As it is in general the case for local quantum field theory, we would like to associate with this observable a physical state, i.e. a BRST cohomology class and more in particular a Hodge-representative in this class. This can be done as follows. Let us take a hemisphere with the field  $\phi_i$  inserted on it at the point  $p$ . In this way we obtain at the boundary a physical state  $|i\rangle_p$  satisfying

$Q^+|i\rangle_p = 0$ . When we change the position of the insertion, the state we will obtain will differ from the former one by  $Q^+$ -exact forms. A simple way to project on the Hodge-representative, will be by gluing the hemisphere to an infinitely long cylinder with fixed perimeter  $\beta$ . Using now relation (1.b), we can project on the harmonic representative by taking the limit

$$|i\rangle = \lim_{T \rightarrow \infty} e^{-TH}|i\rangle_p \quad (13)$$

The state  $|i\rangle$  satisfy

$$Q^+|i\rangle = Q^-|i\rangle = 0 \quad (14)$$

By the construction we have used [4], the state  $|i\rangle$  associated with the observable  $\phi_i$  will in principle depend on the perimeter of the cylinder  $\beta$ . This statement can sound a priori a bit strange. In fact if for different values of  $\beta$  we obtain different harmonic representatives we will be in contradiction with the topological invariance as introduced in equation (5), namely the difference of two harmonic forms is not a  $Q^+$ -exact form and, on the other hand, a change of the perimeter seems to be an innocent geometrical variation. What is the solution to this puzzle? To get the solution we need to understand the perimeter  $\beta$  used to map physical observables into Hodge-representatives as a renormalization group point or scale. In this sense, changes of  $\beta$  will produce in general variations in the coupling constants<sup>2</sup>. Now the cohomology class is defined relative to  $Q^+$  which will depend explicitly on these couplings. Therefore changing  $\beta$  we will get, in general, different harmonic forms in different cohomology classes. After this comment we can try to connect the operator formalism construction presented in section 1.2, with the definition of correlation functions for physical observables.

By means of the sewing procedure we have reduced the problem of defining the states  $|g, n\rangle$  to that of defining a topological metric  $\eta_{ij}$  and the set of elementary three point functions  $C_0^{ijk}$ . Our task will be now to get these building blocks of the TFT directly from the algebra of observables. Let us consider two physical observables  $\phi_i, \phi_j$  inserted on the hemisphere, and let us project on a Hodge-representative by gluing an infinite cylinder of fixed perimeter  $\beta$ . The state  $|i, j\rangle_\beta$  obtained by this procedure is by construction a physical state and can be projected on a basis of  $\mathcal{H}$

$$|i, j\rangle_\beta = \sum C_{ij}^k(\beta)|k\rangle_\beta \quad (15)$$

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<sup>2</sup>A more detailed characterization of the RG in topological field theories will be presented in section 1.6

The constants  $C_{ij}^k(\beta)$  define the cohomology ring structure for a particular set of Hodge-representatives, namely the ones defined at the renormalization point  $\beta$ .

Now we can use these cohomology ring constants, and sewing, to define any correlator of physical observables

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle_0 = \sum_{nm} C_{ij}^n \eta_{nm} C_{kl}^m = \sum_n C_{ij}^n C_{nkl} \quad (16)$$

From (16) we get in particular

$$\eta_{ij} = \langle \phi_i \phi_j \rangle_0 \quad (17)$$

It should be stressed that the sewing metric or topological metric,  $\eta_{ij}$  defined by two point correlators on the sphere, does not coincide with the inner product of  $\mathcal{H}$  relative to which the adjoint of  $Q^+$  is  $Q^-$ . The dependence on the renormalization group point  $\beta$  of these correlators should be constrained to satisfy, as usual in quantum field theory, renormalization group equations

$$\frac{d}{d\beta}(C_g^{i_1 \dots i_n}) = 0 \quad (18)$$

It will be important for the rest of our study to have control on the  $\beta$ -dependence of the Hodge-representative states. To do that, it is convenient to pass from the abstract discussion we are developping until now to some concrete cases of TFT.

## 1.4 Twisting N=2 Super Conformal Field Theories

In the previous section we have presented the general structure of a TFT. To materialize this structure in one concrete case, we will define TFT's associated with  $N = 2$  super conformal field theories (SCFT).

The chiral algebra of a  $N = 2$  SCFT [5] is generated by the identity, the energy momentum tensor  $T(z)$ , two supersymmetric currents  $G^\pm$  and a  $\mathcal{U}(1)$  current  $J(z)$ . In terms of the corresponding Laurent expansions

$$T(z) = \sum_n L_n z^{-n-2}, \quad J(z) = \sum_n J_n z^{-n-1}, \quad G^\pm(z) = \sum_n G_n^\pm z^{-n-3/2} \quad (19)$$

The  $N = 2$  algebra is given by

$$\begin{aligned}
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)J_{r+s} + (c/3)(r^2 - 1/4)\delta_{r+s,0} \\
[L_m, L_n] &= (m-n)L_{m+n} + (c/12)m(m^2 - 1)\delta_{m+n,0} \\
[L_n, G_r^\pm] &= (n/2 - r)G_{n+r}^\pm \quad , \quad [L_n, J_m] = -mJ_{m+n} \\
[J_m, J_n] &= (c/3)m\delta_{m+n,0} \quad , \quad [J_n, G_r^\pm] = \pm G_{n+r}^\pm
\end{aligned} \tag{20}$$

where  $r$  and  $s$  are intergers or half intergers depending if the representation is in the NS or R sectors. The same holds for the antiholomorphic components  $\bar{G}_n^\pm$ ,  $\bar{J}_n$  and  $\bar{L}_n$ .

In order to build a TFT we want, first, to use one of the two SUSY currents to define a BRST charge. This is not possible immediately because the SUSY currents have spin 3/2 instead of 1, as should be the case for defining a BRST-charge. Second, we need an energy-momentum tensor that can be written as an exact form in order to implement topological invariance. The two things can be achieved by twisting [2, 6] the theory, which consists in changing the energy-momentum tensor  $T(z)$  to  $T^t(z)$ , defined by

$$T^t(z) = T(z) + \frac{1}{2}\partial J(z) \tag{21}$$

This change in the energy-momentum tensor corresponds to couple the  $\mathcal{U}(1)$  current to a background gauge field equal to half the spin connection. The net result of this background field is to change the spin  $s$  of any field of charge  $q$  to  $s - q/2$ . This is the effective change of the spin coming from the holonomy contribution for a charge  $q$  coupled to a  $\mathcal{U}(1)$  gauge field equal to one half the spin connection.

After twisting, the SUSY current  $G^+$  of positive charge  $q = 1$  and spin 3/2, becomes a one form and can be used to define a BRST charge

$$Q^+ = \oint G^+(z)dz \tag{22}$$

Moreover, from the algebra relations (20) we get

$$T^t(z) = \oint G^+(w)G^-(z)dw \tag{23}$$

which makes the twisted energy-momentum tensor (21) a  $Q^+$ -exact form.

Therefore, by twisting the  $N = 2$  SCFT we have obtained a topological conformal field theory with two BRST charges

$$Q^+ = \int G^+(z)dz, \quad \bar{Q}^+ = \int \bar{G}^+(z)dz \tag{24}$$



and a traceless energy-momentum tensor  $T^t(z), \bar{T}^t(\bar{z})$  which are, relative to  $Q^+$  and  $\bar{Q}^+$ , exact forms.

Before leaving this section, let us note that we could have done the twist coupling the  $\mathcal{U}(1)$  charge to minus one half of the spin connection. In this case, the  $G^-$  current becomes the BRST charge. This can be done independently in the left and right sectors.

### Physical Hilbert Space and Observables: Chiral Ring [8]

The Hilbert space of the original  $N = 2$  SCFT is, as usual in CFT's, a direct sum of irreps of the chiral algebra. Each irrep is associated with a primary field which represents the observables of the theory and is characterized by a weight  $\Delta$  and a  $\mathcal{U}(1)$  charge  $q$ .

As an example we can mention the case of  $A_{k+1}$  minimal models. The central extension in this case is given by

$$c = \frac{3k}{k+2} \quad (25)$$

with  $k$  an interger number, called level. The irreps fulfilling the Hilbert space are parametrized by

$$\begin{aligned} \text{weights : } \Delta_{l,m} &= \frac{l(l+2) - m^2}{4(k+2)} & l = 0, \dots, k ; m = -l, -l+2, \dots, l-2, l \\ \mathcal{U}(1)\text{charges : } q_m &= \frac{m}{k+2} \end{aligned} \quad (26)$$

Each of these irreps is associated with a primary field  $\phi_{l,m}(z)$ . Denoting  $|l, m\rangle$  the weight vector, the map between observables and states is given by

$$|l, m\rangle = \phi_{l,m}(0)|0\rangle \quad (27)$$

When we twist the theory, the Hilbert space of the  $N = 2$  SCFT collapses into  $Q^+$ -cohomology classes. The best way to understand this truncation is by using a Coulomb gas representation where the BRST charge of the  $N = 2$  SCFT is defined in terms of the screening currents [7]. By the twist we modify the theory in such a way that the energy-momentum tensor becomes, relative to this BRST charge, an exact form.

Our task now will be to associate a Hodge-representative to each cohomology class and to define the corresponding physical observables. As usual, we can take as Hodge-representative the harmonic or vacuum forms of the twisted theory

$$Q^+|i\rangle = Q^-|i\rangle = 0 \quad (28)$$

These Hodge-representatives are precisely the Ramond vacua of the original  $N = 2$  SCFT<sup>3</sup>. In other words, each cohomology class of the twisted theory has as Hodge-representative a Ramond vacua of the untwisted theory. It is well known that the NS and R realizations of a  $N = 2$  superconformal algebra are connected by the spectral flow transformation  $\mathcal{U}_{1/2}$

$$\begin{aligned} \mathcal{U}_{1/2} L_0 \mathcal{U}_{1/2}^{-1} &= L_0 - \frac{J_0}{2} + \frac{c}{24} \\ \mathcal{U}_{1/2} J_0 \mathcal{U}_{1/2}^{-1} &= J_0 - \frac{c}{6} \\ \mathcal{U}_{1/2} G_{\mp 1/2}^{\pm} \mathcal{U}_{1/2}^{-1} &= G_0^{\pm} \end{aligned} \tag{29}$$

The Ramond vacuum states are defined by

$$\begin{aligned} L_0 |i\rangle_R &= \frac{c}{24} |i\rangle_R \\ G_0^+ |i\rangle_R &= 0 \end{aligned} \tag{30}$$

It is easy to see, using (29), that NS w.v. satisfying  $\Delta = q/2$  are one to one related to Ramond vacua (30). These NS w.v. are associated in the  $N = 2$  SCFT with local primary field  $\phi_i$  that verify

$$[G_{-1/2}^+, \phi_i] = 0 \tag{31}$$

Fields satisfying (31) are called chiral fields. Summarizing, each cohomology class of the twisted theory is associated with a chiral primary field. Indeed, equation (31) corresponds to the BRST-invariance condition in the twisted theory, and it can be proved that any general chiral field can be decomposed into the sum of a chiral primary field and a  $Q^+$ -exact one. The consistency of the twisting procedure requires that the operator product expansion for chiral primary fields is, up to  $Q^+$ -exact forms, another chiral primary field. This is in fact the case. In the twisted theory and due to the fact that the energy-momentum tensor is  $Q^+$ -exact we can, up to  $Q^+$ -exact forms, reduce our study of the operator product  $\phi_i(z)\phi_j(w)$  between two chiral primary field to the "topological limit"  $z \rightarrow w$ . In this limit and by  $\mathcal{U}(1)$  charge conservation, the only possibility is another chiral primary field  $\phi_k$  such that

$$\phi_i \phi_j = C_{ij}^k \phi_k \quad , \quad q_k = q_i + q_j \tag{32}$$

We reobtain in this way the ring structure we have already discussed in section 1.3. This ring of observables is known as the chiral ring [8]. Analogously, there exist an antichiral

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<sup>3</sup>For the minimal models (25)-(27), solutions to (28) correspond to  $l = m$ .

ring of observables when we choose  $G^-$  to define a BRST charge, and an spectral flow transformation  $\mathcal{U}_{-1/2}$  connecting antichiral fields with Ramond vacua.

It is nice to see that the topological theory defined by the twisting mechanism implements in a natural way the spectral flow transformation. In fact if we define the Hodge-representative  $|i\rangle$  by inserting on the hemisphere the chiral field  $\phi_i$  and projecting on the zero energy sector by gluing an infinitely long cylinder, the state we get at the boundary will have charge  $q_i - c/6$ , where  $c/6$  comes from the contribution of the twist to the functional integral representation of the state  $|i\rangle$ . Now from (29) we see that this is precisely the charge of the state  $\mathcal{U}_{1/2}\phi_i|0\rangle$  obtained from the NS sector by spectral flow.

The anomaly of the  $\mathcal{U}(1)$  current generated by the twist imposes the following selection rule for non vanishing correlators  $\langle\phi_{i_1}\dots\phi_{i_n}\rangle_g$

$$\sum_i q_i = \hat{c}(1 - g) \quad , \quad \hat{c} = c/3 \quad (33)$$

In particular the topological metric  $\eta_{ij} = \langle j|i\rangle$  defined for Hodge-representatives, i.e. Ramond vacua, will be non vanishing only if

$$q_i + q_j = \hat{c} \quad (34)$$

or in other words, when the sum of the Ramond charges is equal to zero.

## 1.5 Deformations Preserving Topological Invariance: Coupling Constants and the $t\bar{t}$ -equations

Physical observables of an  $N = 2$  SCFT are associated with chiral superfields, of components

$$\Phi_i = (\phi_i^{(0)}(z, \bar{z}), \phi_i^{(1)}(z, \bar{z}), \bar{\phi}_i^{(1)}(z, \bar{z}), \phi_i^{(2)}(z, \bar{z})) \quad (35)$$

where

$$\phi_i^{(2)} = \{Q^-, [\bar{Q}^-, \phi_i^{(0)}]\} \quad (36)$$

and therefore

$$[Q^+, \int_{\Sigma} \phi_i^{(2)}] = 0 \quad (37)$$

Similarly, for antichiral fields  $\bar{\phi}_i$  we can define, relative to the SUSY charge  $Q^+$

$$\bar{\phi}_i^{(2)} = \{Q^+, [\bar{Q}^+, \bar{\phi}_i^{(0)}]\} \quad (38)$$

which trivially implies

$$[Q^+, \int_{\Sigma} \bar{\phi}_i^{(2)}] = 0 \quad (39)$$

Moreover, for  $Q^+$  being the BRST charge,  $\bar{\phi}_i^{(2)}$  becomes a pure BRST field.

Using (36) and (38) we define a deformed theory parametrized by the coupling constants  $(t_i, \bar{t}_i)$  as follows

$$\mathcal{L}(t_i, \bar{t}_i) = \mathcal{L}_0^{N=2} + \sum_i t_i \int_{\Sigma} \phi_i^{(2)} + \sum_{\bar{i}} \bar{t}_{\bar{i}} \int_{\Sigma} \bar{\phi}_{\bar{i}}^{(2)} \quad (40)$$

This deformed theory can be transformed into a TFT again by the twisting mechanism. Let us fix a set of values  $(t_0, \bar{t}_0)$  for the coupling constants. If some of the non vanishing coupling constants correspond to relevant deformations, then the theory defined by (40) will represent a massive deformation of the  $N = 2$  SCFT,  $\mathcal{L}_0^{N=2}$ . For these massive deformations, the only conserved  $\mathcal{U}(1)$  current correspond to the fermion number current (the difference between the left and right charges at the conformal point). The TFT at this point in the space of couplings is obtained by twisting with respect to the conserved fermion number current

$$\mathcal{L}^T = \mathcal{L}(t_i, \bar{t}_i) + \frac{1}{2} \int j A \quad (41)$$

for  $A$  the  $\mathcal{U}(1)$  spin connection and  $j$  the fermion number current. The antitopological twist is defined by

$$\mathcal{L}^{T^*} = \mathcal{L}(t_i, \bar{t}_i) - \frac{1}{2} \int j A \quad (42)$$

Physical observables of (41) are associated with the chiral fields  $\phi_i$  and the ones of (42) with the antichiral fields  $\bar{\phi}_i$ .

If the deformed theory (40) is massive, the  $N = 2$  algebra generated by  $Q^{\pm}$ ,  $\bar{Q}^{\pm}$  will contain non vanishing central terms of the type

$$\{Q^+, \bar{Q}^+\} = \Delta, \quad \{Q^-, \bar{Q}^-\} = \bar{\Delta} \quad (43)$$

The  $N = 2$  algebra (1) is then defined by

$$Q_{\pm} = \frac{1}{\sqrt{2}}(Q^{\pm} + \bar{Q}^{\pm}) \quad (44)$$

After the twisting (41) ((42)),  $Q_+$  ( $Q_-$ ) becomes the corresponding BRST charges.

For each point  $(t, \bar{t})$  in the coupling space, we have defined a BRST charge  $Q_+$  and therefore we can fiber the coupling space by the cohomology ring. The study of this bundle will be the main task for the rest of this section.

