Drell-Yan Cross Section to Third Order in the Strong Coupling Constant

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We present phenomenological results for the inclusive cross section for the production of a lepton pair via virtual photon exchange at next-to-next-to-next-to-leading order in perturbative QCD. In line with the case of Higgs production, we find that the hadronic cross section receives corrections at the percent level, and the residual dependence on the perturbative scales is reduced. However, unlike in the Higgs case, we observe that the uncertainty band derived from scale variation is no longer contained in the band of the previous order.

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The discovery of the Higgs boson at the Large Hadron Collider (LHC) at CERN [1,2] and the absence of signals of new physics has resulted in a new precision collider program. This program was made possible by parallel advances on both the theoretical and experimental sides. On the experimental side, we have seen the use of advanced experimental techniques and an improved and solid understanding of systematic uncertainties. On the theoretical side, we have lived through a revolution in our ability to perform higher-order computations for collider observables. These two things combined have made the LHC a precision machine, where measurements at the percent level will be achieved routinely and compared to theoretical predictions at a similar level of accuracy.

The Drell-Yan (DY) process is the shining example of the precision phenomenology program at the LHC. This process corresponds to the production of a pair of charged leptons with a fixed invariant mass Q^2 from an off shell photon or Z boson in quantum chromodynamics (QCD). Its clean final-state signature makes it an ideal candidate for luminosity measurements and detector calibration. Moreover, the DY process plays a key role in the measurement of parton distribution functions (PDFs) at the LHC and also in many searches for physics beyond the Standard Model. Its importance for the physics program at the LHC makes the DY process one of the main processes for which very precise theoretical predictions are desirable. In particular, we need to have a clear understanding of its perturbative stability and the size of the theoretical uncertainties.

The LHC being a hadron collider, one expects the most important perturbative corrections to arise due to QCD. The next-to-leading-order (NLO) QCD corrections to lepton-pair production were computed already four decades ago in Refs. [3,4]. The next-to-next-to-leading-order (NNLO) corrections were computed in Refs. [5–11] and were supplemented by electroweak (EW) corrections in Refs. [12,13]. Very recently, also mixed QCD-EW corrections have become available [14,15].

In the remainder of this Letter, we focus on corrections in the strong coupling constant. The known results at NLO and NNLO indicated a good convergence of the perturbative series. In particular, the size of the missing higher-order terms was estimated be at the percent level at NNLO. This estimate was obtained by varying the renormalization and factorization scales by a factor of 2 around the hard scale of the process set by the invariant mass of the lepton pair. Given the phenomenological importance of the DY process for the precision physics program at the LHC, it is important to have reliable predictions for the DY process and the associated theoretical uncertainties. In this Letter, we compute for the first time the next-to-next-to-next-to-leading-order (N^3LO) corrections to lepton-pair production from a virtual photon. While the complete description of the DY process at this order will also require the contribution from the off shell Z boson (as well as the interference between the photon and the Z boson), the contribution from virtual photon production already gives valuable information about the size of the QCD corrections and the convergence of the perturbative series. Specifically, we expect the relative size of perturbative corrections for Z boson and virtual photon exchange to be very similar. We have checked this explicitly at lower orders in perturbation theory. In addition, this is motivated by the fact that the analytic partonic cross sections are largely the same (up to differences related to diagrams with an axial vector current coupling to a single fermion trace starting at NNLO). Furthermore, we found in Ref. [16] that the size of Drell-Yan K factors is independent of the type of the electroweak gauge boson through $\mathrm{N^3LO}$. Moreover, for small values of the invariant mass Q^2, the value of the cross section is dominated by the photon contribution, so that our results will provide reliable estimates of the size of the $\mathrm{N^3LO}$ corrections in that region. In the remainder of this Letter, we review the computation of the $\mathrm{N^3LO}$ corrections to the photon contribution to the DY process and discuss its phenomenological implications.