Let  $|i, t, \bar{t}; \beta\rangle$  to be the state defined by inserting on the hemisphere the field  $\phi_i$  and projecting on a zero energy state by gluing the hemisphere to an infinitely long cylinder of perimeter  $\beta$ . This correspond to use for the hemisphere a metric  $g = e^\phi dz d\bar{z}$  with  $\beta = e^\phi$ . Let us now introduce a set of "connections"  $A_i, A_{\bar{i}}$ <sup>4</sup> as follows

$$\begin{aligned}\partial_{t_i}|j, t, \bar{t}; \beta\rangle &= A_{ij}^k |k, t, \bar{t}; \beta\rangle + Q^+ - exact \\ \partial_{\bar{t}_i}|j, t, \bar{t}; \beta\rangle &= A_{\bar{i}j}^k |k, t, \bar{t}; \beta\rangle + Q^+ - exact\end{aligned}\quad (45)$$

Therefore the covariant derivatives are given by

$$D_i = \partial_i - A_i \quad , \quad \bar{D}_{\bar{i}} = \partial_{\bar{i}} - A_{\bar{i}} \quad (46)$$

Using the functional integral representation of  $|i, t, \bar{t}\rangle$  and interpreting the partial derivative  $\partial_i$  as the insertion and integration over the hemisphere of the operator  $\phi_i^{(2)}$ , we can conclude, by contour deformation techniques and equation (37), that  $\partial_i|j, t, \bar{t}; \beta\rangle$  is also a physical state. With the same techniques, it is easy to see that

$$A_{ij}^k = 0 \quad (47)$$

Defining now

$$A_{ijk} = \langle k | \partial_i | j \rangle = A_{ij}^l \eta_{lk} \quad (48)$$

for  $\eta_{lk}$  the topological metric, we can derive, by standard functional integral arguments, curvature equations for the connections  $A_i$ . From (47) we get

$$\partial_{\bar{l}} A_{ijk} = \partial_{\bar{l}} A_{ijk} - \partial_i A_{\bar{l}jk} \quad (49)$$

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<sup>4</sup>Properly speaking these connections are defined by

$$\langle \bar{k} | \partial_i - A_i | j \rangle = 0$$

with  $|\bar{k}\rangle$  the antiholomorphic basis. The connection (45) is then defined by

$$A_{ij}^k = A_{ij\bar{k}} g^{\bar{k}k}$$

with  $g^{\bar{k}k}$  the inverse of the hermitian metric  $g_{i\bar{j}} = \langle \bar{j} | i \rangle$ .

which admits the functional integral representation

$$\langle \phi_k \left( \int_{\Sigma_L} Q^+ \bar{Q}^+ \bar{\phi}_\tau \right) | \left( \int_{\Sigma_R} Q^- \bar{Q}^- \phi_i \right) \phi_j \rangle - \langle \phi_k \left( \int_{\Sigma_L} Q^- \bar{Q}^- \phi_i \right) | \left( \int_{\Sigma_R} Q^+ \bar{Q}^+ \bar{\phi}_\tau \right) \phi_j \rangle \quad (50)$$

where  $\Sigma_L$  and  $\Sigma_R$  represent the two hemispheres glued respectively to infinitely long cylinders of perimeter  $\beta$ . To compute the first component of (50), we contract the SUSY currents. The result is

$$\langle \phi_k \left( \int_{\Sigma_L} \bar{\phi}_\tau \right) | \left( \int_{\Sigma_R} \partial \bar{\partial} \phi_i \right) \phi_j \rangle \quad (51)$$

which can be written as

$$- \langle \phi_k \left( \int_{\Sigma_L} \bar{\phi}_\tau \right) | \left( \oint_C \partial_n \phi_i \right) \phi_j \rangle \quad (52)$$

for  $C$  the boundary of  $\Sigma_R$  and  $\partial_n$  the normal derivative along the cylinder. At the boundary the state defined by inserting  $\phi_j$  is projected on a zero energy state  $|j\rangle$ , therefore, and taking into account that  $\partial_n \phi_i = [H, \phi_i]$ , we can rewrite (52) as

$$- \langle \phi_k \int_{\Sigma_L} \bar{\phi}_\tau | H \oint_C \phi_i |j\rangle = - \int d\tau \langle k | \oint_{C_\tau} \bar{\phi}_\tau H \oint_C \phi_i |j\rangle \quad (53)$$

We have used that the state obtained by inserting  $\phi_k$  and  $\bar{\phi}_\tau$  on the left hemisphere, after gluing the cylinder, is annihilated by  $H$ . The integral over  $\tau$  in (53) is over the length of the left cylinder,  $T$ . Moving  $H$  to the left in (53), we obtain

$$\int d\tau \langle k | (\partial_\tau \oint_{C_\tau} \bar{\phi}_\tau) \oint_C \phi_i |j\rangle \quad (54)$$

The integration in  $\tau$  is now performed easily, getting contributions from the boundaries at  $\tau = 0, T$ . The contribution at  $\tau = T$  cancels with an identical one coming from the second term in (50). Then, we are left with

$$- \int \langle k | \oint_{C_\tau} \bar{\phi}_\tau \exp(-TH) \oint_C \phi_i |j\rangle \quad (55)$$

where the propagation of  $\bar{\phi}_\tau$  along the infinitely long left cylinder, explicitied by the factor  $\exp(-TH)$ , has the usual effect of projecting into the ground states. Therefore (55) can be written in terms of the structure constants of the chiral ring

$$- (\bar{C}_\tau C_i)_{kj} \quad (56)$$

Using the same arguments for the second term in (50), we finally obtain

$$\partial_\tau A_{ij}^k = \beta^2 [C_i, \bar{C}_\tau]_j^k \quad (57)$$

with  $\beta$  the perimeter of the cylinder<sup>5</sup>. Equation (57), first derived by Cecotti and Vafa [4], together with

$$\begin{aligned} [D_i, D_j] &= [\bar{D}_i, \bar{D}_j] = [D_i, \bar{C}_j] = [\bar{D}_i, C_j] = 0 \\ [D_i, C_i] &= [D_j, C_i] \quad , \quad [\bar{D}_i, \bar{C}_j] = [\bar{D}_j, \bar{C}_i] \end{aligned} \tag{58}$$

which can be deduced by similar techniques as (57), are known as the  $t\bar{t}$ -equations. To contract the indices of the topological and antitopological structure constants in (57) we use the metric  $g_{i\bar{j}}$  of the physical Hilbert space  $\mathcal{H}$ , namely

$$\bar{C}_{i\bar{j}}^k = g_{j\bar{n}} \bar{C}_{i\bar{m}}^{\bar{n}} g^{\bar{m}k} \tag{59}$$

where

$$g_{i\bar{j}} = \langle \bar{j} | i \rangle \tag{60}$$

with  $\langle \bar{j} |$  the adjoint of the state  $|j\rangle$ . Recall that for the inner product of  $\mathbb{H}$  introduced in section 1.1, the adjoint of  $Q^+$  is  $Q^-$ . The functional integral representation of the state  $|\bar{j}\rangle$  is obtained, using the twisted lagrangian (42), by inserting on the hemisphere the anti-topological field  $\bar{\phi}_{\bar{j}}$  and projecting in the standard way on the zero energy representative. Using this functional integral representation we can interpret the metric tensor  $g_{i\bar{j}}$  as a topological-antitopological correlator on the sphere, where we glue the two hemispheres through an infinitely long cylinder with fixed perimeter  $\beta$  and where we twist the theory with  $+\frac{1}{2}w$  on the right hemisphere and  $-\frac{1}{2}w$  on the left, for  $w$  the spin connection. Notice that the correlator defined in this way is not a topological correlator. Its geometrical meaning can be derived as follows. From the definition of the connection  $A_i$  we derive

$$\begin{aligned} D_i g_{j\bar{k}} &= 0 \\ D_i \eta_{ij} &= 0 \end{aligned} \tag{61}$$

From (61) and (47), we get

$$\partial_i g_{j\bar{k}} = A_{ij}^l g_{l\bar{k}} \tag{62}$$

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<sup>5</sup>The previous derivation of the  $t\bar{t}$ -equation (57) admits a more geometrical interpretation in the following terms. For  $\Sigma_R$  one can consider fixed  $\phi_i$  at the point 1 and reduce the integral over  $\phi_i$  to integrate the moduli  $\tau^R \in [0, \infty]$ ,  $\phi^R \in [0, 2\pi]$  (see section 2.4). The same for the part  $\Sigma_L$  where one will fix  $\bar{\phi}_{\bar{j}}$  at 1 and represent the integration over the insertion of  $\bar{\phi}_{\bar{j}}$  by the one of  $\tau^L \in [0, \infty]$ ,  $\phi^L \in [0, 2\pi]$ . These computations define two contact terms (see section 2.4). The commutator after interchanging  $\phi_i$  and  $\bar{\phi}_{\bar{j}}$  gives equation (57). Notice the difference in this construction with the definition of the 4-point amplitude on the Riemann sphere where we only count with one moduli parameter.

which means that  $A$  is the connection of the metric  $g$

$$A_{ij}^l = -g_{j\bar{k}}(\partial_i g^{-1})^{\bar{k}l} \quad (63)$$

Using (63) we can rewrite the  $t\bar{t}$ -equation (57) as equations for the metric  $g$

$$\partial_{\bar{t}}(g\partial_i g^{-1})_j^k = \beta^2[C_i, g\bar{C}_{\bar{t}}g^{-1}]_j^k \quad (64)$$

As we will see in section 1.8, the geometrical picture emerging from these equations is closely connected with the special geometry for special Kähler manifolds.

## 1.6 Landau-Ginzburg Description

Let us consider Landau-Ginzburg  $N=2$  quantum field theories. They are characterized by a superpotential  $W$ , the F-term, which is a holomorphic function of  $n$  chiral superfields  $X_A$ , and a D-term defined by a Kähler potential  $K(X_A, \bar{X}_A)$ . Using the canonical potential

$$K(X_A, \bar{X}_A) = \sum_{A=1}^n X_A \bar{X}_A \quad (65)$$

the lagrangian reads

$$\mathcal{L} = \int d^2z d^4\theta \sum_{A=1}^n \theta X_A \bar{X}_A + \int d^2z d^2\theta^+ W(X) + \int d^2z d^2\theta^- \bar{W}(\bar{X}) \quad (66)$$

Defining the superfields

$$\begin{aligned} X_A &= (x_A, \psi_A, \bar{\psi}_A, F_A) \\ \bar{X}_A &= (\bar{x}_A, \rho_A, \bar{\rho}_A, \bar{F}_A) \end{aligned} \quad (67)$$

and after eliminating the F fields, using for that the equations of motion

$$F_A = \frac{\partial \bar{W}}{\partial \bar{X}_A}, \quad \bar{F}_A = \frac{\partial W}{\partial X_A} \quad (68)$$

we get in components<sup>6</sup>

$$\mathcal{L} = \int d^2z (-|\partial x_A|^2 + \psi \bar{\partial} \rho_A + \bar{\psi}_A \partial \bar{\rho}_A - |\partial_A W|^2 + \partial_A \partial_B W \psi_A \bar{\psi}_B + \bar{\partial}_A \bar{\partial}_B \bar{W} \rho_A \bar{\rho}_B) \quad (69)$$

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<sup>6</sup>Here we assume that  $\bar{W}$  is the complex conjugate of  $W$ .



The F-term of lagrangian (66) is given by

$$\mathcal{L}^F = \partial_A \partial_B W \psi_A \bar{\psi}_B \quad (70)$$

and the  $\bar{F}$ -term by

$$\mathcal{L}^{\bar{F}} = \bar{\partial}_A \bar{\partial}_B \bar{W} \rho_A \bar{\rho}_B - |\partial_A W|^2 \quad (71)$$

with the rest defining the D-term.

After twisting the lagrangian  $\mathcal{L}$ , the fields  $\psi_A, \bar{\psi}_A$  will become one forms and  $\rho_A, \bar{\rho}_A$  zero forms. This is, as usual, the net effect of the coupling to a  $\mathcal{U}(1)$  gauge field defined as 1/2 of the spin connection. Moreover, in the twisted theory the  $\bar{F}$  and D-terms become BRST-exact forms and, therefore, we can define the topological field theory by the F-term lagrangian (70). The BRST-cohomology is given by [9, 10]

$$\mathcal{R}_W = \frac{C[X_A]}{[W'(X_A)]} \quad (72)$$

i.e. the set of polynomials in the chiral superfields  $X_A$  modulo the ideal generated by  $W'(X_A)$ .

What is known as the Landau-Ginzburg representation of a TFT is to find a superpotential  $W$  such that (72) coincides with the chiral ring. Given a superpotential  $W$  and a basis  $\{\phi_i(X_A)\}$  of  $\mathcal{R}_W$ , the ring structure constants are defined by

$$\phi_i(X) \phi_j(X) = C_{ij}^k \phi_k(X) \text{ mod } W' \quad (73)$$

It is important at this point to realize the different behaviour under scale transformations of the world sheet metric

$$g \rightarrow \lambda^2 g \quad (74)$$

of the F and  $\bar{F}$  lagrangians (70) and (71). While (70) is invariant under transformation (74), the  $\bar{F}$ -term will transform<sup>7</sup>

$$\mathcal{L}^{\bar{F}} \rightarrow \lambda^2 \mathcal{L}^{\bar{F}} \quad (75)$$

Due to the invariance of (70), the scale transformations will act as automorphisms of the chiral ring (72). This is consistent with the non-renormalization theorems for  $N = 2$  quantum field theories. These theorems, which are mainly based on the holomorphicity

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<sup>7</sup>The transformation law (75) in the twisted theory comes from the fact that  $\rho_A, \bar{\rho}_A$  are, after twisting, zero forms.

of the superpotential, imply that F-terms are not corrected perturbatively and even non-perturbatively. Hence the renormalization group transformation will preserve the chiral ring structure (72) which only depends on F-terms.

Let us denote  $|i, t, \bar{t}; \beta\rangle$  the state associated with the observable  $\phi_i(X_A)$ . The functional integral representation of this state can be formally written like

$$|i, t, \bar{t}; \beta\rangle = \int \prod_{A=1}^n dX_A d\bar{X}_A \phi_i(X) \exp(-\int_H \mathcal{L}^F) \exp(-\int_H \mathcal{L}^{\bar{F}}) \quad (76)$$

where the integration in the exponents is on the hemisphere  $H$  used to define the state. The parameter  $\beta$ , as usual, represents the perimeter of the hemisphere. The transformation  $g \rightarrow \lambda^2 g$  is now interpreted in two complementary ways. First, it changes the non conformal part of the lagrangian in the way described above. Secondly and based on the non renormalization of the superpotential  $W$ , the change  $z \rightarrow \lambda z$ ,  $\theta \rightarrow \lambda^{-1/2} \theta$  amounts to a change  $\int dz^2 d\theta^2 W \rightarrow \lambda \int dz^2 d\theta^2 W$  which can be compensated by changing the couplings  $t_i$ . This change of couplings in terms of the scale  $\lambda$  would define the renormalization group  $\beta$ -functions for the different couplings. Using these two facts and the equations of motion for (69), we obtain [4]

$$\beta^2 \frac{\partial}{\partial \beta^2} |i, t, \bar{t}; \beta\rangle = -(\oint J_0^5 + \frac{n}{2}) |i, t, \bar{t}; \beta\rangle + Q^+ - \text{exacts} \quad (77)$$

with  $\beta^2 \sim \lambda \bar{\lambda}$ ,  $\lambda$  in general complex. The factor  $\frac{n}{2}$  comes from the contribution of the zero modes<sup>8</sup>.

At this point we can compose the previous computation with the one we will perform for the twisted lagrangian (41). In that case the  $\beta$  dependence will come directly from the twist term, which under dilatations transforms as

$$\int j \wedge d\phi \rightarrow \int j \wedge d(\phi + \epsilon) \quad (78)$$

with  $d\phi$  the spin connection for the metric  $g_{z\bar{z}} = e^\phi dz d\bar{z}$  [12].

From equation (77) we can easily obtain the dependence on  $\beta$  of the  $t\bar{t}$ -metric at the conformal point<sup>9</sup>

$$g_{i\bar{i}} \sim (\beta^2)^{-q_i - \frac{n}{2}} \quad (79)$$

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<sup>8</sup>The reader should be aware here that the only non conformal piece of the lagrangian (69) is the  $\bar{F}$ -part.

<sup>9</sup>Recall that for Landau-Ginzburg models  $\hat{c} = \sum_{i=1}^n (1-2q_i)$ , with  $q_i$  the charge of the chiral field  $X_A$  and  $n$  the number of chiral fields. This representation of  $\hat{c}$  can be derived using singularity theory (see[11]).

with  $q_i$  the Ramond charge of the state  $|i\rangle$ .

We can now read equation (77) as defining a connection  $A_{\beta i}^j$ . The  $t\bar{t}$ -equation for this connection is

$$\partial_{\bar{t}} A_{\beta i}^j = \beta^2 [C_W, \bar{C}_{\bar{t}i}^j] \quad (80)$$

where  $C_W$  means multiplication by  $W$  in  $\mathcal{R}_W$ . At the conformal point we get  $\partial_{\bar{t}} A_{\beta i}^j = 0$  since the quasi homogeneity of the superpotential implies  $C_W = 0$ .

### 1.6.1 Landau-Ginzburg Representation

In this subsection we will consider the problem of defining a Landau-Ginzburg representation for the TFT defined by the lagrangian

$$\mathcal{L} = \mathcal{L}_0^{N=2} + \sum_i t_i \int \phi_i^{(2)} \quad (81)$$

with  $\mathcal{L}^{N=2}$  representing a twisted  $N=2$  SCFT. We will consider all  $\bar{t}_i$ -deformations equal to zero.

For a minimal  $N=2$  SCFT at level  $k$  (25)-(27), the chiral ring is defined by

$$\begin{aligned} \phi_i \phi_j &= \phi_{i+j} & i+j &\leq k \\ &= 0 & i+j &> k \end{aligned} \quad (82)$$

This is isomorphic to the ring  $\mathcal{R}_W$  for

$$W = \frac{X^{k+2}}{k+2} \quad (83)$$

with only one chiral superfield  $X$ . The isomorphism is defined by

$$\phi_i \rightarrow X^i \quad (84)$$

We consider now the deformed lagrangian (81) and look for a superpotential  $W(X, t)$  such that the corresponding Landau-Ginzburg lagrangian is equivalent to it. This in particular will mean that

$$\langle \phi_{i_1}(X, t) \dots \phi_{i_s}(X, t) \rangle_{W(X, t)} = \langle \phi_{i_1}(X) \dots \phi_{i_s}(X) \rangle_{\mathcal{L}} \quad (85)$$

for

$$\phi_i(X, t) = -\frac{\partial W}{\partial t_i} \quad (86)$$

and where the l.h.s. of (85) is computed with the Landau-Ginzburg F-term lagrangian<sup>10</sup> and the r.h.s. with the lagrangian (81).

Next we will follow the discussion in ref.[13] for determining  $W(X, t)$ . Let us assume

$$\phi_1 = X \tag{87}$$

and define the perturbed ring structure as

$$X\phi_i = \phi_{i+1} + \sum_j a_{ij} t_{j-i+n+1} \phi_j \tag{88}$$

where we assign  $\mathcal{U}(1)$  charge  $1 - q_i$  to the couplings  $t_i$ . The constants  $a_{ij}$  are given by

$$a_{ij} = \langle \phi_1 \phi_i \phi_j \int \phi_{2k+1-i-j}^{(2)} \rangle_0 \tag{89}$$

We can determine the value of  $a_{ij}$  by the following argument. For the perturbed theory defined by  $t_j=0 \ \forall j \neq 1$ ,  $t_1 \equiv t$ , we get

$$\begin{aligned} \phi_i \phi_j &= \phi_{i+j} & i+j \leq k \\ &= t a_{ij} \phi_{i+j-k-1} & i+j > k \end{aligned} \tag{90}$$

Imposing associativity to (90), with  $a_{ij}$  again given by (89), we obtain

$$\begin{aligned} a_{ij} &= 0 & i+j \leq k \\ &= \mu & i+j > k \end{aligned} \tag{91}$$

for  $\mu$  some undetermined constant. Introducing this solution into (88), we obtain polynomials in  $X$  and  $t$  for representing the chiral fields  $\phi_i$ . The only thing that remains now, is to get the superpotential with respect to which (88) is the ring multiplication. A nice way to interpret (88) is as diagonalizing the matrix  $(C_1)_i^j$  defined by the multiplication rule, namely

$$\phi_1 \phi = C_1^j_i \phi_j = X \phi_i \tag{92}$$

and therefore we can define  $W$  by the characteristic equation determining the eigenvalues of  $(C_1)_i^j$

$$W'(X, t) = \det(X \delta_i^j - C_1^j_i(t)) \tag{93}$$

This conclude the derivation of the superpotential associated with the deformed lagrangian (81). The result however will depends on the renormalization constant  $\mu$ . A

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<sup>10</sup>See subsection 1.6.2 on residue formulae.

change in the scale  $\mu$  can be represented by a change  $t_i \rightarrow \mu t_i$  in the couplings (see equation (88)). For a generic correlator the dependence on  $\mu$  will be  $\mu^s$  with  $s$  the number of integrated fields entering into the correlator. If we also scale the fields  $\phi_i$  as  $\phi_i \rightarrow \mu^{-q_i} \phi_i$ , we get an overall factor  $\mu^{(-\sum q_i)+s}$ . By  $U(1)$ -charge conservation, this factor is equal to  $\mu^{\tilde{c}/2(2-2g)}$  and therefore can be cancelled by introducing the string coupling constant coefficient  $\lambda^{2g-2}$  for  $\lambda = \mu^{-\tilde{c}/2}$ .

### 1.6.2 Residue Formulae

Here we summarize the way to compute correlators in Landau-Ginzburg theory. We will assume for the rest of this section that  $\bar{W}$  is the complex conjugate of  $W$ . We consider the lagrangian

$$\mathcal{L} = \mathcal{L}^D + \mathcal{L}^F + \tilde{\lambda} \mathcal{L}^{\bar{F}} \quad (94)$$

where  $\mathcal{L}^D$ ,  $\mathcal{L}^F$  and  $\mathcal{L}^{\bar{F}}$  were defined in section 1.6. In the infrared limit  $\tilde{\lambda} \rightarrow \infty$  the main contribution to the Landau-Ginzburg action comes from critical configurations [14]

$$\frac{\partial W}{\partial X_A} = 0 \quad (95)$$

and the only contribution to the expectation values will come from zero modes. For the bosonic part of  $\mathcal{L}^{\bar{F}}$ , we get

$$\int \prod dX_A \exp(-\tilde{\lambda} |\partial_i W|^2) \quad (96)$$

which by gaussian integration around the critical points (95), gives us

$$\tilde{\lambda}^{-n} (H \bar{H})^{-1} \quad (97)$$

with  $H = \det(\partial_A \partial_B W)$ , the hessian of  $W$ . The fermionic contribution contains two pieces, one from the constant zero mode  $\rho_A^{(0)}$  and the other from the  $g$  holomorphic one forms  $\psi_A^{(0)}$ , if we are computing the correlator in a genus  $g$  Riemann surface. Hence the fermion zero mode contribution is

$$\tilde{\lambda}^n H^g \bar{H} \quad (98)$$

and therefore we get for the correlator

$$\langle \phi_{i_1} \dots \phi_{i_s} \rangle_W^g = \sum_{crit. points} \phi_{i_1}(X) \dots \phi_{i_s}(X) H^{g-1} \quad (99)$$

where  $\phi_{i_j}$  and  $H$  are evaluated at the critical points. Notice that the final result is  $\tilde{\lambda}$ -independent and therefore we can use (99) as the general definition of Landau-Ginzburg correlators. At genus  $g=0$  and for a "target space" of dimension one, we obtain

$$\langle \phi_{i_1} \dots \phi_{i_s} \rangle_W^{g=0} = \text{res} \left( \frac{\phi_{i_1} \dots \phi_{i_s}}{\partial W} \right) \equiv \frac{1}{2\pi i} \oint_{\Gamma} dX \frac{\phi_{i_1}(X) \dots \phi_{i_s}(X)}{\partial W} \quad (100)$$

with the contour  $\Gamma$  going around the critical points of  $W$  [14], [13]. Notice that at genus zero, in order to get (100), we have already integrated over the  $\rho$ -zero modes. The result (100) is not invariant under the scaling  $W \rightarrow \lambda W$ .

## 1.7 Frobenius Manifolds

The concept of Frobenius manifolds [15] is an useful mathematical tool for formalizing the structure of TFT's. Given a commutative and associative algebra  $A$ , with unity and non-degenerate invariant inner product

$$\langle a, bc \rangle = \langle ab, c \rangle \quad (101)$$

we will say that it is Frobenius if for a basis  $e_i$  ( $i=1, \dots, n$ ) of  $A$ , the tensors  $\eta_{ij}$  and  $C_{ijk}$  defined by

$$\begin{aligned} \langle e_i, e_j \rangle &= \eta_{ij} \\ e_i e_j &= C_{ij}^k e_k \end{aligned} \quad (102)$$

satisfy the following conditions

$$\begin{aligned} \eta_{ij} &= \eta_{ji} \\ C_{ij}^s C_{sk}^l &= C_{is}^l C_{jk}^s \\ C_{ijk} &= C_{ij}^l \eta_{lk} \end{aligned} \quad (103)$$

and for  $e = (e^i)$ , the unit of  $A$

$$e^s C_{sj}^i = \delta_i^j \quad (104)$$

Notice that (103) are the generic conditions that we have imposed on the topological metric  $\eta_{ij}$  and the ring structure constants  $C_{ijk}$  of a TFT.

A Frobenius manifold  $M$  is a manifold which locally is a Frobenius algebra. This means that at each point  $x \in M$ , there exists tensors  $\eta_{ij}(x)$ ,  $C_{ij}^k(x)$  and unity  $e^i(x)$  satisfying conditions (103) and (104). We define the invariant metric on  $M$

$$ds^2 = \eta_{ij} dx^i dx^j \quad (105)$$

relative to which the unit vector is covariantly constant.

For the metric (105), we can define local coordinates  $t_i$  on  $M$  such that [15]

$$\eta_{ij} = c t e \quad (106)$$

We will call these coordinates coupling constants. The tensor  $C_{ijk}(t)$  in these coordinates satisfy the integrability condition (see equation (58))

$$\partial_i C_{jkl} = \partial_j C_{ikl} \quad (107)$$

which means that it can be represented

$$C_{ijk} = \partial_i \partial_j \partial_k F(t) \quad (108)$$

with  $F(t)$  being determined by the following set of equations

$$\begin{aligned} \frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k} \eta^{kl} \frac{\partial^3 F(t)}{\partial t_l \partial t_m \partial t_n} &= \frac{\partial^3 F(t)}{\partial t_i \partial t_m \partial t_k} \eta^{kl} \frac{\partial^3 F(t)}{\partial t_l \partial t_j \partial t_n} \\ \frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k} &= C_{ijk} \end{aligned} \quad (109)$$

For  $\eta_{ij}$  the topological metric and  $C_{ijk}$  the genus zero three point function of a TFT it is easy to derive, by means of standard Ward identities, equations (106)-(107) [13], using for such purpose the lagrangian

$$\mathcal{L} = \mathcal{L}^{(0)} + \sum_i t_i \int \phi_i^{(2)} \quad (110)$$

Thus the coordinates  $t_i$  can be identified with the coupling constants in (110).

As an example we will consider the Frobenius manifold associated with the Landau-Ginzburg superpotential for minimal models (see section 1.6.1). Let  $M$  be defined by the following set of polynomials

$$M = \{W(X, g_i) = X^{k+2} - (k+2) \sum_{i=0}^k g_i X^i\} \quad (111)$$

The invariant inner product will be given by the residue formula derived in the previous section

$$\langle f, g \rangle = \text{res}\left(\frac{fg}{W'}\right) \quad (112)$$

We can find the flat coordinates  $t_i$  using the condition  $\eta_{ij} = cte$  and the inner product (112). The result is

$$t_i = -\frac{1}{k+1-i} \text{res}\left(W^{\frac{k+1-i}{k+2}}\right) \quad (113)$$

which defines the change from the Landau-Ginzburg coordinates  $g_i$  into the coupling constants  $t_i$ . We will come back to equation (113) in a future section.

### $t\bar{t}$ -equations and Topological-Antitopological Fusion

It is known [15] that the  $t\bar{t}$ -equations can be interpreted as the zero curvature condition for the system of linear differential equations

$$\nabla_i \Psi = \bar{\nabla}_{\bar{i}} \Psi = 0 \quad (114)$$

for

$$\begin{aligned} \nabla_i &= \partial_i + (g\partial_i g^{-1}) - \lambda C_i \\ \bar{\nabla}_{\bar{i}} &= \partial_{\bar{i}} - \lambda^{-1} \bar{C}_{\bar{i}} \end{aligned} \quad (115)$$

with  $\lambda$  a spectral parameter.

The mathematical meaning of (114) and (115) as a way to fuse a topological and antitopological theory was pointed out by Krichever in [16]. Given two topological theories characterized by  $C_i$  and  $\bar{C}_{\bar{i}}$  respectively as ring structure constants, we define

$$\begin{aligned} (\partial_i - \lambda C_i)\Phi &= 0 \\ (\partial_{\bar{i}} - \lambda^{-1} \bar{C}_{\bar{i}})\bar{\Phi} &= 0 \end{aligned} \quad (116)$$

with  $\Phi(t, \lambda)$  and  $\bar{\Phi}(t, \lambda^{-1})$ . The essential singularities in (116) are in  $\lambda=0$  and  $\lambda=\infty$ . The  $t\bar{t}$  fusion corresponds to the Riemann-Hilbert problem of defining a functional  $\Psi(\lambda, t, \bar{t})$  such that at  $\lambda=0$  behaves like  $\Phi$  and at  $\lambda=\infty$  like  $\bar{\Phi}$ . The solution to this problem is determined by equations (114) and (115). The  $t\bar{t}$ -equations admit now the interpretation of the isomonodromy equations for (114) and (115).



## 1.8 $t\bar{t}$ -equations and Special Geometry

It is clear that the  $t\bar{t}$ -equations provide an extra geometrical structure on the space of topological field theories. In general the space of couplings constants is a complex manifold with coordinates  $(t_i, \bar{t}_i)$ <sup>11</sup> and we can define in addition to the topological metric  $\eta_{ij}$  the hermitian metric

$$ds^2 = g_{i\bar{j}} dt^i d\bar{t}^{\bar{j}} \quad (117)$$

In section (1.5) we introduced a connection  $A_i$  such that both  $g_{i\bar{j}}$  and  $\eta_{ij}$  are covariantly constant. This connection can be written in terms of  $g_{i\bar{j}}$  as in (62). Moreover,  $A_i$  defines a connection for the bundle obtained by fibering the space of couplings with the BRST cohomology. The  $t\bar{t}$ -equations satisfied by the connection  $A_i$  are, structurally, very similar to the ones defining special Kähler geometry [17]. Before entering into a more detailed technical discussion, let us try to understand intuitively the physical origin of this Kähler structure. The pieces we need for this discussion have been already introduced in the previous sections and are intimately connected with the meaning of renormalization group.

First of all, and reducing the discussion to Landau-Ginzburg theories, we observe two interconnected phenomena

i) A reparametrization  $W \rightarrow \lambda W$  in the superpotential induces a flow of the coupling constants.

ii) A world sheet reparametrization  $g \rightarrow \lambda g$  modifies the  $\bar{F}$ -part of the Landau-Ginzburg lagrangian. Recall that the  $\bar{F}$ -term is the non conformal part of the twisted LG lagrangian.

From i) and ii) follows that a rescaling  $g \rightarrow \lambda g$  of the world sheet metric induces both a change in  $t$  and  $\bar{t}$  couplings. A point of view to understand the physical meaning of the  $t\bar{t}$ -equation is as the lifting of this renormalization group flow on the  $t\bar{t}$  plane to the fiber defined by the set of harmonic or zero energy states. If now we think the "vacuum", the state of Ramond charge  $-\hat{c}/2$ , as defining a line subbundle, i.e. we assume conservation of charge, it is natural to translate the  $(t\bar{t})$  geometry into the characterization of the first Chern class of the vacuum subbundle. Mathematically this picture will become clear if the " $(t\bar{t})$ -plane" defines a Kähler Hodge manifold.

To check this intuitive picture requires to be able to define the vacuum as a line bundle

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<sup>11</sup>Notice the difference between the Frobenius manifold defined in the previous subsection, which contains only the couplings  $(t_i)$ , and the full space of coupling constants  $(t_i, \bar{t}_i)$ .

on the space of theories. This can be done in a very particular case, namely when we are working on the moduli space of a  $N=2$  SCFT. In this case and due to the independent left and right conservation of  $\mathcal{U}(1)$  current we can decompose the bundle defined by the BRST cohomology into different charge sectors. Moreover we get the constraint on the  $t\bar{t}$ -metric

$$g_{i\bar{j}} = 0 \quad \text{if } q_i + q_{\bar{j}} \neq 0 \quad (118)$$

Introducing the unit of the ring by

$$C_{i0}^j = C_{0i}^j = \delta_i^j \quad (119)$$

we get from the  $t\bar{t}$ -equations, reduced to marginal fields

$$[\partial_{\bar{j}}(g\partial_i g^{-1})]_0^0 = C_{i0}^k g_{k\bar{l}} \bar{C}_{\bar{j}0}^{\bar{l}} g^{\bar{0}0} = \frac{g_{i\bar{j}}}{g_{0\bar{0}}} \quad (120)$$

Taking into account that

$$g_{0\bar{0}} = \langle \bar{0} | 0 \rangle \quad (121)$$

we can use (120) to define a Kähler potential and a Zamolodchikov metric [18] as

$$\begin{aligned} G_{i\bar{j}} &\equiv \frac{g_{i\bar{j}}}{g_{0\bar{0}}} \\ K &= -\log \langle \bar{0} | 0 \rangle \end{aligned} \quad (122)$$

in such a way that

$$G_{i\bar{j}} = \partial_{\bar{j}} \partial_i K \quad (123)$$

i.e. the standard definition of Kähler metric. Using (122) we obtain the decomposition of the connection  $A_i = (\partial_i g)g^{-1}$  into two pieces. The first is the Kähler connection on the moduli space, defined as usual by

$$\Gamma_{i\bar{j}}^k = (\partial_i G_{\bar{j}\bar{k}}) G^{\bar{k}k} \quad (124)$$

and a second piece corresponding to the  $\mathcal{U}(1)$  connection of the line bundle generated by the vacuum

$$-\partial_i K \quad (125)$$

which as usual for the Hodge-Kähler manifolds, is defined in terms of the Kähler potential  $K$ . Comparing (79) (section 1.6) with (121) we observe that  $K$  is determined by the contribution of fermionic zero modes.