$\mathrm{N^3LO}$ *corrections to the Drell-Yan process.*—The inclusive cross section for the production of a lepton pair with invariant mass Q^2 can be written as

$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int_0^1 dx_1 dx_2 f_i(x_1, \mu_f) f_j(x_2, \mu_f) \hat{\sigma}_{ij}(z, \mu_r, \mu_f), \tag{1}$$

where the sum runs over all parton flavors, f_i are parton densities, and $\hat{\sigma}_{ij}$ are partonic cross sections. The partonic cross sections depend on the ratio $z = Q^2/s$, where $\sqrt{s}$ is the partonic center-of-mass energy, related to the hadronic center-of-mass energy $\sqrt{S}$ by $s = x_1 x_2 S$ through the two Bjorken momentum fractions $x_{1,2}$. μ_r and μ_f denote the renormalization and factorization scales, respectively. We have computed the partonic cross sections analytically through $\mathrm{N^3LO}$ for all partonic channels. At NLO and NNLO we reproduce the results of Refs. [3–11]. Our computation follows closely the one for the inclusive cross sections for Higgs production in gluon fusion [17–19] and bottom-quark fusion [20]. All relevant Feynman diagrams are generated with QGraf [21] and sorted into scalar integral topologies, which are then reduced to a set of master integrals via integration-by-parts identities [22,23] using an in house code. The master integrals are computed analytically as a function of z using the differential equations method [24–28]. The master integrals contributing to the $\mathrm{N^3LO}$ cross section can be subdivided into several classes. First, there are purely virtual three-loop integrals, which are encoded in the quark form factor up to three loops [29–35]. We have recomputed the purely virtual corrections, and we find perfect agreement with the existing results in the literature. The $\mathrm{N^3LO}$ cross section also receives contributions from partonic subprocesses describing additional final-state radiation. The master integrals describing the emission of a single massless parton at this order in perturbation theory have been computed in Refs. [36–40]. Similarly, the master integrals for double-real virtual and triple-real contributions have been computed in Refs. [17,41–45] as an expansion around the production threshold of the Higgs boson and exactly as a function of z in Ref. [19]. We work exclusively with the master integrals of Ref. [19]. All master integrals have already been evaluated in the context of the $\mathrm{N^3LO}$ corrections to the gluon fusion and bottom-quark fusion cross sections.

The different contributions that we have described are not yet well defined in four space-time dimensions. They are individually ultraviolet (UV) and infrared (IR) divergent, and we regulate both UV and IR using conventional dimensional regularization, i.e., we work in $D = 4 - 2\epsilon$ space-time dimensions. The UV divergences are absorbed by replacing the strong coupling constant by its renormalized value in the modified minimal subtraction $\overline{\mathrm{MS}}$ scheme. The UV counterterm for the strong coupling constant has been computed through five loops in Refs. [46–50]. After UV renormalization, all remaining divergences are of IR origin. They can be absorbed into the definition of the PDFs using mass factorization at $\mathrm{N^3LO}$ [51–53], which involves convoluting lower-order partonic cross sections with the three-loop splitting functions of Refs. [54–56]. All convolutions are computed analytically in z space using the PolyLogTools package [57]. We observe that after UV renormalization and mass factorization, all poles in the dimensional regulator cancel and we obtain finite results for all partonic channels.

In addition to the explicit analytic cancellation of the UV and IR poles, we have performed various checks to establish the correctness of our computation. First, we have reproduced the soft-virtual $\mathrm{N^3LO}$ cross section of Refs. [42,58–61] and the physical kernel constraints of Refs. [62–64] for the next-to-soft term of the quark-initiated cross section. Second, we have checked that our partonic cross sections have the expected behavior in the high-energy limit, which corresponds to $z \to 0$ [65,66]. Finally, we have also checked that all logarithmic terms in the renormalization and factorization scales produced from the cancellation of the UV and IR poles satisfy the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation [67–69].