### The $\hat{c}=3$ case and special geometry

We will introduce first some generalities on special Kähler manifolds. On a Hodge-Kähler manifold we introduce fields  $\phi_{p\bar{p}}$  of Kähler weight  $(p, \bar{p})$  by the following transformation rule

$$\phi_{p\bar{p}} \rightarrow \phi_{p\bar{p}} e^{-\frac{p}{2}f} e^{-\frac{\bar{p}}{2}\bar{f}} \quad (126)$$

with the  $\mathcal{U}(1)$  gauge connection transforming like

$$\partial_i K \rightarrow \partial_i K + \partial_i f \quad (127)$$

for  $f, \bar{f}$  respectively holomorphic and antiholomorphic functions. The covariant derivatives of these fields are defined by

$$\begin{aligned} D_i \phi_{p\bar{p}} &= \left( \partial_i + \frac{p}{2} \partial_i K - \Gamma_i \right) \phi_{p\bar{p}} \\ \bar{D}_{\bar{i}} \phi_{p\bar{p}} &= \left( \partial_{\bar{i}} + \frac{\bar{p}}{2} \partial_{\bar{i}} K \right) \phi_{p\bar{p}} \end{aligned} \quad (128)$$

If  $\phi_{p\bar{p}}$  is covariantly holomorphic

$$\bar{D}_{\bar{i}} \phi_{p\bar{p}} = 0 \quad (129)$$

Then we can define the holomorphic field  $\tilde{\phi}_{p\bar{p}}$  as

$$\tilde{\phi}_{p\bar{p}} = e^{\frac{p}{2}K} \phi_{p\bar{p}} \quad (130)$$

which is a  $(p - \bar{p}, 0)$  field. A Hodge-Kähler manifold is special if there exists a symmetric tensor  $W_{ijk}$  of Kähler weight  $(2, -2)$ , such that [17]

$$\begin{aligned} \bar{D}_{\bar{l}} W_{ijk} &= 0 \\ D_i W_{jkl} &= D_j W_{ikl} \\ R_{i\bar{j}}{}^l{}_k &= G_{k\bar{j}} \delta_i^l + G_{i\bar{j}} \delta_k^l - W_{ikn} \bar{W}_{j\bar{n}\bar{m}} G^{\bar{n}n} G^{\bar{m}l} \end{aligned} \quad (131)$$

Using (130), we can define a holomorphic tensor  $C_{ijk}$  as

$$C_{ijk} = e^{-K} W_{ijk} \quad (132)$$

which has weight  $(4, 0)$ . From the integrability condition (131.b), we can find a "covariant" prepotential  $\hat{S}$  verifying

$$W_{ijk} = D_i D_j D_k \hat{S} \quad (133)$$

with  $\hat{S}$  again of Kähler weight  $(2, -2)$ . Analogously we have

$$\bar{W}_{i\bar{j}\bar{k}} = \bar{D}_i \bar{D}_{\bar{j}} \bar{D}_{\bar{k}} \check{S} \quad (134)$$

with  $\check{S}$  of weight  $(-2, 2)$ . Defining the holomorphic tensor  $\bar{C}_{i\bar{j}\bar{k}}$  of weight  $(0, 4)$  as

$$\bar{C}_{i\bar{j}\bar{k}} \equiv e^{-K} \bar{W}_{i\bar{j}\bar{k}} \quad (135)$$

and using  $e^K \bar{D}_i \check{S} = \partial_i e^K \check{S}$ , we get

$$\bar{C}_{i\bar{j}\bar{k}} = e^{2K} \bar{D}_i \bar{D}_{\bar{j}} \partial_{\bar{k}} S \quad (136)$$

for

$$S \equiv e^K \check{S} \quad (137)$$

The covariant prepotential  $S$  allow us to integrate the special geometry relation (131.c). Let us define

$$\bar{C}_i^{lk} \equiv e^{2K} \bar{C}_{i\bar{l}\bar{k}} G^{\bar{l}l} G^{\bar{k}k} \quad (138)$$

with the property

$$\bar{C}_i^{lk} = \partial_i S^{lk} \quad (139)$$

where

$$S^{lk} = G^{\bar{l}l} \partial_{\bar{l}} (G^{\bar{k}k} \partial_{\bar{k}} S) \quad (140)$$

Using this and the holomorphicity of  $C_{ijk}$ , we can write (131.c) as

$$\begin{aligned} \partial_{\bar{j}} \Gamma_{ik}^l &= G_{k\bar{j}} \delta_i^l + G_{i\bar{j}} \delta_k^l - C_{ikn} \bar{C}_{\bar{j}}^{nl} = \\ &= \partial_{\bar{j}} (\partial_k K \delta_i^l + \partial_i K \delta_k^l - S^{ln} C_{ikn}) \end{aligned} \quad (141)$$

Integrating (141) we obtain

$$\Gamma_{ik}^l = \partial_i K \delta_k^l + \partial_k K \delta_i^l - S^{ln} C_{ikn} + f_{ik}^l \quad (142)$$

with  $f_{ik}^l$  a holomorphic tensor.

After this brief description of the special geometry, our next task will be to identify the Kähler metric  $G_{i\bar{j}}$  with the Zamolodchikov metric and the tensor  $C_{ijk}$  with the three point function. It is only in the particular case  $\hat{c} = 3$ , where we have non vanishing three point functions on the sphere for marginal fields, that all the indices in (141) can be marginal. After identifying the line bundle of the Hodge-Kähler manifold with the one generated by

the vacuum, we can define the state  $|0\rangle$  by a holomorphic section of weight  $(2, 0)$ . In this case the first Chern class can be defined in terms of the norm of  $|0\rangle$  as

$$\partial_i \partial_{\bar{j}} (\log \| |0\rangle \|^2) dz^i \wedge d\bar{z}^j \quad (143)$$

and we get

$$\langle \bar{0} | 0 \rangle = e^{-K} \quad (144)$$

in agreement with equation (122). Summarizing, in order to map the  $t\bar{t}$ -geometry of the moduli space of  $\hat{c}=3$   $N=2$  SCFT's into the special Kähler geometry we need to identify the vacuum state with the trivializing holomorphic section of the Hodge line bundle.

For Landau-Ginzburg theories we can write (144) as

$$\langle \bar{0} | 0 \rangle = \int \prod_{A=1}^n dX_A d\bar{X}_A \exp(W - \bar{W}) \quad (145)$$

which makes explicit the  $t, \bar{t}$  dependence.

## 2 Topological Strings

### 2.1 Topological Gravity and Gravitational Descendants

Topological gravity is the topological theory associated with the moduli space of Riemann surfaces  $\mathcal{M}_{g,n}$ . There are many good reviews on this subject so we will concentrate the discussion on some technical points that will be relevant for our future analysis.

The aim of topological gravity is to get a topological field theory representation of Mumford-Morita cohomology classes [19]. Given a Riemann surface  $\Sigma_{g,n}$  with genus  $g$  and  $n$  marked points, we can consider its cotangent line bundle at one of the points, namely  $p_i$ . When we move the moduli parameters of the surface, this defines a line bundle over  $\mathcal{M}_{g,n}$ . Let's denote by  $\alpha_i$  the first Chern class of this bundle. The physical observables of topological gravity  $\sigma_n(p_i)$ , called gravitational descendants, are one to one related with the n-power of the two form  $\alpha_i$  in such a way that

$$\langle \sigma_{n_1}(p_1) \dots \sigma_{n_s}(p_s) \rangle_g = \int_{\mathcal{M}_{g,s}} \alpha_1^{n_1} \wedge \dots \wedge \alpha_n^{n_s} \quad (146)$$

From (146) we see that these amplitudes will be non vanishing only when it is fulfilled the following selection rule

$$\sum_{i=1}^s n_i = 3g - 3 + n \quad (147)$$

The simplest, from a physical point of view, way to realize this topological field theory is to use the topological gauge theory for the group ISO(2) [20]. After fixing the gauge, the corresponding action is given by

$$S = \int \pi \partial \bar{\partial} \phi + \chi \partial \bar{\partial} \psi + \bar{b} \bar{\partial} c + \bar{b} \partial \bar{c} + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} \quad (148)$$

where  $\phi$  is the Liouville field,  $\psi$  its superpartner,  $\pi$  and  $\chi$  are Lagrange multipliers conjugate to  $\phi$  and  $\psi$ , and  $(b, c)$  and  $(\beta, \gamma)$  are respectively the ghost and superghost fields. The ghost number assignments are the following: zero for  $\phi$ , 1 for  $\psi$ , 1 for  $c$  and 2 for  $\gamma$ . The main ingredient to build a topological field theory is the existence of a supersymmetric charge  $Q_S$  which behaves, BRST-improved, as an exterior derivative of the moduli space under study. It is under  $Q_S$  that all the fields are arranged into supermultiplets.

For the action (148) the BRST charge is defined by

$$Q = Q_S + Q_g \quad (149)$$

being  $Q_S = \oint (\partial \pi \psi + b \gamma)$  and  $Q_g = \oint c ((T_L + \frac{1}{2} T_{gh}) + \gamma (G_L + \frac{1}{2} G_{gh}))$  respectively the  $N=2$  and the the gauge BRST charge.  $T_L$  and  $T_{gh}$  are the energy momentum tensors of the Liouville and the ghost sectors, and  $G_L, G_{gh}$  the corresponding super stress tensors. The topological nature of the action (148) is clear from the following relations

$$\begin{aligned} T_L &= \{Q_S, G_L\} \\ T_{gh} &= \{Q_S, G_{gh}\} \end{aligned} \quad (150)$$

Physical observables are defined by the BRST cohomology of (149). In topological gravity this cohomology turns out to be very simple. In fact all physical observables are given by interger powers of a field  $\gamma_0$

$$\gamma_0 = \{Q, \psi - \bar{\psi}\} = \frac{1}{2} \{Q, \{Q_S - \bar{Q}_S, \phi\}\} \quad (151)$$

where  $Q$  is the total BRST charge, i.e. left plus right. Therefore we can define

$$\sigma_n = \gamma_0^n \quad (152)$$

From (151) it looks that all observables of topological gravity are BRST-trivial. The reason this is not the case is because we are interested in equivariant cohomology where we define the BRST cohomology on gauge invariant objects. This doesn't include  $\gamma_0$ , since  $(\psi - \bar{\psi})$  is not a gauge invariant quantity. We will come back to the discussion on equivariant cohomology later.

The observables  $\sigma_n$  given by (152) are zero forms on the Riemann surface. We can define 1 and 2 forms by the following recursive relations

$$d\sigma_n = \delta_{BRST}\sigma_n^{(1)} \quad , \quad d\sigma_n^{(1)} = \delta_{BRST}\sigma_n^{(2)} \quad (153)$$

The new operators  $\sigma_n^{(1)}$  and  $\sigma_n^{(2)}$  can be integrated respectively over a one dimensional submanifold or the whole surface  $\Sigma$ , giving also BRST invariant objects.

We want now to find a functional integral representation for correlators  $\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_g$ . Instead of presenting the complete derivation we will, qualitatively, motivate the final result.

The first thing will be to write the action (148) in a covariant way. The formulation of the  $(\pi, \phi)$  system in (148) presents problems because the Liouville field  $\phi$  behaves inhomogeneously under coordinate transformations. In order to solve this, we can interpret  $\phi$  as the conformal factor for the metric  $g = e^{\phi}\hat{g}$ . It can be shown that the physical quantities are independent of the metric  $\hat{g}$  chosen. Under these conditions the conjugate field  $\pi$  gets coupled to the scalar curvature  $R(\hat{g})$ . Using  $\int \sqrt{\hat{g}}\hat{R} = 2g - 2$ , we must cancel this background curvature by inserting operators

$$\prod_k e^{\alpha_k \pi(z_k)} \quad (154)$$

in a set of points  $\{z_k\}$  and in such a way that the constants  $\alpha_k$  satisfy

$$\sum_k \alpha_k = 2g - 2 \quad (155)$$

Since the action (148) is supersymmetric, in order to define a measure on  $\mathcal{M}_{g,n}$ , we need to integrate first the superpartners  $\hat{m}_i$  of the moduli parameters  $m_i$ . The integration of the supermoduli can be easily done because in this rigid supersymmetry the supermoduli is split. The integration of the  $(3g-3)$  odd moduli parameters  $\hat{m}_i$  produces the insertion of super stress tensors folded to Beltrami differentials  $\chi_a, \bar{\chi}_{\bar{a}}$

$$\prod_{a, \bar{a}=1}^{3g-3} G(\chi_a) \bar{G}(\bar{\chi}_{\bar{a}}) \quad (156)$$

where  $G = G_L + G_{gh}$ . In a similar way, the integration over the superpartners of the puncture moduli will produce insertions of the supertranslation generator

$$\prod_{i,\bar{i}=1}^s \oint_{C_i} (b + G) \oint_{C_i} (\bar{b} + \bar{G}) \quad (157)$$

with the contour  $C_i$  defined around each puncture.

For the external states we must use

$$|\sigma_n\rangle = \sigma_n P|0\rangle \quad (158)$$

where  $P = c\bar{c}\delta(\gamma)\delta(\bar{\gamma})$  is the puncture operator that reduces the number of diffeomorphisms to those which leaves the puncture fixed. Let's recall that the measure over  $\mathcal{M}_{g,n}$  includes the necessary factors to project out the zero modes of the ghosts  $b\bar{b}$  and  $\delta(\beta)\delta(\bar{\beta})$ . Due to this, when we integrate the moduli of a puncture the operator  $P$  is reduced to 1. Acting now with (157) on (158), we get as net result for the external insertions  $\sigma_n$  in the amplitudes

$$\prod_{i=1}^s \int_{\Sigma} \sigma_{n_i}^{(2)} \quad (159)$$

where  $\sigma_{n_i}^{(2)}$  is given by (153) and has ghost number  $n_i - 1$ .

Combining now (154), (156) and (159) we obtain for the amplitudes the following integral representation

$$\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_g = \int_{\mathcal{M}_{g,s}} \int e^{-S} \prod_k e^{\alpha_k \pi(z_k)} \prod_{a,\bar{a}=1}^{3g-3} G(\chi_a) \bar{G}(\bar{\chi}_{\bar{a}}) \prod_{i=1}^s \int_{\Sigma} \sigma_{n_i}^{(2)} \quad (160)$$

By ghost number counting this will be non vanishing only if

$$\sum_{i=1}^s (n_i - 1) = 3g - 3 \quad (161)$$

in agreement with equation (147).

Let us next consider the coupling of topological matter to topological gravity [19]. The gravitational descendants  $\sigma_n(\phi_i)$  associated with the chiral primary fields are simply defined by

$$\sigma_{n,i} = \phi_i \sigma_n \quad , \quad n \geq 0 \quad (162)$$

Generic amplitudes  $\langle \sigma_{n_1}(\phi_{i_1}) \dots \sigma_{n_s}(\phi_{i_s}) \rangle_g$  are again defined by equation (160). The only additional information we need to take into account is the extra  $\mathcal{U}(1)$  background charge,



modifying the selection rule (161) to

$$\sum_{i=1}^s (q_i + n_i - 1) = 3g - 3 + \hat{c}(1 - g) \quad (163)$$

Therefore, we obtain

$$\langle \sigma_{n_1, i_1} \dots \sigma_{n_s, i_s} \rangle_g = \int_{\mathcal{M}_{g,s}} \int e^{-S} \prod_k e^{\alpha_k \tilde{\pi}(z_k)} \prod_{a, \bar{a}=1}^{3g-3} G(\chi_a) \bar{G}(\bar{\chi}_{\bar{a}}) \prod_{j=1}^s \int_{\Sigma} \sigma_{n_j, i_j}^{(2)} \quad (164)$$

where  $\tilde{\pi}$  is matter-modified conjugate of the Liouville field.

The action  $S$  in (164) is the unperturbed lagrangian  $\mathcal{L}_0^{N=2} + \mathcal{L}^{gr}$ , with  $\mathcal{L}^{gr}$  given by equation (148). Our next task will be to generalize (164) for a perturbed lagrangian. The approach will consist in generalizing the Landau-Ginzburg description to topological matter coupled to topological gravity.

## 2.2 Gravity and Landau-Ginzburg

In this section we will consider a TFT which possesses a Landau-Ginzburg description in terms of a superpotential  $W$ , coupled to topological gravity. This study will help in a better understanding of some results derived in section 1.6.

To start with, we will first present a crucial theorem due to Dijkgraaf-Witten. Let us consider the lagrangian general perturbed lagrangian

$$\mathcal{L} = \mathcal{L}_0^{N=2} + \mathcal{L}^{gr} + \sum_{i=0}^k t_i \int \phi_i^{(2)} + \sum_{n=1}^{\infty} \sum_{i=0}^k t_{i,n} \sigma_n(\phi_i)^{(2)} \quad (165)$$

where  $t_i \equiv t_{i,0}$  and the small phase space is defined by  $t_{i,n} = 0$ ,  $n \geq 0$ . The identity operator, after coupling to gravity becomes the puncture operator  $P$ .

Let us define the expectation values

$$u_i \equiv \langle P \phi_i \rangle, \quad i = 0, \dots, k \quad (166)$$

On the small phase space, we get

$$u_i = \eta_{ij} t_j \quad (167)$$

Taking into account that  $\eta_{ij}$  is invertible we can interpret  $u_i$  as a new set of coordinates on the small phase space. Moreover, for any correlator computed on the small phase space we obtain

$$\langle AB \rangle = R_{AB}(u_0, \dots, u_k) \quad (168)$$

for some functional  $R_{AB}$ . The theorem proved in [21] assures that the correlator  $\langle AB \rangle$  defined on the full phase space, i.e. for the lagrangian (165) with  $t_{i,n} \neq 0$ , is given by the same functional  $R_{AB}$  where now the coordinates  $u_i$  (166) are computed taking into account the couplings  $t_{i,n}$ . The proof of this theorem is based on the topological recursion relations. What we need to show is that

$$\frac{\partial}{\partial t_{i,n}} R_{AB} = \frac{\partial u_l}{\partial t_{i,n}} \frac{\partial R_{AB}}{\partial u_l} = \langle \sigma_n(\phi_i) AB \rangle \quad (169)$$

Using the recursion relations for topological strings [19]

$$\langle \sigma_n(\phi_i) AB \rangle = n \langle \sigma_{n-1}(\phi_i) \phi_l \rangle \langle \phi^l AB \rangle \quad (170)$$

we obtain

$$\begin{aligned} n \langle \sigma_{n-1}(\phi_i) \phi_l \rangle \langle \phi^l AB \rangle &= n \langle \sigma_{n-1}(\phi_i) \phi_l \rangle \frac{\partial R_{AB}}{\partial t_{l,0}} = \\ &= n \langle \sigma_{n-1}(\phi_i) \phi_l \rangle \langle \phi^l P \phi_k \rangle \frac{\partial R_{AB}}{\partial u_k} = \langle \sigma_n P \phi_k \rangle \frac{\partial R_{AB}}{\partial u_k} = \\ &= \frac{\partial u_k}{\partial t_{i,n}} \frac{\partial R_{AB}}{\partial u_k} \end{aligned} \quad (171)$$

which concludes the proof of (168). The practical relevance of this theorem is that allow us to get the form of the correlators on the full phase space by doing a much simpler computation on the small phase space.

As an illustrative example let us compute the string anomalous dimension for the critical points of one matrix models. We start with pure topological gravity, i.e. only one primary field, the puncture  $P$ . The small phase space is the complex line parametrizing the value of the cosmological constant  $t_0$ . In the small phase space we get

$$u = \langle PP \rangle = t_0 \quad (172)$$

and for a generic correlator

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{(i+j+1)!} \langle \sigma_i \sigma_j P^{i+j+1} \rangle t_0^{i+j+1} \quad (173)$$

Using the puncture equation [21]

$$\langle \sigma_i \sigma_j P \rangle = i \langle \sigma_{i-1} \sigma_j \rangle + j \langle \sigma_i \sigma_{j-1} \rangle \quad (174)$$

we can rewrite (173) as

$$\langle \sigma_i \sigma_j \rangle = \frac{1}{i+j+1} u^{i+j+1} \quad (175)$$

The previous theorem tell us that, on the full phase space, (175) is the correct value for  $\langle \sigma_i \sigma_j \rangle$  if we replace  $u$  by the value of  $\langle PP \rangle$  on the full phase space.

Taking into account all couplings we obtain

$$u = t_0 + \sum_{i=1}^{\infty} t_i \langle PP \sigma_i \rangle \quad (176)$$

and from (175)

$$u = t_0 + \sum_{i=1}^{\infty} t_i u_i \quad (177)$$

The  $k$ -th critical point [22] is defined by  $t_1 = 1$ ,  $t_k = -1$ , and from (177) we get

$$u = t_0^{1/k} \quad (178)$$

The string anomalous dimension  $\gamma$  is defined by

$$\langle 1 \rangle = t_0^{2-\gamma} \quad (179)$$

Therefore, using (178) we have Kazakov's result

$$\gamma = -\frac{1}{k} \quad (180)$$

Defining the string coupling constant  $\lambda$  by

$$\langle 1 \rangle = \lambda^{-2} \quad (181)$$

we obtain

$$\lambda^{-2} = t_0^{2+1/k} \quad (182)$$

Dijkgraaf-Witten theorem underlines the equivalence of matrix models and minimal topological strings. In fact, in the matrix model approach [23] we start with the KdV operator

$$L = D^{k+2} + (k+2) \sum_{i=0}^k v_i D^i \quad (183)$$

with  $D = \frac{d}{dX}$  for a formal parameter  $X$ . The KdV flows are defined by

$$\frac{\partial L}{\partial \tilde{t}_p} = [(L^{\frac{p}{k+2}})_+, L] \quad (184)$$

with  $L_+$  the positive powers of  $L$ . Identifying

$$\begin{aligned} v_k &= \langle PP \rangle \\ \tilde{t}_p &= t_{i,n} c_{i,n} \ , \quad p = (k+2)n + i + 1 \\ c_{i,n} &= ((i+1)(i+1+k+2)\dots(i+1+n(k+2)))^{-1} \end{aligned} \quad (185)$$

we obtain from (184)

$$\frac{\partial v_k}{\partial t_{i,n}} = c_{i,n} \text{res}(L^{\frac{(k+2)n+i+1}{k+2}}) \quad (186)$$

Denoting  $\hat{L} \equiv L^{\frac{1}{k+2}}$  and integrating  $X$ , we get

$$\langle P\sigma_n(\phi_i) \rangle = c_{i,n} \text{res}(\hat{L}^{(k+2)n+i+1}) \quad (187)$$

and similar relations for other correlators. From (187) we observe how the correlator on the full phase space is defined by a functional of the  $(k+1)$  "coordinates"  $v_k$  appearing in (183).

It is already clear the strong similarity of the matrix model formula (187) and the residue formula we have derived in the previous chapter for Landau-Ginzburg minimal models. Following references [13, 24, 25], we define the map from matrix models into Landau-Ginzburg theories as follows

$$\begin{aligned} \hat{L}^{k+2} &= W \\ \phi_i &= [\hat{L}^i \partial_X \hat{L}]_+ \\ \sigma_n(\phi_i) &= [\hat{L}^{n(k+2)+i} \partial_X \hat{L}]_+ c_{n-1,i} \\ P &= 1 \end{aligned} \quad (188)$$

which allows to represent the whole gravitational spectrum inside the matter theory<sup>12</sup>. Using this map we will now compare the matrix model expression for correlators with the one we will obtain, from residue formulae, in the Landau-Ginzburg framework.

For correlator  $\langle \phi_i P\sigma_n(\phi_j) \rangle$ , we have in the matrix model formalism

$$\langle \phi_i P\sigma_n(\phi_j) \rangle = \frac{\partial}{\partial t_i} \langle P\sigma_n(\phi_j) \rangle = c_{j,n} \frac{\partial}{\partial t_i} \text{res}(\hat{L}^{(k+2)n+j+1}) \quad (189)$$

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<sup>12</sup>For the extension of this map to  $W$ -gravity see [26].

On the other hand, using (188) and the residue formulae, we get in the Landau-Ginzburg formalism

$$\begin{aligned}\langle \phi_i P \sigma_n(\phi_j) \rangle &= c_{j,n-1} \int \left( \frac{\hat{L}^{n(k+2)+j} \partial_X \hat{L}}{W'} \phi_i \right) dX = \\ &= c_{j,n-1} \int \hat{L}^{j+1+(n-1)(k+2)} \phi_i dX\end{aligned}\quad (190)$$

It is worth recalling that working in the Landau-Ginzburg formalism, we are always at genus zero. If we confine ourselves to the small phase space, we can use the relation

$$\phi_i = \frac{\partial \hat{L}^{k+2}}{\partial t_i} \quad (191)$$

From this we get

$$\begin{aligned}\langle \phi_i P \sigma_n(\phi_j) \rangle &= c_{j,n-1} \int dX \hat{L}^{j+n(k+2)} \partial_i \hat{L} = \\ &= c_{j,n} \frac{\partial}{\partial t_i} \text{res}(\hat{L}^{(k+2)n+j+1})\end{aligned}\quad (192)$$

in agreement with (189). Notice that in principle (189) is a well defined expression on the full phase space and the same for (190) if we replace  $\phi_i$  by  $[\hat{L}^i \partial_X \hat{L}]_+$ , however, only on the small phase space we can use equation (191). We will come back to this point in the next section.

## 2.3 Contact Terms and Gravitational Descendents

Let us consider the correlator  $\langle \phi_i \phi_j \phi_k \int \phi_l^{(2)} \rangle$  in Landau-Ginzburg theories. From the residue formulae we obtain

$$\langle \phi_i \phi_j \phi_k \int \phi_l^{(2)} \rangle = \frac{\partial}{\partial t_l} \int \frac{\phi_i \phi_j \phi_k}{W'} dX = \quad (193)$$

$$= - \int \frac{\phi_i \phi_j \phi_k \phi_l'^2}{W'^2} dX + \int \frac{1}{W'} \left[ \left( \frac{\partial \phi_i}{\partial t_l} \right) \phi_j \phi_k + \dots \right] \quad (194)$$

Using (188) and (191) we get

$$\frac{\partial \phi_i}{\partial t_j} = \frac{\partial}{\partial t_j} [\hat{L}^i \partial_X \hat{L}]_+ = \left[ \frac{\phi_i \phi_j'}{W'} \right]_+ \equiv C(\phi_i, \phi_j) \quad (195)$$

which are known as contact terms. From the matter representation (188) of gravitational descendents we can compute the contact term of a gravitational descendent with a primary field, so we get in particular

$$C(\sigma_n(\phi_i), P) = \left[ \frac{W' \int^X \sigma_{n-1}(\phi_i)}{W'} \right]'_+ = \sigma_{n-1}(\phi_i) \quad (196)$$

which is Saito's recursion relation [27]. To derive (196) we have used the following decomposition of  $\sigma_n(\phi_i)$

$$\sigma_n(\phi_i) = W' \int^X \sigma_{n-1}(\phi_i) + \sum_{l=0}^k a_l \phi_l = c_{i,n-1} [\hat{L}^{(k+2)n+i} \partial_X \hat{L}]_+ \quad (197)$$

The part of  $\sigma_n(\phi_i)$  projecting on chiral primary fields can be easily obtained by noticing that  $W' \int^X \sigma_{n-1}(\phi_i)$  is a pure BRST object, namely

$$res_W(F + GW') = res_W(F) \quad (198)$$

for any functions  $F$  and  $G$ . In fact, from (198) we get

$$\langle \sigma_n(\phi_i) \phi_j P \rangle = \sum_{l=0}^k a_l \langle \phi_l \phi_j P \rangle \quad (199)$$

and from (189)

$$\sum_{l=0}^k a_l \phi_l = \sum_{l=0}^k c_{n,i} \frac{\partial}{\partial t_l} res(\hat{L}^{(k+2)n+i+1}) \phi_{k-l} \quad (200)$$

thus

$$\sum_{l=0}^k a_l \phi_l = \sum_{l=0}^k \frac{\partial}{\partial t_l} \langle P \sigma_n(\phi_i) \rangle \phi_{k-l} \quad (201)$$

The contribution to correlators from the piece of  $\sigma_n(\phi_i)$  projecting on chiral fields originates recursion relations. Let's take the three point function

$$\begin{aligned} \langle \sigma_n(\phi_i) \phi_j \phi_k \rangle &= \sum_{l=0}^k \frac{\partial}{\partial t_l} \langle P \sigma_n(\phi_i) \rangle \langle \phi_{k-l} \phi_j \phi_k \rangle = \\ &= \sum_{l=0}^k \langle P \sigma_n(\phi_i) \phi_l \rangle \langle \phi^l \phi_j \phi_k \rangle \end{aligned} \quad (202)$$

while using (196) we obtain the recursion relation

$$\langle \sigma_n(\phi_i) \phi_j \phi_k \rangle = \sum_{l=0}^k \langle \sigma_{n-1}(\phi_i) \phi_l \rangle \langle \phi^l \phi_j \phi_k \rangle \quad (203)$$

From the definition (195) it is clear that contact terms are symmetric

$$C(\phi_i, \phi_j) = C(\phi_j, \phi_i) \quad (204)$$

Moreover, we can write

$$C(P_i, \phi_j) \equiv A_{ij}^k P_k \quad (205)$$

for  $P_i$  either chiral primary or gravitational descendent. Using again (195) we get

$$C(\phi_k, C(P_i, \phi_j)) = \frac{\partial A_{ij}^l}{\partial t_k} P_l + A_{ij}^l C(\phi_k, P_l) \quad (206)$$

as the rule for composing contact terms.

The reader should notice that the Landau-Ginzburg description of contact terms can not be trivially extended to the computation of contact terms for two gravitational descendents  $C(\sigma_n(\phi_i), \sigma_m(\phi_j))$ . The reason is again that we are assuming relation (191) only on the small phase space.

## 2.4 Integral Representation of the Contact Terms

From the Landau-Ginzburg definition (195) of contact terms, it is clear that the contribution to  $C(\phi_i, \phi_j)$  is the part in the product  $\phi_i \phi_j$  which goes as  $W'F$  for some functional  $F$ . This is a priori a bit paradoxical taking into account that for the matter theory,  $W'F$  is a pure BRST-object which decouples from any correlator. As it was first pointed out by [24, 25], the reason for the contribution of cohomologically trivial objects of the pure matter theory to the contact terms is that they are non-trivial in the equivariant cohomology which rules the spectrum after coupling to gravity. To see this more clearly, let us introduce the following integral representation of the contact terms

$$|C(\phi_i, \phi_j)\rangle = \int_D \phi_i^{(2)} |\phi_j\rangle \quad (207)$$

where  $D$  is an infinitesimal neighborhood of the point where the operator  $\phi_j$  is inserted.

A "sewing"<sup>13</sup> or cancel propagator argument representation of the contact term can be defined working with  $\phi_i$  and  $\phi_j$  inserted on two fixed points of the hemisphere and

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<sup>13</sup>We will refer to "sewing"-representation when the integration of a field over the Riemann surface is transformed into integration over sewing parameters with all punctures fixed.

integrating over the moduli  $(\phi, \tau)$  with  $\phi \in [0, 2\pi]$  and  $\tau \in [0, \infty]$  as follows

$$|C(\phi_i, \phi_j)\rangle = \int_0^\infty d\tau \int_0^{2\pi} d\phi e^{\tau T_+} e^{\phi T_-} G_{0,-} G_{0,+} \phi_i(1) |\phi_j\rangle \quad (208)$$

where we have inserted  $\phi_i$  at the point 1. The notation  $\pm$  refers to light-cone type components and  $G$  in (208) is the superpartner of the energy-momentum tensor. Formally we can interpret (208) (compare with equation (164)) as a computation in the topological matter theory coupled to topological gravity.

Let us now assume that in the product  $\phi_i \phi_j$  there is some piece of the type  $W'F$ . Using the SUSY transformations of the Landau-Ginzburg lagrangian, we can write

$$W'F = Q(\rho_- F) \quad (209)$$

with  $\rho$  the fermionic zero-form and  $Q$  the BRST charge. Now we realize from the Landau-Ginzburg representation of  $G_{0,-}$  [24, 25] that

$$G_{0,-}(\rho_- F) \neq 0 \quad (210)$$

moreover, using the commutation relations between  $Q$  and  $G_{0,-}$  we get from (208)

$$|C(\phi_i \phi_j)\rangle = \int_0^\infty d\tau e^{-\tau T_+} T_+ G_{0,-} |\psi\rangle \quad (211)$$

where  $Q(\rho_- F) \equiv Q\psi$ . After a finite energy regularization we finally obtain

$$|C(\phi_i \phi_j)\rangle = G_{0,-} |\psi\rangle \quad (212)$$

as the result from the "sewing"-representation of (208).

The reader should notice that the key step in the derivation is equation (210), i.e. we have in the product  $\phi_i \phi_j$  a BRST exact state  $Q|\psi\rangle$  such that  $G_{0,-} |\psi\rangle \neq 0$ . Using this fact we can define a notion of equivariant cohomology by the conditions

$$\begin{aligned} Q|\phi\rangle &= 0 \\ G_{0,-}|\phi\rangle &= 0 \end{aligned} \quad (213)$$

and to describe the previous computation by simply saying that the contribution to  $C(\phi_i \phi_j)$  is determined by non trivial elements in the equivariant cohomology (213).

At this point we can make contact with the equivariant cohomology of the topological matter theory coupled to topological gravity. Following [25], we will present a simple



example. Let us consider, for all couplings  $t_i=0$ , the first Landau-Ginzburg gravitational descendant

$$\sigma_1(P) = W'x = Q(\rho_-x) \quad (214)$$

This is non trivial in the equivariant cohomology (213). We want now to compare  $\sigma_1(P)$  in representation (214) with the dilaton of the theory coupled to topological gravity (151)

$$\gamma_0 = \frac{1}{2}\hat{Q}(\partial c + c\partial\phi - \bar{\partial}c - c\bar{\partial}\phi) \quad (215)$$

with  $\hat{Q}$  the  $N=2$  BRST generator of the coupled theory. The equivariant cohomology condition for the coupled theory is defined by the condition

$$(b_0 + G_0)_-|\psi\rangle = 0 \quad (216)$$

with  $G_0$  the total super energy momentum tensor. It is now easy to check that

$$(b_0 + G_0)_-(\rho_-x + \partial c + c\partial\phi - \bar{\partial}c - c\bar{\partial}\phi) = 0 \quad (217)$$

In summary, there exists a map between the "matter" equivariant cohomology defined by (213) and the equivariant cohomology of the topological string obtained after coupling topological matter to topological gravity.

### Comment on the physical meaning of equivariant cohomology in string theory

The physical motivation for the requirement of equivariant cohomology, comes from the operator formalism definition of string amplitudes. As it is well known [3], string amplitudes are defined by associating with a Riemann surface  $\Sigma_{g,s}$  equipped with a set  $\{\xi_s\}$  of local coordinates around the punctures, a state  $|\Sigma_{g,s}\{\xi_s\}\rangle \in \otimes^s \mathcal{H}$  with  $\mathcal{H}$  the Hilbert space of the matter and ghost system. On  $\mathcal{H}$  it is defined a nilpotent BRST operator  $Q$ . Physical amplitudes for a set of s-external states  $|\chi_i\rangle$  are defined by

$$\int_{M_{g,s}} \langle \chi_1 | \dots \langle \chi_s | \prod_{i,\vec{i}=1}^{3g-3} b(\chi_i)\bar{b}(\bar{\chi}_{\vec{i}}) \prod_{j,\vec{j}=1}^s \oint b(V_j)\bar{b}(\bar{V}_{\vec{j}}) |\Sigma_{g,s}\{\xi_s\}\rangle \quad (218)$$

where  $V_j, \bar{V}_{\vec{j}}$  are vector fields over the Riemann surface. To (218) we should impose two requirements

- i) reparametrization invariance
- ii) BRST-invariance

The condition i) implies that (218) should be invariant under any change of local coordinates. This means in particular invariance under change  $V_i \rightarrow V_i + \delta V_i$  with  $\delta V_i$  a vector field that extends holomorphically inside the disc around the puncture and it is zero at the puncture, i.e. a "vertical" vector field referred to the bundle  $\hat{M}_{g,s} \rightarrow M_{g,s}$  with  $\hat{M}_{g,s}$  the augmented moduli space. The requirements i) and ii) are satisfied if we impose the equivariant cohomology condition on external states [28]

$$\begin{aligned} Q|\chi\rangle &= 0 \\ b_{0,-}|\chi\rangle &= 0 \end{aligned} \tag{219}$$

In abstract terms, the ingredients to define the string amplitudes are a couple  $(Q, b)$  such that

$$\begin{aligned} Q^2 &= 0 \\ \{Q, b\} &= T \end{aligned} \tag{220}$$

for  $T$  the total energy-momentum tensor and the physical spectrum being defined by the equivariant cohomology (219).

In standard string theory  $(Q, b)$  are respectively identified with the BRST charge and the  $b$ -ghost. However it is in principle possible to define formally generalizations of string theory where  $(Q, b)$  are more general solutions to (219). One particular case that we will discuss later is to use the  $N=2$  SUSY pair (1)  $(Q^+, Q^-)$ .