Phenomenological results.—In this section we present our phenomenological results for lepton-pair production via an off shell photon at $\mathrm{N^3LO}$ in QCD. The strong coupling is $\alpha_s(m_Z^2) = 0.118$, and we evolve it to the renormalization scale μ_r using the four-loop QCD beta function in the $\overline{\mathrm{MS}}$ scheme assuming $N_f = 5$ active, massless quark flavors. In the remainder of this section, we present our results for the cross section as a function of the invariant mass of the lepton pair, and we discuss the sources of uncertainty that affect it.

Table I contains numerical values for the QCD K factor, i.e., the ratio of the $\mathrm{N^3LO}$ cross section over the NNLO cross section. We observe that, for all values of the invariant mass Q considered, the cross section receives negative corrections at the percent level at LHC center-of-mass energies. We include numerical estimates of the size of the three uncertainties discussed. The central values and scale

TABLE I. Numerical predictions for the QCD K factor at N^{3}LO.

Q/GeV	$K_{\rm QCD}^{\rm N^3LO}$	$\delta(\text{scale})(\%)$	$\delta(\text{PDF}+\alpha_S)(\%)$	$\delta(\text{PDF}-\text{TH})(\%)$	$\{[\sigma^{(0)}_{Z+\gamma^*}]/[\sigma^{(0)}_{\gamma^*}]\}$
30	0.952	$^{+1.5}_{-2.5}$	± 4.1	± 2.7	1.01
50	0.966	$^{+1.1}_{-1.6}$	± 3.2	± 2.5	1.09
70	0.973	$^{+0.89}_{-1.1}$	± 2.7	± 2.4	2.16
90	0.978	$^{+0.75}_{-0.89}$	± 2.5	± 2.4	415
110	0.9811	$^{+0.65}_{-0.73}$	± 2.3	± 2.3	7.4
130	0.983	$^{+0.57}_{-0.63}$	± 2.2	± 2.2	3.5
150	0.985	$^{+0.50}_{-0.54}$	± 2.2	± 2.2	2.6

variation bands for the K-factor are obtained with the zeroth member of the `pdf4lhc15_nnlo_mc` set of [70]. We define

$$K_{\rm QCD}^{\rm N^3LO} = \frac{\sigma^{(3)}(\mu_f = \mu_r = Q)}{\sigma^{(2)}(\mu_f = \mu_r = Q)},$$
$$\delta(X) = \frac{\delta_X(\sigma^{(3)})}{\sigma^{(3)}(\mu_f = \mu_r = Q)}, \qquad (2)$$

where $\sigma^{(n)}(\mu_f = \mu_r = Q)$ is the hadronic cross section including perturbative corrections up to nth order evaluated for $\mu_F = \mu_R = Q$, and $\delta_X(\sigma^{(n)})$ is the absolute uncertainty of the cross section from source X as described below. Furthermore, we show in the last column of Table I the ratio of the leading-order cross section to produce a lepton pair via Z boson and virtual photon exchange [71–74] over exclusively virtual photon exchange.

Let us now analyze the two sources of uncertainty related to the PDFs and the dependence of the cross section on the renormalization and factorization scales. Figure 1 displays the impact of our imprecise knowledge of parton distribution functions and the strong coupling constant on our abilities to predict the DY cross section. The PDFs and the strong coupling constant cannot be computed from first principle, but they need to be extracted from measurements. In order to study the PDF uncertainties, we use the Monte Carlo replica method following the PDF4LHC recommendation [70] that uses 100 different PDF sets to compute the 68% confidence level interval. The strong coupling constant uncertainty is computed using two correlated PDF sets provided by Ref. [70] and is then combined in quadrature with the PDF uncertainty to give $\delta(\text{PDF}+\alpha_S)$. The uncertainty obtained in this way does not yet include the fact that currently all PDF sets are extracted by comparing experimental to predictions at (at most) NNLO level, nor do they include the next order in the DGLAP equation. A fully consistent N^{3}LO calculation, however, would require the use of a complete set of N^{3}LO PDFs. We include an uncertainty δ(PDF-TH) reflecting the fact that currently there are no N^{3}LO PDF sets available.. The estimate of this uncertainty was obtained following the recipe introduced in Ref. [18] that uses half the change of the NNLO cross section in changing from NLO to NNLO PDFs as a measure of uncertainty. As shown in Fig. 1, each of the two uncertainties is of the order of $\pm 2\%$ over the whole range of invariant masses considered.