## 2.5 Gravity and the $t$ -part of the $t\bar{t}$ -equations

A natural way to think about the geometry of the space of TFT's is as an indirect description of the topological matter coupled to topological gravity (see for instance [24]). The logic for this point of view is that any connection in the space of theories should be determined by integrating fields, which already implies to construct forms on the moduli space of the Riemann surface. This is certainly a fruitful approach at least when we work at genus zero. In this spirit we can easily derive the  $t$ -part of the  $t\bar{t}$ -equations from two very natural string postulates

$$\begin{aligned} p.1) \quad & \langle \phi_{i_1} \phi_{i_2} \dots \int \phi_{i_s}^{(2)} \rangle = \langle \int \phi_{i_1}^{(2)} \phi_{i_2} \dots \phi_{i_s} \rangle \\ p.2) \quad & \langle \phi_{i_1} \phi_{i_2} \dots \int \phi_{i_{s-1}}^{(2)} \int \phi_{i_s}^{(2)} \rangle = \langle \phi_{i_1} \phi_{i_2} \dots \int \phi_{i_s}^{(2)} \int \phi_{i_{s-1}}^{(2)} \rangle \end{aligned} \tag{221}$$

Defining

$$\langle \phi_i \phi_j \phi_k \int \phi_l^{(2)} \rangle \equiv \partial_l \langle \phi_i \phi_j \phi_k \exp(\mathcal{L}_0 + \sum_a t_a \int \phi_a^{(2)}) \rangle \quad (222)$$

we will get from (221)

$$[\partial_i, C_j] = 0 \quad (223)$$

$$[\partial_i, \partial_j] = 0$$

which are the  $t$ -part of the  $t\bar{t}$ -equations (see equation 58). Now we can try to use p.1) and p.2) as constraints on the Landau-Ginzburg description of the TFT  $\mathcal{L} = \mathcal{L}_0 + \sum_i t_i \int \phi_i^{(2)}$ . In fact assuming the existence of a superpotential  $W(X, t)$  and some polynomial representation  $\phi_i(X, t)$  of the chiral fields  $\phi_i$ , and using the residue formulae representation

$$\langle \phi_i \phi_j \phi_k \int \phi_l^{(2)} \rangle = \frac{\partial}{\partial t_l} \int \frac{\phi_i \phi_j \phi_k}{W'} dX \quad (224)$$

we can ask ourselves how much information we get from the string postulates (221), and moreover if the Landau-Ginzburg representation (224) satisfies them in a natural way. In fact, this is the case for

$$\begin{aligned} \phi_i &= \frac{\partial W}{\partial t_i} \\ \frac{\partial \phi_i}{\partial t_j} &= C(\phi_i, \phi_j) \end{aligned} \quad (225)$$

with  $C(\phi_i, \phi_j) = C(\phi_j, \phi_i)$ , the symmetric contact terms defined in the previous section. The natural questions now will be

- a) To get a string, i.e. gravitational, interpretation of the  $t\bar{t}$ -part of the  $t\bar{t}$ -equations.
- b) To use the  $t\bar{t}$ -equations as a way to find, at least partially, the Landau-Ginzburg description of more general lagrangians with  $\bar{t}_i$ -couplings different from zero.

## 2.6 Verlinde-Verlinde Contact Term Algebra [20]: Pure Topological Gravity

Saito's recursion relation (196) can be derived, using again cancel propagator arguments, in the context of pure topological gravity. From (208) we get

$$|C(P, \sigma_n)\rangle = \int_0^\infty d\tau \int_0^{2\pi} d\phi e^{\tau T_+} e^{\phi T_-} G_{0,-} G_{0,+} P(1) |\sigma_n\rangle \quad (226)$$

Using the insertions of  $b_0, \bar{b}_0$  and  $\delta(\beta_0), \delta(\bar{\beta}_0)$  associated to the moduli of each puncture, we can set  $P=1$

$$\int_0^\infty d\tau \int_0^{2\pi} d\phi e^{\tau T_+} e^{\phi T_-} G_{0,-} G_{0,+} |\sigma_n\rangle \quad (227)$$

From the representation

$$|\sigma_n\rangle = \gamma_0 |\sigma_{n-1}\rangle \quad (228)$$

equation (215), and the operator product expansion

$$T(z)\phi(w) = \frac{1}{(z-w)^2} + \frac{\partial\phi}{(z-w)} \quad (229)$$

we reobtain Saito's recursion relatio (196)

$$|C(P, \sigma_n)\rangle = |\sigma_{n-1}\rangle \quad (230)$$

Using now

$$\sigma_n = \sigma_n^{(0)} P \quad (231)$$

we can define<sup>14</sup>

$$|C(\sigma_n, \sigma_m)\rangle = \sigma_n^{(0)} \int_D P^{(2)} |\sigma_m\rangle = |\sigma_{n+m-1}\rangle \quad (232)$$

for the rest of the contact terms. Notice that the structure of the contact terms (232) is consistent with ghost number conservation

$$gh(\int \sigma_n^{(2)} |\sigma_m\rangle) = (n-1) + m = gh(|\sigma_{n+m-1}\rangle) \quad (233)$$

In the derivation of (230) and (232) we have not included the curvature factor (154). In order to include these contributions, we can use the following trick. Let us consider the correlator  $\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_g$  for all  $t_i=0$ <sup>15</sup> and satisfying

$$\sum_{i=1}^s (n_i - 1) = 3g - 3 \quad (234)$$

In terms of the string coupling constant  $\lambda$  we know that  $\langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_g$  goes like  $\lambda^{2g-2+n}$  with  $(2g-2+n)$  the number of 3-vertex necessary for the sewing construction of  $\Sigma_{g,s}$ . From recursion relations and the puncture equation (177) we have [21], for all  $t_i=0$ , that

$$\lambda \frac{\partial}{\partial \lambda} = \frac{\partial}{\partial t_1} \rightarrow \sigma_1 \quad (235)$$

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<sup>14</sup>We fix  $\sigma_n$  at the point zero and only integrate the puncture operator.

<sup>15</sup>Here the  $t_i$  are the couplings in pure topological gravity

and the dilaton equation

$$\langle \sigma_1 \sigma_{n_1} \dots \sigma_{n_s} \rangle_g = (2g - 2 + n) \langle \sigma_{n_1} \dots \sigma_{n_s} \rangle_g \quad (236)$$

Defining

$$\hat{\sigma}_n = e^{\frac{2}{3}(n-1)\pi} \sigma_n \quad (237)$$

where the  $\pi$  is the conjugate of the Liouville field, we localize the curvature at the insertion points. Therefore we expect to derive (236) exclusively from the contribution of contact terms

$$\langle \hat{\sigma}_1 \hat{\sigma}_{n_1} \dots \hat{\sigma}_{n_s} \rangle_g = \sum_{i=1}^s C(\hat{\sigma}_1, \hat{\sigma}_{n_i}) \langle \hat{\sigma}_{n_1} \dots \hat{\sigma}_{n_s} \rangle_g = (2g - 2 + n) \langle \hat{\sigma}_{n_1} \dots \hat{\sigma}_{n_s} \rangle_g \quad (238)$$

From (234) we obtain

$$|C(\hat{\sigma}_1, \hat{\sigma}_n)\rangle = \frac{1}{3}(2n + 1)|\hat{\sigma}_n\rangle \quad (239)$$

In general the contact term algebra will be given by

$$\int_D \hat{\sigma}_n^{(2)} |\hat{\sigma}_m\rangle = A_n^m |\hat{\sigma}_{n+m-1}\rangle \quad (240)$$

for certain coefficients  $A_n^m$ . From now on we will omit the superindex (2) in the expression of contact terms (see (240)), in order to simplify notation. Assuming representation (232), we obtain that  $A_n^m$  will depend only on  $m$

$$A_n^m \equiv A_m \quad (241)$$

with  $A_m$  defined by

$$\int_D \hat{P} |\hat{\sigma}_m\rangle = A_m |\hat{\sigma}_{m-1}\rangle \quad (242)$$

From (239) we can conclude that

$$A_m = \frac{1}{3}(2m + 1) \quad (243)$$

To check the previous argument, we should impose the consistency conditions of type (221.b)

$$\int_D \hat{\sigma}_{n_1} \int_D \hat{\sigma}_{n_2} |\hat{\sigma}_{n_3}\rangle = \int_D \hat{\sigma}_{n_2} \int_D \hat{\sigma}_{n_1} |\hat{\sigma}_{n_3}\rangle \quad (244)$$

which implies

$$A_{n_1}^{n_3+n_2-1} A_{n_2}^{n_3} - A_{n_2}^{n_3+n_1-1} A_{n_1}^{n_3} = C_{n_2 n_1} A_{n_2+n_1-1}^{n_3} \quad (245)$$

with

$$C_{n_2 n_1} = A_{n_1}^{n_2} - A_{n_2}^{n_1} \quad (246)$$

The coefficients  $A_n^m$  given by (241) and (243) are clearly a solution with

$$C_{n_2 n_1} = \frac{2}{3}(n_2 - n_1) \quad (247)$$

Notice that Saito's recursion relation (196) becomes now

$$|C(\hat{P}, \hat{\sigma}_n)\rangle = \frac{1}{3}(2n + 1)|\hat{\sigma}_{n-1}\rangle \quad (248)$$

An interesting exercise will be to derive relation (248) directly from the Landau-Ginzburg description.

The consistency of the asymmetric contact term algebra with the string requirement

$$\langle \hat{\sigma}_n \hat{\sigma}_m \prod_{i=1}^s \hat{\sigma}_{n_i} \rangle_g = \langle \hat{\sigma}_m \hat{\sigma}_n \prod_{i=1}^s \hat{\sigma}_{n_i} \rangle_g \quad (249)$$

impose severe constraints on the correlators. From (240), (243) and (249), we conclude

$$\frac{2}{3}(m - n) \langle \hat{\sigma}_{n+m-1} \prod_{i=1}^s \hat{\sigma}_{n_i} \rangle_g = \sum_{i=1}^s \mathcal{R}_{D_i} + \mathcal{R}_\Delta \quad (250)$$

with  $\mathcal{R}_{D_i}$  the commutator of the contact terms of  $\hat{\sigma}_n$  and  $\hat{\sigma}_m$  with the  $\hat{\sigma}_{n_i}$ , and  $\mathcal{R}_\Delta$  the commutator for the node contribution. Equation (250) clearly shows one of the most important results of pure topological gravity, namely that correlators are saturated by the contribution of the boundary of moduli space. The contributions from the nodes  $\mathcal{R}_\Delta$  are of two types. One corresponds to the pinching of a handle, which results in a correlator at genus  $g-1$ . The other corresponds to factorizations of the original surface in two of genus  $g-r$  and  $r$  respectively. Therefore, equation (250) originates recursion relations relating correlators at genus  $g$  and  $g' < g$ . These recursion relations are crucial to show the equivalence between matrix models and topological strings [20, 19, 21].

## 2.7 The Gravitational Meaning of the $t\bar{t}$ -equations

In section 2.5 we have pointed out the equivalence between moving in the space of theories and coupling to topological gravity. Using this approach, we formally associate the  $t$ -part of the  $t\bar{t}$ -equations with the string postulates p.1) and p.2). We can now try to extend our

analysis to the whole "t-t̄-plane" of topological matter theories, i.e. considering a generic family of TFT described by

$$\mathcal{L} = \mathcal{L}^0 + \sum_i t_i \int \phi_i^{(2)} + \sum_{\bar{i}} \bar{t}_{\bar{i}} \int \bar{\phi}_{\bar{i}}^{(2)} \quad (251)$$

and moving not only in  $t$ , but also in the  $\bar{t}$  direction. We will start performing this analysis at genus zero. Intuitively we should expect to get from this study some extra information concerning the contribution to topological amplitudes from the boundary of the moduli space. The logic for this comes from the standard version of the BRST anomaly in string theory. Notice that a variation in  $\bar{t}$ 's naively implies coupling to a pure BRST state.

When we change the couplings  $t_i$ , we are forced to compute correlators of the type  $\langle \phi_{i_1} \dots \phi_{i_s} \int \phi_i^{(2)} \rangle$  and we can always interpret the integration over the position of  $\phi_i^{(2)}$  in a gravitational way, associated with the definition of a Mumford-Morita form on the moduli space of the Riemann sphere with punctures. When we try to do the same for a variation in  $\bar{t}_i$  without changing the way in which we have twisted the lagrangian  $\mathcal{L}^0$ , we immediately find some conceptual problems. The simplest one is the moduli interpretation of the integration over the position of  $\bar{\phi}_{\bar{i}}^{(2)}$ . The reason for this is that  $\bar{\phi}_{\bar{i}}^{(2)}$ , as defined by (38), involves  $Q^+$  and, on the other hand, the integration over the moduli or sewing parameters in the way described in section 2.4 involves the SUSY charge  $Q^-$ , i.e.  $G_{0,-}$ .

The approach we want to present here will consist in interpreting the variation in the  $\bar{t}$ -direction in a standard gravitational way, in equal footing with the way we have interpreted the variation in  $t$ -direction, but at the price of modifying the field  $\bar{\phi}_{\bar{i}}$ . Namely we introduce a new field  $\hat{\phi}_{\bar{i}}$  by<sup>16</sup>

$$\int_D \bar{\phi}_{\bar{i}}^{(2)} |\phi_j\rangle \equiv \int_0^\infty d\tau \int_0^{2\pi} d\phi e^{\tau T_+} e^{\phi T_-} G_{0,-}^- G_{0,+}^- \hat{\phi}_{\bar{i}}(1) |\phi_j\rangle \quad (252)$$

In other words, we use  $G_0^-$  to define the integration over the insertion  $\bar{\phi}_{\bar{i}}$  and we change the field  $\bar{\phi}_{\bar{i}}$  to  $\hat{\phi}_{\bar{i}}$  in order to take into account that, in the l.h.s. of (252),  $\bar{\phi}_{\bar{i}}$  was defined by equation (38) in terms of  $Q^+$ . In a more compact notation we can write (252) as

$$\int_D \bar{\phi}_{\bar{i}}^{(2)} |\phi_i\rangle \equiv |C(\hat{\phi}_{\bar{i}}, \phi_j)\rangle \quad (253)$$

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<sup>16</sup>The reader should notice an important difference between the contact term (252) and the one described in (208) for Landau-Ginzburg models. In the case (208), the couple  $(Q, G)$  we use is not Hodge, i.e.  $G$  has trivial cohomology, while in (252) we use for  $(Q, G)$  the  $N=2$  SUSY Hodge system, i.e.  $G$  has non trivial cohomology. See more on this phenomena in section 2.10.

In order to characterize the operator  $\hat{\phi}_{\bar{i}}$ , we will use the following constructive path [29]

i) We define a  $t\bar{t}$  contact term algebra including contact terms between topological and antitopological fields.

ii) We will impose on this contact term algebra consistency conditions of the type (244)

iii) From both contact terms in the  $t$  and the  $\bar{t}$  direction, we will try to compute the curvature of the  $t\bar{t}$ -”plane”, i.e. to derive the  $t\bar{t}$ -equations.

We have only developed the previous program in the particular case of  $\hat{c}=3$  theories reducing the  $t$  and  $\bar{t}$  deformations to marginal directions, i.e. to the moduli space of the  $\hat{c}=3$   $N=2$ . However we believe that this program can be extended to more general cases.

## 2.8 $t\bar{t}$ -Contact Term Algebra for $\hat{c}=3$ SCFT's

Let us consider the algebra of operators generated by:  $\phi_i, \hat{\phi}_{\bar{i}}$  and the dilaton field  $\sigma_1$  with  $i = 1, \dots, n$  for  $n$  the number of marginal deformations. We define the following contact term algebra [29]

$$\begin{aligned}
\int_D \phi_i |\phi_j\rangle &= \Gamma_{ij}^k |\phi_k\rangle \quad , \quad \int_D \hat{\phi}_{\bar{i}} |\hat{\phi}_{\bar{j}}\rangle = \tilde{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} |\hat{\phi}_{\bar{k}}\rangle \\
\int_D \hat{\phi}_{\bar{i}} |\phi_j\rangle &= G_{j\bar{i}} |\sigma_1\rangle \quad , \quad \int_D \phi_i |\hat{\phi}_{\bar{j}}\rangle = \tilde{G}_{i\bar{j}} |\sigma_1\rangle \\
\int_D \sigma_1 |\phi_i\rangle &= a |\phi_i\rangle \quad , \quad \int_D \phi_i |\sigma_1\rangle = b |\phi_i\rangle \\
\int_D \sigma_1 |\hat{\phi}_{\bar{i}}\rangle &= c |\hat{\phi}_{\bar{i}}\rangle \quad , \quad \int_D \hat{\phi}_{\bar{i}} |\sigma_1\rangle = d |\hat{\phi}_{\bar{i}}\rangle \\
\int_D \sigma_1 |\sigma_1\rangle &= e |\sigma_1\rangle
\end{aligned} \tag{254}$$

In order to take into account the contribution of the curvature and the twist we introduce the operator

$$e^{\frac{1}{2}\tilde{\varphi}(z)} \tag{255}$$

for  $\tilde{\varphi} = \varphi + 2\pi$ , where  $\varphi$  is the operator the bosonizes the  $\mathcal{U}(1)$  current of the  $N=2$  SCFT and  $\pi$  the conjugate of the Liouville field. The contact term algebra for this operator is



defined as follows

$$\begin{aligned} \int_D \phi_i |e^{\frac{1}{2}\bar{\varphi}(z)}\rangle &= A_i |e^{\frac{1}{2}\bar{\varphi}(z)}\rangle \quad , \quad \int_D \hat{\phi}_{\bar{i}} |e^{\frac{1}{2}\bar{\varphi}(z)}\rangle = 0 \\ \int_D \sigma_1 |e^{\frac{1}{2}\bar{\varphi}(z)}\rangle &= a |e^{\frac{1}{2}\bar{\varphi}(z)}\rangle \end{aligned} \quad (256)$$

The undetermined constants appearing in (254) and (256) will be now fixed by imposing consistency conditions (244).

Before entering into a detailed description of the consistency conditions for the algebra (254) (256), we would like to make some comments. The most interesting aspect of (254) is the specific  $t\bar{t}$ -contact term

$$|C(\hat{\phi}_{\bar{i}}, \phi_j)\rangle = G_{j\bar{i}} |\sigma_1\rangle \quad (257)$$

where the topological-antitopological fusion really takes place. We can argue on (257) in the following way. From the definition (252) of the field  $\hat{\phi}_{\bar{i}}$  we can formally write

$$\hat{\phi}_{\bar{i}} \rightarrow Q_{(+)}^+ \text{ " } \frac{1}{Q_{(-)}^-} \text{ " } \bar{\phi}_{\bar{i}} \quad (258)$$

and taking into account the ghost charges of  $Q^+$  and  $Q^-$  to interpret  $\hat{\phi}_{\bar{i}}$ , at least at the level of ghost charges, as having implicitly an  $n=2$  gravitational descendent index. This interpretation as gravitational descendent should be considered only as an heuristic way to motivate (257). Even when (258) is purely formal, we notice that can give a hint on the appearance of the dilaton in (257), because "  $\frac{1}{Q^-}$  " could be interpreted as a would-be  $c$ -ghost field<sup>17</sup>.

After this general comment we proceed to solve the consistency conditions. For this, we will assume

i) That  $G_{i,\bar{j}}$  is invertible.

ii) The value of  $a$  equals -1. This condition is based on the way the dilaton field measures the curvature.

iii) The following derivation rules

$$\begin{aligned} \phi_i \Gamma_{\alpha\beta}^\gamma(t, \bar{t}) &= \partial_i \Gamma_{\alpha\beta}^\gamma(t, \bar{t}) \\ \hat{\phi}_{\bar{i}} \Gamma_{\alpha\beta}^\gamma(t, \bar{t}) &= (-1)^{F(\Gamma_{\alpha\beta}^\gamma)} \partial_{\bar{i}} \Gamma_{\alpha\beta}^\gamma(t, \bar{t}) \quad , \quad F(\Gamma_{\alpha\beta}^\gamma) = q_\gamma - q_\alpha - q_\beta \end{aligned} \quad (259)$$

---

<sup>17</sup>As a marginal comment we notice that for type B-models in the case  $\hat{c}=3$ , we can identify  $b_0^-$  with  $\partial$  and the BRST charge with  $\bar{\partial}$ . Now we can use the fact that  $\partial, \bar{\partial}$  define a Hodge structure defined by the  $(Q^+, Q^-)$   $N=2$  algebra. Using Hodge  $\partial\bar{\partial}$ -lemma we define  $\frac{1}{Q_{(-)}^-}$  as  $\frac{\partial}{\Delta}$  [30]. This is the basic lemma needed to define the kinetic part of the Kodaira-Spencer lagrangian.

where  $\Gamma_{\alpha\beta}^\gamma$  stands for a generic contact term tensor,  $q_\alpha$  for the  $\mathcal{U}(1)$  charge associated to the corresponding field, and which defines the way the operators act on the coefficients appearing in the contact term algebra. Notice that in general these coefficients will depend on the moduli parameters  $(t, \bar{t})$ . The logic for this rule is the equivalence between the insertion of a marginal field and the derivation with respect to the corresponding moduli parameter. For this reason we will not associate any derivative with the dilaton field. The derivation rule (259.b) is forced by the topological interpretation of the  $\bar{t}$  insertions we are using. Once we decide to work with the operators  $\hat{\phi}_{\bar{t}}$  and to define the measure using only  $G^-$  insertions, we must accommodate to this picture the coupling of the spin connection to the  $\mathcal{U}(1)$  current. Since the derivation  $\partial_{\bar{t}}$  corresponds to the insertion of an antitopological field, we need to change, in the neighborhood of the insertion, the sign of the coupling of the  $\mathcal{U}(1)$  current to the background gauge field defined by the spin connection. This fact gives raise to the factor  $(-1)^{F(\Gamma)}$  in (259.b).

Using i), ii) and iii), let us start by analyzing the following consistency condition

$$\int_D \sigma_1 \int_D \phi_i |\phi_j\rangle = \int_D \phi_i \int_D \sigma_1 |\phi_j\rangle \quad (260)$$

Applying the contact term algebra (254), we get

$$b \Gamma_{ij}^k |\phi_k\rangle - \Gamma_{ij}^k |\phi_k\rangle = -2 \Gamma_{ij}^k |\phi_k\rangle \quad (261)$$

which, for a non vanishing  $\Gamma_{ij}^k$ , implies that

$$b = -1 \quad (262)$$

From the condition

$$\int_D \sigma_1 \int_D \phi_i |\sigma_1\rangle = \int_D \phi_i \int_D \sigma_1 |\sigma_1\rangle \quad (263)$$

together with equation (262) and the derivation rules (259), we obtain

$$\partial_i e |\sigma_1\rangle - e |\phi_i\rangle = |\phi_i\rangle \quad (264)$$

being solved by

$$e = -1 \quad (265)$$

To continue the study, we take the condition

$$\int_D \hat{\phi}_{\bar{t}} \int_D \hat{\phi}_{\bar{j}} |\phi_k\rangle = \int_D \hat{\phi}_{\bar{j}} \int_D \hat{\phi}_{\bar{t}} |\phi_k\rangle \quad (266)$$

which leads to

$$(\tilde{\Gamma}_{j\bar{i}}^{\bar{l}} G_{k\bar{l}} + \partial_i G_{k\bar{j}}) |\sigma_1\rangle + d G_{k\bar{j}} |\hat{\phi}_{\bar{i}}\rangle = (\tilde{\Gamma}_{i\bar{j}}^{\bar{l}} G_{k\bar{l}} + \partial_{\bar{j}} G_{k\bar{i}}) |\sigma_1\rangle + d G_{k\bar{i}} |\hat{\phi}_{\bar{j}}\rangle \quad (267)$$

Using that  $G_{i\bar{j}}$  is invertible, and for a general number of marginal deformations, we get from the above equation

$$d = 0 \quad (268)$$

Moreover, the consistency condition

$$\int_D \sigma_1 \int_D \hat{\phi}_{\bar{i}} |\phi_i\rangle = \int_D \hat{\phi}_{\bar{i}} \int_D \sigma_1 |\phi_i\rangle \quad (269)$$

and equation (268) imply that

$$c = 0 \quad (270)$$

From (262), (265), (268) and the consistency condition

$$\int_D \phi_i \int_D \hat{\phi}_{\bar{j}} |\sigma_1\rangle = \int_D \hat{\phi}_{\bar{j}} \int_D \phi_i |\sigma_1\rangle \quad (271)$$

we get easily

$$\tilde{G}_{i\bar{j}} = 0 \quad (272)$$

The next conditions we will analyze involve the curvature operator  $e^{\frac{1}{2}\tilde{\varphi}(z)}$

$$\begin{aligned} \int_D \phi_i \int_D \hat{\phi}_{\bar{j}} |e^{\frac{1}{2}\tilde{\varphi}(z)}\rangle &= \int_D \hat{\phi}_{\bar{j}} \int_D \phi_i |e^{\frac{1}{2}\tilde{\varphi}(z)}\rangle \\ \int_D \phi_i \int_D \phi_{\bar{j}} |e^{\frac{1}{2}\tilde{\varphi}(z)}\rangle &= \int_D \phi_{\bar{j}} \int_D \phi_i |e^{\frac{1}{2}\tilde{\varphi}(z)}\rangle \end{aligned} \quad (273)$$

from which we get, assuming that  $\Gamma_{i\bar{j}}^k$  is symmetric in the lower indices<sup>18</sup>

$$\begin{aligned} G_{i\bar{j}} &= \partial_{\bar{j}} A_i \\ \partial_i A_j &= \partial_j A_i \end{aligned} \quad (274)$$

Equations (274) imply that the metric  $G_{i\bar{j}}$  is Kähler, for a certain potential  $K(t, \bar{t})$

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \quad (275)$$

With this information, we can return to (267) and deduce that the tensor  $\tilde{\Gamma}_{i\bar{j}}^{\bar{k}}$  is symmetric in the lower indices

$$\tilde{\Gamma}_{i\bar{j}}^{\bar{k}} = \tilde{\Gamma}_{j\bar{i}}^{\bar{k}} \quad (276)$$

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<sup>18</sup>The symmetry of  $\Gamma_{i\bar{j}}^k$  will assure that  $\int_D \phi_i \int_D \phi_j |\phi_k\rangle = \int_D \phi_j \int_D \phi_i |\phi_k\rangle$  is satisfied.

Using now

$$\int_D \phi_i \int_D \hat{\phi}_{\bar{j}} |\hat{\phi}_{\bar{k}}\rangle = \int_D \hat{\phi}_{\bar{j}} \int_D \phi_i |\hat{\phi}_{\bar{k}}\rangle \quad (277)$$

we obtain that  $\tilde{\Gamma}_{\bar{j}\bar{k}}^{\bar{l}}$  is only function of the antitopological variables

$$\partial_i \tilde{\Gamma}_{\bar{j}\bar{k}}^{\bar{l}} = 0 \quad (278)$$

Condition (278), together with  $\int_D \hat{\phi}_{\bar{i}} \int_D \hat{\phi}_{\bar{j}} |\hat{\phi}_{\bar{k}}\rangle = \int_D \hat{\phi}_{\bar{j}} \int_D \hat{\phi}_{\bar{i}} |\hat{\phi}_{\bar{k}}\rangle$  allow to impose a vanishing contact term for antitopological operators.

To conclude the study of the consistency conditions we will consider now the relation

$$\int_D \phi_i \int_D \hat{\phi}_{\bar{j}} |\phi_k\rangle = \int_D \hat{\phi}_{\bar{j}} \int_D \phi_i |\phi_k\rangle \quad (279)$$

Using equations (262) and (272), we obtain

$$\begin{aligned} \int_D \phi_i \int_D \hat{\phi}_{\bar{j}} |\phi_k\rangle &= \partial_i G_{k\bar{j}} |\sigma_1\rangle - G_{k\bar{j}} |\phi_i\rangle - G_{i\bar{j}} |\phi_k\rangle + \text{fact terms} \\ \int_D \hat{\phi}_{\bar{j}} \int_D \phi_i |\phi_k\rangle &= -\partial_{\bar{j}} \Gamma_{ik}^{\bar{l}} |\phi_l\rangle + \Gamma_{ik}^{\bar{l}} G_{l\bar{j}} |\sigma_1\rangle \end{aligned} \quad (280)$$

In order to motive the inclusion of factorization terms in (280), let us notice two facts. The necessity of including factorization terms at the level of consistency conditions (244) is already present in the simplest case of contact term algebra, i.e. in pure gravity. Due to the asymmetry of the factors  $A_n^m$  (241), it is not possible to satisfy the relations  $\int \hat{P} \int \hat{\sigma}_n |\hat{P}\rangle = \int \hat{\sigma}_n \int \hat{P} |\hat{P}\rangle$  without taking factorization terms into account. Second, the heuristic argument (258) seems to indicate a hidden gravitational descendent index in the operators  $\hat{\phi}_{\bar{j}}$ . Therefore, and due to the non vanishing correlation function  $C_{ijk}$  at genus zero for three marginal fields, we should consider the possible existence of factorization terms associated to the  $\hat{\phi}_{\bar{j}}$  insertions. We can write them generically as follows

$$\text{fact terms} = B_{\bar{j}}^{l\bar{n}} C_{ikn} |\phi_l\rangle \quad (281)$$

From equations (279)-(281), we obtain that the coefficient  $\Gamma_{ij}^k$  is the connection for the metric  $G_{i\bar{j}}$ , which we already know that is Kähler

$$\Gamma_{ij}^k = (\partial_i G_{j\bar{l}}) G^{\bar{l}k} \quad (282)$$

and a  $(t, \bar{t})$  type equation

$$\partial_{\bar{n}} \Gamma_{ij}^k = G_{i\bar{n}} \delta_j^k + G_{j\bar{n}} \delta_i^k - B_{\bar{n}}^{mk} C_{ijm} \quad (283)$$

The tensor  $B_j^{ln}$  can be derived from the contact term algebra by the following argument. Let's consider the consistency condition on a general string amplitude

$$\langle \hat{\phi}_i \hat{\phi}_j \prod_{l=1}^s \phi_l \rangle_g = \langle \hat{\phi}_j \hat{\phi}_i \prod_{l=1}^s \phi_l \rangle_g \quad (284)$$

from (254) we get

$$(\tilde{\Gamma}_{ij}^k - \tilde{\Gamma}_{ji}^k) \langle \hat{\phi}_k \prod_{l=1}^s \phi_l \rangle_g = \sum_{l=1}^s \mathcal{R}_{D_l} + \sum_{nodes} \mathcal{R}_\Delta \quad (285)$$

where  $\mathcal{R}_{D_l}$  denotes the commutator of the contact terms of  $\hat{\phi}_i$  and  $\hat{\phi}_j$  with  $\phi_l$ , and  $\mathcal{R}_\Delta$  the commutator of those at the nodes. Using now the symmetry of  $\tilde{\Gamma}_{ij}^k$  in the lower indices (276), we can conclude

$$\sum_{l=1}^s \mathcal{R}_{D_l} = \sum_{nodes} \mathcal{R}_\Delta = 0 \quad (286)$$

The contribution at a node associated with the factorization of the surface, will be defined by the tensor  $B_j^{\alpha\beta}$  as follows

$$\begin{aligned} \langle \hat{\phi}_i \hat{\phi}_j \prod_{l \in S} \phi_l \rangle_{g,\Delta} &= \sum_{r=0}^g \sum_{X \cup Y = S} [ B_j^{\alpha\beta} G_{\alpha\bar{i}} \langle \sigma_1 \prod_{l \in X} \phi_l \rangle_r \langle \phi_\beta \prod_{n \in Y} \phi_n \rangle_{g-r} + \\ &+ \partial_i B_j^{\alpha\beta} \langle \phi_\alpha \prod_{l \in X} \phi_l \rangle_r \langle \phi_\beta \prod_{n \in Y} \phi_n \rangle_{g-r} ] \end{aligned} \quad (287)$$

where  $S$  refers to the set of all punctures,  $X$  and  $Y$  is a partition of it, and the tensor  $B$  can be chosen symmetric in the upper indices. Using now (286) we get

$$\begin{aligned} B_i^{\alpha\beta} G_{\alpha\bar{j}} &= B_j^{\alpha\beta} G_{\alpha\bar{i}} \\ \partial_i B_j^{\alpha\beta} &= \partial_j B_i^{\alpha\beta} \end{aligned} \quad (288)$$

By an analogous argument, we find from condition (279) and for a general string amplitude

$$\partial_i B_j^{\alpha\beta} + B_j^{\alpha\gamma} \Gamma_{i\gamma}^\beta + B_j^{\gamma\beta} \Gamma_{i\gamma}^\alpha - 2\partial_i K B_j^{\alpha\beta} = 0 \quad (289)$$

Let's define  $B_j^{\alpha\beta} = B_{j\bar{\alpha}\bar{\beta}} e^{2K} G^{\bar{\alpha}\alpha} G^{\bar{\beta}\beta}$ . Then, equations (288) and (289) imply that  $B_{i\bar{j}\bar{\beta}}$  is proportional to the three point correlation function for the antitopological fields. Substituting this information into equation (283), we obtain the  $(t, \bar{t})$ -equation

$$\partial_{\bar{n}} \Gamma_{ij}^k = G_{i\bar{n}} \delta_j^k + G_{j\bar{n}} \delta_i^k - \bar{C}_{\bar{n}}^{mk} C_{ijm} \quad (290)$$

Notice that in order to get the special geometry relation (290) from the contact term algebra, it was necessary to make use of the derivation rule (259.b). From (290) we can

conclude that the metric  $G_{i\bar{j}}$  is the Zamolodchikov metric for the marginal deformations, therefore obtaining the special geometry of the moduli space of  $N = 2$ ,  $\hat{c} = 3$  SCFT's presented in section 1.8.

From the previous result we observe that the combined action on  $t$  defined by the contact terms  $\Gamma_{ij}^k$ , and on  $\bar{t}$  characterized by  $G_{i\bar{j}}$ , produces the whole  $t\bar{t}$ -connection, concluding for the case  $\hat{c}=3$  the steps i), ii) and iii) of section 2.7.

## 2.9 Holomorphic Anomaly: the Genus Zero Case

Let us write the  $t\bar{t}$ -equation in the condensed way

$$[D_i, D_{\bar{j}}] = -[C_i, \bar{C}_{\bar{j}}] \quad (291)$$

If now we interpret  $D_i, D_{\bar{j}}$  as defining the motion in the space of theories

$$\begin{aligned} C_{i_1 \dots i_s; j}^0 &\equiv \langle \phi_{i_1} \dots \phi_{i_s} \int \phi_j^{(2)} \rangle \equiv D_j \langle \phi_{i_1} \dots \phi_{i_s} \rangle \\ C_{i_1 \dots i_s; \bar{j}}^0 &\equiv \langle \phi_{i_1} \dots \phi_{i_s} \int \bar{\phi}_{\bar{j}}^{(2)} \rangle \equiv D_{\bar{j}} \langle \phi_{i_1} \dots \phi_{i_s} \rangle \end{aligned} \quad (292)$$

we get

$$D_{\bar{j}} C_{i_1 \dots i_s; \bar{i}}^0 = [D_i, D_{\bar{j}}] C_{i_1 \dots i_s}^0 + D_i C_{i_1 \dots i_s; \bar{j}}^0 \quad (293)$$

and even if we start with holomorphic correlators  $C_{i_1 \dots i_s}^0$  for a topological field theory, we will find for the correlators of a neighborhood theory defined by  $C_{i_1 \dots i_s; i}^0$  an anomalous contribution coming from (291). This anomalous contribution, first discovered by [30], is known as holomorphic anomaly. The physical origin of this anomaly is associated with the fact that derivatives with respect to the couplings of pure BRST operators are not any more pure BRST. In principle the anomaly (293), at least at genus zero, is a general fact independently of the value of  $\hat{c}$ . However it is only for the special case  $\hat{c}=3$  that we can interpret this anomaly using the tools we have introduced in the previous section. For  $\hat{c}=3$  and reducing to marginal  $t$  and  $\bar{t}$  deformations, the only non-vanishing correlators at genus zero are of the form

$$C_{i_1 i_2 i_3; j_1 \dots j_s}^0 = \langle \phi_{i_1} \phi_{i_2} \phi_{i_3} \int \phi_{j_1}^{(2)} \dots \int \phi_{j_s}^{(2)} \rangle_0 \quad (294)$$

and therefore all of them should define measures on the moduli space of Riemann surfaces with  $n+3$  punctures. In other words, the study of these correlators is strictly equivalent to couple the matter theory to topological gravity. Using (292), correlators can be expressed in terms of the three point functions

$$C_{i_1 i_2 i_3; j_1 \dots j_s}^0 = D_{j_s} \dots D_{j_1} C_{i_1 i_2 i_3}^0 \quad (295)$$

Their anomalous piece can be computed applying successive times equation (293) together with the  $t\bar{t}$ -equation (291). In particular the  $t\bar{t}$ -equation can be seen as the simplest case of the holomorphic anomaly, i.e. for the four point function  $C_{i_1 i_2 i_3; j}$ .