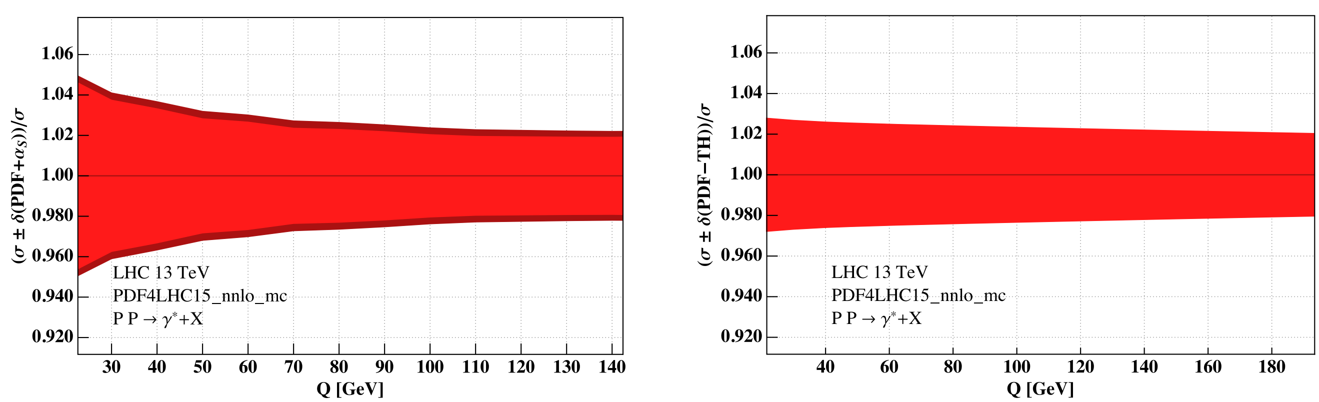

FIG. 1. The light red area in the left plot represents the PDF uncertainty, the dark red area corresponds to the combination in quadrature of PDF + α_s uncertainty. The right plot shows the uncertainty on the cross section due to missing N^{3}LO PDFs.

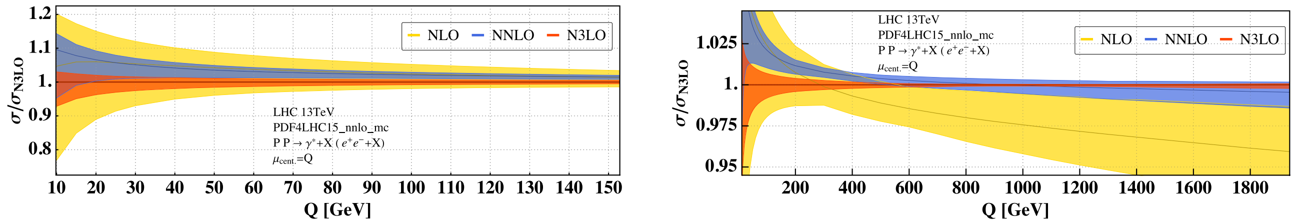

FIG. 2. The cross section as a function of the invariant mass Q^2 of the lepton pair for small (left) and large (right) values of Q.