## 2.10 Higher Genus and Quantum Geometry

Until now we have reduced our discussion to the case of genus zero. It is in this reduced framework where we have connected the geometry of the space of theories with the physics of topological strings. Once we have topological matter coupled to topological gravity, nothing prevent us a priori for computing higher genus amplitudes. It is on the basis of these amplitudes that some form of quantum geometry should appear in the future.

One of the more important facts we have observed in the study of pure topological gravity are the recursion relations, by means of which we can construct genus  $g$  amplitudes in terms of genus  $(g-1)$  amplitudes. The origin of these recursion relations is the zero contribution from the bulk. It would be certainly important to generalize these type of recursion relations to generic topological strings. A way to begin this project, initiated in [30], is to generalize the holomorphic anomaly to higher genus amplitudes.

For the case  $\hat{c}=3$  a generic correlator  $C_{i_1 \dots i_s}^g$  for marginal fields at genus  $g$  is defined by

$$C_{i_1 \dots i_s}^g = \int_{M_{g,s+1}} \left\langle \oint_{C_{z_1}} G^- \bar{G}^- \phi_{i_1} \dots \oint_{C_{z_s}} G^- \bar{G}^- \phi_{i_s} \prod_{\substack{j, \bar{j}=1 \\ j, \bar{j}=1}}^{3g-3} G^-(\chi_j) \bar{G}^-(\bar{\chi}_{\bar{j}}) \right\rangle \quad (296)$$

with  $\chi_j, \bar{\chi}_{\bar{j}}$  the Beltrami differentials and where  $\oint_{C_{z_i}} G^- \bar{G}^- \phi_i = \phi_i^{(2)}$  (see equation (36)). The correlator (296) have the same structure as a correlator in the bosonic string provided we interpret the  $G^-$ 's as the  $b, \bar{b}$ -ghosts. The difference however is that, as we have already mention in section 2.1, the factor  $\prod_{j, \bar{j}=1}^{3g-3} G^-(\chi_j) \bar{G}^-(\bar{\chi}_{\bar{j}})$  is coming from the integration over the supermoduli and therefore we are forced a priori to define the string measures using

a pair  $(Q, b)$  which in addition to the standard requirement  $\{Q, b\} = T$  defines Hodge structure, i.e. the cohomology of  $Q$  is isomorphic to the cohomology of the field  $b$ . We have already feel this fact in the computations at genus zero in the definition of contact terms. From a physical point of view the first implication of defining string amplitudes using a  $(Q, b)$  system which at the same time satisfies the  $N=2$  algebra, i.e. it is Hodge, is that the propagators

$$\frac{b_{0,+}b_{0,-}}{L_0 + \bar{L}_0} \quad (297)$$

which we are going to associate with the sewing operators in order to define the string amplitudes, project out all zero energy states. These simple reasoning seems a priori to prevent any consistent way to define genus  $g$  amplitudes for external zero energy states in terms of amplitudes at genus  $(g-1)$  for again external zero energy states. The holomorphic anomaly can be extended to genus  $g$  amplitudes and partially solves this puzzle.

We will derive the anomaly for correlators at any genus  $g$  using again the contact term algebra introduced in section 2.8. Let us remember the expression of the  $t\bar{t}$ -amplitudes with the help of the formal operators  $\hat{\phi}_{\bar{j}}$

$$\begin{aligned} \partial_{\bar{t}} C_{i_1 \dots i_s}^g &= \\ &= \int_{\mathcal{M}_{g,s+1}} \langle \oint_{C_z} G^- \bar{G}^- \hat{\phi}_{\bar{t}} \prod_{i=1}^s \oint_{C_{z_i}} G^- \bar{G}^- \phi_i \prod_{a, \bar{a}=1}^{3g-3} G^-(\chi_a) \bar{G}^-(\bar{\chi}_{\bar{a}}) \rangle_{\Sigma_{g,s+1}} = \\ &= \langle \hat{\phi}_{\bar{t}} \prod_{i \in S} \phi_i \rangle_g \end{aligned} \quad (298)$$

where  $S$  notes the set of all punctures and we have introduced the last equality to simplify the notation. The contributions to (298) can be written:

$$\langle \hat{\phi}_{\bar{t}} \prod_{i \in S} \phi_i \rangle_g = \sum_{i \in S} R_{D_i} + \sum_{nodes} R_{\Delta} \quad (299)$$

where  $R_{D_i}$  is the contact term of  $\hat{\phi}_{\bar{t}}$  with the  $\phi_i$  insertion, and  $R_{\Delta}$  the contact term contribution that factorize the surface through a node. Let's start by analyzing the  $R_{D_i}$  boundaries:

$$\begin{aligned} \sum_{i \in S} R_{D_i} &= \sum_{i \in S} \langle \hat{\phi}_{\bar{t}} \prod_{j \in S} \phi_j \rangle_{D_i} = \sum_{i \in S} G_{i\bar{i}} \langle \sigma_1 \prod_{j \neq i} \phi_j \rangle = \\ &= \sum_{i \in S} G_{i\bar{i}} (2-2g-s+1) \langle \prod_{j \neq i} \phi_j \rangle \end{aligned} \quad (300)$$

The internal nodes  $\Delta$  are associated to the two types of boundaries of a Riemann surface of genus  $g$  and  $s$  punctures. The first one, we will note it as  $\Delta_1$ , comes from pinching a



handle, leading to a surface of genus  $g-1$ :

$$\langle \hat{\phi}_{\bar{t}} \prod_{i \in S} \phi_i \rangle_{g, \Delta_1} = \frac{1}{2} B'_{\bar{t}}{}^{\alpha\beta} \langle \phi_\alpha \phi_\beta \prod_{i \in S} \phi_i \rangle_{g-1} \quad (301)$$

where the factor  $\frac{1}{2}$  should be added to reflect the equivalency between the order in which the two new insertions  $\phi_\alpha$  are integrated. The factorization tensor  $B'$  satisfies the same set of equations (288) and (289) that the tensor  $B$ , thus it is also proportional to the three point correlation function. With an appropriate choice of normalization of the string amplitudes, the proportionality constant between both factorization tensors can be set equal to one [31].

The second ones, noted  $\Delta_2$ , come from the factorization of the surface into two surfaces of genus  $r$  and punctures in the subset  $X$ , and genus  $g-r$  and punctures in  $Y$  respectively:

$$\langle \hat{\phi}_{\bar{t}} \prod_{i \in S} \phi_i \rangle_{g, \Delta_2} = \frac{1}{2} \sum_{r=0}^g \sum_{X \cup Y = S} \bar{C}_{\bar{t}}{}^{\alpha\beta} \langle \phi_\alpha \prod_{j \in X} \phi_j \rangle_r \langle \phi_\beta \prod_{k \in Y} \phi_k \rangle_{g-r} \quad (302)$$

Collecting now equations (300), (301) and (302), we obtain the equation for the  $\bar{t}$ -dependence of any string amplitude:

$$\begin{aligned} \partial_{\bar{t}} \langle \prod_{i \in S} \phi_i \rangle_g &= \frac{1}{2} \bar{C}_{\bar{t}}{}^{\alpha\beta} \langle \phi_\alpha \phi_\beta \prod_{i \in S} \phi_i \rangle_{g-1} + \\ &+ \frac{1}{2} \sum_{r=0}^g \sum_{X \cup Y = S} \bar{C}_{\bar{t}}{}^{\alpha\beta} \langle \phi_\alpha \prod_{j \in X} \phi_j \rangle_r \langle \phi_\beta \prod_{k \in Y} \phi_k \rangle_{g-r} + \\ &+ \sum_{i \in S} G_{i\bar{t}} (2-2g-s+1) \langle \prod_{j \neq i} \phi_j \rangle_g \end{aligned} \quad (303)$$

Notice that in our derivation of the holomorphic anomaly from the contact term algebra we have only considered the contact terms of the antitopological operator  $\hat{\phi}_{\bar{t}}$  with the rest of the operators  $\phi_i$  but not the contact terms among the operators  $\phi_i$  themselves. This is equivalent to define the correlators  $\langle \prod \phi_i \rangle$  by covariant derivatives of the generating functional. There are however some aspects of the previous derivation that should be stressed at this point.

1) The correlators  $\langle \prod \phi_i \rangle$  for topological operators can not be determined by the contact term algebra, by contrast to what happen in topological gravity. In fact from the contact term algebra we can only get relations of the type:

$$(\Gamma_{ij}^k - \Gamma_{ji}^k) \langle \phi_k \prod_{l \in S} \phi_l \rangle = \sum_{l \in S} \mathcal{R}_{D_l} + \sum_{nodes} \mathcal{R}_\Delta \quad (304)$$

which does not imply ( $\Gamma_{ij}^k - \Gamma_{ji}^k = 0$ ) anything on the surface contribution. Moreover they are compatible with making all contact terms  $R_{D_i}$  equal to zero by covariantization.

2) If in the computation of  $\langle \hat{\phi}_{\bar{t}} \prod_{i \in S} \phi_i \rangle$  we take into account all contact terms, i.e. contact terms between the  $\phi_i$  operators, we will find, as a consequence of the derivation rules (259) and the  $(t, \bar{t})$  equations (290), that the holomorphic anomaly is cancelled, reflecting the commutativity of ordinary derivatives  $[\partial_{\bar{t}_i}, \partial_j] = 0$ .

3) We should say that from the contact term algebra we can not prove, at least directly, that the correlators  $\langle \hat{\phi}_{\bar{t}} \prod_{i \in S} \phi_i \rangle$  are saturated by contact terms. The fact we have proved is that the contact term contribution dictated by the contact term algebra (254) (256) is precisely the holomorphic anomaly.

4) The curvature of the initial surface is augmented by two units in both processes of pinching a handle or factorizing the surface. In order to take this into account, the two insertions  $\phi_\alpha, \phi_\beta$  generated in these processes should include, in addition, an extra unit of curvature. Therefore, the total balance of curvature for the new insertions is zero. This can be seen as the reason for the zero contact term between the dilaton field  $\sigma_1$  and the antitopological operators  $\hat{\phi}_{\bar{t}}$  (see equations (268) and (270)).

To finish this section, we will notice an important property of the holomorphic anomaly equation (303). Using the covariant prepotential  $S$ ,  $S^i = G^{\bar{i}i} \partial_{\bar{i}} S$ ,  $S^{ij} = G^{\bar{i}i} \partial_{\bar{i}} S^j$ , introduced in section 1.8, we can integrate (303) [30]. From this we get in particular a Feynman diagram description of part of the boundary contributions to  $C_{i_1 \dots i_s}^g$ . Let us stress the appearance in the Feynman rules of a new field, with the defining properties of the dilaton. This fact has the origin in the pieces depending linearly on the curvature of the Riemann surface in the expression of the holomorphic anomaly (303), or from the point of view of the algebra (254), in the contact term between a topological and a antitopological field (257).

It would be interesting at this point to reinterpret the Feynman propagator  $S^{ij}$  as a regularization of the Kodaira-Spencer propagator  $\frac{\bar{\partial}^{\dagger} \partial}{\Delta}$  and to connect this regularization with an effective and operative way for reproducing, using the cancel propagator argument, the contact term algebra.

## 2.11 Final Comments

In this section we will collect some concrete questions which we believe would be worth to consider in more detail.

- i) A direct derivation, using cancel propagator arguments, of the  $t\bar{t}$ -connection.
- ii) To find a Landau-Ginzburg description of topological matter theories with  $t$  and  $\bar{t}$  couplings different from zero.
- iii) A direct derivation of the renormalization group "  $\beta$ -functions"  $t_i(\beta)$ ,  $\bar{t}_i(\beta)$  for  $\beta$  the world-sheet scale.
- iv) To extend the holomorphic anomaly for correlators involving gravitational descendants and to massive topological field theories.
- v) To find an effective regularization of the "Kodaira-Spencer" propagator  $\frac{\bar{\partial}^i \partial}{\Delta}$  in a way consistent with the holomorphic anomaly.
- vi) Based on the connection between  $t\bar{t}$ -equation and the thermodynamic Bethe ansatz (TBA) [12], namely TBA as integral representation of  $t\bar{t}$ -equations for massive models, to study, from the  $t\bar{t}$ -geometry, the integrability of the corresponding solitonic infrared theory.
- vii) To study in a more systematic way properties of strings defined for a pair  $(Q, b)$  which satisfy Hodge relations, i.e. strings with non-trivial  $b$ -cohomology.

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## References

- [1] E. Witten, *Comm. Math. Phys.* 117 (1988) 353
- [2] E. Witten, *Comm. Math. Phys.* 118 (1988) 411
- [3] L. Alvarez-Gaume, C. Gomez, G. Moore and C. Vafa, *Nucl. Phys.* B303 (1988) 455
- [4] S. Cecotti and C. Vafa, *Nucl. Phys* B367 (1991) 359
- [5] N. Warner, "N = 2 Supersymmetric Integrable Models and Topological Field Theories", lectures given at Trieste Summer School 1992, hep-th 9301088
- [6] T. Eguchi and S. K. Yang, *Mod. Phys. Lett*, A5 (1990) 1693
- [7] G. Felder, *Nucl. Phys.* B317 (1989) 215
- [8] W. Lerche, C. Vafa and N. Warner, *Nucl. Phys.* B324 (1989) 427
- [9] C. Vafa and N. Warner, *Phys. Lett* B218 (1989) 51
- [10] E. J. Martinec, "Criticality, Catastrophes and Compactifications", in *Physics and Mathematics of Strings, Memorial Volume for Vadim Knizhnik*, ed. L. Brink, D. Friedan and A. M. Polyakov, World Scientific, 1990
- [11] V. I. Arnold, *Singularity Theory*, London Mathematical Lecture Notes Series, Vol. 53 (Cambridge U.P., Cambridge, 1981)
- [12] S. Cecotti P. Fendley, K. Intriligator and C. Vafa, *Nucl. Phys* B386 (1992) 405
- [13] R. Dijkgraaf, H. Verlinde and E. Verlinde, *Nucl. Phys.* B352 (1991) 59
- [14] C. Vafa, *Mod. Phys. Lett.* A6 (1990) 337
- [15] B. Dubrovin, *Comm. Math. Phys.* 152 (1993) 539
- [16] I. Krichever, *Soviet Math. Dokl.* 22 (1980) 79
- [17] B. de Witt and A. van Proeyen, *Nucl. Phys.* B245 (1984) 89;  
A. Strominger, *Commun. Math. Phys.* 133 (1990) 163;  
L. Castellani, R. D'Auria and S. Ferrara, *Phys. Lett.* B241 (1990) 57; *Class. Quant. Grav.* 1 (1990) 317; R. D'Auria, S. Ferrara and P. Frè, *Nucl. Phys.* B359 (1991) 705

- [18] A.B. Zamolodchikov, J.E.T.P. Lett. 43 (1986) 730
- [19] E. Witten, Nucl. Phys. B340 (1990) 281
- [20] H. Verlinde and E. Verlinde, Nucl Phys. B348 (1991) 457
- [21] R. Dijkgraaf and E. Witten, Nucl. Phys B342 (1990) 486
- [22] E. Brézin and V. Kazakov, Phys. Lett B236 (1990) 144;  
M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635;  
D. J. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127; Nucl. Phys. B340 (1990) 333
- [23] M. Douglas, Phys. Lett. B238 (1990) 176
- [24] A. Losev, "Descendants Constructed from Matter Field and K. Saito Higher Residue Pairing in Landau-Ginzburg Theories Coupled to Topological Gravity", TPI-MINN-92/40-T; "Structures of K. Saito Theory of Primitive Form in Topological Theories Coupled to Topological Gravity", preprint ITEP April 94
- [25] T. Eguchi, H. Kanno, Y. Yamada and S.K. Yang, Phys. Lett. B305 (1993) 235
- [26] W. Lerche and A. Sevrin, "On the Landau-Ginzburg Realization of Tological Gravities", preprint CERN-TH.7210/94, hep-th 9403183; W. Lerche and N. Warner, "On the Algebraic Structure of Gravitational Descendants in  $CP(n-1)$  Coset Models", preprint CERN-TH.7442/94, hep-th 9409069
- [27] K. Saito, "On the Periods of Primitive Integrals", Harvard Lectures Notes (1980)
- [28] J. Distler and P. Nelson, Comm. Math. Phys. 138 (1991) 273
- [29] C. Gómez and E. López, Phys. Lett B334 (1994) 323
- [30] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer Theory of Gravity, and Exact Results for Quantum String Amplitudes", HUTP93/A025; "Holomorphic Anomaly in Topological Field Theories", HUTP93/A008
- [31] K. Li, Nucl. Phys. B354 (1991) 711; Nucl. Phys. B354 (1991) 725