Figure 2 shows the value of the NLO, NNLO, and N^{3}LO cross sections normalized to the central N^{3}LO value as a function of the invariant mass Q^2 of the lepton pair. The bands indicate the dependence of the cross section at different orders on the choice of the renormalization and factorization scales. We choose Q as a central scale and increase and decrease both scales independently by a factor of 2 with respect to the central scale while maintaining $\frac{1}{2} \leq \mu_R/\mu_F \leq 2$. We observe that at N^{3}LO the cross section depends only very mildly on the choice of the scale. In particular, for small and very large invariant masses, the dependence on the scale is substantially reduced by inclusion of N^{3}LO corrections compared to NNLO. Remarkably, however, we find that, for invariant masses $50 \lesssim Q \lesssim 400$ GeV, the bands obtained by varying the renormalization and factorization scales at NNLO and N^{3}LO do not overlap for the choice of the central scale Q that is conventionally chosen in the literature. This is in stark contrast to the case of the N^{3}LO corrections to the inclusive cross section for Higgs production in gluon and bottom-quark fusion [17,19,20], where the band obtained at N^{3}LO was always strictly contained in the NNLO band (for reasonable choices of the central scales). We note that this behavior does not depend on our choice of the central scale, but we observe the same behavior when the central scale is chosen as $Q/2$. Since this is a new feature that has not been observed so far for inclusive N^{3}LO cross section, we analyze it in some detail.

Figure 3 shows the dependence of the cross section for an invariant mass $Q = 100$ GeV on one scale, with the other held fixed at the central scale $Q = 100$ GeV. The bands are again obtained by varying the scale by a factor of 2 up and down around the central scale. We see that in both cases the NNLO and N^{3}LO bands do not overlap. Furthermore, we see that for the μ_R dependence the width of the band is substantially reduced when going from NNLO to N^{3}LO. For the μ_F dependence, however, the width of the band is increasing from NNLO to N^{3}LO. We note that this statement depends on the choice of the value of Q^2 considered as well as the center-of-mass energy of the hadron collider. It would be interesting in how far this observation is related to the missing N^{3}LO PDFs (keeping in mind that in that case one could not disentangle completely the PDF-TH and scale uncertainties anymore).

Figure 4 shows the relative contribution of the different partonic channels as a function of the invariant mass Q^2 to the N^{3}LO correction of the DY cross section. We see that the cross section is dominated by the $q\bar{q}$, qg, and gg channels. While the qg channel gives a large and positive contribution, the $q\bar{q}$ channel (and to a lesser extent also the gg channel) gives a negative contribution that largely cancels the contribution from the qg channel. The same cancellation happens already in the case of the NNLO corrections to an even larger extent. Given the sizeable cancellation of different partonic initial-state contributions, small numerical changes in the parton distribution functions will have an enhanced effect on the prediction of the DY cross section. Consequently, estimating and improving on the sources of uncertainties related to parton distribution functions considered in Fig. 1 is of great importance.

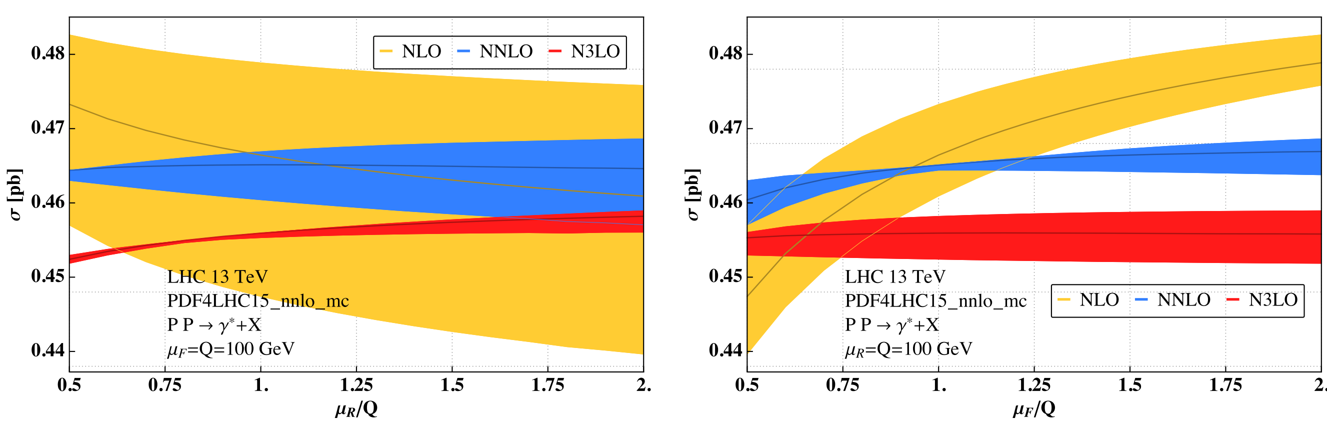

FIG. 3. Dependence of the cross section on either μ_F or μ_R with the other scale held fixed.

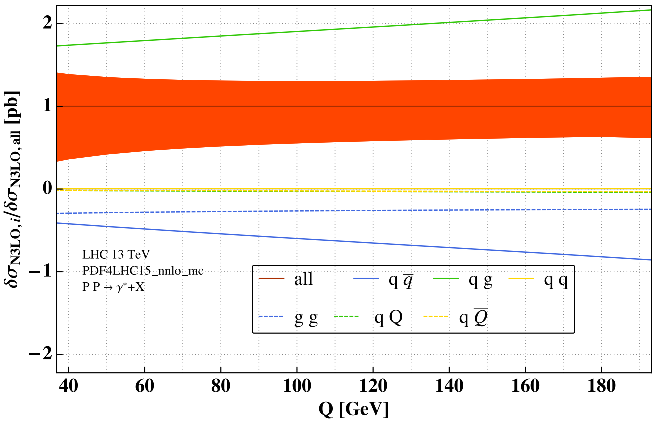

FIG. 4. Contributions of the different partonic channels at N^{3}LO, as a function of the invariant mass Q and normalized to the N^{3}LO cross section. The red band correspond to the variation of the N^{3}LO correction under variation of the factorization and renormalization scale as described in the text.

Conclusions.—We have presented for the first time the complete computation of the N^{3}LO corrections in QCD for the production of a lepton pair from a virtual photon. Our main findings are percent level corrections to the hadronic cross section and an overall reduction of dependence on the perturbative scales. The size of these corrections is consistent with N^{3}LO corrections to Higgs boson production in gluon fusion [17–19] and bottom-quark fusion [20] and indicates the importance of N^{3}LO corrections to LHC processes for phenomenology conducted at the percent level.

In the region of small invariant masses where the contribution from the Z boson is small, $Q \lesssim 50$ GeV, the photon contribution computed here is the dominant part of the cross section. For other kinematic regions, we expect the K factor of the Z boson contribution to behave qualitatively very similarly to the photon contribution, and our results provide essential information. We see from Fig. 2 that our computation substantially reduces the dependence of the cross section on the renormalization and factorization scales. In contrast to the corrections to Higgs boson production, however, the shift of the predicted value of the DY cross section due to the inclusion of N^{3}LO corrections is not contained in the naive scale variation bands of NNLO predictions for all values of Q. We emphasize that this should not be interpreted as an indication of a breakdown of perturbative QCD, but rather as a sign that uncertainty estimates based on a purely conventional variation of the scales should be taken with a grain of salt. Moreover, we observe an intricate pattern of large cancellations of contributions from different partonic initial states at NNLO and N^{3}LO. This implies a large sensitivity of the cross section on relatively small shifts in parton distribution functions. In combination with the fact that the DY process is a key ingredient for the determination of PDFs, this motivates us to push for parton distributions determined from N^{3}LO cross sections in the future. It also hints at an intricate entanglement of PDFs and the structure of QCD cross sections, so that the uncertainty estimate obtained from scale variation cannot be completely disentangled from the PDF-TH uncertainties. The perturbative uncertainty should rather be seen as the combination of the two. Finally, we believe that our findings warrant a critical revision of the strategy to assess perturbative uncertainties and the consequences thereof on PDF determination, etc.

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